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ON THE MONADICITY OVER GRAPHS OF CATEGORIES WITH LIMITS

*by G.M. KELLY and I.J. LE CREURER**

Résumé : Donnée une classe petite $\mathcal{M}$ de catégories petites, notons $\mathbf{Cat}_{\mathcal{M}}$ la catégorie dont les objets sont les petites catégories qui admettent, pour tout $M \in \mathcal{M}$, des M-limites (choisies), et dont les morphismes sont les foncteurs qui préservent (strictement) ces limites; notons **Gph** la catégorie des graphes (petits); et notons $U : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Gph}$ le foncteur d'oubli qui envoie une catégorie à M-limites sur son graphe sous-jacent. Pour certaines classes $\mathcal{M}$ il est connu que ce foncteur U est monadique; mais les démonstrations emploient pour chacune de ces $\mathcal{M}$ une astuce différente. Nous démontrons que U est au moins "de descente" si chaque $M \in \mathcal{M}$ est une catégorie librement engendrée par un graphe, et que U est alors monadique quand ce graphe est acyclique.

1 Introduction

As in the abstract above, we consider a small class $\mathcal{M}$ of small categories, and write $\mathbf{Cat}_{\mathcal{M}}$ for the category whose objects are small categories with (chosen) M-limits for each $M \in \mathcal{M}$, and whose morphisms are those functors which strictly preserve these chosen limits.

Although functors preserving chosen limits *strictly* would seem to be of scant mathematical interest, there is a reason for considering them: namely, the monadicity of the forgetful functor $W : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$.

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This monadicity, already studied by Lair in [8], [9], and [10], is a special case of the monadicity of the forgetful functor for structures given by everywhere-defined operations (with arities in the base category), subjected to equations between derived operations. A modern account of this monadicity, adapted to our present context, was sketched in [6, Section 8] and developed more fully in [7]; but these ideas are also related to the notion of "algebras over a syntax on a base-category", initiated by Coppey in [3] and developed further by Coppey and Lair in [4].

In the present case, the structure we are to place on a category A is that of having M-limits for each $M \in \mathcal{M}$. To give these is just to give a right adjoint $L : A^M \to A$ to the diagonal $\Delta : A \to A^M$. Equivalently, we are to give a functor $L : A^M \to A$, a natural transformation (the unit of the adjunction) $\rho : 1 \to L\Delta$, and a natural transformation (the counit) $\sigma : \Delta L \to 1$, satisfying the "triangular equations" $L\sigma.\rho L = 1$ and $\sigma\Delta.\Delta\rho = 1$. To give the functor L on objects is to give an object $L(d)$ of A for each "diagram" $d : M \to A$; this is an (object-type) operation of arity M. To give L on morphisms is to give an arrow $L(\phi) : L(d) \to L(e)$ in A for each arrow $\phi : d \to e : M \to A$ in A^M; since ϕ may be seen as a diagram $M \times \mathbf{2} \to A$, where $\mathbf{2}$ is the arrow category $\{0 \to 1\}$, this is an (arrow-type) operation of arity $M \times \mathbf{2}$. The equations (between arrows) $L(\psi\phi) = L(\psi)L(\phi)$ and $L(1) = 1$, asserting the functoriality of L, are of the respective arities $M \times \mathbf{3}$ and M, where $\mathbf{3}$ is the ordered set $\{0 \to 1 \to 2\}$. To give the components $\rho(a) : a \to L\Delta(a)$ of ρ for $a \in A$ is to give an arrow-type operation of arity $\mathbf{1}$, since a is equally a diagram $a : \mathbf{1} \to A$; and the naturality condition for ρ with respect to $f : a \to b$ in A is an arrow type equation of arity $\mathbf{2}$. Similarly, to give the components $\sigma(d) : L\Delta(d) \to d$ of σ is to give an arrow-type operation of arity M, while the naturality of σ is an arrow-type equation of arity $M \times \mathbf{2}$. Finally, the triangular equations are arrow-type equations of respective arities M and 1. This gives the desired monadicity of $W : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$, since the class $\mathcal{M}$ is small. (If it were not, W might fail to have a left adjoint H; in other words, the free $\mathcal{M}$-complete category HA on a small category A might fail to be small.)

