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MULTI-BIMODELS

by *Enrico M. VITALE*

RESUME. L'auteur étudie les équivalences entre sous-catégories multi-réflexives de catégories de préfaisceaux covariants. En utilisant une notion convenable de multi-bimodèle, il obtient une généralisation des théorèmes classiques de Eilenberg-Watts et de Morita relatifs aux catégories de modules. L'exemple motivant est donné par les catégories localement multi-présentables, c'est-à-dire les catégories esquissables par des esquisses à limites et coproduits.

Introduction

In [1], Adamek and Borceux have established a very general Morita theory for sketches. Two sketches $\mathcal{S}$ and $\mathcal{T}$ are called Morita-equivalent if their categories $\text{Mod}\mathcal{S}$ and $\text{Mod}\mathcal{T}$ of Set-valued models are equivalent. In [1], Morita-equivalent sketches are classified by means of mutually inverse bimodels, where an $\mathcal{S}$ - $\mathcal{T}$ -bimodel is a model of $\mathcal{S}$ in a certain subcategory $\hat{\mathcal{T}}$ of the functor category $[\text{Mod}\mathcal{T}, \text{Set}]$. In [1] a great attention is devoted to (connected)-limit-coproduct sketches, since in this case the category $\hat{\mathcal{T}}$ admits a more explicit description : it is equivalent to the dual of the product-completion $\prod(\text{Mod}\mathcal{T})$ of $\text{Mod}\mathcal{T}$.

The particular case of Morita-equivalent limit sketches was firstly studied in [4] following a different approach. In [4] a Morita theorem is established using *only* the fact that for a limit sketch $\mathcal{S}$, the category $\text{Mod}\mathcal{S}$ is reflective in the functor category $\text{Set}^{\mathbb{S}}$ (where $\mathbb{S}$ is the small category underlying the sketch $\mathcal{S}$).

The aim of this note is to improve the method used in [4] to recapture the case of limit-coproduct sketches, because for such a sketch $\mathcal{S}$, the category $\text{Mod}\mathcal{S}$ is multi-reflective in $\text{Set}^{\mathcal{S}}$. Even if we do not rise the level of generality of [1], the advantage of this method is that we obtain not only a Morita theorem (corollary 2.8 below), but also a theorem which is the direct generalisation of the Eilenberg-Watts theorem characterizing colimit-preserving functors between module categories. Moreover, since our definition of multi-bimodel is at a non-doctrinaire level, techniques are quite different from those used in [1].

Another approach to Morita theory for sketches, based on the so-called generic model of a sketch, is contained in [6].

To support intuition, we recall here the classical Morita theory (all details can be found in [3]). Let A and B be two unital rings, and $A\text{-mod}$ and $B\text{-mod}$ the corresponding categories of left modules. Any A - B -bimodul M induces a pair of adjoint functors

$$M \otimes_B -: B\text{-mod} \longrightarrow A\text{-mod} \quad \text{Lin}_A(M, -): A\text{-mod} \longrightarrow B\text{-mod}$$

with $M \otimes_B -$ left adjoint to $\text{Lin}_A(M, -)$. The Eilenberg-Watts theorem states that any colimit-preserving functor $F: B\text{-mod} \longrightarrow A\text{-mod}$ is isomorphic to one of the form $M \otimes_B -$ for a suitable bimodule M . As a consequence, the categories $A\text{-mod}$ and $B\text{-mod}$ are equivalent iff there exist a A - B -bimodule M and a B - A -bimodule N such that $M \otimes_B N$ is isomorphic to A and $N \otimes_A M$ is isomorphic to B .

