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CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES *Volume XXXV (1994)*

ACCESSIBLE EMBEDDINGS AND THE SOLUTION-SET CONDITION

by H. HU and M. MAKKAI

Dedicated to the memory of Jan Reiterman

Résumé. Pour chaque catégorie localement présentable, on démontre qu'une sous-catégorie accessiblement plongée d'une catégorie localement présentable est accessible si et seulement si elle satisfait la condition de l'ensemble-solution.

Introduction

The connection between accessibility and the solution set condition has been pointed out in several papers, including [7] and [9]. J. Adámek and J. Rosický have recently proved that each locally presentable category has the following property: a full subcategory closed under products and κ-filtered limits (for some infinite regular cardinal κ) is accessible if the embedding satisfies the solution set condition, see [3]. It is well-known that a category is locally presentable iff it is accessible and complete, or equivalently cocomplete (cf. [7]), but no assumption is made of the existence of any particular limits for accessible categories. One is impelled to ask: does the above mentioned result hold for any full subcategory closed under κ-filtered colimits? In this paper we provide an affirmative answer to this question. We will show that, for any locally presentable category $\mathbf{B}$, a full subcategory $\mathbf{A}$ of $\mathbf{B}$ closed under κ-filtered colimits is accessible iff the embedding satisfies the solution set condition. The proof of this result is a modification of the proof of the above mentioned result of J. Adámek and J. Rosický in [3].

1 Generalities on Accessible Categories

Let κ be an infinite regular cardinal. Recall that a category $\mathbf{A}$ is κ-filtered if for any graph $\mathbf{G}$ of cardinality less than κ, any diagram $D : \mathbf{G} \to \mathbf{A}$ has a cocone on it. $\mathbf{A}$ has κ-filtered colimits, if $\mathbf{A}$ has colimits of all diagrams whose domain is a κ-filtered category. Another concept is limit of κ-diagram, where a κ-diagram is a diagram whose domain category is of size less than κ.

An object A of a category $\mathbf{A}$ is said to be κ-presentable if the representable functor $\mathbf{A}(A, -)$ preserves κ-filtered colimits existing in $\mathbf{A}$. The full subcategory of $\mathbf{A}$ whose objects are the κ-presentable ones is denoted by $\mathbf{A}_\kappa$. Note that for any category, a κ-colimit of a diagram in which the objects are κ-presentable is κ-presentable itself (see [5] and [7]).

Definition 1.1 *([7]) A category* $\mathbf{A}$ *is* κ*-accessible if:*

(i) $\mathbf{A}$ *has* κ*-filtered colimits;*

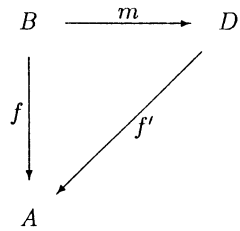
(ii) There is a small full subcategory $\mathbf{C}$ *of* $\mathbf{A}_\kappa$ *such that every object of* $\mathbf{A}$ *is a* κ*-filtered colimit of a diagram of objects in* $\mathbf{C}$.

A category is accessible if it is κ*-accessible for some infinite regular cardinal* κ.

Recall from [4] that a category is called locally κ-presentable if it is locally small, cocomplete, and has a small strong generator consisting of κ-presentable objects. A theorem (Theorem 6.1.4. in [7]) says that an accessible category is complete iff it is cocomplete. P. Gabriel and F. Ulmer have shown in [4] that a category $\mathbf{B}$ is locally κ-presentable if it is equivalent to the category of the form $L_\kappa(\mathbf{C}, \mathbf{Set})$, the category of all functors preserving κ-limits; here $\mathbf{C}$ is a small category with κ-limits, and $\mathbf{Set}$ denotes the category of small sets.

Let $\mathbf{B}$ be a κ-accessible category, and $\mathbf{A}$ a full subcategory of $\mathbf{B}$. $\mathbf{A}$ is said to be κ-accessibly embedded if $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$. $\mathbf{A}$ is accessibly embedded if it is λ-accessibly embedded for some regular cardinal λ with $\lambda \geq \kappa$.