In fact the results in [7] are given for *enriched* categories; and a similar argument to that above can be carried out in the **Cat**-enriched world, leading to the conclusion that the forgetful $W : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$, now seen as a 2-functor between 2-categories, is 2-monadic. Structures defined by 2-monads were extensively studied in [1], wherein are developed the chief results on the more-interesting *non-strict* morphisms of such structures, using the strict ones as a necessary starting-point to produce the 2-monad. However we do not follow this direction in the present paper: for here we shall be concerned with the composite forgetful functor $U = VW : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Gph}$, where **Gph** is the category of (small) graphs and where $V : \mathbf{Cat} \to \mathbf{Gph}$ sends a category to its underlying graph; and in this context it is pointless to treat $\mathbf{Cat}_{\mathcal{M}}$ and **Cat** as 2-categories, since **Gph** is a mere category with no non-trivial 2-cells.

The study of categories monadic over **Gph** was initiated by Lair in [9] and [10] and by Burroni in [2], and continued by various authors in [6], [11], [4], and [5]. In particular, $V : \mathbf{Cat} \to \mathbf{Gph}$ is monadic; let us write G for its left adjoint, and $S = VG$ for the corresponding monad on **Gph**. Our present concern is with the monadicity, for various $\mathcal{M}$, of $U = VW : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Gph}$. Let us write H for the left adjoint of W, and $R = WH$ for the corresponding monad on **Cat**; then U has the left adjoint $F = HG$, with unit say $\eta : 1 \to UF$ and counit $\varepsilon : FU \to 1$. We write T for the monad UF on **Gph**, and $K : \mathbf{Cat}_{\mathcal{M}} \to T$–Alg for the comparison functor to the category of T-algebras; so that U is monadic precisely when K is an equivalence. Recall that the functor U is said to be *of descent type* when K has the weaker property of being fully faithful. We may write $U_{\mathcal{M}}$ for U when we wish to emphasize $\mathcal{M}$.

It follows from the results of [7] that $U = U_{\mathcal{M}}$ is monadic precisely when the structure of an $\mathcal{M}$-complete category A with the underlying graph X can be expressed in terms of everywhere-defined operations on X, and equations between derived operations, the arities now being *graphs*. This is equally to ask that the existence of $\mathcal{M}$-limits in a category A should be expressible by operations on A, and equations between derived operations, each of whose arities is *a free category on a graph*. Note that the presentation by operations and equations given in the second paragraph of this Introduction has free categories for its arities *only* when each $M \in \mathcal{M}$ is a *discrete* category; this allows us to infer the monadicity of U when, for example, the class of $\mathcal{M}$-limits is that of finite products - a case discussed by Lair in [9] and [10]. Yet U may well be monadic in cases where not every $M \in \mathcal{M}$ is discrete: the point is that one may be able to find a *different* presentation of $\mathcal{M}$-limits by operations and equations, this time with each arity a free category on a graph. Burroni did this in [2] for the case of terminal objects (covered by the above, since here M is empty and hence discrete), but also for the case of pullbacks, where M is $\{\bullet \longrightarrow \bullet \longleftarrow \bullet\}$; this latter case was also treated by Dubuc and Kelly [6], and was revisited by Cury in [5]. It follows from the first sentence of this paragraph that, when the class $\mathcal{M}$ is a union $\bigcup \mathcal{M}_i$ for which each $U_{\mathcal{M}_i} : \mathbf{Cat}_{\mathcal{M}_i} \to \mathbf{Cat}$ is monadic, then $U_{\mathcal{M}} : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$ is monadic. Thus we may conclude from Burroni's results that U is monadic when $\mathcal{M}$ consists of all finite categories (so that to admit $\mathcal{M}$-limits is to admit all finite limits); or equally where $\mathcal{M}$ consists of all *finitely presentable* categories, since this gives the same category $\mathbf{Cat}_{\mathcal{M}}$.

However, if $U_{\mathcal{M}}$ is monadic, the results of [7] do not allow us to conclude that $U_{\mathcal{N}}$ is monadic when $\mathcal{N} \subset \mathcal{M}$; and indeed, as we shall see, there is a finite category P for which $U_{\{P\}}$ not only fails to be monadic, but is not even of descent type. Thus if E is the finite category $\{\bullet \rightrightarrows \bullet\}$, so that $\mathbf{Cat}_E$ consists of categories with equalizers, we cannot conclude from Burroni's results that $U_{\{E\}}$ is monadic; yet it is so, a suitable presentation with free categories for its arities having been given by MacDonald and Stone in [11]. Indeed, Coppey and Lair have established in [4] the monadicity of U when $\mathcal{M}$ is any class of finitely-presentable categories *containing* the category E above.

