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CATEGORIES OF TOPOLOGICAL SPACES WITH SUFFICIENTLY MANY SEQUENTIALLY CLOSED SPACES

*by Dikran DIKRANJAN & Jan PELANT**

RESUME. Pour une classe $\mathcal{P}$ d'espaces topologiques, un espace X de $\mathcal{P}$ est dit séquentiellement $\mathcal{P}$-fermé si, pour tout $\mathcal{P}$-espace Y contenant X comme sous-espace, X est séquentiellement fermé dans Y. On dit que $\mathcal{P}$ a assez d'espaces séquentiellement fermés si pour tout X de $\mathcal{P}$ il existe un espace Y $\mathcal{P}$-séquentiellement fermé contenant X comme sous-espace relativement à toutes les itérations de fermeture séquentielle.

Dans cet article, les auteurs prouvent: a) La catégorie **Tych** des espaces de Tychonoff n'a pas assez d'espaces séquentiellement fermés; b) les catégories **US**, resp. **SUS**, d'espaces topologiques dans lesquels toute suite convergente a un unique point limite (resp. un unique point d'accumulation) ont assez d'espaces séquentiellement fermés.

On sait que les espaces séquentiellement **Tych**-fermés et les espaces séquentiellement **SUS**-fermés sont exactement les espaces dénombrablement compacts, et que les espaces séquentiellement **US**-fermés coïncident avec les espaces séquentiellement compacts. On en déduit l'existence de certaines extensions dénombrablement compactes ou pseudocompactes.

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Introduction.

The study of the extensions of topological spaces has always been one of the main points of interest in point-set topology (the pioneer work was done by Alexandroff, Fomin and Katětov). In particular, spaces which have no proper extension within a class $\mathcal{P}$ of spaces (the so called *$\mathcal{P}$-closed spaces*) have been extensively studied (see for example the survey [BPS]).

When $\mathcal{P}$ is a class determined by a low separation axiom $\mathcal{P}$, $\mathcal{P}$-closed spaces are quite rare. In fact, no T_0-closed spaces exist and every T_1-closed space is finite (for more examples see [DGo]). This phenomenon suggests a substitution of the usual notion of density and closedness by density and closedness with respect to the sequential closure σ and its idempotent hull σ^∞. Such an approach was adopted in [DGo] and [Go1], where the notion of a *sequentially $\mathcal{P}$-closed* space was introduced ($X \in \mathcal{P}$ is sequentially $\mathcal{P}$-closed if for each embedding $X \hookrightarrow Y$ with $Y \in \mathcal{P}$, X is sequentially closed in Y). In case $\mathcal{P}$ is closed with respect to taking (sequentially closed) subspaces, X is sequentially $\mathcal{P}$-closed iff X has no proper extension $Y \in \mathcal{P}$ such that X is σ-dense (resp. σ^∞-dense) in Y. Obviously every $\mathcal{P}$-closed space is sequentially $\mathcal{P}$-closed. To give further examples we need the following notation: **Top** will denote the category of topological spaces and continuous maps, all subcategories of **Top** considered below are full, so that they are described by their objects.

$\mathbf{Top}_0$ - T_0-spaces

$\mathbf{Top}_1$ - T_1-spaces

US - topological spaces in which every convergent sequence has a unique limit point

SUS - topological spaces in which every convergent sequence has a unique accumulation point

Haus(Comp) - topological spaces in which every compact subspace is Hausdorff

Haus - Hausdorff topological spaces

Ury - Urysohn spaces (distinct points have disjoint closed neighbourhoods)

Reg - regular T_1-spaces

$S(n)$ - $S(n)$-spaces in the sense of Viglino [V2] (see the definition below)

$S(\alpha)$ - $S(\alpha)$-spaces in the sense of Porter and Votaw [PV] (see the definition below)

FHaus - functionally Hausdorff topological spaces (continuous real valued functions separate points)

Tych - Tychonov spaces (=completely regular T_1-spaces)

Tqc - spaces with trivial quasi-components (every point is intersection of clopen sets)

0-dim - zero-dimensional T_1-spaces (with respect to ind).

A (strongly) epireflective subcategory of **Top** is a subcategory closed under taking subspaces (equipped with finer topologies) and products. It is easy to see that every epireflective subcategory of **Top** containing at least one non-indiscrete space contains **0-dim**. Thus every strongly epireflective subcategory of **Top** containing at least one non-singleton space contains **Tqc**. Moreover, every epireflective subcategory of **Top** contained in $\mathbf{Top}_0$ and containing at least one non-singleton space contains **0-dim**.

Let α be an infinite ordinal, a topological space X is said to satisfy the axiom $S(\alpha)$ (Porter and Votaw [PV]) if for each pair of distinct points x_0, x_1 in X there are transfinite sequences of open sets $\{U_\gamma : \gamma \in \alpha\}$ and $\{V_\gamma : \gamma \in \alpha\}$ such that:

$$x_0 \in \cap\{U_\gamma : \gamma \in \alpha\} \wedge x_1 \in \cap\{V_\gamma : \gamma \in \alpha\},$$

$(\gamma_0, \gamma_1 \in \alpha \wedge \gamma_0 < \gamma_1) \Rightarrow \overline{U_{\gamma_1}} \subset U_{\gamma_0} \wedge \overline{V_{\gamma_1}} \subset V_{\gamma_0}$, and

$(\forall \gamma \in \alpha)\, U_\gamma \cap V_\gamma = \emptyset$.

Let $\alpha = n$ be a finite ordinal, a topological space X is said to satisfy the axiom $S(n)$ (Viglino [V2]) if for each pair of distinct points x_0, x_1 in X there is a finite sequence of open sets $\{U_k : 1 \leq k \leq n\}$ such that

$x_0 \in U_n \wedge x_1 \notin \overline{U_1} \wedge (1 \leq k < l \leq n \Rightarrow \overline{U_l} \subset U_k)$.

A space X is said to be countably compact if X does not contain any countable infinite closed discrete set. Let us note this notion will be applied only to T_1-spaces in this paper.

