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RELATIONAL MONOIDS, MULTIRELATIONS, AND QUANTALIC RECOGNIZERS

by Kimmo I. ROSENTHAL

RESUME. Dans cet article, on montre l'équivalence de 3 structures algébriques à première vue assez différentes. Ce sont les monoïdes relationnels (i.e. des monoïdes dans la catégorie autonome des ensembles et relations), les "quantales-parties" (ensembles de parties munis d'une structure de quantale), et les "multipréordres avec factorisation" (une sorte de multirelation, qui généralise la notion de relation). Cette équivalence est utilisée pour proposer une nouvelle approche des langages "context-free", en utilisant les quantales pour reconnaître ces langages.

The notion of multirelation and the equivalence between relational monoids and multipreorders with factorizations was established in the work of Ghilardi and Meloni [2] on n-ary connectives in logic. In [9], it was shown that the work of Walters [12], [13], [14] on a categorical approach to context-free languages was equivalent to an approach using multirelations. We shall see that Chomsky normal form for a context-free grammar is captured by the notion of a multipreorder with factorizations, which by the equivalence mentioned above leads us to relational monoids, and from there to power quantales. A *power quantale over a set* X refers to having a unital quantale structure on the power set $P(X)$ of a set X. (Quantales are partially ordered algebraic structures which have received much attention in recent years. For a detailed treatment of quantales, see Rosenthal [6].) From this we obtain the notion of a quantalic recognizer of a language and we show that context-free languages in an alphabet A are precisely the languages recognized by certain quantales equipped with the structure of a $P(A^*)$-algebra, where $P(A^*)$ is the free quantale on the free monoid A^* of A.

In the first section of the paper, we establish the equivalence of the category of power quantales with the category of relational monoids and homomorphisms, i.e. the category of monoid objects in the symmetric, monoidal (i.e. autonomous) category *Rel* of sets and relations. We then, in the next section, develop the theory of multirelations and multipreorders (following Ghilardi and Meloni [2]), culminating with the definition of a multipreorder with factorizations, which turns out to be equivalent to the notion of relational monoid, and hence power quantale. We describe how one can associate a multipreorder with factorizations to every multipreorder. This construction will turn out to be related to Chomsky normal form for context-free grammars on an alphabet A (described as certain kinds of multirelations) allowing us to concentrate on multipreorders with factorization for deriving context-free languages. This leads to our defining *quantalic recognizers for A* to be power quantales equipped with a $P(A^*)$-algebra structure. The main theorem is that these quantales serve as recognizers for the context-free languages.

§1. Power quantales and relational monoids

Quantales are complete lattices with an associative binary operation $\circ$, which preserves sups in both variables. Examples of these structures abound in algebra and analysis (various quantales of ideals) and include such fundamental mathematical structures as relations on a set. They have recently sparked much interest in theoretical computer science via their connections with linear logic ([6]), process semantics [1], and automata theory [7], [8]. For an overview of quantale theory, see Rosenthal [6] (for a briefer introduction, there is [5]).

Let us now formally give the definition of quantale and power quantale, and then look at the examples that will be of interest to us.

Definition 1.1 *A quantale is a complete lattice Q equipped with an associative binary operation $\circ$ such that for all $a \in Q$, $\{b_\alpha\} \subseteq Q$, we have that*

$$a \circ (sup_\alpha b_\alpha) = sup_\alpha (a \circ b_\alpha) \text{ and } (sup_\alpha b_\alpha) \circ a = sup_\alpha (b_\alpha \circ a)$$

Q is called unital if there exists $1 \in Q$ such that $1 \circ a = a = a \circ 1$ for all $a \in Q$.

Definition 1.2 *If Q and S are unital quantales, a function $f : Q \to S$ is a homomorphism iff it preserves sups, $\circ$ and 1.*

Now, let us turn our attention to power quantales. We begin with a definition.

Definition 1.3. *A power quantale on a set X refers to a unital quantale structure on the power-set $P(X)$.*

Examples

1) $P(X)$ viewed as a Boolean algebra is a quantale under the operation of intersection $\cap$.

2) If M is a monoid in *Sets*, then $P(M)$ is a quantale under concatenation of subsets, i.e. $A \circ B = \{a \cdot b | a \in A, b \in B\}$. The unit is given by $\{e\}$, where e is the identity element of M.