During a visit to Sydney in early 1995, Carboni raised the question of these positive monadicity results: is each one, with its finding of a suitable presentation, an isolated "clever trick", or is there a more rational common basis to them? The present article provides at least a partial answer, by proving the following four results – the last of which contains all the known positive cases of monadicity:

(A) There is a finite category P for which $U_{\{P\}}$ is not of descent type.

(B) However $U_{\mathcal{M}}$ is of descent type whenever each category $M \in \mathcal{M}$ is the free category GX on a graph X.

(C) Even then, $U_{\mathcal{M}}$ need not be monadic; in fact $U_{\{GY\}}$ is not monadic when Y is the graph with one vertex $*$ and one edge $y : * \to *$.

(D) Yet $U_{\mathcal{M}}$ is monadic when each $M \in \mathcal{M}$ is the free category GX

on *a finite acyclic* graph X (where X is said to be acyclic when GX has no endomorphisms except identities).

2 U need not be of descent type

Let X be the graph with three vertices α, β, γ and with the three edges $\phi : \alpha \to \beta$, $\psi : \alpha \to \beta$, and $\theta : \beta \to \gamma$, and let P be the finite category generated by X, subject to the single equation $\theta\phi = \theta\psi$. We consider now the composite U of $W : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$ and $V : \mathbf{Cat} \to \mathbf{Graph}$, where $\mathcal{M}$ consists of P alone; and we begin by calculating the monad $T = UF = VWHG$.

For $X \in \mathbf{Gph}$, a functor $P \to GX$ is given by a diagram in GX of the form

$$a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{\ h\ } c\ ,$$

where $hf = hg$. Since GX is the free category on X, this equation implies the equation $f = g$; so that the functor $P \to GX$ has the limit a, the limit-cone having 1_a for its a-component. Since any functor $GX \to A$ into a category A with $\mathcal{M}$-limits preserves the limit above, GX is in fact the free $\mathcal{M}$-complete category on itself; so that, if the choices of limits in $\mathbf{Cat}_{\mathcal{M}}$ are suitably made, we have $WHGX = GX$. Thus $TX = VWHGX = VGX$, so that the monad T on $\mathbf{Gph}$ coincides with the monad $S = VG$. Hence T–Alg $= S$–Alg $= \mathbf{Cat}$, and the comparison functor $K : \mathbf{Cat}_{\mathcal{M}} \to T$–Alg coincides with $W : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$. Since a general functor between $\mathcal{M}$-complete categories does not preserve $\mathcal{M}$-limits, W is not fully faithful; and so U is not of descent type.

3 U is of descent type when each $M \in \mathcal{M}$ is free on a graph

Recall that, if $W : \mathcal{A} \to \mathcal{C}$ is a faithful functor, a morphism $e : C \to A$ in $\mathcal{A}$ is said to be *W-final* if, whenever a morphism $t : WA \to WB$ in $\mathcal{C}$ is such that the composite $t.We : WC \to WB$ is of the form Wg for some $g : C \to B$, then t is of the form Ws for some $s : A \to B$. An easy argument shows that e is certainly W-final if e is a coequalizer in

$\mathcal{A}$ and We is epimorphic in $\mathcal{C}$. We shall use the following result of Kelly and Power, from [7, Section 3]:

Lemma 3.1 *Let $U = VW$ where $W : \mathcal{A} \to \mathcal{C}$ is faithful and $V : \mathcal{C} \to \mathcal{G}$ is of descent type; and let U have a left adjoint F with counit $\varepsilon : FU \to 1$. Then U is of descent type if and only if each $\varepsilon A : FUA \to A$ is W-final.*

For the reader's convenience, we shall sketch here the proof of the "if" part, which is what we shall use below. The comparison functor $K : \mathcal{A} \to T$–Alg sends $A \in \mathcal{A}$ to the algebra $(UA, U\varepsilon A : UFUA \to A)$; whence it easily follows that K is fully faithful if each εA is U-final. In the circumstances above, W-finality of the εA suffices for this; because each $W\varepsilon A$ is V-final. This follows from the remark preceeding the lemma: for $VW\varepsilon A = U\varepsilon A$ is a retraction, whence $W\varepsilon A$ is a coequalizer because V is of descent type — for more details see [7, Section 2].