For $\mathcal{P} \subseteq$ **SUS** every countably compact $\mathcal{P}$-space is sequentially $\mathcal{P}$-closed and for $\mathcal{P} =$ **SUS** the converse is also true (see Proposition 1.1 or [DGo, Theorem 1.3]). Analogously, every sequentially compact $\mathcal{P}$-space is sequentially $\mathcal{P}$-closed for $\mathcal{P} \subseteq$ **US**. For $\mathcal{P} =$ **US**, the converse is also true (see Proposition 1.1 or [DGo, Theorem 1.3]). The sequentially $\mathcal{P}$-closed spaces are characterized in [Go] in many cases ($\mathcal{P} =$ **Haus**, $S(n)$, **Reg** etc.). They coincide with the countably compact spaces for $\mathcal{P} =$ **Tych**, metrizable spaces, normal spaces, paracompact spaces etc. [Go1].

In view of the above described phenomenon it is natural to ask whether there are enough sequentially closed spaces. First of all, we need to give a precise form of this question. Take $\mathcal{P} =$ **Tych**, now sequentially closed means countably compact, so that every $X \in \mathcal{P}$ admits an extension Y which is even compact. However, this cannot be considered as a satisfactory answer since different notions of density and closedness appear. This is why we give the following

DEFINITION. *The class $\mathcal{P}$ has sufficiently many sequentially closed spaces if for each $X \in \mathcal{P}$ there exists an embedding $X \hookrightarrow Y$ such that Y is sequentially $\mathcal{P}$-closed and X is σ^∞-dense in Y.*

Note that $\mathcal{P} =$ **Top**$_1$ has not sufficiently many sequentially closed spaces, since each sequentially closed space in $\mathbf{Top_1}$ is finite. Really, every infinite T_1-space X admits a proper one-point T_1 extension X_f (corresponding to the Fréchet filter in X, X_f is both compact and sequentially compact) and X is moreover σ-dense in X_f. This can be extended to other categories of weak Hausdorff spaces (as the categories $\mathcal{C}_\alpha$, defined by Hoffmann [Hof] for any infinite cardinal α, having as objects the T_1-spaces which do not contain copies of the space of α points provided with the cofinite topology). It was mentioned in [Gol] that a counterexample of Herrlich can easily witness that **Reg** has not sufficiently many sequentially closed spaces. We give here appropriate constructions which yield that neither **Reg** nor **Tych** have sufficiently many sequentially closed spaces, while **US** and **SUS** have sufficiently many sequentially closed spaces. For **Tych** and **SUS** the sequentially closed spaces are precisely the countably compact ones so that we discuss also the existence of certain countably compact or pseudocompact extensions. The principal results are announced in Section 1 and the proofs are given in Section 3. In Section 2 we offer a more general categorical setting based on closure operators.

Acknowledgments. Some aspects of the general problem considered here were discussed by the first named author and Iv. Gotchev. In particular, he obtained independently a proof of a part of Theorem 1.2. Thanks are due also to the referee for his helpful suggestions.

1. Main results.

If one takes as starting point the condition that a certain class of compact-like spaces in $\mathcal{P}$ should be sequentially $\mathcal{P}$-closed, then one gets the following separation axioms for $\mathcal{P}$, where $\mathbf{N}_\infty$ denotes the one-point Alexandroff compactification of the discrete space $\mathbf{N}$.

1.1 PROPOSITION.

A) *For a strongly epireflective subcategory $\mathcal{P}$ of* **Top** *the following conditions are equivalent:*

a) every sequentially compact (resp. countably compact) $\mathcal{P}$-space is sequentially $\mathcal{P}$-closed;

b) $\mathbf{N}_\infty$ *is sequentially $\mathcal{P}$-closed (resp. $\mathcal{P}$-closed);*

c) $\mathcal{P} \subseteq \mathbf{US}$ *(resp.* $\mathcal{P} \subseteq \mathbf{SUS}$*).*

B) *If* $\mathcal{P} = \mathbf{US}$ *(resp.* $\mathcal{P} = \mathbf{SUS}$*) then sequential $\mathcal{P}$-closedness coincides with sequential (resp. countable) compactness for $\mathcal{P}$-spaces.*

Proof. A) c) $\Rightarrow$ a) $\Rightarrow$ b) is obvious. Let b) hold. We show first that $\mathcal{P} \subseteq \mathbf{Top}_1$. Assume not, then either $\mathcal{P} = \mathbf{Top}$ or $\mathcal{P} = \mathbf{Top}_0$. It is easy to see that neither sequentially $\mathcal{P}$-closed nor $\mathcal{P}$-closed spaces exist. This contradicts b) and proves $\mathcal{P} \subseteq \mathbf{Top}_1$. To prove the sequentially compact version of the implication b) $\Rightarrow$ c) we assume that $\mathcal{P} \not\subseteq \mathbf{US}$. Then there exists a space $X \in \mathcal{P}$ and a converging sequence $\{x_n\}$ with two distinct limit points $x \neq y$. Note that the set $S = \{x_n : n \in \mathbf{N}\}$ is infinite since $X \in \mathbf{Top}_1$. Consequently $x, y \notin S$ can be assumed without loss of generality. Now the subspace Y of X with underlying set $\{x, y\} \cup S$ is in $\mathcal{P}$. Denote by Y' the space obtained from Y by declaring all points of S isolated. Then $Y' \in \mathcal{P}$, $Y' \backslash \{y\} \cong \mathbf{N}_\infty$ and $Y' \backslash \{y\}$ is σ-dense in Y'. This contradicts b).

To prove the version in brackets of the implication b) $\Rightarrow$ c) assume that $\mathcal{P} \not\subseteq \mathbf{SUS}$. Then there exists a space $X \in \mathcal{P}$ and a converging sequence $x_n \longrightarrow x$ with accumulation points $y \neq x$. With S as above the case of finite S is resolved as before. Otherwise define Y and Y' as before to obtain a similar contradiction with b).