3) the quantale $Rel(X)$ of relations on X is a power quantale, when viewed as the set $P(X \times X)$ with the usual composition of relations. The unit is the diagonal relation Δ.

We shall usually denote all quantale operations by $\circ$, unless the context requires otherwise.

Let **PowerQuant** denote the category of powerquantales and homomorphisms.

Since a homomorphism is a sup-preserving map $F : P(X) \to P(Y)$, it comes from a relation $U : X \to Y$, where $(x,y) \in U$ iff $y \in F(\{x\})$. Thus, for $A \subseteq X$, $F(A) = \{y \mid \text{there exists } a \in A \text{ with } (a,y) \in U)\}$.

We shall now see that the notion of power quantale is equivalent to that of a monoid in the autonomous category *Rel* of sets and relations. We shall refer to these as relational monoids.

Definition 1.4 *A relational monoid consists of a set X together with a relation* $\mu : X \times X \to X$ *and a subset* $\Omega \subseteq X$ *satisfying*
1) (associativity) for all $y, w, u \in X$ *there exists* $z \in X$ *such that* $((z,y) \sim_\mu x$ *and* $(w,u) \sim_\mu z$ *iff there exists* $v \in X$ *such that* $((w,v) \sim_\mu x$ *and* $(u,y) \sim_\mu v$
2) (identity) (a) for all $x \in X$ *there exists* $e \in \Omega$ *such that* $(x,e) \sim_\mu x$ *and for all* $x \in X$ *there exists* $f \in \Omega$ *such that* $(f,x) \sim_\mu x$
(b) for all $x, y \in X, e \in \Omega$, *if* $(x,e) \sim_\mu y$, *then* $x = y$ *and if* $(e,x) \sim_\mu y$, *then* $x = y$.

We shall sometimes write $((x,y),z) \in \mu$ or suppress mention of μ, simply writing $(x,y) \sim z$.

The notion of homomorphism is the usual one, expressed internally in *Rel* using relational composition. In the following definition, we write this out explicitly for the benefit of the reader.

Definition 1.5. *Let X and Y be relational monoids. A relational monoid homomorphism is a relation* $U : X \to Y$ *such that*
(1) $\Omega_Y = \{y \in Y \mid \exists e \in \Omega_X \text{ with } (e,y) \in U\}$
(2) For all $x_1, x_2 \in X, y \in Y$, *there exists* $a \in X$ *such that* $((x_1,x_2),a) \in \mu_X$ *and* $(a,y) \in U$ *iff there exists* $y_1, y_2 \in Y$ *such that* $(x_1,y_1) \in U, (x_2,y_2) \in U$ *and* $((y_1,y_2),y) \in \mu_Y$.

We thus obtain the category **RelMon** of relational monoids.

Theorem 1.1. *The categories* **PowerQuant** *and* **RelMon** *are equivalent.*

Proof: If X is a relational monoid with multiplication μ and identity Ω, define a power quantale structure on $P(X)$ by $A \circ B = \{x \in X \mid \text{there exists } a \in A, b \in B \text{ with } (a,b) \sim_\mu x\}$. Ω becomes the unit element for $\circ$. On the level of morphisms, we have indicated that relations $U : X \to Y$ correspond to sup-preserving maps $P(X) \to P(Y)$. It is not hard to see that $U : X \to Y$ is a monoid homomorphism in *Rel* precisely when the corresponding $P(X) \to P(Y)$ is a homomorphism of unital quantales. This process is evidently functorial, and to see that it defines an equivalence, we observe that the relational monoid structure on X can be recovered from $P(X)$ by stipulating that $((x,y),z) \in \mu$ iff $z \in \{x\} \circ \{y\}$. •

If X is a relational monoid with operation μ and identity Ω_X, and Y is a relational monoid with operation η and identity Ω_Y, then there is a relational monoid structure on $X \times Y$, defined as follows.

$(x_1, y_1), (x_2, y_2) \sim (x, y)$ iff $(x_1, x_2) \sim_\mu x$ and $(y_1, y_2) \sim_\eta y$. It is not hard to check that $\Omega_X \times \Omega_Y$ serves as the identity for this new operation.

Using the equivalence of Theorem 1.1., we can define a tensor product of power quantales $P(X) \otimes P(Y)$, by $P(X) \otimes P(Y) \cong P(X \times Y)$, where if $A, B \subseteq X \times Y$, then $A \circ B = \{(x, y) |$ there exists $(a_1, a_2) \in A, (b_1, b_2) \in B$ with $(a_1, b_1) \sim x$ and $(a_2, b_2) \sim y\}$.

