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# WEAK SUBOBJECTS AND WEAK LIMITS IN CATEGORIES AND HOMOTOPY CATEGORIES

*by Marco GRANDIS*

**Résumé**. Dans une catégorie donnée, un *sousobjet faible*, ou *variation*, d'un objet A est défini comme une classe d'équivalence de morphismes à valeurs dans A, de façon à étendre la notion usuelle de sousobjet. Les sousobjets faibles sont liés aux limites faibles, comme les sousobjets aux limites; et ils peuvent être considérés comme remplaçant les sousobjets dans les catégories "à limites faibles", notamment la catégorie d'homotopie HoTop des espaces topologiques, où il forment un treillis de *types de fibration* sur l'espace donné. La classification des variations des groupes et des groupes abéliens est un outil important pour déterminer ces types de fibration, par les foncteurs d'homotopie et homologie.

## Introduction

We introduce here the notion of weak subobject in a category, as an extension of the notion of subobject. A *weak subobject*, or *variation*, of an object A is an equivalence-class of morphisms with values in A, where $x \sim_A y$ if there exist maps u, v such that x = yu, y = xv; among them, the *monic variations* (having some representative which is so) can be identified to subobjects. As a motivation for the name, a morphism $x: X \to A$ is commonly viewed in category theory as a *variable element* of A, parametrised over X (e.g., see Barr - Wells [1], 1.4). The dual notion is called a *covariation*, or *weak quotient*, of A.

We claim that variations are important in homotopy categories, where they are well linked to weak limits, much in the same way as, in "ordinary" categories, subobjects are linked to limits. Nevertheless, the study of weak subobjects in *ordinary* categories, like abelian groups or groups, is interesting in itself and relevant to classify variations in homotopy categories of spaces, by means of homology and homotopy functors.

In fact, subobjects, defined as equivalence classes of monics $x: \bullet \rightarrowtail A$ (under the same equivalence relation as above), are well related with limits: the inverse image $f^*(y)$ along $f: A \to B$ of a subobject $y: \bullet \rightarrowtail B$, given by the pullback of $y$ along $f$, is precisely determined as a subobject of $A$ (and only depends on $y$ as a subobject of $B$). Thus, subobjects are important in "ordinary" categories, where limits exist, and the common abuse of denoting a subobject by any of its representatives does not lead to errors.

Consider now the *homotopy category* Ho**Top** = **Top**/$\simeq$ of topological spaces modulo homotopy (equivalent to the category of fractions of **Top** which inverts homotopy equivalences). Ho**Top** lacks ordinary pullbacks, but does have *weak pullbacks*, just satisfying the existence part of the usual universal property: they are provided by *homotopy pullbacks* in **Top**. Of course, a weak pullback $(P, h_1, h_2)$ of two morphisms $(f_1, f_2)$ having the same codomain is just determined up to a pair of maps $u, v$ consistent with $(h_1, h_2)$

(1) $u : P \rightleftarrows Q : v \qquad\qquad h_i.vu = h_i.$

Now, mimicking in Ho**Top** (or in any category with weak pullbacks) our previous argument on subobjects: "the" inverse image along $f: A \to B$ of a variation $y: \bullet \to B$, given by the weak pullback of $y$ along $f$, is determined as a variation of $A$, and can still be written $f^*(y)$, up to a similar abuse of denoting a variation by any of its representative. It is also of interest to note that each variation of a space $A$ in Ho**Top** *can be represented by a fibration*, forming a (possibly large) lattice Fib(A) of *types of fibrations* over $A$ (2.1).

In a second paper, whose main results are sketched in 1.8, it will be shown that *variations in* **A** *can be identified to* (distinguished) *subobjects in* Fr**A**, the *free category with epi-monic factorisation system* over **A**, extending the Freyd embedding of the stable homotopy category of spaces in an abelian category [11].

Classifying variations seems often to be a difficult task. After some trivial cases, where they reduce to subobjects, our main results here cover some classes of finitely generated abelian groups; a complete classification within this category should be attainable. Such results are the obvious tool to separate homotopy variations of spaces, via homology and homotopy functors, and are a first step in the study of the problem for CW-spaces; this is only achieved here in a very particular case, for a cluster of circles (3.4). As partial results, it may be interesting to note that there is a sequence of homotopy variations of the 2-sphere $\mathbf{S}^2$, defined over the torus, which is *collapsed by all* $\pi_i$ *and separated by* $H_2$; and a sequence $\mathbf{S}^3 \to \mathbf{S}^2$ generated by the Hopf fibration, which is collapsed by homology, but separated by $\pi_3$, and can not be realised over closed surfaces (2.4). Of course, the choice of the ground-category is crucial to obtain good classifications; e.g., *finitely generated* abelian variations or covariations always yield *countable* lattices, whereas any prime order group **Z**/p has *at least a continuum* of abelian variations and a *proper class* of abelian covariations (1.5-6). Further study should show which restriction on spaces are more productive (see 2.2).

Concerning literature, *normal* or *regular* variations have appeared in Eckmann-Hilton [9] and Freyd [13-14], under the equivalent form of "principal right ideals" of maps, to deal with weak kernels or weak equalisers. Recently, Lawvere [21] has considered a "proof-theoretic power set $\mathcal{P}_{\mathbf{C}}(A)$", defined as the "poset-reflection of the slice category **C**/A", which amounts to the present ordered set $\mathrm{Var}_{\mathbf{C}}(A)$ of weak subobjects. The Freyd embedding for stable homotopy [10-12] and his results on the non-concreteness of homotopy categories [13-14] are also relevant for the present study. Weak limits have been recently used in Carboni-Grandis [6] and Carboni-Vitale [7]. For homotopy pullbacks and homotopy limits, see Mather [23], Bousfield-Kan [5], Vogt [24]. A different approach to "subobjects" in homotopy categories is given by Kieboom [18]. The author gratefully acknowledges helpful remarks from F.W. Lawvere and P. Freyd.

*Outline.* The first section deals with generalities; some classifications of variations and covariations are given for sets, pointed sets, vector spaces (1.4), abelian groups (1.5-6) and groups (1.7). Homotopy variations are introduced in Section 2, and studied for the circle, the sphere and the projective plane. In Section 3 we deal in general with various transformations of weak subobjects, under direct and inverse images, or adjunction, or product-decompositions. These tools are used to classify the homotopy variations of a cluster of circles (3.4) and, in Section 4, to give further classifications of finitely generated abelian variations, in particular for all cyclic groups (4.3); some open problems are listed in 4.6.

## 1. Variations and covariations in a category

**A** is a category. A *variation* in **A** is an extension of the notion of subobject.

**1.1. Definition.** *For an object* A *of* **A**, *a* variation, *or* weak subobject, *of* A *will denote a class of morphisms with values in* A, equivalent over A *with respect to mutual factorisation*

(1) $x \sim_A y$ *iff* *there exist* u, v *such that* $x = yu$, $y = xv$

$$(2) \qquad \begin{array}{ccc} X & \xrightarrow{x} & A \\ {\scriptstyle u}\downarrow\uparrow{\scriptstyle v} & & \| \\ Y & \xrightarrow{y} & A \end{array}$$

In other words, x and y generate the same principal right ideal of maps with values in A (or, also, are connected by morphisms $x \to y \to x$ in the slice-category **A**/A of *objects over* A). By a standard abuse of notation, as for subobjects, a variation $\xi = [x]_A$ will be generally denoted by any of its representatives x. Note that the *domain* of the variation $x: X \to A$ is only determined up to a pair of maps $u: X \to Y$, $v: Y \to X$ such that $x.vu = x$ (x *sees* its domain as a retract of Y; and symmetrically, $x' = xv: Y \to A$ sees its domain as a retract of X).

The (possibly large) set of variations of A is written Var(A). It is ordered by the relation $x \leq y$, meaning that x factors through y (independently of representatives). The *identity variation* $1_A$ is the maximum of Var(A). If two variations x, y of A have a weak pullback, then its diagonal is a representative of their meet $x \wedge y$. However, a weak pullback is more than really needed; e.g., the trivial meet $x = x \wedge x$ always exists, whereas a weak pullback of (x, x) need not; further, if it exists, its diagonal is certainly equivalent to x as a weak subobject, but need not be linked to it by an iso (see 2.3.2). If **A** has an initial object, Var(A) has also a minimum $0_A: \bot \to A$; similarly, if **A** has finite (or small) sums, then Var(A) has finite (or small) joins, computed in the obvious way (weak sums are sufficient)

(3) $\vee(x_i: X_i \to A) = x: \Sigma_i X_i \to A.$

A variation x will be said to be *epi* if it has a representative which is so in **A**, or equivalently if all of them are so. Such a variation need not be the identity: $x \sim_A 1_A$ iff $x: X \to A$ is a retraction (a split epi); a split epi onto A should thus be viewed as giving the same *information with values in* A as $1_A$, with redundant duplication (see also 2.3). The object A has only one epi variation (namely, $1_A$) iff it is projective in **A**; more generally, an epimorphism $p: P \twoheadrightarrow A$ defined over a projective object provides the *least epi variation* of A.