1 Notations

For a category $\mathcal{A}$, we denote

$$\eta_{\mathcal{A}}: \mathcal{A} \longrightarrow \prod(\mathcal{A})$$

its product-completion. If $\mathcal{B}$ is a category with products and $F: \mathcal{A} \longrightarrow \mathcal{B}$ is an arbitrary functor, we write $F^*: \prod(\mathcal{A}) \longrightarrow \mathcal{B}$ for the $\prod$ -extension (product-preserving extension) of F ; it is the essentially unique product-preserving functor making commutative the following diagram

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{\eta_{\mathcal{A}}} & \Pi(\mathcal{A}) \\
 & \searrow F & \swarrow F^* \\
 & \mathcal{B} &
 \end{array}$$

(when we say that a diagram of functors is commutative, we mean commutative up to isomorphisms). Given a functor $G: \mathcal{A} \longrightarrow \mathcal{B}$, we write $\Pi(G): \Pi(\mathcal{A}) \longrightarrow \Pi(\mathcal{B})$ for the Π -extension of $G \cdot \eta_{\mathcal{B}}: \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \Pi(\mathcal{B})$.

A functor $F: \mathcal{A} \longrightarrow \Pi(\mathcal{B})$ is also called a multi-functor $F: \mathcal{A} \mapsto \mathcal{B}$. The composition of two multi-functors $F: \mathcal{A} \mapsto \mathcal{B}$ and $G: \mathcal{B} \mapsto \mathcal{C}$ is given by $F \cdot G^*: \mathcal{A} \longrightarrow \Pi(\mathcal{B}) \longrightarrow \Pi(\mathcal{C})$. Up to isomorphisms, this composition is associative and the unit $\eta_{\mathcal{A}}: \mathcal{A} \longrightarrow \Pi(\mathcal{A})$ acts as identity. For more details on multi-functors and multi-adjoints the reader can see [2] and [5].

In what follows $\mathbb{T}, \mathbb{S}, \dots$ are small categories. Given a small category $\mathbb{T}$, $\text{Mod}\mathcal{T}$ is a chosen multi-reflective subcategory of the functor category $\text{Set}^{\mathbb{T}}$, $i_{\mathbb{T}}: \text{Mod}\mathcal{T} \longrightarrow \text{Set}^{\mathbb{T}}$ is the full inclusion and $R_{\mathbb{T}}: \text{Set}^{\mathbb{T}} \mapsto \text{Mod}\mathcal{T}$ its left multi-adjoint. If $\varphi: \mathbb{T}^{op} \longrightarrow \Pi(\text{Mod}\mathcal{T})$ is a functor, in the following diagram

$$\begin{array}{ccc}
 \mathbb{T}^{op} & \xrightarrow{Y_{\mathbb{T}}} & \text{Set}^{\mathbb{T}} \\
 & \searrow \varphi & \swarrow \hat{\varphi} \\
 & \Pi(\text{Set}^{\mathbb{S}}) &
 \end{array}$$

$\hat{\varphi}$ is the left Kan-extension of φ along the Yoneda embedding $Y_{\mathbb{T}}$, and $\text{Hom}(\varphi, -): \Pi(\text{Set}^{\mathbb{S}}) \longrightarrow \text{Set}^{\mathbb{T}}$ is the right adjoint of $\hat{\varphi}$. If X is an object of $\Pi(\text{Set}^{\mathbb{S}})$ and T is an object of $\mathbb{T}$, $\text{Hom}(\varphi, -)(X)(T)$ is given by the hom-set $\text{Hom}[\varphi(T), X]$. Note that the Kan-extension $\hat{\varphi}$ exists because $\Pi(\text{Set}^{\mathbb{S}})$ is cocomplete.

(We will usually omit subscripts in $\eta_{\mathcal{A}}, i_{\mathbb{T}}, R_{\mathbb{T}}$ and $Y_{\mathbb{T}}.$)

2 Multi-bimodels

Definition 2.1 Let $M: \mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ be a multi-functor. We say that M is a multi-bimodel if the functor

$$\text{Hom}(M, -): \text{Mod}\mathcal{S} \longrightarrow \text{Set}^{\mathbb{T}}$$

given by $\text{Hom}(M, -)(G)(T) = \text{Hom}[M(T), \eta(G)]$ for G in $\text{Mod}\mathcal{S}$ and T in $\mathbb{T}$, factors through the full inclusion $i: \text{Mod}\mathcal{T} \longrightarrow \text{Set}^{\mathbb{T}}$.