Recall from [3] that a full subcategory $\mathbf{A}$ of $\mathbf{B}$ is said to be cone-reflective if the inclusion functor $\mathbf{A} \to \mathbf{B}$ satisfies the solution set condition, i.e., for each object B of $\mathbf{B}$ there exists a small cone $< r_i : B \to A_i >_{i \in I}$ with $A_i \in \mathbf{A}$ such that for any $A \in \mathbf{A}$, every morphism $B \to A$ factors through some r_i. Let $\mathbf{D}$ be a set of objects of $\mathbf{A}$. We say that $\mathbf{D}$ weakly reflects B (in $\mathbf{A}$) if for every $A \in \mathbf{A}$ and $f : B \to A$ there is $D \in \mathbf{D}$ and a factorization



where m and f' are some morphisms. Note that $\mathbf{A}$ is cone-reflective in $\mathbf{B}$ iff for every $B \in \mathbf{B}$, there is a small set $\mathbf{D} \subset \mathbf{A}$ weakly reflecting B. If $\mathbf{B}$ is accessible, then this is equivalent to saying that for every $B \in \mathbf{B}$ there is κ such that $\mathbf{D}_\kappa = \mathbf{A} \bigcap \mathbf{B}_\kappa$ weakly reflects B. Note that, of course, if $\kappa < \kappa'$ and $\mathbf{D}_\kappa$ weakly reflects B, so does $\mathbf{D}_{\kappa'}$.

The following lemma can be found in [7] (Lemma 1.1.2.).

Lemma 1.2 *Suppose that* $\mathbf{J}$ *is* κ*-filtered and the functor* $F : \mathbf{I} \to \mathbf{J}$ *satisfies that for every* $J \in \mathbf{J}$*, there exists* I *in* $\mathbf{I}$ *and a morphism* $J \to F(I)$*. If* F *is full and faithful, then* $\mathbf{I}$ *is* κ*-filtered and* F *is final, i.e., for any diagram* $\Sigma : \mathbf{J} \to \mathbf{A}$*,* $colim\Sigma$ *exists if and only if* $colim\Sigma \circ F$ *exists and the canonical morphism* $colim\Sigma(F) \to colim\Sigma$ *is an isomorphism.*

Proposition 1.3 *Let* $\mathbf{B}$ *be a* κ*-accessible category, and* $\mathbf{A}$ *a* κ*-accessibly embedded subcategory of* $\mathbf{B}$*. If every* $B \in \mathbf{B}_\kappa$ *is weakly reflected in* $\mathbf{A}$ *by* $\mathbf{D} = \mathbf{A} \bigcap \mathbf{B}_\kappa$*, then* $\mathbf{A}$ *is* κ*-accessible.*

Proof: $\mathbf{A}$ has κ-filtered colimits, by assumption. For any $A \in \mathbf{A}$, we have a canonical diagram $G : \mathbf{B}_\kappa/A \to \mathbf{B}$, and $A = colimG$, by Proposition 2.1.5. in [7]. This colimit is κ-filtered. Let $\mathbf{D} = \mathbf{A} \bigcap \mathbf{B}_\kappa$. Since $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, all objects in $\mathbf{D}$ are κ-presentable in $\mathbf{A}$. We have a full and faithful functor $F : \mathbf{D}/A \to \mathbf{B}_\kappa/A$. Let $G' : \mathbf{D}/A \to \mathbf{B}$ be the canonical diagram. Given an object $f : B \to A$ in $\mathbf{B}_\kappa/A$, by assumption, there is a factorization $f = f' \circ m$, with $f' : D \to A$ in $\mathbf{D}/\mathrm{A}$. By Lemma 1.2, A is the κ-filtered colimit $colimG'$ in $\mathbf{B}$, and as a consequence, also in $\mathbf{A}$. Thus, $\mathbf{A}$ is κ-accessible.

2 Main Theorem and Some Remarks

In what follows, κ, λ and subscripted variants of them always denote infinite cardinals.

The proof of the following theorem follows closely the lines of the corresponding proof in [3].