Dually, a *covariation*, or *weak quotient*, $[x]^A$ of A is a class of morphisms starting from A, *equivalent under* A ($x \sim^A y$ iff there exist u, v such that $x = uy$, $y = vx$). The identity covariation is the maximum of Cov(A), written $1^A$; its representatives coincide with the split monics $A \rightarrowtail \bullet$. If **A** has terminal object, Cov(A) has also a minimum $0^A: A \to \top$. Weak pushouts give meets of covariations, weak products give joins, $\vee(x_i: A \to X_i) = x: A \to \Pi_i X_i$. A *monic* *covariation* is represented by monomorphisms. The zero-object 0, when it exists, has only one variation (any $X \to 0$ is a split epi) and only one covariation.

Plainly, the sets Var(A) and Cov(A) are *small* for every locally small category having a small set of isomorphism types of objects. This condition is not necessary, by far (see 1.4), but provides various concrete examples: given a concrete category $U: \mathbf{A} \to \mathbf{Set}$, where U is faithful and has *small fibres* (every small set has a small set of **A**-objects over it), take the full subcategory of **A** whose objects have underlying set bounded by a given cardinal.

**1.2. Variations and subobjects.** Two *monics* x, y in A are the same variation iff they are equivalent in the usual sense, i.e., define the same subobject of A (in 1.1.2, u and v are reciprocal isomorphisms, uniquely determined). Thus, the ordered set Sub(A) of subobjects of A can be embedded in Var(A), and we may define a variation $x: X \to A$ to be a *subobject* if it has some monic representative $m: M \to A$ (all the other representatives are then given by the split extensions of M, and *include* all monics equivalent to m).

If **A** has unique epi-monic factorisations, every weak subobject x has a well-defined *image*, or carrier im(x), namely the subobject represented by the image of any representative of x, and Sub(A) is a retract of Var(A). Then, x is an epi variation iff im(x) = 1. Given a subobject $m: M \to X$, the variations x of X whose image is m are in bijective correspondence with the epi variations y of M, via $y \mapsto my$. Thus, in a category with unique epi-monic factorisation, the variations of A are determined by *subobjects of* A together with *epi variations of the latter*; and a non-split epi over A provides a variation which no monic can give.

**1.3. Variations and functors.** Let **A** be a *full* subcategory of **B** containing A. Two **A**-morphisms $x: X \to A$, $y: Y \to A$ are equivalent in **A** iff they are so in **B**, and we shall identify each **A**-variation with the **B**-variation containing it, writing $\mathrm{Var}_{\mathbf{A}}(A) \subset \mathrm{Var}_{\mathbf{B}}(A)$ (the embedding reflects the order); of course, A may admit other **B**-variations, not representable in **A** (see 1.7). If **B** has weak pullbacks and weak finite sums and **A** is closed in **B** with respect to (some representatives of) them, then $\mathrm{Var}_{\mathbf{A}}(A)$ is a sublattice of $\mathrm{Var}_{\mathbf{B}}(A)$.

More generally, every functor $U: \mathbf{A} \to \mathbf{B}$ induces two monotonic mappings

(1) $\mathrm{Var}(U): \mathrm{Var}_{\mathbf{A}}(A) \to \mathrm{Var}_{\mathbf{B}}(UA)$, $\quad\quad$ $\mathrm{Cov}(U): \mathrm{Cov}_{\mathbf{A}}(A) \to \mathrm{Cov}_{\mathbf{B}}(UA)$

which reflect the order (and are injective) whenever U is full and faithful; they are surjective if U is full and representative (essentially surjective on objects). Var(U) is also surjective for a category of fractions $U: \mathbf{A} \to S^{-1}\mathbf{A}$ admitting a right calculus [15], since then every $S^{-1}\mathbf{A}$-variation $y = as^{-1}$ has a representative in $\mathbf{A}$, namely a. Both mappings (1) will often be written as U.

Now, before progressing with the abstract theory, we give some examples where weak subobjects and weak quotients can be easily classified.

**1.4. Trivial examples.** We noticed that, in a category with unique epi-monic factorisations, it suffices to consider epi variations of subobjects and monic covariations of quotients (1.2).

In **Set**, every epi splits, by the axiom of choice, and the unique epi variation of a set A is the identity: *variations and subobjects coincide*. As to covariations, note that a monic $x: A \to X$ splits except for $A = \emptyset \neq X$. Thus, the covariations of a non-empty set coincide with its quotients, whereas $\emptyset$ has *two* covariations, the identity and $0^{\emptyset}: \emptyset \to \{*\}$, which is smaller and is not a quotient.

Similarly, in any category with epi-monic factorisations where all epis split (i.e., every object is projective), weak subobjects and subobjects coincide. This property, and its dual as well, hold in the category $\mathbf{Set}^{\top}$ of pointed sets, or in any category of vector spaces (over a fixed field), or also in a category of relations Rel(**A**) over a well-powered abelian category. In all these cases, the sets Var(A) and Cov(A) are always small. In the latter, the subobjects (and quotients) of an object A can be identified to the *subquotients* of A with respect to **A**.

**1.5. Abelian variations.** Consider now the category **Ab** of abelian groups. (Similar arguments can be developed for modules over any principal ideal domain.) We discuss here the variations of free abelian groups and **Z**/2 (any prime-order group behaves as the latter). More general results are given in Section 4.

We write $\mathbf{Ab}_{fg}$ the (abelian) full subcategory of finitely generated objects A (fg-abelian groups, for short) and $Var_{fg}(A)$ the subset of *fg-variations* of A, having representatives in $\mathbf{Ab}_{fg}$; we already know it is a sublattice of the lattice Var(A) of all *abelian variations* of A (1.3). The structure theorem of fg-abelian groups proves that $Var_{fg}(A)$ is always small, actually *countable*: indeed, there are countably many isomorphism types of fg-abelian groups X, and each of them provides countably many homomorphisms $x: X \to A$.

To study $Var_{fg}(A)$, it suffices to consider epi variations of subobjects. And we shall repeatedly use the decomposition 1.1.3 of the variation x

(1) $x = \vee x|X_i, \qquad X = \oplus X_i$

given by a (finite) decomposition of its domain X in indecomposable cyclic groups, infinite or primary.

First, the abelian variations of a *free* abelian group F coincide with its subobjects (and are all finitely generated when F is so). Indeed, if $x: X \to F$ is surjective, then it splits and $x \sim 1$; otherwise, apply the same argument to Im(x), which is also free. In particular, the lattice $\mathrm{Var}(\mathbf{Z}) = \mathrm{Sub}(\mathbf{Z})$ is distributive and noetherian (every ascending chain stabilises); the variations of $\mathbf{Z}$, coinciding with its subobjects, can also be represented by its "positive" endomorphisms

(2) $x_n: \mathbf{Z} \to \mathbf{Z}, \qquad x_n(a) = n.a \quad (n \geq 0)$

$x_m \leq x_n$ iff $m\mathbf{Z} \subset n\mathbf{Z}$, iff n divides m.

The two-element group $A = \mathbf{Z}/2$ has two subobjects, 0 and 1. There is precisely one non-surjective variation, the subobject 0 (provided by the unique variation of the zero-object). But there are infinitely many fg-variations of A; in fact, consider the natural morphism $x_n$ (sending $\bar{1}$ to $\bar{1}$, surjective for $n > 0$)

(3) $x_n: \mathbf{Z}/2^n \to \mathbf{Z}/2 \qquad n \in [0, \infty]$

$1 = x_1 > x_2 > x_3 > \ldots > x_\infty > x_0 = 0$

(including the natural projection $x_\infty: \mathbf{Z} \to \mathbf{Z}/2$, by setting $2^\infty\mathbf{Z} = 0$). These weak subobjects form a totally ordered set V, anti-isomorphic to the ordinal $\omega+2$ (thus, every subset of V has a maximum). The inequalities above are strict, because (for $n \in \mathbf{N}$) any morphism $u: \mathbf{Z}/2^n \to \mathbf{Z}/2^{n+1}$ takes the generator to an element 2a, which is killed by $x_{n+1}$; and the unique morphism $\mathbf{Z}/2^n \to \mathbf{Z}$ is null.

There are no other fg-variations; indeed, any $x: X \to \mathbf{Z}/2$ can be decomposed as above, $x = \vee x|X_i$, and the restriction $x|X_i$ is the null variation unless the order of $X_i$ is infinite or a power of 2; thus, all $x|X_i$ fall in our previous set and have a maximum there. Our lattice is again distributive and noetherian.

On the other hand, $\mathbf{Z}/2$ has also *non-finitely generated* variations, and at least a continuum of them. For any *multiplicative* part M of the ring $\mathbf{Z}$ (a multiplicative submonoid containing $-1$), consisting of *odd* numbers, consider the (surjective) variation $y_M$ defined over the subring $M^{-1}\mathbf{Z} = \{h/m \mid h \in \mathbf{Z}, m \in M\}$ of the rational numbers (the ring of fractions of $\mathbf{Z}$ inverting all $m \in M$)

(4) $y_M: M^{-1}\mathbf{Z} \to \mathbf{Z}/2, \qquad y_M(h/m) = \bar{h}$

$x_\infty \leq y_M \leq x_n \qquad \mathbf{Z} \to M^{-1}\mathbf{Z} \to \mathbf{Z}/2^n \qquad (n < \infty)$

(where $M^{-1}\mathbf{Z} \to \mathbf{Z}/2^n$ is induced by the natural projection $\mathbf{Z} \to \mathbf{Z}/2^n$).