In other words, consider the composite functor

$$\varphi_M: \mathbb{T}^{op} \xrightarrow{M} \prod(\text{Mod}\mathcal{S}) \xrightarrow{\prod(i)} \prod(\text{Set}^{\mathcal{S}}) \quad ;$$

we say that M is a multi-bimodel if the functor $\text{Hom}(\varphi_M, -)$ factors as in the following diagram

$$\begin{array}{ccc} \text{Set}^{\mathbb{T}} & \xleftarrow{i} & \text{Mod}\mathcal{T} \\ \uparrow \text{Hom}(\varphi_M, -) & & \uparrow \\ \prod(\text{Set}^{\mathcal{S}}) & \xleftarrow{\prod(i)} \prod(\text{Mod}\mathcal{S}) \xleftarrow{\eta} & \text{Mod}\mathcal{S} \end{array}$$

We call $\text{Lin}(M, -): \text{Mod}\mathcal{S} \longrightarrow \text{Mod}\mathcal{T}$ the requested factorization. Note that if it exists, it is essentially unique. The key property of a multi-bimodel is the following one.

Proposition 2.2 With the previous notations, consider the composite functor

$$M \otimes -: \text{Mod}\mathcal{T} \xrightarrow{i} \text{Set}^{\mathbb{T}} \xrightarrow{\hat{\varphi}_M} \prod(\text{Set}^{\mathcal{S}}) \xrightarrow{R^*} \prod(\text{Mod}\mathcal{S}) \quad ;$$

if M is a multi-bimodel, then $M \otimes -$ is left multi-adjoint to $\text{Lin}(M, -)$.

Proof: Consider the unit $\eta: \text{Mod}\mathcal{S} \longrightarrow \prod(\text{Mod}\mathcal{S})$, an object G in $\text{Mod}\mathcal{S}$ and an object F in $\text{Mod}\mathcal{T}$. The proof easily reduces to the following natural bijections :

$$\begin{aligned} M \otimes F &= R^*(\varphi_M(i(F))) \longrightarrow \eta(G) \quad \text{iff} \\ \varphi_M(i(F)) &\longrightarrow \prod(i)(\eta(G)) \quad \text{iff} \\ i(F) &\longrightarrow \text{Hom}[\varphi_M, \prod(i)(\eta(G))] = i(\text{Lin}[M, G]) \quad \text{iff} \\ F &\longrightarrow \text{Lin}[M, G] \end{aligned}$$

■

Remark : The previous proposition can equivalently stated saying that the $\prod$ -extension $M^* \otimes -: \prod(\text{Mod}\mathcal{T}) \longrightarrow \prod(\text{Mod}\mathcal{S})$ of $M \otimes -$ is left adjoint to $\prod(\text{Lin}(M, -))$. In fact, the following general fact can be proved. Consider two functors $G: \mathcal{B} \longrightarrow \mathcal{A}$ and $F: \mathcal{A} \longrightarrow \prod(\mathcal{B})$, and the $\prod$ -extension $F^*: \prod(\mathcal{A}) \longrightarrow \prod(\mathcal{B})$; F is a left multi-adjoint of G iff F^* is a left adjoint of $\prod(G)$.

We need another preliminary fact on multi-bimodels.

Proposition 2.3 *Let $M: \mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ be a multi-bimodel ; then*

(i) *the following diagram is commutative*

$$\begin{array}{ccccc} \mathbb{T}^{op} & \xrightarrow{Y} & \text{Set}^{\mathbb{T}} & \xrightarrow{R} & \prod(\text{Mod}\mathcal{T}) \\ & \searrow M & & \nearrow M^* \otimes - & \\ & & \prod(\text{Mod}\mathcal{S}) & & \end{array}$$

- (ii) $M^* \otimes -$ preserves colimits and products ;
- (iii) $M^* \otimes -$ is the unique (up to isomorphisms) functor which satisfies the two previous conditions.

To prove this proposition, we need an easy lemma.