Theorem 2.1 *Let* $\mathbf{B}$ *be a locally presentable category, and* $\mathbf{A}$ *an accessibly embedded subcategory of* $\mathbf{B}$*. If* $\mathbf{A}$ *is cone-reflective, then it is accessible.*

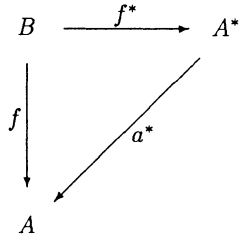
Proof: We may assume that $\mathbf{B}$ is a functor category $(\mathbf{C}, \mathbf{Set})$, for some small category $\mathbf{C}$. The reason is that every locally presentable category is a reflective subcategory of a functor category $(\mathbf{C}, \mathbf{Set})$ with $\mathbf{C}$ small, and the inclusion functor is accessibly embedded. If λ is a regular cardinal bigger than the cardinality of $\mathbf{C}$ and $\aleph_0$, then a functor $F \in \mathbf{B}$ is λ-presentable in $\mathbf{B}$ iff the cardinality of $\coprod_{C \in \mathbf{C}} F(C)$ is less than λ. It easily follows that if $\mu = sup_{i<\nu}\kappa_i$ with $\kappa_i \leq \kappa_j$ for $i < j < \nu$, and $B \in \mathbf{B}_{\mu+}$, then we can write B as a colimit of a ν-chain, $B = colim_{i<\nu} B_i$, with $B_i \in \mathbf{B}_{\kappa_i}$.

Let κ be a regular cardinal such that $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$. Let us define κ_i for $i < \kappa$ by transfinite induction. Let $\kappa_0 = \kappa$. Given $0 < i < \kappa$, having defined κ_j for all $j < i$, for i limit, let κ_i be a regular cardinal bigger than κ_j

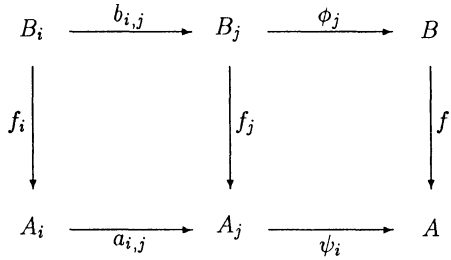
for all $j < i$, and for $i = j + 1 < \kappa$, let κ_{j+1} be a regular cardinal $\geq \kappa_j$ such that all objects in $\mathbf{B}_{\kappa_j}$ have a weak reflection in $\mathbf{D}_{\kappa_{j+1}}$; since each $\mathbf{B}_{\kappa'}$ is small, and since every $B \in \mathbf{B}$ has a weak reflection in $\mathbf{D}_{\kappa'}$, for some κ', such κ_{j+1} clearly exists.

Let $\mu = sup_{i<\kappa}\kappa_i$, and $\lambda = \mu^+$. We claim that every $B \in \mathbf{B}_\lambda$ has a weak reflection in $\mathbf{D}_\lambda$. Since $\mathbf{B}$ is clearly κ'-accessible for all $\kappa' \geq \aleph_0$, in particular, for $\lambda = \kappa'$, and $\lambda > \kappa$, by Proposition 1.3, this will suffice for the proof of the theorem.

Let $B \in \mathbf{B}_\lambda$. As above, let us represent B as the colimit of a κ-chain $(b_{i,j} : B_i \to B_j)_{i<j<\kappa}$, with $B_i \in \mathbf{B}_{\kappa_i}$. Let $\phi_i : B_i \to B$ be the colimit coprojection. Let $A \in \mathbf{A}$ and $f : B \to A$ be arbitrary; we want to find $A^* \in \mathbf{D}_\lambda$ with a factorization

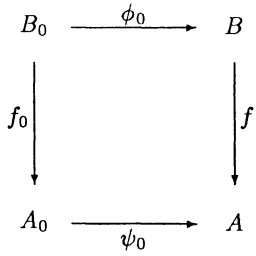


By transfinite induction on $i < \kappa$, we will define objects $A_i \in \mathbf{D}_{\kappa_{i+1}}$ and morphisms $a_{i,j} : A_i \to A_j$, $f_i : B_i \to A_i$, $\psi_i : A_i \to A$ such that the morphisms $a_{i,j}$ form a chain, the morphisms ψ_i form a compatible cocone, and the following diagram



commutes.