§2. Multirelations and multipreorders with factorization

In this section, we consider the notion of a *multirelation* on a set, and various refinements of it. Much of this section comes from the work of Ghilardi and Meloni [2]. The idea of a multirelation and the operation of substitution owe, of course, a debt to the work of Lambek [4] on multicategories. Multirelations can be elegantly described as the multigraph morphisms from a rather simple multigraph to the multigraph *Rel* of sets and relations (see [2] or [9], [10], [11]), however we eschew that approach in order to avoid having to introduce the notion of multigraph, which we shall not need in what follows.

We shall use X^n to denote the n-fold cartesian product $X \times X \times .... \times X$ of a set X with itself and shall use Greek letters to stand for elements of X^n, e.g. $\alpha = (x_1, x_2, ..., x_n)$.

Definition 2.1. *Let X be a set. A multirelation M on X consists of the union $\bigcup M_n$ of sets M_n, where $M_n \subseteq X^n \times X$ for $n \geq 0$.*

We refer to M_n as the $n-level$ of M. We write $(\alpha, x) \in M$, whenever we wish to discuss a typical element of the multirelation M, without specifically referring to its level. Sometimes we may write $\alpha \sim x$, when M is clear from the context. Given an n-tuple α, we shall use α_i to refer to its ith component and if β is an m-tuple, we shall use $\alpha|_i\beta$ to denote the $(m + n - 1)$-tuple obtained by replacing α_i by β. We shall also use concatenation $\alpha \cdot \beta$ to denote the juxtaposition of an n-tuple α and an m-tuple β to produce a new $(n + m)$-tuple.

If α is an n-tuple of elements of X, we use $[\alpha]$ to refer to the corresponding word $\alpha_1\alpha_2....\alpha_n$ in the free monoid X^* generated by X.

We shall use $Multi(X)$ to denote the set of multirelations on X.We can define an ordering $\preceq$ on $Multi(X)$ by specifying that $M \preceq K$ iff $M_n \subseteq K_n$ for all $n \geq 0$. Note that $Multi(X)$ is in fact a complete lattice under this ordering, as it is clearly closed under both arbitrary suprema and infima, by taking unions and intersections at each level.

Definition 2.2 *If M is a multirelation on X and K is a multirelation on Y, a relational map of multirelations is a relation $U : X \rightarrow Y$ such that for all*

$x_1, ...x_n \in X (n \geq 0)$ *and for all* $y \in Y$, *there exists* $x \in X$ *with* $(x_1, ...x_n) \sim_M x$ *and* $(x, y) \in U$ *iff there exists* $y_1, ...y_n \in Y$ *with* $(x_i, y_i) \in U$ *for all* $n \geq 0$, *and* $(y_1, ...y_n) \sim_K y$.

Note that if $n = 0$, we have that $y \in K_0$ iff there exists $x \in M_0$ with $(x, y) \in U$.

We denote the resulting category of multirelations by **Multirel**.

We now wish to describe the operation of substitution (composition) for multirelations. Let M and N be multirelations on X. We define a new multirelation $M[N]$ as follows.

$(\alpha, x) \in M[N]$ iff there exists β and γ such that $(\beta, x) \in M$, $(\gamma, \beta_i) \in N$ and $\alpha = \beta|_i\gamma$.

Definition 2.3. *If M and N are multirelations on a set X, we refer to $M[N]$ as the substitution of N into M (or as the composition of M with N).*

Note that if M and N are ordinary binary relations on X, then $M[N]$ is just the usual composition of relations $M \circ N = \{(z, x) | \exists y \text{ with } (y, x) \in M, (z, y) \in N\}$.

Also, note that the identity for substitution is given by the multirelation Δ, where Δ_1 is the diagonal (x, x) on the set $X \times X$. Δ_n is empty for all $n \neq 1$ and so Δ only lives at level 1.

In analogy with the theory of binary relations, we can talk about a multirelation being reflexive and transitive, leading to the notion of a multipreorder.

Definition 2.4. *A multirelation M on a set X is called a multipreorder if and only if M satisfies that $\Delta \preceq M$ and $M[M] \preceq M$.*

For a detailed look at the role of multipreorders in studying logic with n-ary connectives, see the work of Ghilardi and Meloni [2].