Lemma 2.4

- 1) consider two functors $G: \Pi(\mathcal{B}) \longrightarrow \mathcal{A}$ and $F: \mathcal{A} \longrightarrow \Pi(\mathcal{B})$, the Π -extension F^* and the composite $G \cdot \eta: \Pi(\mathcal{B}) \longrightarrow \mathcal{A} \longrightarrow \Pi(\mathcal{A})$; if F is a left adjoint of G , then F^* is a left adjoint of $G \cdot \eta$;
- 2) the unit $\eta: \mathcal{A} \longrightarrow \Pi(\mathcal{A})$ preserves all colimits which turn out to exist in $\mathcal{A}$;
- 3) if $I: \mathcal{A} \longrightarrow \mathcal{B}$ is a full and faithful functor, then also $\Pi(I)$ is full and faithful ;
- 4) if $I: \mathcal{A} \longrightarrow \mathcal{B}$ is full and faithful and has a left multi-adjoint $R: \mathcal{B} \mapsto \mathcal{A}$, then $I \cdot R \simeq \eta: \mathcal{A} \longrightarrow \Pi(\mathcal{A})$.

Proof of proposition 2.3: (i) : observe that the following diagram is commutative

$$\begin{array}{ccc}
 \Pi(\text{Set}^{\mathcal{T}}) & \xleftarrow{\Pi(i)} & \Pi(\text{Mod}\mathcal{T}) \\
 \text{Hom}(\varphi_M, -) \cdot \eta \uparrow & & \uparrow \Pi(\text{Lin}(M, -)) \\
 \Pi(\text{Set}^{\mathcal{S}}) & \xleftarrow{\Pi(i)} & \Pi(\text{Mod}\mathcal{S})
 \end{array}$$

Since all the functors involved preserve products (three of them by definition, and $\text{Hom}(\varphi_M, -) \cdot \eta$ by lemma 2.4), this commutativity can be checked precomposing with the unit $\eta: \text{Mod}\mathcal{S} \longrightarrow \Pi(\text{Mod}\mathcal{S})$.

Passing to left adjoints, we obtain the commutativity of the following diagram (use lemma 2.4 and proposition 2.2)

$$\begin{array}{ccc}
 \Pi(\text{Set}^{\mathcal{T}}) & \xrightarrow{R^*} & \Pi(\text{Mod}\mathcal{T}) \\
 \varphi_M^* \downarrow & & \downarrow M^* \otimes - \\
 \Pi(\text{Set}^{\mathcal{S}}) & \xrightarrow{R^*} & \Pi(\text{Mod}\mathcal{S})
 \end{array}$$

and, precomposing with the unit $\eta: \text{Set}^{\mathbb{T}} \longrightarrow \Pi(\text{Set}^{\mathbb{T}})$, we have the commutativity of the following diagram

$$\begin{array}{ccc} \text{Set}^{\mathbb{T}} & \xrightarrow{R} & \Pi(\text{Mod}\mathcal{T}) \\ \varphi_M \downarrow & & \downarrow M^* \otimes - \\ \Pi(\text{Set}^{\mathbb{S}}) & \xrightarrow{R^*} & \Pi(\text{Mod}\mathcal{S}) \end{array}$$

Finally, precomposing with the Yoneda embedding $Y: \mathbb{T}^{op} \longrightarrow \text{Set}^{\mathbb{T}}$, we obtain the requested commutativity. In fact $Y \cdot \hat{\varphi}_M \simeq \varphi_M$ because Y is full and faithful, $\varphi_M = M \cdot \Pi(i)$ by definition of φ_M , and $\Pi(i) \cdot R^* \simeq \text{id}$ because $\Pi(i)$ is a full and faithful right adjoint of R^* .

(ii) : $M^* \otimes -$ preserves products by definition and colimits because, by proposition 2.2, it has a right adjoint.

