CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

RODIANI VOREADOU

Some remarks on the subject of coherence

Cahiers de topologie et géométrie différentielle catégoriques, tome 15, nº 4 (1974), p. 399-417

<http://www.numdam.org/item?id=CTGDC_1974__15_4_399_0>

CAHIERS DE TOPOLOGIE
ET GEOMETRIE DIFFERENTIELLE

Vol. XV-4

SOME REMARKS ON THE SUBJECT OF COHERENCE *

by Rodiani VOREADOU

Introduction

In the search for a passage from the result of [3] to a complete solution of the coherence problem for closed categories, we observed that a careful study of the proof of Proposition 7.8 of [3] suggests the very existence of a class $\mathfrak{M}$ of allowable graphs (in the sense of [3]), which permits an improvement of the results of [3] and [4] by replacing the restriction to allowable natural transformations between proper shapes by the restriction to the allowable natural transformations with graphs not in $\mathfrak{M}$. The corresponding extensions of the results of [3] and [4] are given in § 1, with all the necessary details of proofs. In the case of closed categories, the present result (Theorem 2 of § 1) may be weaker than the theorem in [6], but it has the advantages of simplicity and of being obtained by a method which seems to be directly applicable to other situations, where a cut-elimination theorem (of the kind described in [2]) together with a Kelly-MacLane ([3], [4]) method of proof have been, or can be, used. For the latter situations, some conjectured coherence theorems as resulting from the application of the method of § 1 are stated in paragraph 2.

§ 3 contains an observation on the possible relationship between coherence and some extensions of the notion of graph of a natural transformation.

§ 4 is a statement (without full proofs) of further results on closed categories, this time with respect to the non-commutativity of certain diagrams. The example given in [3] (and mentioned in § 3), of a non-commutative diagram in closed categories, is one of a whole class of pairs (h, h') of allowable (with respect to closed categories) natural transformations with $\Gamma h = \Gamma h'$ and $h \neq h'$. A class $\mathfrak{W}$ of such pairs is

---

* Conférence donnée au Colloque d'Amiens (1973)

described in § 4. By Theorem 2 of § 1, such pairs must have graphs in $\mathfrak{M}$. It is indicated in § 4 that, for every $\xi \in \mathfrak{M}$, there is a pair (h, h') of allowable natural transformations with $\Gamma h = \Gamma h' = \xi$ and $h \neq h'$. In this sense, Theorem 2 of § 1 is the best coherence result for closed categories obtainable by using graphs only.

The terminology and notation of [3], [4] and [6] will be used.

1. An extension of the Kelly-MacLane theorems on coherence for closed categories and for closed naturalities.

Let $\mathfrak{M}_0$ be the class of those allowable graphs which can be written in the form $< >$ in two ways:

$$\xi : S \xrightarrow{x} ([B,C] \otimes A) \otimes D \xrightarrow{<f> \otimes 1} C \otimes D \xrightarrow{g} T$$

$$\xi : S \xrightarrow{x'} ([B',C'] \otimes A') \otimes D' \xrightarrow{<f'> \otimes 1} C' \otimes D' \xrightarrow{g'} T$$

with 1° B and B' non-constant,

2° f and f' not of the form $< >$,

3° $[B,C]$ associated with a prime factor of A' via $x' x^{-1}$,

4° $[B',C']$ associated with a prime factor of A via $x(x')^{-1}$

but cannot be written in any of the forms «central», $\otimes$ or π.

Let $\mathfrak{M}$ be the smallest class of allowable graphs satisfying:

$\mathfrak{M}1$. $\mathfrak{M}_0$ is contained in the class,

$\mathfrak{M}2$. if $f : T \to S$ is in the class and $u : T' \to T$, $v : S \to S'$ are central, then $v f u : T' \to S'$ is in the class,

$\mathfrak{M}3$. if at least one of allowable $f : A \to C$ and $g : B \to D$ is in the class then $f \otimes g : A \otimes B \to C \otimes D$ is in the class,

$\mathfrak{M}4$. if $f : A \otimes B \to C$ is in the class, so is $\pi(f) : A \to [B, C]$,

$\mathfrak{M}5$. if at least one of allowable $f : A \to B$ and $g : C \otimes D \to E$ is in the class, then $g(<f> \otimes 1) : ([B, C] \otimes A) \otimes D \to E$ is in the class.

Then, working as for Proposition 6.2 of [3], we prove that, *for each* $\xi \in \mathfrak{M}$, *at least one of the following is true:*

*(*1)* $\xi \in \mathfrak{M}_0$,
*(*2)* $\xi = y(f \otimes g)x$, *with* x *and* y *central,* f *and* g *allowable and non-trivial and at least one of* f *and* g *is in* $\mathfrak{M}$,
*(*3)* $\xi = y\pi(f)$ *with* y *central and* $f \in \mathfrak{M}$,
*(*4)* $\xi = g(\langle f \rangle \otimes 1)x$ *with* x *central,* f *and* g *allowable and at least one of* f *and* g *is in* $\mathfrak{M}$.