There are several natural examples of multipreorders.

1) Let S be a monoid with binary operation $\cdot$. Define a multirelation M_S by $((m_1, m_2, ...m_k), m) \in (M_S)_k$ iff $m_1 \cdot m_2 \cdot .... \cdot m_k = m$. More generally, if S is a partially ordered monoid, we could change the multirelation to $m_1 \cdot m_2 \cdot .... \cdot m_k \leq m$.

2) Generalizing (1), we could consider a small and locally small category **C**, let X be the set of morphisms of **C** and define $M_{\mathbf{C}}$ by $((f_1, f_2, ...f_k), f) \in (M_{\mathbf{C}})_k$ iff $f_1 \circ f_2 \circ .... \circ f_k = f$, where $\circ$ denotes the composition in the category. If **C** is a locally partially ordered bicategory, that is to say the hom-sets are partially ordered with composition respecting the ordering, then we can replace $=$ by $\leq$.

Denote the n-fold substitution of M into itself, $M[M[M...M]..]$ by M^n. It is not hard to see that if M is a multirelation on X, then the *multipreorder generated by M* is given by $M^* = \bigcup M^n$, where $n \geq 0$ and M^0 is understood to be the multirelation Δ.

The notion of multipreorder can be refined further by discussing multipreorders with factorization. Let us begin with the definition.

Definition 2.5. *A multipreorder M on a set X is called a multipreorder with factorizations if and only if*
1) $(x,y) \in M \Longrightarrow x = y$ *for all* $x, y \in X$.
2) $(\alpha \cdot \beta, x) \in M \Longrightarrow \exists y, z \in X$ *such that* $(\alpha, y) \in M, (\beta, z) \in M$ *and* $(y \cdot z, x) \in M$ *for all* α, β *with* $[\alpha], [\beta] \in X^*$.

Note that we are allowing $[\alpha], [\beta]$ to be the empty word in X^* Of course, (2) extends in a natural way to $(\gamma_1 \cdot \ldots \cdot \gamma_n, x) \in M \Longrightarrow \exists x_1, \ldots x_n \in X$ with $(\gamma_i, x_i) \in M$ and $(x_1 \cdot \ldots \cdot x_n, x) \in M$.

Examples

1) Clearly, the multipreorder associated to any monoid S (described earlier), is a multipreorder with factorizations.

2) Consider the following multipreorder on $X \times X$, where X is a set.
$((x_1, y_1), \ldots (x_n, y_n)) \sim (x, y)$ iff $y_1 = x_2, y_2 = x_3, \ldots . y_{n-1} = x_n$ and $x_1 = x, y_n = y$ for $n \geq 1$ and at the 0-level of $\sim$, we pick out the subset of all pairs (x, x). This is clearly a multipreorder with factorizations.

We have the following lemma, whose proof follows directly from the above definition, which says that a multipreorder with factorizations is completely determined by what happens at levels M_2 and M_0, of the multirelation.

Lemma 2.1 *Suppose a multipreorder M on X satisfies that $(x,y) \in M \Longrightarrow x = y$ for all $x, y \in X$. Then, M is a multipreorder with factorizations iff $M = (M_2 \cup M_0)^*$*

Thus, example (2) above can be described by $((x,y),(y,z)) \sim (x,z)$. We shall refer to this as the *relational multipreorder with factorizations on* X, although technically speaking it is on $X \times X$.

We shall denote by **Mpf** the category of multipreorders with factorization with multirelation morphisms as maps.

Theorem 2.1. *There is an equivalence of categories* **Mpf** $\cong$ **RelMon**.

Proof: Given a multipreorder with factorizations M on X, we consider the relation $M_2 : X \times X \to X$ together with the subset $M_0 \subseteq X$. Suppose $\exists z \in X$ such that $((z,y) \sim_\mu x$ and $(w,u) \sim_\mu z$. Since M is a multipreorder, it follows that $(w \cdot u \cdot y, x) \in M$. Now apply factorization to the factors w and $u \cdot y$ to obtain that $\exists v \in X$ such that $((w,v) \sim_\mu x$ and $(u,y) \sim_\mu v$. The other part of associativity follows similarly.
For the identity laws, (a) follows directly from the definition of multipreorder with factorization (Def.2.1. (2)) where α or β are allowed to represent the empty word.