(iii) : let $G: \Pi(\text{Mod}\mathcal{T}) \longrightarrow \Pi(\text{Mod}\mathcal{S})$ be a functor which preserves colimits and products and such that $Y \cdot R \cdot G$ is isomorphic to M . We have $Y \cdot R \cdot G \simeq Y \cdot R \cdot (M^* \otimes -)$, but R preserves colimits (because it factors as $R = \eta \cdot R^*: \text{Set}^{\mathbb{T}} \longrightarrow \Pi(\text{Set}^{\mathbb{T}}) \longrightarrow \Pi(\text{Mod}\mathcal{T})$, R^* preserves colimits because it is a left adjoint, and η preserves colimits by lemma 2.4) and Y is dense, so that we can deduce $R \cdot G \simeq R \cdot (M^* \otimes -)$. (Here we have used that $\Pi(\text{Mod}\mathcal{S})$ is cocomplete, which is the case because it is reflective in the cocomplete category $\Pi(\text{Set}^{\mathbb{S}})$.) This implies $i \cdot R \cdot G \simeq i \cdot R \cdot (M^* \otimes -)$, that is $\eta \cdot G \simeq \eta \cdot (M^* \otimes -)$ (lemma 2.4). Since both G and $M^* \otimes -$ preserve products, this implies that G and $M^* \otimes -$ are isomorphic. ■

Now we can give two basic examples of multi-bimodels.

Proposition 2.5

1) consider the composite

$$M = Y \cdot R : \mathbb{T}^{op} \longrightarrow \text{Set}^{\mathbb{T}} \longrightarrow \Pi(\text{Mod}\mathcal{T}) ;$$

M is a multi-bimodel and $M^* \otimes -$ is isomorphic to the identity functor on $\Pi(\text{Mod}\mathcal{T})$;

2) let $M: \mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ be a multi-bimodel; consider two functors $\alpha: \text{Mod}\mathcal{S} \longrightarrow \prod(\text{Mod}\mathcal{R})$ and $\beta: \text{Mod}\mathcal{R} \longrightarrow \text{Mod}\mathcal{S}$, with α left multi-adjoint to β ; the composite

$$N = M \cdot \alpha^*: \mathbb{T}^{op} \longrightarrow \prod(\text{Mod}\mathcal{S}) \longrightarrow \prod(\text{Mod}\mathcal{R})$$

is a multi-bimodel and $N^* \otimes -$ is isomorphic to $(M^* \otimes -) \cdot \alpha^*$.

Proof: 1) : we will prove that the diagram

$$\begin{array}{ccc} \text{Set}^{\mathbb{T}} & \xleftarrow{i} & \text{Mod}\mathcal{T} \\ \uparrow \text{Hom}(\varphi_M, -) & & \uparrow \text{id} \\ \prod(\text{Set}^{\mathbb{T}}) & \xleftarrow[\prod(i)]{} \prod(\text{Mod}\mathcal{T}) \xleftarrow{\eta} & \text{Mod}\mathcal{T} \end{array}$$

is commutative. This means that $\text{Lin}(M, -)$ is the identity functor and then $M^* \otimes -$, being left adjoint to $\prod(\text{Lin}(M, -))$, is isomorphic to the identity functor. Let F be an object of $\text{Mod}\mathcal{T}$ and T an object of $\mathbb{T}$:
 $\text{Hom}[\varphi_M(T), \prod(i)(\eta(F))] = \text{Hom}[\prod(i)(R(Y(T))), \prod(i)(\eta(F))] =$
 $= \text{Hom}[R(Y(T)), \eta(F)] = \text{Hom}[Y(T), i(F)] = (iF)(T).$

2) : we will prove that the diagram

$$\begin{array}{ccc} \text{Set}^{\mathbb{T}} & \xleftarrow{i} & \text{Mod}\mathcal{T} \\ \uparrow \text{Hom}(\varphi_N, -) & & \uparrow \text{Lin}(M, -) \\ & & \text{Mod}\mathcal{S} \\ & & \uparrow \beta \\ \prod(\text{Set}^{\mathbb{R}}) & \xleftarrow[\prod(i)]{} \prod(\text{Mod}\mathcal{R}) \xleftarrow{\eta} & \text{Mod}\mathcal{R} \end{array}$$