Now, using the above result that each element of $\mathfrak{M}$ is of at least one of the forms (*1)-(*4), we prove, by induction on rank, the following analog of Theorem 2.1 of [3] :

THEOREM 1. *There is a finite test for deciding whether an allowable graph is in* $\mathfrak{M}$.

Theorem 2.4 of [3] can now be extended to :

THEOREM 2. *If* $h, h' ; T \to S$ *are allowable natural transformations and* $\Gamma h = \Gamma h' \notin \mathfrak{M}$, *then* $h = h'$.

To prove Theorem 2, follow [3] until the end of the Remark on p. 130 ; then add the three propositions below; using these, the proof of Theorem 2 is done by an induction similar to that of the proof of Theorem 2.4 in [3] .

PROPOSITION A. *Lemma 7.5 of* [3] *holds with (a),(b) and (d) only.*

PROPOSITION B. *Let* $h : P \otimes Q \to M \otimes N$ *be an allowable morphism in* $\underline{H}$. *Suppose that the graph* Γh *is of the form* $\xi \otimes \eta$ *for graphs* $\xi : P \to M$ *and* $\eta : Q \to N$. *Then there are allowable morphisms* $p : P \to M$, $q : Q \to N$ *such that* $h = p \otimes q$, $\Gamma p = \xi$, $\Gamma q = \eta$.

PROPOSITION C. *Let* $h : ([Q, M] \otimes P) \otimes N \to S$ *be an allowable morphism in* $\underline{H}$, *with* $[Q, M]$ *non-constant. Suppose that* $\Gamma h \notin \mathfrak{M}$ *and* Γh *is of the form* $\eta(\langle \xi \rangle \otimes 1)$ *for graphs* $\xi : P \to Q$, $\eta : M \otimes N \to S$. *Suppose finally that* ξ *cannot be written in the form*

$$P \xrightarrow{\omega} ([F, G] \otimes E) \otimes H \xrightarrow{\rho(\langle \sigma \rangle \otimes 1)} Q$$

for any graphs ω, ρ, σ *with* ω *central. Then there are allowable morphisms* $p : P \to Q$, $q : M \otimes N \to S$ *such that*

$$h = q(\langle p \rangle \otimes 1),\ \Gamma p = \xi,\ \Gamma q = \eta.$$

PROOF OF PROPOSITION B. Proposition B is the analog of Proposition 7.6 of [3] and we follow a similar proof. Work as in the proof of Proposition 7.6 in [3], until case $\langle\rangle$. If h is of type $\langle\rangle$, with the notation of Proposition 6.2 of [3], we may suppose (as in [3]) that $[B, C]$ is associated via x with a prime factor of P. Let A_P (resp. A_Q) be an iterated $\otimes$-product of the prime factors of A associated with prime factors of P (resp. Q) via x. There is a central $z: A \to A_P \otimes A_Q$. If $r(A_Q) = 0$, then (since none of the prime factors of Q is constant) $A_Q = I$ and all the prime factors of A are associated via x with prime factors of P and this case is done in [3]. If $r(A_Q) > 0$, then, since the mate under Γh of any element of $v(A_Q)$ is in $v(A_Q)$ by the hypotheses and the form of h, the composite

$$\begin{tikzcd} A_P \otimes A_Q \arrow[r, "z^{-1}"] & A \arrow[r, "f"] & B \arrow[r, "b^{-1}"] & B \otimes I \end{tikzcd}$$

has graph of the form $\sigma \otimes \tau$, so, by induction, $b^{-1} f z^{-1}$ is $s \otimes t$ for allowable $s: A_P \to B$ and $t: A_Q \to I$ with $\Gamma s = \sigma$ and $\Gamma t = \tau$. Then, with x' being the composite of x with the obvious central

$$([B, C] \otimes A) \otimes D \to (([B, C] \otimes A_P) \otimes D) \otimes A_Q,$$

$$h = b(g \otimes 1)((\langle s \rangle \otimes 1) \otimes t)x' = b(g(\langle s \rangle \otimes 1) \otimes t)x',$$

i.e. h is of the form $\otimes$ and this case is already done. ■

PROOF OF PROPOSITION C. Proposition C is the analog of Proposition 7.8 of [3] and we follow a similar proof. As in [3], we may assume that P, N and S have no constant prime factors. Work as in the proof of Proposition 7.8 in [3], for the cases of h being central or of type π.

If h is of the form $y(f \otimes g)x$ for allowable $f: A \to C$ and $g: B \to D$ and central x and y, we may (as in [3]) suppose that $[Q, M]$ is associated via x with a prime factor of A. Let P_A (resp. P_B) be an iterated $\otimes$-product of the prime factors of P associated via x with prime factors of A (resp. B). There is a central $z: P \to P_A \otimes P_B$. If $r(P_B) = 0$, then (since P has no constant prime factors) $P_B = I$ and all the prime factors of P are associated with prime factors of A via x; continue as in [3] for this case, observing that $\Gamma(fs) \notin \mathfrak{M}$ (in [3]). If $r(P_B) > 0$, let ϕ be the composite

$$b^{-1} h((1 \otimes z^{-1}) \otimes 1) u^{-1} : (([Q, M] \otimes P_A) \otimes N) \otimes P_B \to S \otimes I,$$

where u is the obvious central

$$([Q, M] \otimes (P_A \otimes P_B)) \otimes N \to (([Q, M] \otimes P_A) \otimes N) \otimes P_B \, .$$