For $i = 0$, we let $A_0 \in \mathbf{D}_{\kappa_1}$, $f_0 : B_0 \to A_0$ and $\psi_0 : A_0 \to A$ such that

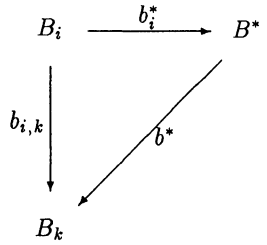


commutes; these items are obtained from a suitable factorization of the morphism $f \circ \phi_0 : B_0 \to A$, possible by the choice of κ_1 and $B_0 \in \mathbf{D}_{\kappa_0}$.

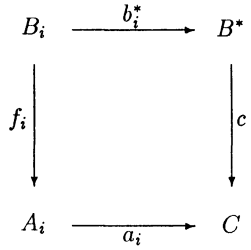
Fix k, $0 < k < \kappa$, and assume that all items with indices $< k$ have been defined. Let $C = colim(a_{i,j} : A_i \to A_j)_{i<j<k}$ with coprojections $a_i : A_i \to C$, and $B^* = colim(b_{i,j} : B_i \to B_j)_{i<j<k}$ with coprojections $b_i^* : B_i \to B^*$.

Since $A_i \in \mathbf{B}_{\kappa_i} \subset \mathbf{B}_{\kappa_k}$, and $\mathbf{B}_{\kappa_k}$ is closed under $< \kappa \leq \kappa_k$-sized colimits, $C \in \mathbf{B}_{\kappa_k}$. Similarly, $B_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_{\kappa_k}$, and so $B^* \in \mathbf{B}_{\kappa_k}$.

By the universal property of B^* , we have $b^* : B^* \to B_k$ such that

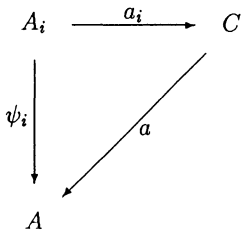


commute, and $c : B^* \to C$ such that

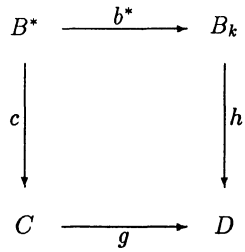


commute, for all $i < k$.

By the universal property of C, we have $a : C \to A$ such that



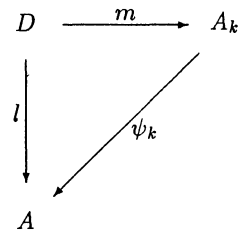
commute for all $i < k$. We form the pushout of c and b^*:



Since $B^*, B_k, C \in \mathbf{B}_{\kappa_k}$, we have $D \in \mathbf{B}_{\kappa_k}$. We prove that $a \circ c = f \circ \phi_k \circ b^*$ by showing that

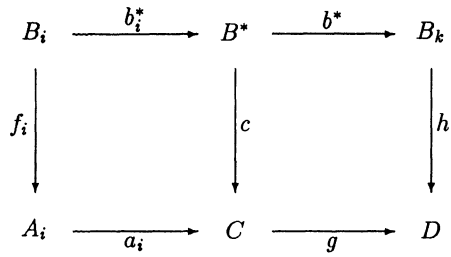
$$\begin{aligned} a \circ c \circ b_i^* &= a \circ a_i \circ f_i \\ &= \psi_i \circ f_i \\ &= f \circ \phi_i \\ &= f \circ \phi_k \circ b^* \circ b_i^* \end{aligned}$$

for each projection b_i^*. By using the universal property of pushout D, we have a unique morphism $l : D \to A$ such that $a = l \circ g$ and $f \circ \phi_k = l \circ h$. Since $D \in \mathbf{B}_{\kappa_k}$, and every object in $\mathbf{B}_{\kappa_k}$ is weakly reflected by $\mathbf{D}_{\kappa_{k+1}}$, there is $A_k \in \mathbf{D}_{\kappa_{k+1}}$ with $\psi_k : A_k \to A$ such that the diagram

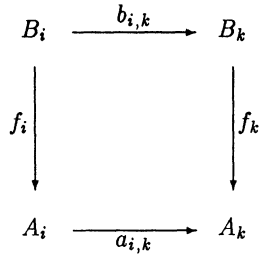


commutes. We have defined the items A_k and ψ_k.