Then, $M = \{1, -1\}$ gives $y_M = x_\infty$. Excluding this case, $y_M$ is not finitely generated: it does not precede $x_\infty$ (any homomorphism $M^{-1}\mathbf{Z} \to \mathbf{Z}$ is null, since its image is divisible by any $m \in M$) nor follow any $x_n$ with $n < \infty$ (any $\mathbf{Z}/2^n \to M^{-1}\mathbf{Z}$ is null, since its image is a torsion group). All the variations $y_M$ are distinct: $y_M \leq y_{M'}$ iff $M \subset M'$. As any subset of the set of (positive) odd prime numbers spans a distinct $M$, our variations form a non-noetherian, complete lattice having the cardinal of the continuum; its maximum is the $2\mathbf{Z}$-localisation of $\mathbf{Z}$.

**1.6. Abelian covariations.** Weak quotients in **Ab** can be studied, in part, in a dual way. For instance, the abelian covariations of a divisible abelian group, like **Q**, **Q/Z**, **R**, **R/Z**, coincide with its quotients (divisible abelian groups are the injective objects of **Ab**, closed under quotients).

Again, the lattice $\mathrm{Cov}_{fg}(A)$ of fg-covariations of an fg-abelian group is countable. And every weak quotient $x: A \to X$ can be decomposed as a join

(1) $x = \vee (pr_i.x: A \to X_i), \qquad X = \oplus X_i$

of covariations with values in an indecomposable cyclic group. If $A$ is torsion, we may clearly omit the free component in $\oplus X_i$ and assume that also $X$ is torsion.

The fg-covariations of the prime-order group $\mathbf{Z}/p$ form a totally ordered set anti-isomorphic to the ordinal $\omega+1$; all of them are monic, except for $x_0: \mathbf{Z}/p \to 0$

(2) $x_n: \mathbf{Z}/p \rightarrowtail \mathbf{Z}/p^n, \qquad x_n(\overline{1}) = \overline{p}^{n-1} \qquad (n \in [1, \infty[)$

$1 = x_1 > x_2 > x_3 > \ldots > x_0 = 0$

(compare with the list of variations in 1.5.3, anti-isomorphic to $\omega+2$: here, we have no contribution from **Z**).

But $\mathbf{Z}/p$ has also non-finitely generated covariations, for instance $y: \mathbf{Z}/p \rightarrowtail \mathbf{Q}/\mathbf{Z}$, $y(\overline{1}) = [1/p]$, and actually *a proper class of abelian covariations*. Indeed, as shown by Freyd ([14], p. 30), one can construct a family of p-primary torsion abelian groups $(G_\alpha)$, indexed over the ordinals, each with a distinguished element $\lambda_\alpha \in G_\alpha$, $\lambda_\alpha \neq 0$, $p\lambda_\alpha = 0$ which is annihilated by any homomorphism $G_\alpha \to G_\beta$, for $\alpha > \beta$. Therefore, all the covariations $y_\alpha: \mathbf{Z}/p \to G_\alpha$, $\overline{1} \mapsto \lambda_\alpha$ are distinct.

The fg-covariations of **Z** can be viewed as monic covariations of its quotients. One should thus classify first the monic fg-covariations of all cyclic groups.

**1.7. Group variations.** Also in the category **Gp** of groups, *the weak subobjects of a free group coincide with its subobjects*, by the Nielsen-Schreier theorem: any subgroup of a free group is free. But things are more complicated than in the abelian case, because a subgroup of a free group of finite rank may have countable rank (see Kurosh [20], § 36).

Consider the full subcategory $\mathbf{Gp}_{fg}$ of *finitely generated* groups (fg-groups). The set $Var_{fg}(G) \subset Var(G)$ of fg-variations (having representatives in $\mathbf{Gp}_{fg}$) of an fg-group G has *at most the cardinal number of a continuum*. In fact, the free group on n generators is countable and the set of its quotients is at most a continuum, whence the same holds for the set of isomorphism types of fg-groups (this is *indeed* a continuum, by a theorem of B.H. Neumann [20], § 38); and each type X yields countably many homomorphisms $X \to G$. For a finite group G, it may be useful to consider the $\wedge$-semilattice $Var_f(G) \subset Var_{fg}(G)$ of *finite* variations; this is countable because the set of isomorphism types of finite groups is so, by the classical Cayley theorem (finite groups can be embedded in symmetric groups).

Consider now the full embedding $\mathbf{Ab} \subset \mathbf{Gp}$. For an abelian group A, the set of abelian variations Var(A) is embedded in the set $Var_{\mathbf{Gp}}(A)$ of its *group-variations*. Every group-variation $y: G \to A$ has an obvious *abelian closure* ${}^{ab}y$: $ab(G) \to A$, the least abelian variation of A following y; y itself is abelian iff it is equivalent to ${}^{ab}y$. $Var(A) \subset Var_{\mathbf{Gp}}(A)$ is thus a retract (with the induced order).

The group-variations of $\mathbf{Z}$, which is also free as a group, coincide with its subobjects and are all abelian. Instead, $\mathbf{Z}/2$ has also non-abelian fg-variations. For instance, if $G = \pi_1(\mathbf{K})$ is the fundamental group of the Klein bottle, generated by two elements a, b under the relation a+b = b–a, consider the following homomorphism y and its abelianised variation ${}^{ab}y$, equivalent to 1

(1) $y: G \to \mathbf{Z}/2, \qquad y(a) = 1, \qquad y(b) = 0$

$\quad {}^{ab}y = pr_1: \mathbf{Z}/2 \oplus \mathbf{Z} \to \mathbf{Z}/2$

but $y \not\sim 1$, since G has no element of order two, actually no torsion at all (it is isomorphic to the semidirect product $\mathbf{Z} \rtimes \mathbf{Z}$, with action $k*h = (-1)^k.h$).

A sequence of *finite non-abelian* group variations of $\mathbf{Z}/2$ can be easily constructed, over the semidirect product $G_n = \mathbf{Z}/2^n \rtimes \mathbf{Z}/4$ ($n \in [2, \infty]$), given by a similar (well defined!) action of $\mathbf{Z}/4$ over $\mathbf{Z}/2^n$: $\lambda*a = (-1)^\lambda.a$. On the other hand, any variation of an abelian group $y: S_n \to A$ defined over a symmetric group is abelian, because $ab(S_n) = \mathbf{Z}/2$ for $n \geq 2$ ([20], p. 102), so that $S_n \twoheadrightarrow ab(S_n)$ is a split epi.

**1.8. Formal aspects.** We end this section with a sketch of some general results, relevant for a better understanding of weak subobjects, which will be proved and developed elsewhere.

Any category $\mathbf{A}$ has an obvious embedding in its category of morphisms $\mathbf{A}^2$.

The latter is equipped with a factorisation system (E, M), which decomposes the map $f = (f', f''): x \to y$ as $f = (f', 1).(1, f'')$, through the object $\overline{f} = f''x = yf'$ (the diagonal of the square f)

$$
(1) \qquad \begin{array}{ccccc}
X' & = & X' & \xrightarrow{f'} & Y' \\
{\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle y} \\
X'' & \xrightarrow{f''} & Y'' & = & Y''
\end{array}
$$

making $\mathbf{A}^2$ into the *free category with factorisation system*, or *factorisation completion* of **A**.

Consider now the quotient $\mathrm{Fr}\mathbf{A} = \mathbf{A}^2/\mathrm{R}$, introduced by Freyd to embed the stable homotopy category of spaces into an abelian category [11]: two parallel morphisms $f, g: x \to y$ of $\mathbf{A}^2$ are R-equivalent whenever their diagonals, $\overline{f}$ and $\overline{g}$, coincide. One proves the following facts. The factorisation system of $\mathbf{A}^2$ induces an *epi-monic* system over FrA, which becomes the *free category with epi-monic factorisation system*, or *epi-monic completion* of **A**. The *distinguished subobjects* of A in FrA coincide with the *weak subobjects* of A in **A**. If **A** has products and weak equalisers (as Ho**Top** and various other homotopy categories), FrA is complete (and dually). Also the links between factorisation systems and pseudo-algebras for the 2-monad $\mathbf{A} \mapsto \mathbf{A}^2$, studied in Coppey [8] and Korostenski-Tholen [19], can be translated for *epi-monic* factorisation systems and the induced 2-monad $\mathbf{A} \mapsto \mathrm{Fr}\mathbf{A}$.

## 2. Variations for spaces

Homotopy variations, in **Top**/$\simeq$ and $\mathbf{Top}^{\mathrm{T}}/\simeq$, are considered. A deep study is beyond the purpose of this paper; we just aim to show the interplay of homology and homotopy functors in separating homotopy variations.

**2.1. Homotopy variations.** Consider a category **A** equipped with a congruence $f \simeq g$ (an equivalence relation between parallel morphisms, consistent with composition), which may be viewed as a sort of homotopy relation, since our main examples are of this type. The quotient category $\mathbf{A}/\simeq$ has the same objects and equivalence classes $[f]: A \to B$ of morphisms of **A** as arrows. A $\simeq$*-equivalence* $f: A \to B$ is a morphism whose induced class [f] is iso: there is some $g: B \to A$

such that $gf \simeq 1$, $fg \simeq 1$.

A $\simeq$*-variation* of A in **A** is just a variation in the quotient category $\mathbf{A}/\simeq$. But it will be simpler to take its representatives in **A**, as morphisms $x: \bullet \to A$ modulo the equivalence relation

(1) $x \simeq_A y$ iff there exist u, v such that $x \simeq yu$, $y \simeq xv$.