B) This is Theorem 1.3 (resp. Theorem 1.4) in [DGo]. □

Let us note that a similar relation holds between the class of compact spaces and the category **Haus(Comp)**: this is the largest epireflective subcategory $\mathcal{P}$ of **Top** containing all compact spaces and such that they are $\mathcal{P}$-closed [Hof, Corollary 1.3 b)].

Proposition 1.1 explains our choice of **US** and **SUS** in the following

1.2 THEOREM. **SUS** *and* **US** *have sufficiently many sequentially* *closed spaces.*

The proofs of this and the next two theorems are given in §3.

1.3 THEOREM. **Tych** *has not sufficiently many sequentially closed spaces.*

We note that for **SUS** and **Tych** the problem of embeddings into sequentially closed spaces becomes a problem of embeddings into countably compact spaces. Let us note that this cannot be resolved for categories **Tqc** $\subseteq \mathcal{P} \subseteq$ **Haus** (in particular, for all $S(\alpha)$) because of the following

1.4 EXAMPLE. There exists a space $X \in$ **Tqc** which does not admit an embedding $X \hookrightarrow Y \in$ **Haus**, such that Y is countably compact. Take as X the Cantor set and fix a converging sequence $x_n \longrightarrow x$ in X such that $x \notin F = \{x_n : n \in \mathrm{N}\}$. Now equip X with the coarsest topology σ containing the usual topology τ of X and having the set F as a closed set. Now X is H-closed (=**Haus**-closed) since the semiregularization of σ is τ, so compact. On the other hand, F is a closed discrete set in (X, σ), so (X, σ) is not countably compact, thus there exists no countably compact Hausdorff space Y containing X as a subspace, since X would be a closed subspace of Y.

The above example gives no answer for the category **Tych**, in fact, even compact extensions are available here, in which the starting space is obviously κ-dense, with κ the compact closure [AF] (compare with 1.3). Let us emphasize, that in all these cases countable compactness yields sequential $\mathcal{P}$-closedness.

Arhangel'skij [A] showed that each Tychonov space X can be embedded in a Hausdorff countably compact space $\tilde{X}$ such that $t(X) = t(\tilde{X})$ and asked if every Tychonov space with countable tightness admits a countably compact Tychonov extension with countable tightness. Nogura showed [N] that there is a Tychonov first countable space which cannot be embedded in a Hausdorff countably compact Fréchet space. In particular, under the assumption $b = \omega_1$, he gave an example of a Tychonov first countable space which cannot be embedded in a regular countably compact T_1-space with countable tightness. The notion of *countably-compact-ification* was introduced by Morita [Mo] (a space Z is a countably-compact-ification of a space X if Z is countably compact, X is dense in X and every closed countably compact subset of X is closed also in Z). He also characterized the M-spaces admitting countably-compact-ifications; an example of a normal locally compact M-space without any countably-compact-ification was provided by D. Burke and van Douwen [BD].

Recall that a space X is said to be pseudocompact if every locally finite family of open sets is finite. It was shown by Gotchev [Go1] that sequentially $\mathcal{P}$-closed spaces are often pseudocompact. We have the following counterpart of Theorem 1.3 and Example 1.4 concerning pseudocompact extensions. As a by-product we obtain a new example witnessing that **Reg** has not sufficiently many sequentially closed spaces (see [Go1]).

1.5 THEOREM. *Every Hausdorff space X admits a pseudocompact Hausdorff extension Y such that X is σ-dense in Y. There exists a zero-dimensional space X having no pseudocompact regular extension Y such that X is σ^∞-dense in Y. In particular,* **Reg** *has not sufficiently many sequentially closed spaces.*

Note that even a closed subspace of a sequentially closed space need not be sequentially closed, therefore the existence of embeddings into sequentially closed spaces does not guarantee the existence of sufficiently many sequentially closed spaces in the sense of 1.1 as the case $\mathcal{P} =$ **Tych** shows.

2. A categorical look – absolutely closed spaces.

The notions of closedness and density with respect to an appropriate general notion of a closure operator give a possibility of a more general approach to absolute closedness. A *closure operator* C in **Top** assigns to every topological space X and each subset M of X a subset $c(M)$ of X according to the following

2.1 DEFINITION. *A collection of functions*

$$C = \{c_X : 2^X \to 2^X\}_{X \in \mathbf{Top}}$$

is a closure operator of **Top** *if:*

i) $(\forall A \subseteq X) c_X(A) \supseteq A$

ii) $c_X(\emptyset) = \emptyset$

iii) for each continuous map $f: X \longrightarrow Y$ *and* $A \subseteq X$ $f(c_X(A)) \subseteq c_X(f(A))$.

We write simply $c(A)$ instead of $c_X(A)$ when no confusion is possible. In the terminology of [DG1] (see also [DT]) these are grounded closure operators of **Top** with respect to the class of embeddings. Note that a closure operator need not be *idempotent* nor *additive* (i.e. $c(c(A)) = c(A)$ or $c_X(A \cup B) = c_X(A) \cup c_X(B)$, for $A, B \subseteq X$, need not always be true).

We say that a map $f: X \longrightarrow Y$ in **Top** is *C-dense* (resp. *C-closed*) if $c_Y(f(X)) = Y$ (resp. $c_Y(f(X)) = f(X)$).

The ordinary closure K in **Top**, defined by $K(M) = \overline{M}$ is a closure operator in the above sense, it is idempotent and has many other pleasant properties. Other examples are σ and κ mentioned in Section 1 and Velichko's θ-closure operator defined for any topological space X and $M \subseteq X$ as $cl_\theta M = \{x \in X$: each closed neighbourhood of x meets $M\}$ ([**V1**]).

A partial order between closure operators is defined by setting $C \leq D$ whenever $c(M) \subseteq d(M)$ always holds. Composition CD between closure operators C and D is defined by $cd(M) = c(d(M))$. For every ordinal α we define the α-th iteration of C as the closure operator C^α given recursively by $C^1 = C$, $C^{\alpha+1} = CC^\alpha$ and $c^\alpha(M) = \cup\{c^\beta(M) : \beta < \alpha\}$ for limit α. The *idempotent hull* C^∞ of a closure operator C is the finest idempotent closure operator coarser than C, it is defined by $c^{\infty}(M) = \cup\{c^{\alpha}(M) : \alpha \in \mathbf{Ord}\}$.