For (b) of the identity laws, if $(e, x) \sim_M y$, since multipreorders are closed under composition and $e \in M_0$, we can replace e by the empty word yielding $x \sim_M y$, which forces $x = y$. Similarly, for the other part.

The notion of relational map of multirelations, when restricted to level 2, captures precisely the definition of monoid homomorphism. For $n = 0$, $y \in K_0$ iff $\exists x \in M_0$ with $(x, y) \in U$ is the desired property for identities. To see that this functor is an equivalence, if $\mu : X \times X \to X$ and $\Omega \subseteq X$ define a relational monoid structure on X, consider $(\mu \cup \Omega)^*$ and apply Lemma 2.1. to obtain a multipreorder with factorizations. A relational monoid homomorphism becomes a map in **Mpf**. Associativity for X yields factorizations at level 2, which easily extend to other levels and condition (1) for factorization follows from the identity laws. This functor is clearly inverse to the functor **Mpf** $\longrightarrow$ **RelMon** described above. •

It follows by combining Theorems 1.1. and 2.1., that there is an equivalence of categories between the category **PowerQuant** of power quantales and the category **Mpf**. For example, the relational multipreorder with factorizations on X described in Example 2 corresponds to the power quantale $Rel(X)$ of relations on X.

We now wish to present a construction of how we can obtain a multipreorder with factorizations from a given multipreorder in a very natural way. This construction will prove to be very important in the next section when we consider context-free grammars and languages.

Suppose that M is a multipreorder on a set X. Then, it is closed under composition. It follows that if $(x, y) \in M_1$, then x can be substituted in for any y that appears at some n-level of M. In other words if $(\beta, z) \in M$ with $\beta_i = y$, then $(\beta|_i x, z) \in M$.

Thus, since we are already starting with a preorder, we can freely remove level 1 from M without losing any information, since the fact that M is closed under composition guarantees us that we still have all the relationships arising from $(x, y) \in M$. In addition to removing level 1, let us replace the base set X by X^*, the free monoid on X. Recall that if $\beta = (x_1, x_2, \ldots . x_n)$ is an n-tuple of elements of X, we use $[\beta]$ to denote the corresponding word in the free monoid X^*. Given such a β, define a sequence as follows.

Let $\beta^1 = \beta$, $\beta^2 = (x_2, \ldots . x_n)$, $\beta^3 = (x_3, \ldots . x_n)$, etc. with finally $\beta^n = (x_n)$

Let us define a new multipreorder $\rho(M))$ on X^* (with β in the following as above).

1) If $(x_1, x_2, \ldots . x_n, x) \in M$, then $([x_1] \cdot [x_2] \cdot \ldots \cdot [x_n], [x]) \in \rho(M)$
2) define new elements of $\rho(M)$ as follows :
$([x_1] \cdot [\beta^2], [x]) \in \rho(M)$
$([x_2] \cdot [\beta^3], [\beta^2]) \in \rho(M)$
$([x_3] \cdot [\beta^4], [\beta^3]) \in \rho(M)$
....
$([x_{n-1}] \cdot [x_n], [\beta^{n-1}]) \in \rho(M)$

Notice that we have only added relations at level 2 in the multipreorder $\rho(M)$ on X. To see that we now have a multipreorder with factorizations, observe that if we have a relation in $\rho(M)$, which does not live at level 2, such as $([x_1]\cdot[x_2]\cdot\ldots\cdot[x_n],[x])$, then this can be viewed as a composite of a sequence of level 2 elements of $\rho(M)$, namely
$([x_1]\cdot[\beta^2],[x])$ composed with $([x_2]\cdot[\beta^3],[\beta^2])$ yields $([x_1]\cdot[x_2]\cdot[\beta^3],[x])$.
This in turn when composed with $([x_3]\cdot[\beta^4],[\beta^3])$ gives rise to $([x_1]\cdot[x_2]\cdot[x_3]\cdot[\beta^4],[x])$.

Continuing in this vein, we arrive at the original element $([x_1]\cdot[x_2]\cdot\ldots\cdot[x_n],[x])$ of $\rho(M)$, showing that the elements of level 2 (together with those of level 0) generate the multipreorder $\rho(M)$, thus making it a multipreorder with factorizations.

It is not hard to see that given $x \in X$, there are no new relations added relative to x; by this we mean that the only relations in $\rho(M)$ of the form $([\gamma],x)$ are the ones required to be by the definition, namely the relations $([x_1]\cdot[x_2]\cdot\ldots\cdot[x_n],[x])$ where $(x_1,x_2,\ldots x_n,x)\in M$. The use of the $[\beta^i]$ above will not impact on these.