We have thus the ordered set $\mathrm{Var}_\simeq(A)$. As a quotient of Var(A), it can be more manageable. Further, it is $\simeq$-invariant: each $\simeq$-equivalence $f: A \to B$ induces an isomorphism of ordered sets $\mathrm{Var}_\simeq(A) \to \mathrm{Var}_\simeq(B)$.

Thus, for a space X, we consider first of all the ordered set $\mathrm{Var}_\simeq(X)$ of its *homotopy variations*, in the *homotopy category* Ho**Top** = **Top**/$\simeq$ of topological spaces modulo homotopy (which can be equivalently realised as the category of fractions of **Top** which inverts homotopy equivalences [15]).

This ordered set is a lattice. In fact, **Top** has sums, consistent with homotopies, and homotopy pullbacks, which implies that the quotient **Top**/$\simeq$ has sums and weak pullbacks. Moreover, each homotopy variation *can be represented by a fibration*, because every map in **Top** factors through a homotopy equivalence followed by a fibration; it is thus more evocative to think of $\mathrm{Var}_\simeq(X)$ as the lattice Fib(X) of *types of fibrations* over X. Dual facts hold for the lattice $\mathrm{Cov}_\simeq(X)$ = Cof(X) of homotopy covariations, or *types of cofibrations* starting from X. Every homology functor $H_n$: **Top** $\to$ **Ab** can be used to represent homotopy variations as abelian variations and, in particular, distinguish them. But Fib(X) can be large. In fact, Freyd [13] proves that Ho**Top** is not concrete showing that a space may have a proper class of *regular variations* (called "generalised regular subobjects"); of course, a regular variation of X is a weak equaliser of some pair f, g: $X \to Y$.

All this can be repeated or adapted for pointed spaces; and much of this can be adapted to various other "categories with homotopies" (see [16-17] and references therein), as chain complexes, diagrams of spaces, spaces under (or over) a space, topological monoids, etc.

**2.2. CW-spaces.** It is important to restrict the class of spaces we are considering, to obtain more homogeneous sets of variations, which one might hopefully classify. The comprehensive investigation of homotopy types in Baues [3] is of help in choosing relevant full subcategories of $\mathbf{Top}^\mathsf{T}/\simeq$ and **Top**/$\simeq$; we follow his notation for such subcategories, with some adaptation.

A standard object of study in Algebraic Topology is the category **CW** of CW-spaces X, i.e. pointed spaces having the homotopy type of a *connected* CWcomplex, with pointed maps; the variations of X in $\mathbf{CW}/\simeq$ will be called *cw-variations*. Because of the Cellular Approximation Theorem, $\mathbf{CW}/\simeq$ is equivalent to pointed connected CW-complexes, with homotopy classes of pointed cellular maps (G.W. Whitehead [25], II.4.5). A crucial J.H.C. Whitehead theorem says that *here* a map $f: X \to Y$ which is a weak homotopy equivalence ($\pi_i(f)$ iso, for all i) is also a homotopy equivalence ([25], V.3.5).

$\mathrm{Var_{cw}}(X)$ *is a sublattice of the lattice* $\mathrm{Var}_{\simeq}(X)$ of all homotopy variations of X (1.3), because **CW** is closed in $\mathbf{Top}^{\mathrm{T}}$ under finite sums and homotopy pullbacks ([9], thm. 2.13). But again, $\mathrm{Var_{cw}}(X)$ may be large, as its subset of *normal variations* (weak kernels) [14].

More particularly, n-*type variations* ($n \geq 1$) are given by the full subcategory **n-type** of $\mathbf{Top}^{\mathrm{T}}/\simeq$ consisting of CW-spaces X with $\pi_i(X) = 0$ for $i > n$. For $n = 1$, the fundamental group gives an equivalence of categories ([3], 2.5)

(1) $\pi_1$: **1-type** $\to$ **Gp**

so that *the 1-type variations of* X *are classified by the group variations of* $\pi_1 X$.

On the other hand, in connection with homology functors, one can consider the full subcategory $\mathbf{M}^n$ of $\mathbf{Top}/\simeq$ consisting of the *Moore spaces* of degree $n \geq 1$, i.e. CW-spaces X with reduced homology $\tilde{H}_i(X) = 0$ for $i \neq n$ ([3], § 1.3); X is said to be a Moore space of type (A, n) if $H_n(X) \cong A$; for $n \geq 2$, these conditions determine the homotopy type of X, written M(A, n) ([3], Lemma 1.3.1). Again for $n \geq 2$, the homology functor $H_n: \mathbf{M}^n \to \mathbf{Ab}$ is representative and full, but not faithful ([3], same lemma), and induces surjections

(2) $H_n: \mathrm{Var}_{\mathbf{M}^n}(X) \twoheadrightarrow \mathrm{Var}_{\mathbf{Ab}}(H_n X)$;

the suspension identifies $\mathrm{Var}_{\mathbf{M}^n}(X)$ with $\mathrm{Var}_{\mathbf{M}^{n+1}}(\Sigma X)$, consistently with $H_n$.

Some cw-variations of $\mathbf{S}^1$, $\mathbf{S}^2$, $\mathbf{P}^2$ are described below; further results will be given in 3.3-4.

**2.3. The circle.** The group variations of **Z**, coinciding with its abelian variations $x_n: \mathbf{Z} \to \mathbf{Z}$ (1.5.2; 1.7), have corresponding cw-variations of the pointed circle $\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$

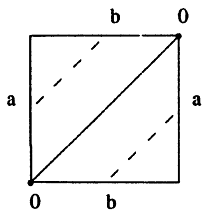
(1) $y_n: \mathbf{S}^1 \to \mathbf{S}^1$, $\quad y_n[\lambda] = [n\lambda] \quad (n \geq 0)$

$\pi_1(y_n) = x_n$, $\quad y_m \leq y_n$ iff n divides m

which realise them through $\pi_1$. This shows that the order relations considered above, which obviously hold, are the only possible ones.

This sequence classifies the 1-type variations of the circle (by 2.2.1); but we have actually got *all its cw-variations* (3.4). For $n > 0$, $y_n$ is the covering map of $\mathbf{S}^1$ of degree n; the universal covering map $p: \mathbf{R} \to \mathbf{S}^1$ corresponds to the weak subobject $y_0$, also represented by $\{*\} \to \mathbf{S}^1$.

It is also interesting to note that the homotopy pullback of two "standard" representatives $y_n$ need not give a standard representative of their meet, even up to homotopy, but may contain "redundant duplication of information". For instance, the (ordinary) pullback of the fibration $y_2: \mathbf{S}^1 \to \mathbf{S}^1$ with itself, homotopy equivalent to the standard homotopy pullback, is the subspace $\{[\lambda, \mu] \mid [2\lambda] = [2\mu]\}$ of the torus $\mathbf{S}^1 \times \mathbf{S}^1 = \mathbf{R}^2/\mathbf{Z}^2$

(2) 


consisting of the union of two disjoint circles, the "solid" and the "dotted" one. The diagonal of the pullback can thus be described as $y = y_2.pr_1: \mathbf{S}^1 \times \mathbf{S}^0 \to \mathbf{S}^1$ and determines the same variation $y_2 = y_2 \wedge y_2$; but the domain of y is not homotopy equivalent to $\mathbf{S}^1$. (The variation $y_2$ amounts to an information, *turn twice*, which is not modified by having two copies of it.)

**2.4. The sphere.** The suspension of the homotopy variations of the circle yields a sequence of cw-variations of the sphere, the standard mappings of degree $n \geq 0$

(1) $s_n: \mathbf{S}^2 \to \mathbf{S}^2$, $\qquad s_m \leq s_n$ iff $n \mid m$

realising the abelian variations $x_n: \mathbf{Z} \to \mathbf{Z}$ via $H_2$ and $\pi_2$. There are no other homotopy variations of $\mathbf{S}^2$ defined over itself, since any endomap of $\mathbf{S}^2$ is homotopic to $s_n$ or $s_{-n} = s_n.s_{-1}$. But it is easy to construct other cw-variations of $\mathbf{S}^2$ defined over compact manifolds.

Consider first the map $f: \mathbf{T}^2 \to \mathbf{S}^2$ defined over the torus $\mathbf{T}^2 = \mathbf{I}^2/R$, by collapsing the boundary of the standard square $\mathbf{I}^2$ to a point. $H_2(f)$ is an isomorphism (use the projection $\mathbf{I}^2 \to \mathbf{T}^2$ as a cubical generator of $H_2(\mathbf{T}^2)$), but f is trivial on all homotopy groups: $\pi_1(\mathbf{S}^2) = 0$, while $\mathbf{T}^2$ is aspherical: $\pi_i(\mathbf{T}^2) = \pi_i(\mathbf{S}^1 \times \mathbf{S}^1) = 0$ for $i \neq 1$. We obtain thus a new sequence of variations $s_n f: \mathbf{T}^2 \to \mathbf{S}^2$, realising the abelian variations $x_n: \mathbf{Z} \to \mathbf{Z}$ via $H_2$: *this sequence is separated* *by* $H_2$ *and collapsed by all* $\pi_i$, whence all its terms are distinct variations, different from the previous ones for $n > 0$.