In analogy with sequentially $\mathcal{P}$-closed spaces, one can consider *absolutely* $(C,\mathcal{P})$*-closed spaces* $X \in \mathcal{P}$ such that for every $Y \in \mathcal{P}$ containing X as a subspace, X is C-closed in Y (so that absolutely $(\sigma,\mathcal{P})$-closed is the same as sequentially $\mathcal{P}$)-closed). This notion becomes especially significant when the closure operator C and the category $\mathcal{P}$ are related. In case $\mathcal{P} = \Delta(C)$ - the category of spaces with C-closed diagonal, the term *absolutely* C*-closed space* will be used ([DG3]). Now the absolutely σ-closed spaces are precisely the sequentially $\mathbf{US}$)-closed spaces since $\mathbf{US} = \Delta(\sigma)$.

The most significant class of examples was introduced by Salbany [S] who attached to any subcategory $\mathbf{A}$ of $\mathbf{Top}$ a *regular closure operator* $\mathbf{c_A}$ in the following way. For $X \in \mathbf{Top}$ and $M \in 2^X$ set

$$\mathbf{c_A}(M) = \cap\{Eq(f,g) : f,g : X \to A \in \mathbf{A}, f_{|M} = g_{|M}\}, \tag{1}$$

where $Eq(f,g) = \{x \in X : f(x) = g(x)\}$.

In case $C = \mathbf{c}_{\mathcal{P}}$ is the regular closure operator associated to a strongly epireflective subcategory $\mathcal{P}$ one can show that $\mathcal{P} = \Delta(\mathcal{P})$ ([GH]). Here we adopt the term *absolutely* $\mathcal{P}$*-closed space* instead of absolutely $\mathbf{c}_{\mathcal{P}}$-closed ([DG2]). If K denotes the usual Kuratowski closure, then $\Delta(K) = \mathbf{Haus}$ and $\mathbf{c_{Haus}} = K$ on $\mathbf{Haus}$, so all three notions - absolutely $(K,\mathbf{Haus})$-closed space, absolutely K-closed space and absolutely $\mathbf{Haus}$-closed space coincide here with the notion of H-closed space ([AU]).

In case $\mathcal{P} = T_0$ the regular closure $F = \mathbf{c}_{\mathcal{P}}$ is the well known front closure ([B]) and absolutely T_0-closed spaces coincide with absolutely F-closed spaces, and coincide with the sober spaces ([DG3]).

The absolutely $\mathbf{Haus(Comp)}$-closed spaces were characterized by Gotchev [Go2]. In particular, he showed, that every $\mathbf{Haus(Comp)}$-closed space is compact, while there exists non-compact absolutely

Haus(Comp)-closed space. Let us mention, that the regular closure of **Haus(Comp)** coincides with κ^∞ in **Haus(Comp)**, where κ is (as before) the compact-closure [AS]. Moreover, due to a recent result in [D2], **Haus(Comp)** $= \Delta(\kappa)$, thus absolutely κ-closed spaces coincide with absolutely **Haus(Comp)**-closed spaces.

The above observation can be generalized by means of the notion of a semiregular closure operator C (see Definition 2.2 below). Semiregularity is a good substitute of regularity – the natural closure operators available in **Top** are usually semiregular and rarely happen to be regular. For a space X and subspace M of X e denote by $X \coprod_M X$ the adjunction space with respect to M and by

$$(2) \qquad k_1, k_2 : X \longrightarrow X \coprod_M X$$

the canonical embeddings of X in $X \coprod_M X$.

2.2 DEFINITION. (*a*) ([D2], Definition 4.2) *A closure operator ϱ is semiregular, if for every $X \in \Delta(C)$ and C-closed subspace M of X the morphisms k_1 and k_2 in (2) are C-closed.*

(*b*) ([D2], Definition 4.3) *For $X \in$ **Top** and $M \in 2^X$ set in the notation of (2) $\tilde{c}(M) = k_1^{-1}(c(k_2(X)))$.*

If the closure operator C is additive, then $\tilde{C}$ is a semiregular closure operator; moreover, C is semiregular iff $C^\infty = \tilde{C}^\infty$ ([D2], Corollary 4.11 and Lemma 4.4). Every regular closure operator is semiregular ([D2], Lemma 4.1). Most of the known closure operators are semiregular (for a general approach to semiregularity and further examples and non-examples see Section 6 in [D2]).

2.3 THEOREM. *Let C be a closure operator of* **Top**. *Then every absolutely $\Delta(C)$-closed space is absolutely C-closed.*

a) If C is additive, then absolutely $\Delta(C)$-closed spaces and absolutely $\tilde{C}$-closed spaces coincide.

b) If C is additive and semiregular, then absolutely C-closed spaces *and absolutely $\Delta(C)$-closed spaces coincide.*

Proof. The first part follows from the fact that every $\Delta(C)$-closed subspace is also C-closed (cf. Lemma 2.1 in [D2]).

a) According to a general result in [D2, Theorem 4.8] for an additive closure operator C, $\mathbf{c}_{\Delta(C)}$ coincides with $\tilde{C}^{\infty}$ on $\Delta(C)$.

b) Assume C is additive and semiregular. According to Theorem 4.9 of [D2] $\mathbf{c}_{\Delta(C)}$ coincides with C^{∞} on $\Delta(C)$. This proves b). □

Let us mention that all particular cases considered above are corollaries of this theorem, since all three closure operators were additive and semiregular. According to [D2, Section 6] θ-closure is semiregular, so item b) of Theorem 2.1 yields that the absolutely θ-closed spaces coincide with the absolutely **Ury**-closed spaces (note that $X \in$ **Ury** iff the diagonal of $X \times X$ is θ-closed). According to [DG2] a Urysohn space X is absolutely **Ury**-closed iff each open U-cover of X admits a finite subcover (a cover $\{U_\alpha\}$ of X is a U-cover if the family $\{X \backslash cl_\theta(X \backslash U_\alpha)\}$ is still a cover of X). An extension of this characterization to $S(n)$ (resp. $S(\alpha)$) can be found in [DG2] (resp. [D1]).