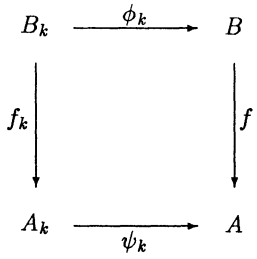
Next, we define $f_k = m \circ h : B_k \to A_k$ and $a_{i,k} = m \circ g \circ a_i : A_i \to A_k$. Note that the diagrams



commute for all $i < k$; and $b_{i,k} = b^* \circ b_i^*$. Then the diagrams

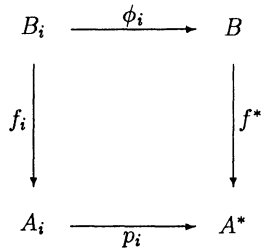


commute for all $i < k$, and the diagram

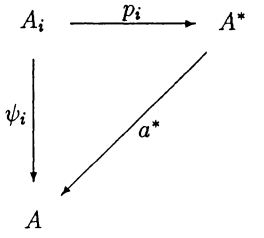


commutes. It is clear that $\psi_i = a_{i,k} \circ \psi_k$ and $a_{i,k} = a_{i,j} \circ a_{j,k}$ hold for all $i < j < k$. This completes the construction.

Put $A^* = colim(a_{i,j} : A_i \to A_j)_{i<j<\kappa}$ with coprojections $p_i : A_i \to A^*$. Since $\mathbf{A}$ is closed under κ-filtered colimits in $\mathbf{B}$, $A^* \in \mathbf{A}$. Also, since $A_i \in \mathbf{B}_{\kappa_{i+1}} \subset \mathbf{B}_\lambda$ and $\kappa < \lambda$, we have that $A^* \in \mathbf{B}_\lambda$; that is, $A^* \in \mathbf{D}_\lambda$. By the construction above, we have $f^* : B \to A^*$ such that the diagrams



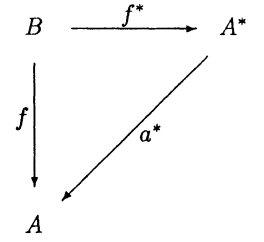
commute for all $i < \kappa$; also, we have $a^* : A^* \to A$ such that the diagrams



commute for all $i < \kappa$; hence we have that

$$f \circ \phi_i = \psi_i \circ f_i = a^* \circ p_i \circ f_i = a^* \circ f^* \circ \phi_i$$

for all $i < \kappa$. Since $< \phi_i >_{i<\kappa}$ is a colimit cocone, we conclude that the diagram



commutes. This completes the proof.

Definition 2.2 *For any locally presentable category* **B**, *an accessibly embedded subcategory* **A** *of* **B** *is accessible iff* **A** *is cone-reflective.*

Proof: By Theorem 2.1 and Proposition 2.1.8. in [7].

Remark 2.3 *Let* **A** *be an accessibly embedded subcategory of* **B** *closed under limits. J. Adámek and J. Rosický have shown in [2] that* **A** *is a reflective subcategory of* **B**. *Thus* **A** *is locally presentable.*

Remark 2.4 *Recall from [1] that Vopěnka's principle is the following statement: the category* **Gra** *of graphs does not have a large discrete full subcategory. It has been shown in [8] that Vopěnka's principle is equivalent to the following statement: every accessibly embedded subcategory of a locally presentable category is accessible. Thus, assuming Vopěnka's principle, every accessible embedding of a locally presentable category satisfies the solution set condition.*

Remark 2.5 *Let* **A** *be an accessible full subcategory of an accessible category* **B**. *Suppose that the inclusion functor from* **A** *to* **B** *satisfies the solution-set condition. J. Rosický and W. Tholen have recently proved that the inclusion functor is accessible (see Theorem 3.10 in [9]). Also, they have proved in [9] that Vopěnka's principle is equivalent to the the following statement: a functor between accessible categories is accessible if and only if it satisfies the solution-set condition.*

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