The map $g: \mathbf{P}^2 \to \mathbf{S}^2$, defined over the real projective plane in a similar way, yields a new homotopy variation. In fact, this map is *trivial in reduced homology* (obviously) *as well as on homotopy groups*: $\pi_i(g) = 0$ for all $i$ (obvious for $i = 1$; otherwise, the covering map $p: \mathbf{S}^2 \to \mathbf{P}^2$ has fibre $\mathbf{S}^0$, whence $\pi_i(p)$ is iso for $i > 1$; and $gp: \mathbf{S}^2 \to \mathbf{S}^2$ is contractible, since $H_2(\mathbf{P}^2) = 0$). But $g$ *is not contractible*, as $H_2(g; \mathbf{Z}/2)$ is an isomorphism (of two-element groups).

The Hopf fibration $h: \mathbf{S}^3 \to \mathbf{S}^2$ induces an iso on $\pi_3$; it yields a third sequence of "Hopf" variations $s_n h: \mathbf{S}^3 \to \mathbf{S}^2$, realising $x_n: \mathbf{Z} \to \mathbf{Z}$ via $\pi_3$. *This sequence is separated by* $\pi_3$ *and collapsed by* $\pi_1, \pi_2$ *and all homology functors with arbitrary coefficients*, whence disjoint from the previous variations ($n > 0$). Moreover, *this sequence can not be realised over closed surfaces*: if $X$ is so, then $\pi_3(X) = 0$ unless $X$ is the sphere or the projective plane (all the other closed surfaces are aspherical, see [25], V.4.2); but the first case only gives the sequence $(s_n)$; and any map $g': \mathbf{P}^2 \to \mathbf{S}^2$ has all $\pi_i(g') = 0$, as proved above for $g$.

**2.5. The real projective plane.** Also $\mathbf{P}^2$ has infinitely many homotopy variations. It is easy to exhibit a sequence $(y_n)$ of them, realising via $\pi_1$ all the fg-abelian variations $(x_n)$ of $\pi_1(\mathbf{P}^2) = \mathbf{Z}/2$ ($n \in [0, +\infty]$, 1.5.3). Setting apart the null variation $y_0: \{*\} \to \mathbf{P}^2$, let $n > 0$. After noting that $x_n$ is "produced" by the left-hand square of the following diagram, by taking cokernels

$$(1) \qquad \begin{array}{ccccc} \mathbf{Z} & \xrightarrow{2^n} & \mathbf{Z} & \xrightarrow{p_n} & \mathbf{Z}/2^n \\ {\scriptstyle 2^{n-1}}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle x_n} \\ \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \xrightarrow{x_\infty} & \mathbf{Z}/2 \end{array} \qquad (x_\infty = p_1)$$

consider the mapping cone of the standard endomap of the circle of degree $2^n$

$$(2) \quad P_{2n} = C(2^n: \mathbf{S}^1 \to \mathbf{S}^1)$$

a Moore-space of type $(\mathbf{Z}/2^n, 1)$, which can be realised as the quotient of the disk under the obvious action of $\mathbf{Z}/2^n$ on its boundary; it is called a "pseudo-projective plane", because $P_2 = \mathbf{P}^2$. We can thus replace the diagram (1) above by the following one, where the right-hand square is obtained by taking mapping cones (homotopy cokernels in $\mathbf{Top}^\mathrm{T}$, and weak cokernels in $\mathrm{Ho}\mathbf{Top}^\mathrm{T}$); note that $\pi_1(q_n)$ is our previous homomorphism $p_n$, whence also $\pi_1(y_n) = x_n$

$$(3)\qquad \begin{array}{ccccc} S^1 & \xrightarrow{2^n} & S^1 & \xrightarrow{q_n} & P_{2n} \\ {\scriptstyle 2^{n-1}}\downarrow & & {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle y_n} \\ S^1 & \xrightarrow{2} & S^1 & \xrightarrow{y_\infty} & P_2 = \mathbf{P}^2 \end{array} \qquad (y_\infty = q_1)$$

We already noticed that the covering map $p\colon \mathbf{S}^2 \to \mathbf{P}^2$ induces isomorphisms $\pi_i(p)$, for $i > 1$ (2.4); thus, it turns the endomaps $s_n\colon \mathbf{S}^2 \to \mathbf{S}^2$ of the sphere (2.4.1) into a new sequence of cw-variations $ps_n\colon \mathbf{S}^2 \to \mathbf{P}^2$ of the projective plane, which realise the abelian variations of $\mathbf{Z}$ through $\pi_2$ and are annihilated by $\pi_1$. Similarly, from the "Hopf" variations $s_nh\colon \mathbf{S}^3 \to \mathbf{S}^2$ (2.4), we get a third sequence $ps_nh\colon \mathbf{S}^3 \to \mathbf{P}^2$ realising the weak subobjects of $\mathbf{Z}$ via $\pi_3$ and annihilated by $\pi_1$, $\pi_2$. The obvious mapping $\mathbf{K} \to \mathbf{P}^2$ defined over the Klein bottle (think of $\mathbf{K}$ as a quotient of the square, and collapse "two parallel edges") realises the non-abelian variation $\pi_1(\mathbf{K}) \to \pi_1(\mathbf{P}^2)$ considered in 1.7.1, and is a variation of $\mathbf{P}^2$ distinct from all the previous ones. (A classification of the maps $\mathbf{P}^2 \to \mathbf{P}^2$ can be found in [2], III, Appendix B.)

Finally, it would not be difficult to exhibit variations of $\mathbf{P}^2$ which realise, via $\pi_1$, the non-finitely generated variations $y_M\colon M^{-1}\mathbf{Z} \to \mathbf{Z}/2$ considered in 1.5.4. For instance, if $M = M_k$ is the multiplicative part of $\mathbf{Z}$ spanned by an *odd* integer $k > 1$, then $M_k{}^{-1}\mathbf{Z} = \mathbf{Z}[1/k]$ is the direct limit of the sequence

$$(4)\qquad \mathbf{Z} \subset \tfrac{1}{k}\mathbf{Z} \subset \tfrac{1}{k^2}\mathbf{Z} \subset \tfrac{1}{k^3}\mathbf{Z} \subset \ldots$$

and $y_{M_k}$ can be realised as a map $z_k\colon V_k \to \mathbf{P}^2$, defined over the homotopy direct limit of the following diagram (a sort of *k-adic funnel*)

$$(5)\qquad \mathbf{S}^1 \xrightarrow{k} \mathbf{S}^1 \xrightarrow{k} \mathbf{S}^1 \xrightarrow{k} \mathbf{S}^1 \longrightarrow \ldots$$

## 3. Transformations of variations

Some formal transformations of weak subobjects are considered, which are of help for further classifications (3.4, Section 4).

**3.1. Transfer.** Let us resume the abstract situation, assuming that our category **A** *has weak pullbacks*. Then every morphism $f\colon A \to B$ defines a covariant connection (or adjunction) $f_* \dashv f^*$ between the meet-semilattices

(1) $f_*: \mathrm{Var}(A) \rightleftarrows \mathrm{Var}(B) : f^*$

$f_*(x) = fx, \quad f^*(y) = $ weak pullback of $y$ along $f \qquad (1 \leq f^*f_*;\ f_*f^* \leq 1)$

$$(2) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle x}\uparrow & & \uparrow{\scriptstyle fx} \\ X & = & X \end{array} \qquad\qquad (3) \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle f^*y}\uparrow & * & \uparrow{\scriptstyle y} \\ \bullet & \longrightarrow & Y \end{array}$$

which actually satisfies the *right-exactness* condition $f_*f^*(y) = y \wedge f_*(1) = y \wedge f$ (the diagonal of the weak pullback in (3)). If $f$ is monic, the square (2) is a pullback and $f^*f_*(x) = x$, whence $\mathrm{Var}(A)$ is a retract of $\mathrm{Var}(B)$.

Since weak pullbacks can be pasted, we have a functor with values in the category of (possibly large) semilattices and their right-exact connections

(4) $\mathrm{Var}: \mathbf{A} \to \mathbf{SLR}, \qquad A \mapsto \mathrm{Var}(A), \qquad f \mapsto \mathrm{Var}(f) = (f_*, f^*).$

If $\mathbf{A}$ has pullbacks, the counterimages of subobjects agree with the counterimages of variations. If $\mathbf{A}$ has unique epi-monic factorisations, the direct image $\mathrm{im}(fx)$ of a subobject $x$ is the carrier of the covariant transfer $fx$ (as a variation).

Dually, in a category *with weak pushouts*, we have a transfer functor for covariations, $\mathrm{Cov}: \mathbf{A} \to \mathbf{SLR}^{op}$

(5) $f_\bullet: \mathrm{Cov}(A) \rightleftarrows \mathrm{Cov}(B) : f^\bullet, \qquad f^\bullet \dashv f_\bullet$

$f^\bullet(y) = yf, \qquad f_\bullet(x) = $ weak pushout of $x$ along $f$.

**3.2. Adjunctions and reflections.** Transformations of weak subobjects produced by functors (1.3) have already been used.

Consider now an adjunction $F \dashv U$, with unit $u$ and counit $v$

(1) $F: \mathbf{Y} \rightleftarrows \mathbf{A} : U, \qquad u: 1 \to UF, \qquad v: FU \to 1$

$(y: Y \to UA) \mapsto ({}^Fy = vA.Fy: FY \to A)$

$(a: FY \to A) \mapsto (a^U = Ua.uY: Y \to UA)$

and recall that the functor $U$ is faithful (resp. full; full and faithful) if and only if all its counit maps $vA: FUA \to A$ are epi (resp. split monic, iso) [22]. The last case is called a *reflection*; we have considered above the full embedding $U: \mathbf{Ab} \to \mathbf{Gp}$ and its reflector $ab \dashv U$ (1.7).