We summarize the above in a proposition.

Proposition 2.1 *If M is a multipreorder, then $\rho(M)$ is a multipreorder with factorizations.*

§3 Context-free grammars and languages and quantalic recognizers

We wish to describe an algebraic way of presenting the notions of context-free grammar and language using multirelations, ultimately using the equivalence of the categories **Mpf** and **PowerQuant** to arrive at the notion of a quantalic recognizer for context-free languages. This approach is inspired by the work of Walters on a categorical approach to context-free grammars and languages using multigraphs and the free category with products on a multigraph [12],[13]. In [9], it was observed that this approach was equivalent to using the notion of a multirelation with constants.

We shall begin with this definition.

Definition 3.1. *Let X and A be sets. A context-free grammar with alphabet A, (or a multirelation with constants A), is a multirelation $M \in Multi(X)$ together with a function $\mu : A \to P(X)$ assigning to every $a \in A$ a subset $\mu(a)$ of X.*

Let M^*, as before, denote the multipreorder generated from M. A typical element $(\beta,x)\in M^*$ arises from M via a finite sequence of substitutions, utilizing Definition 2.3.

To see the connection with the traditional approach ([3]), the elements of M, (α,x), correspond to productions $x \leftarrow \alpha$ and we have $x\in\mu(a)$ iff there is a production $x \leftarrow a$.

Elements $(\beta,x)\in M^*$ are referred to as derivations of x from β.

We can now present a definition of context-free language very simply in terms of the above definition.

Definition 3.2. *Let (M, μ) be a context-free grammar on a set X with alphabet A. Let $x \in X$. The context-free language associated to x is the subset L_x of the free monoid A^* defined by $a_1a_2...a_n \in L_x$ iff there exists $\beta \in X^n$ with $(\beta, x) \in M^*$ and $\beta_i \in \mu(a_i)$.*

A moment's reflection shows that this coincides with the usual definition.

Recall from §2, that to every multipreorder M, we can associate a multipreorder with factorizations, which we denoted $\rho(M)$. $\rho(M)$ has as its underlying set X^*, the free monoid on the set X. Define $\rho(\mu) : A \to P(X^*)$ by $\rho(\mu)(a) = \mu(a)$; that is $\rho(\mu)$ picks out subsets of X^* consisting only of words of length one, i.e. $\rho(\mu)(a)$ has no elements of length ≥ 2 and if $x \in X$, then $x \in \mu(a)$ iff $[x] \in \rho(\mu(a))$

We shall refer to $(\rho(M), \rho(\mu))$ as the *Chomsky normal form* of (M, μ). It is not hard to see that the usual prescription of Chomsky normal form for context-free grammars (e.g. see [3]) coincides with our construction of multipreorders with factorization from a given multipreorder and the ensuing $(\rho(M), \rho(\mu))$. As the following lemma indicates, the end result with regard to context-free languages is the same.

Lemma 3.1. *Let (M, μ) be a context-free grammar on a set X with alphabet A. The context-free language associated to x for this grammar is the same as the context-free language of x calculated according to the grammar $(\rho(M), \mu)$.*

To understand what is going on in the above lemma, note that if $[\beta]$ and $[\alpha]$ are words in X with $[\beta] = \beta_1\beta_2.....\beta_n$ and $[\alpha] = \alpha_1\alpha_2.....\alpha_m$, then we have

$([\beta]\cdot[\alpha], x) \in \rho(M)$ iff $(\beta\cdot\alpha, x) \in M$ iff $([\beta_1][\beta_2].....[\beta_n][\alpha_1][\alpha_2].....[\alpha_m], x) \in \rho(M)$ iff $(\beta_1\beta_2.....\beta_n\alpha_1\alpha_2.....\alpha_m, x) \in M$.

The above definition of L_x shows that we will obtain the same language whether we do our calculations with (M, μ) or with $(\rho(M), (\rho(\mu))$.

Thus, for the purposes of language recognition, it suffices to restrict our attention to context-free grammars, which are mpf's. So, let us suppose that we have a context-free grammar (M, μ) on a set X with alphabet A, where M is a multipreorder with factorizations. Hence, from Theorem 2.1., it follows that there is the structure of a power quantale on $P(X)$.