is commutative. This means that $\text{Lin}(N, -) = \beta \cdot \text{Lin}(M, -)$ and then $N^* \otimes -$, being left adjoint to $\prod(\text{Lin}(N, -))$, is isomorphic to $(M^* \otimes -) \cdot \alpha^*$. Let F be an object of $\text{Mod}\mathcal{R}$ and T an object of $\mathbb{T}$:

$$\begin{aligned} \text{Hom}[\varphi_N(T), \prod(i)(\eta(F))] &= \text{Hom}[\prod(i)(\alpha^*(M(T))), \prod(i)(\eta(F))] = \\ &= \text{Hom}[\alpha^*(M(T)), \eta(F)] = \text{Hom}[M(T), \prod(\beta)(\eta(F))] = \\ &= \text{Hom}[M(T), \eta(\beta(F))] = \text{Hom}[\prod(i)(M(T)), \prod(i)(\eta(\beta(F)))] = \\ &= \text{Hom}[\varphi_M(T), \prod(i)(\eta(\beta(F)))] = \text{Lin}[M(T), \beta(F)]. \end{aligned} \quad \blacksquare$$

We are ready to construct the composition of multi-bimodels.

Proposition 2.6 *Let $M: \mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ and $N: \mathbb{S}^{op} \mapsto \text{Mod}\mathcal{R}$ be two multi-bimodels; the composite*

$$M \cdot (N^* \otimes -): \mathbb{T}^{op} \longrightarrow \prod(\text{Mod}\mathcal{S}) \longrightarrow \prod(\text{Mod}\mathcal{R})$$

is a multi-bimodel. We call this multi-bimodel the composition $M \otimes N$ of M and N . Composition of multi-bimodels is associative and the multi-bimodel

$$Y \cdot R: \mathbb{T}^{op} \longrightarrow \text{Set}^{\mathbb{T}} \longrightarrow \prod(\text{Mod}\mathcal{T})$$

acts as identity (all up to isomorphisms).

Proof: Since $N^* \otimes -$ is left adjoint to $\prod(\text{Lin}(N, -))$, we can use the second part of proposition 2.5 and we have that $P = M \cdot (N^* \otimes -)$ is a multi-bimodel and $P^* \otimes - \simeq (M^* \otimes -) \cdot (N^* \otimes -)$. The rest of the statement easily follows from proposition 2.3. $\blacksquare$

The announced generalizations of the Eilenberg-Watts theorem and of the Morita theorem are now two simple corollaries of the previous analysis.

Corollary 2.7 *There is a bijection between isomorphism classes of multi-bimodels*

$$\mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$$

and isomorphism classes of left multi-adjoints

$$\text{Mod}\mathcal{T} \mapsto \text{Mod}\mathcal{S}.$$

This bijection preserves composition and identities.

Proof: Given a multi-bimodel $M: \mathbb{T}^{op} \longrightarrow \prod(\text{Mod}\mathcal{S})$ we obtain the functor $M \otimes -: \text{Mod}\mathcal{T} \longrightarrow \prod(\text{Mod}\mathcal{S})$ left multi-adjoint to the functor $\text{Lin}(M, -): \text{Mod}\mathcal{S} \longrightarrow \text{Mod}\mathcal{T}$.

Conversely, given a left multi-adjoint $\alpha: \text{Mod}\mathcal{T} \longrightarrow \prod(\text{Mod}\mathcal{S})$, then the composite

$$Y \cdot R \cdot \alpha^*: \mathbb{T}^{op} \longrightarrow \text{Set}^{\mathbb{T}} \longrightarrow \prod(\text{Mod}\mathcal{T}) \longrightarrow \prod(\text{Mod}\mathcal{S})$$

is a multi-bimodel. The rest of the statement easily follows from propositions 2.3 and 2.5. $\blacksquare$

Corollary 2.8 *The categories $\text{Mod}\mathcal{T}$ and $\text{Mod}\mathcal{S}$ are equivalent if and only if there exist two multi-bimodels $M: \mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ and $N: \mathbb{S}^{op} \mapsto \text{Mod}\mathcal{T}$ such that $M \otimes N \simeq Y_{\mathbb{T}} \cdot R_{\mathbb{T}}$ and $N \otimes M \simeq Y_{\mathbb{S}} \cdot R_{\mathbb{S}}$.*