First, $U$ induces the transformation $U: \mathrm{Var}_{\mathbf{A}}(A) \to \mathrm{Var}_{\mathbf{Y}}(UA)$ (1.3).

Backwards, the mapping $y \mapsto {}^F y$ is monotonic at the level of arrows (with values in UA), hence well defined and monotonic at the level of weak subobjects. We obtain an *associated adjunction* between the ordered sets of variations

(2) $^F(-) : \mathrm{Var}_\mathbf{Y}(UA) \rightleftarrows \mathrm{Var}_\mathbf{A}(A) : U$

$[y] = [U(^Fy).uY] \leq [U(^Fy)], \qquad [^F(Ua)] = [vA.FUa] = [a.vA'] \leq [a]$

so that $^Fy$ is the least variation $a \in \mathrm{Var}_\mathbf{A}(A)$ such that $y \leq Ua$, say the **A**-*closure* of y. We say that $y: Y \to UA$ is an **A**-*type* variation if it coincides with $U(^Fy)$, or equivalently with some Ua.

*If the adjunction is a reflection,* also the associated adjunction of weak subobjects is so: for any variation $a: A' \to A$ we have $^F(Ua) = a$, and $\mathrm{Var}_\mathbf{A}(A)$ is a retract of $\mathrm{Var}_\mathbf{Y}(UA)$. But it is sufficient that each counit map $vA: FUA \to A$ be a split epi; in this case, any section $\bar{v}A$ of vA gives

(3) $[a] = [a.vA'.\bar{v}A'] = [^F(Ua).\bar{v}A'] \leq [^F(Ua)]$.

**3.3. Groups and CW-spaces.** If **CW** is the category of pointed (connected) CW-spaces and pointed maps (2.2), the previous construction can be applied to the fundamental group functor over $\mathbf{CW}/\simeq$ and its right adjoint $K_1$

(1) $\pi_1 : \mathbf{CW}/\simeq \;\rightleftarrows\; \mathbf{Gp} : K_1, \qquad\qquad \pi_1 \dashv K_1.$

The existence of $K_1$ is a consequence of the Eilenberg - Mac Lane space $K_1(G) = K(G, 1)$ of a group G (a cw-space with $\pi_1(K(G, 1)) \cong G$ and trivial higher homotopy groups), combined with a second classical result: if X is a 1-type pointed space ($\pi_i(X) = 0$ for $i \neq 1$) and Y a CW-space, then $\pi_1$ induces a bijection $[Y, X] \to \mathbf{Gp}(\pi_1(Y), \pi_1(X))$ ([25], thm. V.4.3). Any isomorphism vG: $\pi_1 K_1(G) \to G$ is thus a universal arrow from $\pi_1$ to G, proving the existence of the right adjoint $K_1$ [22]; the latter is determined up to functorial isomorphism (and strictly determined by an arbitrary choice of all K(G, 1) and all vG).

Since the counit is iso, our reflection yields an adjoint retraction for variations

(2) $\mathrm{Var}_{\mathbf{Gp}}(G) \rightarrowtail \mathrm{Var}_{cw}(K_1 G) \twoheadrightarrow \mathrm{Var}_{\mathbf{Gp}}(G)$

$(x: G' \to G) \;\mapsto\; (K_1 x: K_1 G' \to K_1 G)$

$(y: Y \to K_1 G) \;\mapsto\; (^\pi y: \pi_1 Y \to G)$

$^\pi(K_1 x) = x, \quad y \leq K_1(^\pi y).$

The cw-variation $y: Y \to K_1 G$ can be viewed as a *spatial variation* of G, since $K_1$ is full and faithful. It has a *group-closure* $^\pi y: \pi_1 Y \to G$, the least group-variation of G such that $y \leq K_1(^\pi y)$; our y will be called a *group-type* variation of $K_1G$ if it coincides with $K_1({}^{\pi}y)$, or equivalently, with some $K_1(x)$.

Note also that, for any CW-space $X$, $\pi_1$: **CW**/$\simeq$ $\to$ **Gp** induces a lattice-epimorphism

(3) $\pi_1\colon \mathrm{Var}_{cw}(X) \twoheadrightarrow \mathrm{Var}_{\mathbf{Gp}}(\pi_1 X)$

since $\pi_1$ preserves weak pullbacks, as a left adjoint, and pointed sums.

**3.4. Clusters of circles.** We can now prove that *the cw-variations of* $\mathbf{S}^1$ *are all of group-type*, classified by the canonical maps $y_n\colon \mathbf{S}^1 \to \mathbf{S}^1$ of degree $n \geq 0$ (2.3.1). In fact, let $y\colon Y \to K_1\mathbf{Z}$ be a cw-variation, $x = {}^{\pi}y\colon \pi_1 Y \to \mathbf{Z}$ its group closure, with $\mathrm{Im}(x) = n\mathbf{Z}$ and $x \sim x_n\colon \mathbf{Z} \to \mathbf{Z}$. Because of the adjunction $\pi_1 \dashv K_1$, we already know that $y \leq K_1({}^{\pi}y) = K_1(x_n) = y_n$. But $y_n \leq y$, as there is a loop $\lambda\colon \mathbf{S}^1 \to Y$ such that $(\pi_1 y)([\lambda]) = n$, and $y.\lambda \simeq y_n$.

This argument can be easily extended to an arbitrary *cluster of circles* $\Sigma_I \mathbf{S}^1$ (a categorical sum of pointed circles over a small set $I$, i.e. a disjoint union of circles with base-points identified). The group variations of the free group $F = *_I \mathbf{Z}$ coincide with its subgroups $x_H\colon H \rightarrowtail F$ (1.7; free, but possibly not finitely generated, even for a finite $I$), and yield an isomorphic lattice of group-type variations for the cluster $K_1F = \Sigma_I \mathbf{S}^1$

(1) $y_H = K_1 x_H\colon K_1 H \to \Sigma_I \mathbf{S}^1, \qquad \pi_1(y_H) = x_H, \qquad (y_H \leq y_{H'} \text{ iff } H \subset H').$

Again, *there are no other cw-variations*. Let $y\colon Y \to K_1F$ be a cw-variation, $x = {}^{\pi}y\colon \pi_1 Y \to F$ its group closure, with $\mathrm{Im}(x) = H$ and $x \sim x_H$, $y \leq K_1({}^{\pi}y) = K_1(x_H) = y_H$. To show that $y_H \leq y$, choose a basis $B \subset H$ and for each $b \in B$ a loop $\lambda_b\colon \mathbf{S}^1 \to Y$ such that $(\pi_1 y)([\lambda_b]) = b$. We have thus a map $\lambda\colon K_1 H = \Sigma_B \mathbf{S}^1 \to Y$ such that $y.\lambda \simeq y_H$.

**3.5. Product variations.** Coming back to the abstract situation, consider an object $A = A_1 \times A_2$ in a category **A** with finite products and zero object, like an abelian category or **Gp**, $\mathbf{Set}^{\mathsf{T}}$, $\mathbf{Top}^{\mathsf{T}}$, $\mathbf{Top}^{\mathsf{T}}/\simeq$. It will be useful to understand to which extent the weak subobjects of $A$ can be reduced to variations of its factors. We always have monotonic mappings

(1) $\varphi\colon \mathrm{Var}(A_1) \times \mathrm{Var}(A_2) \to \mathrm{Var}(A), \qquad (x_1, x_2) \mapsto x_1 \times x_2\colon X_1 \times X_2 \to A_1 \times A_2$

(2) $\psi\colon \mathrm{Var}(A) \to \mathrm{Var}(A_1) \times \mathrm{Var}(A_2), \qquad x \mapsto (pr_1.x, pr_2.x)$

with $\psi\varphi = 1$, showing that $\mathrm{Var}(A_1) \times \mathrm{Var}(A_2)$ *is a retract of* $\mathrm{Var}(A_1 \times A_2)$ (among ordered sets).

(In fact, if $x \leq y$ as maps with values in $A_1 \times A_2$, the same holds for their projections. Similarly, if $x_i = y_i u_i$: $X_i \to A_i$, then $x_1 \times x_2 = (y_1 \times y_2)(u_1 \times u_2)$. This shows that $\psi$ and $\varphi$ are well defined on variations and monotonic. The composite $\psi\varphi$ turns a pair of variations $(x_1, x_2)$, with $x_i$: $X_i \to A_i$, into the pair $y_i = x_i.pr_i$: $X_1 \times X_2 \to A_i$, which factors through the former; also the converse holds, as $x_i = y_i.in_i$; the injections $in_i$: $X_i \to X_1 \times X_2$ being provided by the zero-object.)

Say that a variation $x \in Var(A)$ is a *product variation* if it belongs to $Im(\varphi)$, or equivalently if $\varphi\psi(x) = x$ (within variations, of course). Say that the product $A_1 \times A_2$ *has (only) product variations* if all its weak subobjects are so ($\varphi$ is surjective); or equivalently, if the above mappings are reciprocal; or also, if $\psi$ is injective. Of course, the notion of a product variation is *relative to a given decomposition* of an object A as a product $A_1 \times A_2$; in particular, we shall see that each weak subobject of the non-cyclic four-element group A is a product variation with respect to a *suitable* representation of A as $\mathbf{Z}/2 \oplus \mathbf{Z}/2$, but not with respect to a fixed one (4.4-5).