We now apply the lemma to our case of $W : \mathbf{Cat}_{\mathcal{M}} \to \mathbf{Cat}$ and $V : \mathbf{Cat} \to \mathbf{Gph}$, where each $M \in \mathcal{M}$ is of the form GX for a graph X. We are to prove $W\varepsilon A : WFUA \to WA$ to be W-final. Suppose then that $t : WA \to WB$ has $t.W\varepsilon A = Wg$ for some $g : FUA \to B$; we are to show that t is of the form Ws, or equally that t preserves $\mathcal{M}$-limits.

With $M = GX \in \mathcal{M}$, consider any functor $f : GX \to WA$. Since $G \dashv V$, there is a corresponding morphism $\overline{f} : X \to VWA = UA$ of graphs. Write $\overline{h}$ for the composite graph-morphism

$$X \xrightarrow{\overline{f}} UA \xrightarrow{\eta UA} UFUA$$

where $\eta : 1 \to UF$ is the unit of the adjunction; noting that, since $U\varepsilon A.\eta UA = 1$, the composite

$$X \xrightarrow{\overline{h}} UFUA \xrightarrow{U\varepsilon A} UA$$

is $\overline{f}$. It follows that, if $\overline{h}$ in turn corresponds under the adjunction $G \dashv V$ to $h : GX \to WFUA$, then the composite

$$GX \xrightarrow{h} WFUA \xrightarrow{W\varepsilon A} WA$$

is the functor f we began with.

Our desired conclusion, that t preserves $\mathcal{M}$-limits, is now obtained as follows. We have

$$\begin{array}{rcll} t.\lim f & = & t.\lim(W\varepsilon A.h) & \\ & \cong & t.W\varepsilon A.\lim h & \text{because } W\varepsilon A \text{ preserves } \mathcal{M}-\text{limits} \\ & = & Wg.\lim h & \text{since } t.W\varepsilon A = Wg \\ & \cong & \lim(Wg.h) & \text{because } Wg \text{ preserves } \mathcal{M}-\text{limits} \\ & = & \lim(t.W\varepsilon A.h) & \\ & = & \lim(tf). & \end{array}$$

4 $U_{\{GY\}}$ is not monadic when Y is the loop with one vertex and one edge

For this $\mathcal{M} = \{GY\}$, a functor $GY \to A$ is just an endomorphism e in A, and A is $\mathcal{M}$-complete precisely when each such endomorphism has a limit — that is, a universal arrow f with $ef = f$.

We now describe, in the language of Mac Lane's book [12], a V-split fork

$$A_1 \underset{j}{\overset{p,\ q}{\rightleftarrows}} A_2 \underset{i}{\overset{r}{\rightleftarrows}} A_3$$

in **Cat**. Each category A_i is generated by a graph X_i, subject to some relations. Each X_i has three objects a_i, b_i, c_i, and edges $e_i : a_i \to a_i$, $f_i : b_i \to a_i$, and $g_i : c_i \to a_i$; in addition, X_1 has another edge $g'_1 : c_1 \to a_1$. The relations describing A_1 are ${e_1}^2 = 1$ and $e_1 f_1 = f_1$; those describing A_2 are ${e_2}^2 = 1$ and $e_2 f_2 = f_2$; and those describing A_3 are ${e_3}^2 = 1$, $e_3 f_3 = f_3$, and $e_3 g_3 = g_3$. Each of the functors p, q, r and each of the graph morphisms i, j is the "identity on objects", in a loose manner of speaking — we mean more precisely that $pa_1 = a_2$, $pb_1 = b_2$, $pc_1 = c_2$, and so on. Their effects on arrows are given by

$pe_1 = e_2$, $pf_1 = f_2$, $pg_1 = g_2$, $pg'_1 = g_2$;
$qe_1 = e_2$, $qf_1 = f_2$, $qg_1 = g_2$, $qg'_1 = e_2 g_2$;
$re_2 = e_3$, $rf_2 = f_3$, $rg_2 = g_3$;

$ie_3 = e_2$, $if_3 = f_2$, $ig_3 = g_2$;
$je_2 = e_1$, $jf_2 = f_1$, $jg_2 = g_1$, $j(e_2g_2) = g_1'$.