Proof: It follows from the previous corollary using the following general fact : two categories $\mathcal{A}$ and $\mathcal{B}$ are equivalent iff their product-completions $\prod(\mathcal{A})$ and $\prod(\mathcal{B})$ are equivalent. $\blacksquare$

3 Comparison with related results

I - We want to compare the classification given in corollary 2.8 with that given in theorem 5.6 of [4]. In [4] a bimodel is a functor $M: \mathbb{T}^{op} \longrightarrow \text{Mod}\mathcal{S}$ such that the functor $\text{Hom}(M, -): \text{Mod}\mathcal{S} \longrightarrow \text{Set}^{\mathbb{T}}$, given by $\text{Hom}(M, -)(G)(T) = \text{Hom}[M(T), G]$ for G in $\text{Mod}\mathcal{S}$ and T in $\mathbb{T}$, factors through the full inclusion $i: \text{Mod}\mathcal{T} \longrightarrow \text{Set}^{\mathbb{T}}$.

We will use the following fact.

Lemma 3.1 *Given two functors $i: \mathcal{A} \longrightarrow \mathcal{B}$ and $r: \mathcal{B} \longrightarrow \mathcal{A}$, r is a left adjoint of i iff $r \cdot \eta: \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \prod(\mathcal{A})$ is a left multi-adjoint of i .*

When the chosen subcategory $\text{Mod}\mathcal{T}$ is reflective in $\text{Set}^{\mathbb{T}}$, and not only multi-reflective, we obtain theorem 5.6 of [4] from corollary 2.8 via the following proposition.

Proposition 3.2

- 1) *there is a bijection between bimodels $\mathbb{T}^{op} \longrightarrow \text{Mod}\mathcal{S}$ in the sense of definition 4.1 in [4] and multi-bimodels $\mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ which factor through the unit $\eta: \text{Mod}\mathcal{S} \longrightarrow \Pi(\text{Mod}\mathcal{S})$;*
- 2) *if $\text{Mod}\mathcal{T}, \text{Mod}\mathcal{S} \dots$ are reflective in the appropriate categories of functors, then the previous bijection preserves composition and identities.*

Proof: 1) : it is easy to check that, given the composite functor $M \cdot \eta: \mathbb{T}^{op} \longrightarrow \text{Mod}\mathcal{S} \longrightarrow \Pi(\text{Mod}\mathcal{S})$, M is a bimodel iff $M \cdot \eta$ is a multi-bimodel (in both directions one uses that the unit η is full and faithful).
 2) : let $r: \text{Set}^{\mathbb{T}} \longrightarrow \text{Mod}\mathcal{T}$ the reflector. The previous lemma says that the identity bimodel $Y \cdot r: \mathbb{T}^{op} \longrightarrow \text{Set}^{\mathbb{T}} \longrightarrow \text{Mod}\mathcal{T}$ corresponds to the identity multi-bimodel $Y \cdot R: \mathbb{T}^{op} \longrightarrow \text{Set}^{\mathbb{T}} \longrightarrow \Pi(\text{Mod}\mathcal{T})$.

The key to prove that the bijection of point 1) also preserves composition is to observe that the following diagram is commutative (where $N = M \cdot \eta$ is the multi-bimodel corresponding to a bimodel M and $M \otimes -$ is the left Kan-extension of M along $Y \cdot r$)

$$\begin{array}{ccc}
 \text{Mod}\mathcal{T} & \xrightarrow{\eta} & \Pi(\text{Mod}\mathcal{T}) \\
 M \otimes - \downarrow & & \downarrow N^* \otimes - \\
 \text{Mod}\mathcal{S} & \xrightarrow{\eta} & \Pi(\text{Mod}\mathcal{S})
 \end{array}$$