## 4. Further classifications of abelian variations

Finally, we classify the abelian fg-variations of finite cyclic groups (4.2-3) and of $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ (4.4-5). The main tools are the notion of product variation (3.5; 4.1) and, again, the structure theorem of fg-abelian groups.

**4.1. Abelian product variations.** Recall the notion of product variation (3.5). In **Ab** or **Gp**, it is plain that any power $A \times A$ $(A \neq 0)$ has non-product variations: the diagonal $d$: $A \to A \times A$ is not so, as $d < 1_{A \times A} = \varphi\psi(d)$ $(im(d) < 1)$.

On the other hand, if the abelian groups $A_1$ and $A_2$ are annihilated by two *coprime* integers a and b ($aA_1 = 0$, $bA_2 = 0$), then all the abelian variations of $A_1 \oplus A_2$ are product variations (and all fg-abelian variations are product of fg-abelian ones). Starting from a variation $x = \langle x_1, x_2 \rangle$: $X \to A_1 \oplus A_2$, we obtain $y = x_1 \oplus x_2$: $X \oplus X \to A_1 \oplus A_2$, and $x = yd \leq y$ through the diagonal $d$: $X \to X \oplus X$. Choose now two integers h, k such that $ha + kb = 1$, so that the multiplication by ha kills $A_1$ while the multiplication by kb induces the identity on $A_1$. Consider the morphism $v$: $X \oplus X \to X$, $v(\lambda, \mu) = kb.\lambda + ha.\mu$ (here, we need X commutative). Then $xv = y$, as

(1) $xv.in_1 = \langle kb.x_1, kb.x_2 \rangle = \langle x_1, 0 \rangle = (x_1 \oplus x_2).in_1$

and similarly $xv.in_2 = (x_1 \oplus x_2).in_2$.

Applying this to the structural decomposition of any finite abelian group A as a (finite) direct sum of its indecomposable (primary cyclic) components, we have

(2) $\mathrm{Var}(A) \cong \Pi \mathrm{Var}(A_i), \qquad \mathrm{Var}_{fg}(A) \cong \Pi \mathrm{Var}_{fg}(A_i) \qquad (A = \oplus A_i).$

**4.2. Primary cyclic groups.** We proceed now to classify the abelian fg-variations of a finite, indecomposable group $A = \mathbf{Z}/p^r$ (p prime), showing that they form a distributive noetherian lattice. All the subobjects of $\mathbf{Z}/p^r$ are p-primary cyclic groups, and form a finite chain of length r+1

(1) $x_k\colon \mathbf{Z}/p^k \rightarrowtail \mathbf{Z}/p^r, \qquad x_k(\overline{1}) = \overline{p^{r-k}} \qquad (0 \leq k \leq r)$

$0 = x_0 < x_1 < \ldots < x_r = 1.$

One shows as above, in 1.5.3, that the surjective fg-variations of $\mathbf{Z}/p^k$ $(k > 0)$ form the totally ordered set of natural homomorphisms

(2) $y_{kn}\colon \mathbf{Z}/p^n \twoheadrightarrow \mathbf{Z}/p^k \qquad (0 < k \leq n \leq \infty)$

$1 = y_{kk} > y_{k,k+1} > y_{k,k+2} > \cdots > y_{k\infty}$

and it follows (by 1.2) that each non-null fg-variation of $\mathbf{Z}/p^r$ is of the type

(3) $x_{kn} = x_k.y_{kn}\colon \mathbf{Z}/p^n \twoheadrightarrow \mathbf{Z}/p^k \rightarrowtail \mathbf{Z}/p^r \qquad (0 < k \leq r;\ k \leq n \leq \infty)$

$x_{kn}(\overline{1}) = \overline{p^{r-k}} \qquad \mathrm{im}(x_{kn}) = x_k = x_{kk}.$

We do not yet know that such variations are indeed different. But we can organise them as follows, an arrow meaning $\leq$ (in fact, $<$)

$$
(4)\qquad
\begin{array}{ccccccccc}
 & & x_1 & \longrightarrow & x_2 & \longrightarrow \ldots \longrightarrow & x_{r-1} & \longrightarrow & x_r = 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & x_{12} & \longrightarrow & x_{23} & \longrightarrow \ldots \longrightarrow & x_{r-1,r} & \longrightarrow & x_{r,r+1} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & x_{13} & \longrightarrow & x_{24} & \longrightarrow \ldots \longrightarrow & x_{r-1,r+1} & \longrightarrow & x_{r,r+2} \\
 & & \ldots & & \ldots & \ldots & \ldots & & \ldots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & x_{1\infty} & \longrightarrow & x_{2\infty} & \longrightarrow \ldots \longrightarrow & x_{r-1,\infty} & \longrightarrow & x_{r\infty}
\end{array}
$$

the inequalities being provided by the following morphisms between their domains, either induced by p-multiplication (the horizontal ones) or by the identity of $\mathbf{Z}$

(5)
$$\begin{array}{ccccccccccc}
 & & \mathbf{Z}/p & \xrightarrow{p} & \mathbf{Z}/p^2 & \xrightarrow{p} & \ldots \longrightarrow & \mathbf{Z}/p^{r-1} & \xrightarrow{p} & \mathbf{Z}/p^r \\
 & & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\
 & & \mathbf{Z}/p^2 & \xrightarrow{p} & \mathbf{Z}/p^3 & \xrightarrow{p} & \ldots \xrightarrow{p} & \mathbf{Z}/p^r & \xrightarrow{p} & \mathbf{Z}/p^{r+1} \\
 & & \ldots & & \ldots & & \ldots & \ldots & & \ldots \\
 & & \uparrow & & \uparrow & & & \uparrow & & \uparrow \\
0 & \longrightarrow & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z} & \xrightarrow{p} & \ldots \xrightarrow{p} & \mathbf{Z} & \xrightarrow{p} & \mathbf{Z}
\end{array}$$

Now, it is not difficult to show that no other inequalities, not considered in (4), may subsist, using the fact that $x \leq y$ implies $\mathrm{im}(x) \leq \mathrm{im}(y)$. Note also that each weak subobject in (4) may be read in (5), as the composite of any string starting from the corresponding position in (5) and ending in $\mathbf{Z}/p^r$.

**4.3. Cyclic groups.** The previous two points completely characterise the lattice of fg-variations of any *finite cyclic group* (the infinite case being already known): if $n = \Pi_i \, p_i^{r_i}$ is the decomposition of $n > 0$ in primary factors

(1) $\mathbf{Z}/n \cong \oplus_i \, \mathbf{Z}/p_i^{r_i}$, $\qquad \mathrm{Var}_{fg}(\mathbf{Z}/n) \cong \Pi_i \, \mathrm{Var}_{fg}(\mathbf{Z}/p_i^{r_i})$

and again, *this lattice is distributive and noetherian.*

From the description of weak subobjects of $\mathbf{Z}/p^r$ (4.2.4), it follows that any *finite* variation of $\mathbf{Z}/n$ can be be represented over a *cyclic* group, as

(2) $x(a, b) = (\mathbf{Z}/b \twoheadrightarrow \mathbf{Z}/a \rightarrowtail \mathbf{Z}/n)$

for a pair of positive integers $(a, b)$ such that

- $a \mid n$ ($\mathbf{Z}/a \rightarrowtail \mathbf{Z}/n$ is the multiplication by $n/a$)
- $a \mid b$ and these two numbers have the same prime factors ($\mathbf{Z}/b \twoheadrightarrow \mathbf{Z}/a$ is the natural projection).

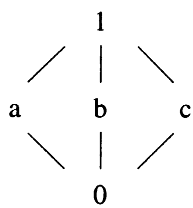
Moreover $x(a, b) \leq x(a', b')$ iff $a \mid a'$ and $b'$ divides $b.a'/a$

(3)
$$\begin{array}{ccccc}
\mathbf{Z}/b & \twoheadrightarrow & \mathbf{Z}/a & \xrightarrow{n/a} & \mathbf{Z}/n \\
{\scriptstyle a'/a}\downarrow & & {\scriptstyle a'/a}\downarrow & & \| \\
\mathbf{Z}/b' & \twoheadrightarrow & \mathbf{Z}/a' & \xrightarrow{n/a'} & \mathbf{Z}/n
\end{array}$$

**4.4. The variations of the Klein group, Part I.** Let us begin the classification of the fg-abelian variations of the Klein group $A = \mathbf{Z}/2 \oplus \mathbf{Z}/2$.

We shall use the family $(x_n)$ of fg-abelian variations of $\mathbf{Z}/2$ (1.5.3), which is here convenient to re-index over the ordinal sum $[1, \infty'] = [1, \infty] + \{\infty'\}$, letting $x_{\infty'}: 0 \to \mathbf{Z}/2$ be the least variation, previously written $x_0$.