For non-semiregular C, item b) of Theorem 2.3 is no more true as Example 2.5 in [DGo] shows with $C = \sigma$: there exists a sequentially compact **US**-space which is not absolutely **US**-closed (note that sequentially compact **US**-spaces are sequentially **US**-closed, i. e. absolutely σ-closed).

If $\mathcal{P}$ is such that $\mathbf{c}_{\mathcal{P}}$ coincides with K on $\mathcal{P}$, then necessarily $\mathcal{P} \subseteq$ **Haus** ([S]). Now obviously absolutely $\mathcal{P}$-closed spaces coincide with $\mathcal{P}$-closed ones. This occurs for $\mathcal{P} =$ **Haus**, **Reg**, **Tych**, **0-dim**.

In general for $\mathcal{P} \subseteq$ **Haus** the closure operator $\mathbf{c}_{\mathcal{P}}$ is coarser than K, that is why every absolutely $\mathcal{P}$-closed spaces is $\mathcal{P}$-closed. The next example shows that this may occur independently on the fact that $\mathbf{c}_{\mathcal{P}}$ coincides with K on $\mathcal{P}$.

2.4 EXAMPLE. A space $(X,\tau) \in \mathbf{FHaus}$ is **FHaus**-closed iff the space (X,τ_w) is compact, where τ_w denotes the (Tychonov) topology on X generated by all continuous real-valued functions of (X,τ) ([Ba] or [BPS]). In such a case (X,τ) is also absolutely **FHaus**-closed. In fact, if (X,τ) is a subspace of $(Y,\tau') \in \mathbf{FHaus}$, then the inclusion map $i : (X,\tau_w) \hookrightarrow (Y,\tau'_w)$ is continuous, so that the compactness of (X,τ_w) implies i is an embedding. Again by the compactness of (X,τ_w) it follows that $i(X)$ is closed in (Y,τ'_w). It remains to note that $\mathbf{c}_{\mathbf{FHaus}}$ coincides on Y with τ'_w ([DT]).

In most of the known cases the absolutely $(C,\mathcal{P})$-closed spaces are stable under taking continuous images (with sober spaces being an exception). It should be noted that for topological groups or modules this property is heavily missing even in the case of K (see [DTö] or [DU]).

Finally, one can extend the problem of the existence of sufficiently many sequentially closed spaces to the general case by means of the following

2.5 DEFINITION. *Let C be a closure operator of* **Top**. *The class $\mathcal{P}$ has enough absolutely $(C,\mathcal{P})$-closed spaces if for each $X \in \mathcal{P}$ there exists an embedding $X \hookrightarrow Y$ such that Y is absolutely $(C,\mathcal{P})$-closed and X is C^∞-dense in Y.*

3. Proofs of the main theorems.

In this Section we consider only T_1-spaces. Recall that $\mathbf{Top_1}$ contains the categories **US**, **SUS**, **Haus**, **Reg**, **Tych** and **0-dim** which are relevant to our main results.

3.1 LEMMA. *For every topological space X there exist topological spaces $\tilde{X}$ and $\widehat{X}$ containing X as a subspace, such that :*

a) X is σ-dense in $\tilde{X}$ and $\widehat{X}$;

b) every sequence in X has a subsequence converging in $\widehat{X}$;

c) every (countably) infinite set in X has an accumulation point in $\tilde{X}$;

*d) if $X \in$ **US**, then $\widehat{X} \in$ **US**;*

*e) if $X \in$ **SUS**, then $\tilde{X} \in$ **SUS**.*

Moreover, X is sequentially compact (resp. countably compact) iff $X = \widehat{X}$ (resp. $X = \tilde{X}$).

Proof. Consider the family $\mathcal{S}_X$ of all countable subsets of X with no converging subsequences (hence sequentially closed in X) and let $\mathcal{D}_X$ be the subfamily of $\mathcal{S}_X$ consisting of all discrete closed countable subsets D of X. Clearly X is sequentially compact (resp. countably compact) iff $\mathcal{S}_X = \emptyset$ (resp. $\mathcal{D}_X = \emptyset$). Fix a maximal almost disjoint family $\mathcal{N}$ in $\mathcal{D}_X$ and choose a maximal almost disjoint family $\mathcal{M}$ in $\mathcal{S}_X$ containing $\mathcal{N}$. Now define $\widehat{X}$ to be the set $X \cup \mathcal{M}$ equipped with the following topology: X is open in $\widehat{X}$ and basic neighbourhoods of points $D \in \mathcal{M}$ are $\{D\} \cup U$ where U is an open subset of X such that $D \backslash U$ is finite. Now let $\tilde{X}$ be the subspace of $\widehat{X}$ with underlying set $X \cup \mathcal{N}$.

Clearly, for every $D \in \mathcal{M}$ and any enumeration of D, the sequence D converges to the point $\{D\}$ of $\widehat{X}$. This proves a). For every sequence S in X with no converging subsequences, there exists a $D \in \mathcal{M}$ which has an infinite intersection with S. Hence the subsequence $D \cap S$ of S converges in $\widehat{X}$. This proves b). Analogously c) can be proved.

Assume that $X \in$ **US**. To prove that also $\widehat{X} \in$ **US** consider a converging sequence $x_n \longrightarrow x$ in $\widehat{X}$. If $x = \{D\} \in \mathcal{M}$, then according to Lemma 1.2 in [DGo] the sequence $\{x_n\}$ is definitely contained in D. Thus no other point $\{D'\} \in \mathcal{M}$ can be a limit point of the sequence. On the other hand, by the choice of $\{D\} \in \mathcal{M}$ the sequence $\{x_n\}$ has no converging subsequences in X, so in particular it has no limit points in X.