that is $N^* \otimes - = \Pi(M \otimes -)$. ■

II (Adamek, Borceux) - Let $\mathcal{T}$ and $\mathcal{S}$ be two limit sketches, $\mathbb{T}$ the small category underlying the sketch $\mathcal{T}$ and $\text{Mod}\mathcal{S}$ the usual category of Set-valued model of $\mathcal{S}$. A functor $\mathbb{T}^{op} \longrightarrow \text{Mod}\mathcal{S}$ is a bimodel in the sense of definition 4.1 in [4] iff it is a $\mathcal{T}$ -model in $(\text{Mod}\mathcal{S})^{op}$ (cf. section 7 in [4]). Categories sketchable by limit sketches are exactly locally presentable categories.

More in general, locally multipresentable categories are exactly categories sketchable by limit-coproduct sketches. The category of Set-valued models of such a sketch is multi-reflective in the category of Set-valued functors defined on the small category underlying the sketch.

To end this note, we show that our notion of multi-bimodel specializes to the notion of bimodel used in [1] to classify Morita-equivalences between limit-coproduct sketches. If $\mathcal{T}$ and $\mathcal{S}$ are two such sketches, in [1] a $\mathcal{T}$ - $\mathcal{S}$ -bimodel is a $\mathcal{T}$ -model in $(\Pi(\text{Mod}\mathcal{S}))^{op}$, where $\text{Mod}\mathcal{S}$ is the category of Set-valued models of $\mathcal{S}$.

Proposition 3.3 *Let $\mathcal{T}$ and $\mathcal{S}$ be two limit-coproduct sketches. A functor $\mathbb{T}^{op} \longrightarrow \Pi(\text{Mod}\mathcal{S})$ is a multi-bimodel iff it is a $\mathcal{T}$ -model in $(\Pi(\text{Mod}\mathcal{S}))^{op}$.*

Proof: The proof is a “ Π -fication” of the proof of the first proposition in section 7 of [4], making use of the following general fact : let A be an object of a category $\mathcal{A}$ and consider the unit $\eta: \mathcal{A} \longrightarrow \Pi(\mathcal{A})$; the hom-functor $\text{Hom}(-, \eta(A)): \Pi(\mathcal{A})^{op} \longrightarrow \text{Set}$ preserves coproducts. ■

Example 3.4

Let $\mathcal{T}$ be the sketch over the following poset $\mathbb{T}$:

$$x_1 \longrightarrow y \longleftarrow x_2$$

with no cones and with the discrete cocone over $\{x_1, x_2\}$. Then $\text{Mod}\mathcal{T} \simeq \text{Set} \times \text{Set}$, so $\mathcal{T}$ is Morita-equivalent to the sketch $\mathcal{S}$ with two objects, no nonidentity maps, no cones and no cocones.

The multi-bimodel $M: \mathbb{T}^{op} \mapsto \text{Mod}\mathcal{S}$ inducing the equivalence $\text{Mod}\mathcal{T} \simeq \text{Mod}\mathcal{S}$ is given by $M(x_1) = \eta((*, \emptyset))$, $M(x_2) = \eta((\emptyset, *))$ and $M(y) = M(x_1) \times M(x_2)$. This equivalence can not be induced by a $\mathcal{T}$ -model in $(\text{Mod}\mathcal{S})^{op}$.

References

- [1] J. ADAMEK, F. BORCEUX : *Morita equivalence of sketches*, Mathematik-Arbeitspapiere Universität Bremen (1997).
- [2] J. ADAMEK, J. ROSICKY : *Locally presentable and accessible categories*, Cambridge University Press (1994).
- [3] H. BASS : *Algebraic K-Theory*, W.A. Benjamin Inc. (1968).

- [4] F. BORCEUX, E.M. VITALE : *On the notion of bimodel for functorial semantics*, Applied Categorical Structures 2 (1994), pp. 283-295.
- [5] Y. DIERS : *Catégories localisables*, thesis, Paris VI (1977).
- [6] CH. LAIR : *Catégories qualifiables et catégories esquissables*, Diagrammes 17 (1987), 153 p.

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