For A, we have to distinguish between *relative* facts, inherent to our presentation of the group A as a product, and *absolute* facts, invariant under automorphisms of A. To begin with, it is an absolute fact that the lattice of subobjects of A is the "elementary modular, non-distributive lattice", which has three atoms (the three non trivial cyclic subgroups)

(1) 

$(a = 1 \oplus 0: \mathbf{Z}/2 \to A)$
$(b = 0 \oplus 1: \mathbf{Z}/2 \to A)$
$(c = \langle 1, 1 \rangle: \mathbf{Z}/2 \to A)$

while the description of the atoms in brackets is relative to our presentation; according to it, a and b are product variations, while c is not; from an absolute point of view, there is no distinction between the atoms, since the automorphisms u, v permute them (and c is a product variation for other decompositions of A)

$$(2) \quad u: A \to A, \quad u(\lambda, \mu) = (\lambda+\mu, \mu), \quad ua = a, \quad ub = c, \quad uc = b$$
$$v: A \to A, \quad v(\lambda, \mu) = (\lambda, \lambda+\mu), \quad va = c, \quad vb = b, \quad vc = a.$$

Now, each weak subobject of A is produced by a surjective variation of one subobject. For a generic *fg*-abelian variation $x: X \to A$, the structural decomposition $x = \vee x|X_i$ (1.5.1) shows that we can start from studying indecomposable variations, necessarily of type $x: \mathbf{Z}/2^n \to A$. Since such a weak subobject has a cyclic image, and can not be surjective, we deduce that x is either null or belongs to precisely one of the following totally ordered sets $(n < \infty')$,

$$(3) \quad a_n = ax_n: \mathbf{Z}/2^n \to A \qquad (a_n = x_n \oplus 0)$$
$$b_n = bx_n: \mathbf{Z}/2^n \to A \qquad (b_n = 0 \oplus x_n)$$
$$c_n = cx_n: \mathbf{Z}/2^n \to A$$

and that elements of different families are never comparable. Thus, every fg-abelian variation of A can be expressed as a join of zero, one, two or three indecomposable ones, picked in different families. But different "expressions" of these joins may give the same element; a precise description of the variations is given below.

**4.5. Theorem.** *The finitely generated abelian variations of* $A = \mathbf{Z}/2 \oplus \mathbf{Z}/2$ *are classified in the following disjoint lists of distinct elements, based on the three families of indecomposable variations* $a_n, b_n, c_n$ (4.4.3)

(1) $e_n = a_n \vee b_n = a_n \vee c_n = b_n \vee c_n \ (= x_n \oplus x_n)$ $\qquad n \in [1, \infty']$

(2) $e^1_{nk} = a_n \vee b_k = a_n \vee c_k \ (= x_n \oplus x_k)$ $\qquad n < k, \text{ in } [1, \infty']$

(3) $e^2_{nk} = b_n \vee a_k = b_n \vee c_k \ (= x_k \oplus x_n)$ $\qquad n < k, \text{ in } [1, \infty']$

(4) $e^3_{nk} = c_n \vee a_k = c_n \vee b_k$ $\qquad n < k, \text{ in } [1, \infty']$.

*Setting* $e^1_{nn} = e^2_{nn} = e^3_{nn} = e_n$, *the order relation is described as follows, for* $n \leq k$ *and* $n' \leq k'$ *in* $[1, \infty']$, *and* $i \neq j$ *in* $\{1, 2, 3\}$

(5) $e^i_{nk} \leq e^i_{n'k'}$ $\qquad$ iff $\qquad n \geq n', \ k \geq k'$

$e^i_{nk} \leq e^j_{n'k'}$ $\qquad$ iff $\qquad n \geq k'$ (i.e. $n' \leq k' \leq n \leq k$).

*The lattice* $\mathrm{Var_{fg}}(A)$ *is modular and can be described as the union of three triangles* $T_i = \{e^i_{nk} \mid n \leq k\}$ $(i = 1, 2, 3)$ *along their common "oblique" edge* $(e_n)$

$$(6) \qquad \begin{array}{ccccccccc}
e^i_{1\infty'} & \longrightarrow & e^i_{1\infty} & \longrightarrow \ \cdots & e^i_{1n} & \longrightarrow \ \cdots & e^i_{12} & \longrightarrow & e_1 = 1 \\
\uparrow & & \uparrow & \cdots & \uparrow & \cdots & \uparrow & & \\
e^i_{2\infty'} & \longrightarrow & e^i_{2\infty} & \longrightarrow \ \cdots & e^i_{2n} & \longrightarrow \ \cdots & e_2 & & \\
\cdots & & \cdots & & \cdots & & & & \\
\uparrow & & \uparrow & \cdots & \uparrow & & & & \\
e^i_{n\infty'} & \longrightarrow & e^i_{n\infty} & \longrightarrow \ \cdots & e_n & & & & \\
\cdots & & \cdots & & & & & & \\
\uparrow & & \uparrow & & & & & & \\
e^i_{\infty\infty'} & \longrightarrow & e_\infty & & & & & & \\
\uparrow & & & & & & & & \\
0 = e_{\infty'} & & & & & & & &
\end{array}$$

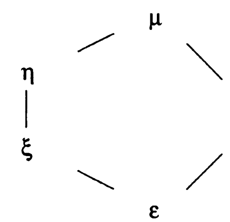
*the union of any two of them forming a distributive sublattice isomorphic to* $\mathrm{Var}(\mathbf{Z}/2) \times \mathrm{Var}(\mathbf{Z}/2)$. *In particular,* $T_1 \cup T_2$ *consists of the weak subobjects* $x_m \oplus x_n$ *listed in* (1)-(3), *the product variations with respect to our presentation. The "free" vertices* $e^i_{1\infty'}$ *of the three triangles are the atomic subobjects* a, b, c.

**Proof.** The indecomposable variations have been classified above (4.4.3). First, it is easy to see that the definition (1) of the sequence $e_n$ is correct: the joins of any pair of the three indecomposable variations $a_n$, $b_n$, $c_n$ (with the same index) coincide.

It follows that, in any join of two indecomposable variations of different families, we can replace the one having the highest index with the corresponding one in the third family; e.g., for $n < k$, $a_n \vee b_k = a_n \vee a_k \vee b_k = a_n \vee a_k \vee c_k = a_n \vee c_k$ so that also our definition of the remaining lists (2)-(4) is correct. It follows also that, in any join of three indecomposable variations of different families, we can omit the one having the highest index. All the weak subobjects are thus listed in (1)-(4).

But the product variations, of type $x_m \oplus x_n$ ($m, n \in [1, \infty']$), are known to form a lattice isomorphic to $\mathrm{Var}(\mathbf{Z}/2) \times \mathrm{Var}(\mathbf{Z}/2)$, so that the order between two of them is indeed described by (5), for $i, j = 1, 2$. Now, the automorphisms $u_*$, $v_*$ of $\mathrm{Var}_{fg}(A)$ (4.4.2) permute our families ($ua_n = a_n$, $ub_n = c_n$, $uc_n = b_n$; $va_n = c_n$, $vb_n = b_n$, $vc_n = a_n$) and show that (5) is globally correct.

Finally, suppose that $\mathrm{Var}_{fg}(A)$ is not modular. Then, it contains a sublattice isomorphic to the "elementary non-modular lattice" (Birkhoff [4])

(7) 


The generators $\xi, \eta, \zeta$ are in different lists (2) - (4) (otherwise, they would be in a distributive sublattice of $\mathrm{Var}_{fg}(A)$), and we may take $\xi = e^1_{nk}$, $\eta = e^2_{n'k'}$, $\zeta = e^3_{n''k''}$. Since $\xi < \eta$ and $\zeta$ is not comparable with them, we have $n' < k' \leq n < k$; $n' < k''$; $n'' < k$. It is now easy to show that $\xi \vee \zeta$ and $\eta \vee \zeta$ can not coincide.

**4.6. Extensions and problems.** It is easy to extend these results (4.4-5) to any square power $A = \mathbf{Z}/p \oplus \mathbf{Z}/p$ of a prime-order group. The lattice of subobjects of $A$ has now $p+1$ atoms, the non-trivial cyclic subgroups (these subgroups have order $p$ and induce a partition over $A - \{0\}$); the lattice of weak subobjects is the union of $p+1$ triangles as above (4.5.6), each pair of them forming a sublattice isomorphic to $\mathrm{Var}(\mathbf{Z}/p) \oplus \mathrm{Var}(\mathbf{Z}/p)$.

By 4.1, we also know the lattice of fg-variations of a direct sum

(1) $B = \oplus_i (\mathbf{Z}/p_i)^{n_i}$, $\qquad\qquad \mathrm{Var}_{fg}(B) = \Pi_i \mathrm{Var}_{fg}(\mathbf{Z}/p_i)^{n_i}$

where $(p_i)$ is a finite family of distinct primes and $n_i = 1, 2$. In particular we know the fg-variations of a square $(\mathbf{Z}/n)^2$, provided that the integer n is *square-free* (its decomposition in prime factors has no exponent > 1).

The following problems on fg-abelian variations or group variations (1.7) arise.

a) Characterise the lattice of fg-abelian variations of $(\mathbf{Z}/p^r)^n$, for p prime. (Then, applying the decomposition 4.1.2, one would get a classification of fg-variations for any finite abelian group.)

b) Is $\mathrm{Var}_{fg}(A)$ a noetherian modular lattice, for every fg-abelian group A?

c) Classify the finite (or fg-) group variations of $\mathbf{Z}/n$.

d) Deduce information about the cw-variations of the lens space $L^\infty(n) = K(\mathbf{Z}/n, 1)$ and the $\mathbf{M}^k$-variations of the Moore space $M(\mathbf{Z}/n, k)$.

e) Classify the homotopy variations within closed surfaces (see 2.4-2.5).

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