One verifies at once that p, q, r are indeed functors and that we have $rp = rq$ and, at the level of graph-morphisms, $ri = 1$, $qj = 1$, and $pj = ir$; so this is indeed a V-split fork — indeed a V-split coequalizer — in **Cat**.

What is more, A_1 and A_2 lie in $\mathbf{Cat}_{\mathcal{M}}$, the endomorphisms e_1 and e_2 having the limits f_1 and f_2; and these limits are preserved by p and q, which are accordingly morphisms in $\mathbf{Cat}_{\mathcal{M}}$. If U were monadic, we could conclude from Beck's theorem that A_3 too was $\mathcal{M}$-complete: but this is false, since clearly the endomorphism e_3 has no limit.

5 U is monadic when each $M \in \mathcal{M}$ is free on a finite acyclic graph

Finally, we establish the result of the heading above, which encompasses all the positive results given by Burroni or by Mac Donald and Stone. Recall that the graph X is said to be *acyclic* when the free category GX on X has no endomorphisms except identities. Let us write $|X|$ for the set of vertices of X, which is the set of objects of GX. Because GX is a category, the relation "there is some arrow $x \to y$ in GX" is a preorder relation $x \lhd y$ on $|X|$; and because X is acyclic, it is in fact a (partial) order relation. By choosing a minimal element x_1 of $|X|$ with respect to this preorder, and then a minimal element x_2 of $|X| - \{x_1\}$, and so on, we enumerate the elements of the finite set $|X|$ as $\{x_1, x_2, \ldots, x_n\}$ in such a way that $x_i \lhd x_j$ implies $i \leq j$. In fact, since X is acyclic, we have more: if there is an edge from x_i to x_j, then $i < j$. Finally we simplify the notation further still, by writing just i for the vertex x_i. So now the vertex-set of X is $\{1, 2, \ldots, n\}$, and $i < j$ whenever there is a non-identity arrow $i \to j$ in GX.

In the second paragraph of the Introduction, we gave a presentation of M-limits in the category A in terms of operations on A and equations between these; the various arities occuring were the categories M, $M \times \mathbf{2}$, $M \times \mathbf{3}$, $\mathbf{1}$, and $\mathbf{2}$. In the present case, where each $M \in \mathcal{M}$ has the form GX, each of these arities is free on a graph except $M \times \mathbf{2}$ (the arity for the limit-functor $L : A^M \to A$ as given on morphisms, and also the arrow for the naturality condition on the counit σ), and $M \times \mathbf{3}$ (the arity for the functoriality equation $L(\psi\phi) = L(\psi)L(\phi)$). We now complete the proof by so modifying the presentation as to avoid these arities $GX \times \mathbf{2}$ and $GX \times \mathbf{3}$, in favour of others that are free categories on graphs.

The point is that, because $M = GX$ with X finite and acyclic, the giving of $L(\phi) : L(d) \to L(e)$ for a natural transformation $\phi : d \to e : GX \to A$ can be reduced to the giving of $L(\phi)$ for those special ϕ having all but one of their components $\phi_i : d(i) \to e(i)$ (for $i \in |X| = \{1, 2, \ldots, n\}$) equal to an identity. This is so because a general $\phi : d \to e : GX \to A$ can be written as a composite of such special ones; explicitly, ϕ is the composite

$$d = d^0 \xrightarrow{\theta^1} d^1 \xrightarrow{\theta^2} d^2 \xrightarrow{\theta^3} \ldots \xrightarrow{\theta^{n-1}} d^{n-1} \xrightarrow{\theta^n} d^n = e, \tag{1}$$

where the functors $d^i : GX \to A$ (or equivalently the graph-morphisms $d^i : X \to A$) are given on objects by

$$\begin{aligned} d^i(j) &= d(j) \ \text{ for } i \leq n-j, \\ &= e(j) \ \text{ for } i > n-j, \end{aligned}$$

and are given on the edge $u : j \to k$ of X (whose existence entails $j < k$) by

$$\begin{aligned} d^i(u) &= d(u) && \text{for } i \leq n-k, \\ &= \phi_k d(u) = e(u)\phi_j && \text{for } n-k < i \leq n-j, \\ &= e(u) && \text{for } i > n-j; \end{aligned}$$

while the natural transformations θ^i are given by

$$\begin{aligned} (\theta^i)_{n-i+1} &= \phi_i, \\ (\theta^i)_j &= 1 \quad \text{for } j \neq n-i+1, \end{aligned}$$

so that each has at most one non-identity component.