If $x \in X$, then by $X \in$ **US** the sequence has no other limit points in X. The above argument shows that such a sequence cannot have limit points out of X. Therefore $\widehat{X} \in \mathbf{US}$.

Now assume that $X \in \mathbf{SUS}$. To prove that also $\tilde{X} \in \mathbf{SUS}$ consider a converging sequence $x_n \longrightarrow x$ in $\tilde{X}$. If $x = \{D\} \in \mathcal{N}$, then according to Lemma 1.2 in [DGo] the sequence $\{x_n\}$ is definitely contained in D. Thus no other point $\{D'\} \in \mathcal{N}$ can be an accumulation point of the sequence and by the choice of $\{D\} \in \mathcal{N}$ the sequence $\{x_n\}$ has no accumulation points in X. Thus $\tilde{X} \in \mathbf{SUS}$ in this case.

If $x \in X$, then by $X \in \mathbf{SUS}$ the sequence has no other accumulation points in X. Thus the subset $F = \{x\} \cup \{x_n : n \in N\}$ of X is closed and for each $D \in \mathcal{N}$ the intersection $F \cap D$ is finite. Thus for the open subset $U = X \backslash F$ of X the set $\{D\} \cup U$ is a neighbourhood of $\{D\}$ avoiding the sequence $\{x_n\}$. This argument shows that the sequence $\{x_n\}$ cannot have accumulation points out of X. Therefore $\tilde{X} \in \mathbf{SUS}$. □

There is no hope to establish a stronger separation property of $\tilde{X}$ even if $X \in \mathbf{Haus}$ (take for example the space X in Example 1.3). On the other hand, $X \in \mathbf{SUS}$ does not guarantee $\widehat{X} \in \mathbf{SUS}$. In fact, for every compact non sequentially compact Hausdorff space X (for example $X = \beta\mathrm{N}$, the Stone-Čech compactification of N) $X = \tilde{X} \neq \widehat{X}$ and obviously the latter space is not even $\mathbf{SUS}$ since X is sequentially $\mathbf{SUS}$-closed and σ-dense in $\widehat{X}$.

Proof of Theorem 1.2.

For a space X consider the ordinal chains of spaces $\{X_\alpha\}$ and $\{X^\alpha\}$ defined as follows:

a) $X_0 = X$, $X_{\alpha+1} = \tilde{X}_\alpha$ and $X_\beta = \bigcup\{X_\alpha : \alpha < \beta\}$ for limit β has the final topology with respect to all inclusions $X_\alpha \hookrightarrow X_\beta$, $\alpha < \beta$;

b) $X^{(0)} = X$, $X^{(\alpha+1)} = \widehat{X^{(\alpha)}}$ and $X^{(\beta)} = \bigcup\{X^{(\alpha)} : \alpha < \beta\}$ for limit β has the final topology with respect to all inclusions $X^{(\alpha)} \hookrightarrow$

$X^{(\beta)}$, $\alpha < \beta$.

The space X_{ω_1} is countably compact, while $X^{(\omega_1)}$ is sequentially compact. In fact, let $Z \subseteq X_{\omega_1}$ be countably infinite. Then $Z \subseteq X_\alpha$ for some $\alpha < \omega_1$. By Lemma 3.1 c) Z has an accumulation point in $X_{\alpha+1}$. Analogously, if $Z \subseteq X^{(\omega_1)}$ then $Z \subseteq X^{(\alpha)}$ for some $\alpha < \omega_1$. By Lemma 3.1 b) Z has a converging subsequence in $X^{(\alpha+1)}$.

If $X \in$ **SUS** (resp. $X \in$ **US**), then every $X_\alpha \in$ **SUS** (resp. $X^{(\alpha)} \in$ **US**). We prove it by transfinite induction. The step from X_α to $X_{\alpha+1}$ (resp. from $X^{(\alpha)}$ to $X^{(\alpha+1)}$) follows from the above lemma. If for a limit ordinal β all $X_\alpha \in$ **SUS** (resp. $X^{(\alpha)} \in$ **US**) for $\alpha < \beta$, then for every converging sequence $x_n \longrightarrow x$ in X_β (resp. $X^{(\beta)}$) there exists $\alpha < \beta$ such that $x \in X_\alpha$ (resp. $x \in X^{(\alpha)}$), so the inductive hypothesis applies since X_α (resp. $X^{(\alpha)}$) is open in X_β (resp. $X^{(\beta)}$), so contains almost all x_n. □

Proof of Theorem 1.3.

The main step of the proof given below implies that for no epireflective subcategory $\mathcal{P}$ of **Top** containing a non-singleton space and consisting of Hausdorff spaces, the following extension problem has a solution:

For every $X \in \mathcal{P}$ there exists a countably compact space $Y \in \mathcal{P}$ containing X as a σ^∞-dense subspace.

Main Step. *There exists a Tychonov zero-dimensional pseudocompact space X such that every countably compact space Y containing X as a σ^∞-dense subspace is non Hausdorff.*

Fix a regular cardinal $\rho > 2^{\mathfrak{c}}$ and set $Z = \beta\rho$. Let $\mathcal{D}$ be a disjoint family of countable subsets of ρ with $|\mathcal{D}| = \rho$. For each $D \in \mathcal{D}$ fix a countable discrete set $S_D \subset \overline{D}^Z \setminus D$ and put $V_D = \overline{S_D}^Z \setminus S_D$. Let

$$W = \bigcup\{V_D : D \in \mathcal{D}\} \text{ and } X = Z \setminus W.$$

Claim 1. The space X is pseudocompact.

Proof. Suppose X is not pseudocompact, then there exists a continuous unbounded function $f : X \to \mathbf{R}$. It is easy to conclude that for some $D_0 \in \mathcal{D}$ the restriction of f on $S_0 = S_{D_0}$ is unbounded. We may and shall assume that there are $s_n \in S_0$ and neighbourhoods U_n of s_n such that :

1) $f(\overline{U_n}) \subseteq [n, +\infty)$;

2) $\forall n \in \omega$ U_n is clopen;

3) $n \neq m \Rightarrow U_n \cap U_m = \emptyset$;

4) $\overline{\cup\{U_n : n \in \omega\}}^Z \cap S_0 = \{s_n : n \in \omega\}$.