This power quantale structure is given by $A \circ B = \{x \in X | \exists a \in A, b \in B$ with $(ab, x) \in M\}$ where A, B are subsets of X.

Since A^* is the free monoid on the set A, and in turn, $P(A^*)$ is the free quantale on the monoid A^*, we have that the information of having a map $\mu : A \to P(X)$ is equivalent to having a monoid homomorphism $\mu : A^* \to P(X)$, which in turn is equivalent to a quantale homomorphism $\mu : P(A^*) \to P(X)$.

We can think of this quantale homomorphism $\mu : P(A^*) \to P(X)$ as endowing the quantale $P(X)$ with the structure of a $P(A^*)$-algebra, in the sense that μ leads to a definition of an action of $P(A^*)$ on $P(X)$ analogous to the usual notion of an algebra from commutative algebra.

For our current purposes, it suffices to deal with the map $\mu : P(A^*) \to P(X)$. This leads to the following definition.

Definition 3.3. *A quantalic recognizer for the alphabet A is a pair $(P(X), \mu)$, where $P(X)$ is a power quantale and $\mu : P(A^*) \to P(X)$ is a quantale homomorphism.*

Let $(P(X), \mu)$ be a quantalic recognizer for the alphabet A. Given $x \in X$, then $Q_x = \{\sigma \in A^* | x \in \mu(\sigma)\}$ is called the *language recognized by* $(P(X), \mu)$.

Definition 3.4. *A language L in the alphabet A is called quantally recognizable if and only if there exists a quantalic recognizer for A, which recognizes L.*

Note that in the above definition, if $\sigma = a_1 a_2 .... a_n$, then the statement $x \in \mu(\sigma)$ is equivalent to saying that $x \in \mu(a_1) \circ \mu(a_2) \circ .. \circ \mu(a_n)$, where $\circ$ denotes the quantale operation of the power quantale $P(X)$. The following theorem is the main result we have been leading up to.

Theorem 3.1. *Let A be a set. Then, the set of context-free languages in the alphabet A coincides with the set of quantally recognizable languages in the alphabet A.*

Proof: Let L be the context-free language associated to x, where $x \in X$ and (M, μ) is a context-free grammar on X. Thus, a word σ is in L precisely if $\sigma = a_1 a_2 .... a_n$ and there is $(\beta, x) \in M^*$ with $\beta_i \in \mu(a_i)$ for all $i = 1, 2....n$. By Lemma 3.1., it suffices to assume that M is a multipreorder with factorizations on X. In the corresponding power quantale structure on $P(X)$, $x \in \{\beta_1\} \circ ... \circ \{\beta_n\}$ and since $\beta_i \in \mu(a_i)$, it follows that $\sigma \in L$ precisely if $x \in \mu(a_1) \circ .... \circ \mu(a_n)$, in other words L is recognized by the quantalic recognizer $(P(X), \mu)$ and thus is quantally recognizable.

Conversely, suppose L is quantally recognizable by the quantalic recognizer $(P(X), \mu)$, with $L = Q_x$ for some $x \in X$. To the power quantale $P(X)$ we can associate a multipreorder with factorizations M on X and the quantale homomorphism $\mu : P(A^*) \to P(X)$ restricts to a map $\mu : A \to P(X)$. Thus we have a context-free grammar with alphabet A. A word $\sigma = a_1 a_2 .... a_n$ is in L if and only if $x \in \mu(\sigma)$ if and only if $x \in \{\beta_1\} \circ ... \circ \{\beta_n\}$ where $\beta_i \in \mu(a_i)$. If β is the n-tuple in X with β_i as its i^{th} component, then $x \in \{\beta_1\} \circ ... \circ \{\beta_n\}$ if and only if $(\beta, x) \in M$ and thus L is the context-free language of x relative to the context-free grammar (M, μ). •

We should note that by the equivalence of power quantales and multipreorders with factorization, via relational monoids, it follows that if $P(X)$ denotes the power quantale obtained from a context-free grammar (M, μ), with M a multipreorder with factorizations on a set X, then $P(X)$ recognizes precisely the context-free languages L_x generated by the grammar (M, μ). Thus, there is an underlying equivalence of categories between context-free grammars and their recognizers. To someone with a good grasp of the relational calculus, the use of multirelations and their substitution operation provides an effective and efficient way to describe context-free grammars and languages. It is hopeful that this approach will lead to some interesting observations about recognizers.

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