Instead of giving the operations $L(\phi)$, therefore, it suffices to give operations $L^i(\phi) : L(d) \to L(e)$ for $1 \leq i \leq n$, where $L^i(\phi)$ is defined only for those $\phi : d \to e$ having $\phi_j = 1$ for $j \neq i$; then, if ϕ has the canonical factorization $\phi = \theta^n\theta^{n-1} \ldots \theta^1$ as in (5.1), we re-find $L(\phi)$ as

$L^n(\theta^n)L^{n-1}(\theta^{n-1})\ldots L^1(\theta^1)$. Moreover the arity of the operation L^i is clearly the free category GX^i on the graph X^i described as follows: the vertices of X^i are those vertices of X other than i, along with two further vertices i' and i''; to each edge $u : j \to k$ of X with $j \neq i$ and $k \neq i$, there is a corresponding edge $u : j \to k$ of X^i; to each edge $v : j \to i$ of X (where we necessarily have $j < i$), there is a corresponding edge $v : j \to i'$ of X^i; to each edge $w : i \to k$ of X (where we necessarily have $i < k$), there is a corresponding edge $w : i'' \to k$ of X^i; and finally, besides the above, X^i has one further edge $* : i' \to i''$.

In ridding ourselves of the operation $L(\phi)$ of arity $GX \times \mathbf{2}$ in favour of the operations $L^i(\phi)$ of arities GX^i, we have also ridded ourselves of the need for an equation of arity $GX \times \mathbf{2}$ to express the naturality of the counit σ of the adjunction: for it suffices for this naturality that σ be natural only with respect to the special ϕ with at most one non-identity component, and this is expressed by one equation of arity GX^i for each i.

It remains to ensure the equation $L(\psi\phi) = L(\psi)L(\phi)$, which as it stands has arity $GX \times \mathbf{3}$. Certainly we must have for each i the special case $L^i(\psi\phi) = L^i(\psi)L^i(\phi)$ of this, where ϕ_j and ψ_j are identities for $j \neq i$; and this is an equation of arity GX^{ii}, where X^{ii} is the following graph. Its vertices are those of X other than i, along with three new vertices i', i'', and i'''; to each edge $u : j \to k$ of X with $j \neq i$ and $k \neq i$ there is a corresponding edge $u : j \to k$ of X^{ii}; to each edge $v : j \to i$ of X there is a corresponding edge $v : j \to i'$ of X^{ii}; to each edge $w : i \to k$ of X there is a corresponding edge $w : i''' \to k$ of X^{ii}; and finally, besides the above, X^{ii} has an edge $i' \to i''$ and an edge $i'' \to i'''$. (If we think of the passage from X to X^i as a general process sending a graph X and one of its vertices i to new graph with i replaced by a pair i', i'', then we may see X^{ii} as $(X^i)^{i''}$.)

To see what further equations between the L^i are needed, consider in the functor-category A^{GX} a morphism $\phi : d \to e$ where ϕ_k is an identity for $k \neq i$, and a morphism $\psi : e \to f$ where ψ_k is an identity for $k \neq j$, and suppose that $i < j$. Observe that to give $d, e, f : GX \to A$ along with such morphisms ϕ and ψ is equally to give a graph-morphism $h : X^{ij} \to A$, where X^{ij} is the graph described as follows. Its vertices are the vertices of X other than i and j, along with vertices i', i'', j', and j''. To each edge $u : k \to m$ of X where k is neither i nor j and m

is neither i nor j, there is a corresponding edge $u : k \to m$ of X^{ij}; to each edge $v : k \to i$ of X there is a corresponding edge $v : k \to i'$ of X^{ij}; to each edge $w : k \to j$ of X with $k \neq i$, there is a corresponding edge $w : k \to j'$ of X^{ij}; to each edge $x : i \to k$ of X with $k \neq j$, there is a corresponding edge $x : i'' \to k$ of X^{ij}; to each edge $y : j \to k$ of X, there is a corresponding edge $y : j'' \to k$ of X^{ij}; to each edge $z : i \to j$ of X, there is a corresponding edge $z : i'' \to j'$ of X^{ij}; and finally, besides the above, X^{ij} has edges $* : i' \to i''$ and $\dagger : j' \to j''$. (In the language of the final sentence of the last paragraph, X^{ij} is $(X^i)^j$.) The reader will find it easy to express d, e, f, ϕ and ψ explicitly in terms of $h : X^{ij} \to A$.