For each $n \in \omega$, pick up $q_n \in U_n \cap \rho$. Put $Q = \{q_n : n \in \omega\}$. Because of 3) and 4), $\overline{Q}^Z \cap \overline{S_0}^Z = \emptyset$, so $\overline{Q}^Z \subseteq X$ which is impossible by 1). □

Assume that Y is a countably compact space containing X as a σ^∞-dense subspace.

a) If some point $y \in Y \setminus X$ belongs to infinitely many sets $\overline{S_{D_n}}^Y$, then Y is not Hausdorff.

Let $D_1, ..., D_n, ...$ be distinct members of $\mathcal{D}$ such that $y \in \overline{S_{D_n}}^Y$ for each n. Let $\mathcal{F}_n$ be the filter on S_{D_n} of traces of the neighbourhoods of y on S_{D_n} and let Φ_n be an ultrafilter on S_{D_n} containing $\mathcal{F}_n$. Denote by z_n the point of V_{D_n} corresponding to this ultrafilter. Then the closure A of the set $\{z_n : n \in \mathrm{N}\}$ taken in Z is not contained in W. In fact, assume, that $A \subseteq W$, then clearly A is contained also in the (disjoint) union of the clopen sets $\overline{D_n}$. Then by the compactness of A only finitely many of them cover A - a contradiction in view of $z_n \notin \overline{D_k}$ for $k \neq n$.

Choose $p \in A \setminus W$, then $p \in X$, while $y \notin X$, so that $y \neq p$. Let us see now that these points cannot be separated by disjoint neighbourhoods in Y. In fact, let V be an open neighbourhood of p in

Y. Then there exists an open neighbourhood V' of p in Z such that $V' \cap X = V$. Now $z_n \in V'$ for some $n \in \mathbf{N}$. Therefore, being V' also an open neighbourhood of z_n we have $V' \cap S_{D_n} \in \Phi_n$. On the other hand, each open neighbourhood U of y satisfies $U \cap S_{D_n} \in \mathcal{F}_n \subset \Phi_n$. Hence $U \cap V \supseteq U \cap V \cap S_{D_n} \neq \emptyset$.

From now on we assume that every point $y \in Y \backslash X$ belongs to finitely many sets $\overline{S_D}^Y$. This yields

$$|Y \backslash X| \geq \rho. \tag{1}$$

In fact, by the countable compactness of Y, there exists a point $y_D \in \overline{S_D}^Y \backslash S_D$ for every $D \in \mathcal{D}$. By our assumption the correspondence $\mathcal{D} \longrightarrow Y \backslash X$ defined by the assignment $D \mapsto y_D$ is finitely many-to-one, hence (1) results by the choice of $\mathcal{D}$.

Next we prove that:

b) if there are infinitely many $D \in \mathcal{D}$ such that there exists a sequence T_D in S_D converging in Y, then Y is not Hausdorff.

Assume that $D_1, ..., D_n, ...$ are distinct members of $\mathcal{D}$ such that for each n there exists a sequence T_n in S_{D_n} converging in Y. Put $t_n = \lim T_n$ and let v be an accumulation point of the set $\{t_n : n \in \mathbf{N}\}$. Arguing as above we find a point $p \in A \backslash W$, where

$$A = \bigcap \{ \overline{\bigcup \{T_n : n \in F\}}^Z : F \in \mathcal{F}\},$$

$\mathcal{F}$ being the filter on $\mathbf{N}$ generated by the filter base $F(U) = \{n : T_n \backslash U \text{ is finite}\}$, where U is a neighbourhood of v in Y. If $p = v$ we chose $p' \in A \backslash W$ and $p' \neq v$. This is possible, since the argument showing that $A \not\subseteq W$ gives also $|A \backslash W| > 1$. So we obtain again two distinct points in Y which cannot be separated by disjoint neighbourhoods.

From now on we assume that there are finitely many $D \in \mathcal{D}$ such that there exists a sequence T_D in S_D converging in Y. Denote by $\mathcal{D}'$ this finite subset of $\mathcal{D}$.

To finish the proof assume that Y is Hausdorff, we will show that then $|Y\backslash X| \leq 2^{\mathfrak{c}}$ results, in contradiction with (1).

Now we use that fact, that $Y = \sigma_Y^\infty(X)$ and we show, that $Y\backslash X$ is contained in the closure of the countable set $E = \bigcup\{S_D : D \in \mathcal{D}'\}$. Therefore, $|Y\backslash X| \leq 2^{\mathfrak{c}}$, since E is countable and Y is Hausdorff by assumption.

Pick $y \in Y\backslash X$, then $y \in \sigma_Y^\infty(X)$. Since $K = K^\infty = KK^\infty \geq K\sigma^\infty$, $y \in \sigma_Y^\infty(X)$ yields $y \in K\sigma(X)$. Hence each open neighbourhood U of y in Y meets $\sigma_Y(X)$. If $z \in U \cap \sigma_Y(X)$, and $z = \lim x_n, x_n \in X$, then almost all x_n are in U. This proves that every open neighbourhood U of y in Y contains a sequence T in X converging in Y. Since Y is Hausdorff such a sequence can be contained only in $\cup\{S_D : D \in \mathcal{D}\}$. It is not possible to have all intersections $T \cap S_D$ finite, since then T would have also an accumulation point in X, which contradicts Hausdorffness of Y. Thus T must be contained in a finite union of the sets S_D. On the other hand, if $T \cap S_D$ is infinite for some $D \in \mathcal{D}$, then clearly $D \in \mathcal{D}'$. This proves, that $y \in \overline{E}$. □

Remark. In the next proof the countable compactness of Y, used in the proof of (1) to produce the points y_D, will be weakened to pseudocompactness (with X replaced by a subspace X_1).

We have produced a zerodimensional pseudocompact space X which shows that the countably compact extension problem mentioned in the beginning of the proof of Theorem 1.4 has no solution. To a completely different effect leads the assumption that Y is pseudocompact. In this case we can prove that for every Tychonov space X there exists a pseudocompact Hausdorff extension $Y = P_X$ such that X is sequentially dense in P_X.

Proof of Theorem 1.5. Here we adopt the following characterization of pseudocompact spaces: X is pseudocompact iff every locally finite family of non-empty open sets in X is finite.

Let $\mathcal{L}_X$ denote the set of all locally finite countably infinite families $D = \{D_n\}_{n=1}^{\infty}$ of open pairwise disjoint sets of X. Note that X is pseudocompact iff $\mathcal{L}_X = \emptyset$. For each $D \in \mathcal{L}_X$ define a one-point extension X_D of X by letting X to be open in X_D and declaring the extra-point $p(D)$ of X_D to have basic neighbourhoods $U_n = p(D) \cup \bigcup_{k>n} D_k$. Define analogously the extension $X_{\mathcal{F}}$ when $\mathcal{F}$ is a subfamily of $\mathcal{L}_X$ – now $X_{\mathcal{F}} \setminus X$ is discrete and for $D \in \mathcal{L}_X$ the subspace $X \cup \{p(D)\}$ of $X_{\mathcal{F}}$ has the same topology as X_D. One can easily prove that X is σ-dense in $X_{\mathcal{F}}$, as well as the following:

Claim 2. If X is Hausdorff and $D, D' \in \mathcal{L}_X$ then the following assertions are equivalent:

a) the extension $X_{\{D,D'\}}$ is Hausdorff.

b) $D \in \mathcal{L}_{X_{\{D'\}}}$, i.e., the family D is locally finite in $X_{\{D'\}}$.

b′) $D' \in \mathcal{L}_{X_{\{D\}}}$.

c) there exist n, m such that $(\bigcup_{k>n} D_k) \cap \bigcup_{k>m} D'_k = \emptyset$.

For $D, D' \in \mathcal{L}_X$ satisfying the above equivalent condition we say briefly: D and D' are *almost disjoint.* Families $\mathcal{F} \subseteq \mathcal{L}_X$ consisting of pairwise almost disjoint members will be called *almost disjoint.* By a standard application of Zorn's lemma one can prove that every almost disjoint family $\mathcal{F} \subseteq \mathcal{L}_X$ is contained into a *maximal almost disjoint family* $\mathcal{M} \subseteq \mathcal{L}_X$, i.e., there exits no almost disjoint family $\mathcal{N} \subseteq \mathcal{L}_X$ properly containing $\mathcal{M}$. Now $X_{\mathcal{M}}$ is pseudocompact since $\mathcal{L}_{X_{\mathcal{M}}} = \emptyset$, i.e. every locally finite family of open sets of $X_{\mathcal{M}}$ is finite.

Now take ρ, Z and $\mathcal{D}$ as in the proof of Theorem 1.3. Set

$$X_1 = Z \setminus \bigcup \{\overline{D}^Z \setminus D : D \in \mathcal{D}\}.$$

The space X_1 is obviously a subspace of the space X defined in the proof of Theorem 1.4, so that X_1 is zero-dimensional. Now we prove that X_1 does not admit a pseudocompact Urysohn extension Y such that X_1 is σ^∞-dense in Y.

First we prove the following :

Claim 3. *There are finitely many $D \in \mathcal{D}$ such that there exists a sequence T_D in D converging in Y.*

Assume that $D_1, ..., D_n, ...$ are distinct members of $\mathcal{D}$ such that there exists for each n a sequence $T_n \to t_n$, $T_n \subseteq D_n$, converging in Y. Let $T_n = \{w_j^n : j \in \omega\}$ and put $O_n = \{w_{2j+1}^n : j \in \omega\}$, $E_n = \{w_{2j}^n : j \in \omega\}$. For each n, pick ultrafilters $\sigma_n \in \overline{O_n}^Z \setminus O_n$ and $\eta_n \in \overline{E_n}^Z \setminus E_n$. Choose an ultrafilter $\mathcal{F}$ on N and define $\sigma = \Sigma_{\mathcal{F}}\sigma_n$ and $\eta = \Sigma_{\mathcal{F}}\eta_n$. Then $\{\sigma, \eta\} \subseteq X_1$. Moreover, σ and η cannot be separated by disjoint closed neighbourhoods in Y. In fact, take a neighbourhood U of σ and a neighbourhood V of η in Y. Then there is $n_0 \in \omega$ such that $|O_{n_0} \cap U| = |E_{n_0} \cap V| = \omega$, so we obtain $t_{n_0} \in \overline{U} \cap \overline{V}$.

As in the proof of Theorem 1.4 we can prove that every point $y \in Y \setminus X_1$ belongs to finitely many sets $\overline{D}^Y$ (replace S_D in that proof by D). This yields (in analogy to (1)) $|Y \setminus X_1| \geq \rho$. Now the countable compactness of Y, used in the proof of (1) to produce the points y_D, is replaced by pseudocompactness. In fact, every $D \in \mathcal{D}$ is clopen and discrete in X. Since Y is T_1 and X_1 is dense in Y, each point of D is open in Y, so that D is actually a countable family of open subsets of Y. By the pseudocompactness of Y there exists a point $y_D \in \overline{S_D}^Y \setminus S_D$. By our previous remark, the correspondence $\mathcal{D} \longrightarrow Y \setminus X$ defined by the assignment $D \mapsto y_D$ is finitely many-to-one, hence $|Y \setminus X_1| \geq \rho$ results by the choice of $\mathcal{D}$.

Claim 3 immediately yields $|Y \setminus X| \leq 2^{\mathfrak{c}}$ in contradiction to what we have just established.

To finish the proof of Theorem 1.5 it remains to note that sequentially **Reg**-closed spaces are pseudocompact ([Go1]). □

Remark. Let us note that the space X_1 admits a pseudocompact Urysohn (in fact, zerodimensional) extension, namely the space X. However, X_1 is σ-closed in X.

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