We now observe that there is in A^{GX} a unique commutative square

$$\begin{array}{ccc} d & \xrightarrow{\overline{\psi}} & g \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\phi}} \\ e & \xrightarrow[\psi]{} & f \end{array} \tag{2}$$

wherein $\overline{\phi}_i = \phi_i$, $\overline{\psi}_i = \psi_i$, and all the other components of $\overline{\phi}$ and of $\overline{\psi}$ are identities; for we are forced to define the graph-morphism $g : X \to A$ as follows, and then the reader will easily verify that the given $\overline{\phi}$ and $\overline{\psi}$ are indeed natural transformations. On the vertices, $g(k) = e(k)$ if $k \neq i$ and $k \neq j$, while $g(i) = d(i)$ and $g(j) = f(j)$. On an edge $u : k \to m$, we have $g(u) = e(u)$ if k is neither i nor j and m is neither i nor j; while $g(u) = d(u)$ if $m = i$, or if $k = i$ and $m \neq j$; and similarly $g(u) = f(u)$ if $k = j$, or if $m = j$ and $k \neq i$; and finally $g(u)$ is the common value $\psi_j d(u) = f(u)\phi_i$ when $k = i$ and $m = j$.

If we are to have $L(\psi\phi) = L(\psi)L(\phi)$, we must certainly in the circumstances of (2) have the equation

$$L^j(\psi)L^i(\phi) = L^i(\overline{\phi})L^j(\overline{\psi}), \tag{3}$$

which is an equation of arity GX^{ij}. Moreover the equations (3), along with our earlier equations $L^i(\psi\phi) = L^i(\psi)L^i(\phi)$, suffice to give $L(\psi\phi) = L(\psi)L(\phi)$ in general. For suppose that ϕ has the canonical factorization $\phi = \theta^n\theta^{n-1}\dots\theta^1$ as in (1), while ψ has a similar factorization $\psi = \xi^n\xi^{n-1}\dots\xi^1$. Then repeated use of (2) allows us to rewrite $\psi\phi = \xi^n\xi^{n-1}\dots\xi^1\theta^n\theta^{n-1}\dots\theta^1$ in the form $\overline{\overline{\xi^n}}\ \overline{\overline{\theta^n}}\ \overline{\overline{\xi^{n-1}}}\ \overline{\overline{\theta^{n-1}}}\dots\overline{\overline{\xi^1}}\ \overline{\overline{\theta^1}}$, where all the components of $\overline{\overline{\xi^i}}$ and $\overline{\overline{\theta^i}}$ except the i-th are identities. It follows from our definition of L as a derived operation that $L(\psi\phi) = L^n(\overline{\overline{\xi^n}}\ \overline{\overline{\theta^n}}) \dots L^1(\overline{\overline{\xi^1}}\ \overline{\overline{\theta^1}})$. The equations $L^i(\psi\phi) = L^i(\psi)L^i(\phi)$ translate this into $L(\psi\phi) = L^n(\overline{\overline{\xi^n}})\ L^n(\overline{\overline{\theta^n}}) \dots L^1(\overline{\overline{\xi^1}})\ L^1(\overline{\overline{\theta^1}})$; and now the equations (3) allow us to retrace our steps from $\xi^n\xi^{n-1}\dots\xi^1\theta^n\theta^{n-1}\dots\theta^1$ to $\overline{\overline{\xi^n}}\ \overline{\overline{\theta^n}}\ \overline{\overline{\xi^{n-1}}}\ \overline{\overline{\theta^{n-1}}} \dots \overline{\overline{\xi^1}}\ \overline{\overline{\theta^1}}$, this time with the appropriate L^i inserted, to get

$$L^n(\overline{\overline{\xi^n}})\ L^n(\overline{\overline{\theta^n}}) \dots L^1(\overline{\overline{\xi^1}})\ L^1(\overline{\overline{\theta^1}}) = L^n(\xi^n) \dots L^1(\xi^1)\ L^n(\theta^n) \dots L^1(\theta^1),$$

which is the desired $L(\psi\phi) = L(\psi)L(\phi)$.

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