The mate of an element of $v(P_B)$ under $\Gamma\phi$ is in $v(P) + v(Q)$ by hypothesis and in $v(B) + v(D) + v(S)$ by the form of h, so it must be in $v(P_B)$. So $\Gamma\phi$ is of the form $\sigma \otimes \tau$ where $\sigma : ([Q, M] \otimes P_A) \otimes N \to S$ is the obvious restriction of Γh; by Proposition B, $\phi = s \otimes t$ for allowable

$$s : ([Q, M] \otimes P_A) \otimes N \to S \quad \text{and} \quad t : P_B \to I.$$

$r(P_B) > 0$ implies $r(s) < r(h)$; $\sigma = \eta(\langle \xi' \rangle \otimes 1)$, where $\xi' : P_A \to Q$ is the obvious restriction of ξ, and $\sigma \notin \mathfrak{M}$; so, by induction, $s = q(\langle p' \rangle \otimes 1)$ for allowable $p' : P_A \to Q$ and $q : M \otimes N \to S$. Take $p = b(p' \otimes t) z : P \to Q$. Then

$$h = h\phi u((1 \otimes z) \otimes 1) = h(s \otimes 1)(1 \otimes t) u((1 \otimes z) \otimes 1)$$
$$= h(q \otimes 1)((\langle p' \rangle \otimes 1) \otimes 1)(1 \otimes t) u((1 \otimes z) \otimes 1)$$
$$= q(\langle p' \rangle \otimes 1) h(1 \otimes t) u((1 \otimes z) \times 1)$$
$$= q(\langle p' \rangle \otimes 1) hu((1 \otimes (1 \otimes t)) \otimes 1)((1 \otimes z) \otimes 1) = (\text{by Theorem 4.9 of}$$
$$[3]) = q(\langle p' \rangle \otimes 1)((1 \otimes b) \otimes 1)((1 \otimes (1 \otimes t)) \otimes 1)((1 \otimes z) \otimes 1)$$
$$= q(\langle p' \rangle \otimes 1)((1 \otimes b(1 \otimes t) z) \otimes 1)$$
$$= q(\langle b(p' \otimes t) z \rangle \otimes 1) = q(\langle p \rangle \otimes 1)$$

and $\Gamma p = \xi$, $\Gamma q = \eta$.

If h is of type $\langle\rangle$, we distinguish cases (i), (ii), (iii) as in [3], and use the notation of [3]. We may assume that f is not of the form $\langle\rangle$ and $[B, C]$ is not constant (see the proof of Theorem 2.4 in [3]) in subcase I of case (ii) below.

Case (i). $[Q, M]$ is associated with $[B, C]$ via x. Then, as in [3], $B = Q$, $M = C$ and h is

$$([Q,M]\otimes P)\otimes N\xrightarrow{x}([Q,M]\otimes A)\otimes D\xrightarrow{(1\otimes f)\otimes 1}([Q,M]\otimes Q)\otimes D\xrightarrow{e\otimes 1}M\otimes D\xrightarrow{g}S$$

with $f:A\to Q$. Let P_A (resp. P_D, resp. N_A, resp. N_D) be an iterated $\otimes$-product of the prime factors of P (resp. P, resp. N, resp. N) associated with prime factors of A (resp. D, resp. A, resp. D) via x. There are central $z:P\to P_A\otimes P_D$ and $w:N\to N_A\otimes N_D$. The mate of an element of $v(P_D)$ under Γh is in $v(P)+v(Q)$ by hypothesis and by the form of h, in $v(M)+v(D)+v(S)$, so it must be in $v(P_D)$. The mate of an element of $v(N_A)$ under Γh is in $v(M)+v(N)+v(S)$ by hypothesis and in $v(A)+v(Q)$ by the form of h, so it must be in $v(N_A)$.

Subcase I: $r(P_D\otimes N_A)=0$. Then (since no prime factor of P or N is constant) $P_D=I$, $N_A=I$ and all prime factors of P (resp. N) are associated with prime factors of A (resp. D) via x; continue as in [3] for this case.

Subcase II: $r(P_D)>0$ and $r(N_A)>0$. Let ϕ be the composite

$$ab^{-1}b^{-1}h((1\otimes z^{-1})\otimes w^{-1})u^{-1}:$$

$$(([Q,M]\otimes P_A)\otimes N_D)\otimes(P_D\otimes N_A)\longrightarrow S\otimes(I\otimes I)$$

where u is the obvious central

$$([Q,M]\otimes(P_A\otimes P_D))\otimes(N_A\otimes N_D)\to(([Q,M]\otimes P_A)\otimes N_D)\otimes(P_D\otimes N_A)\ .$$

$\Gamma\phi$ is of the form $\sigma\otimes\tau$ where $\sigma:([Q,M]\otimes P_A)\otimes N_D\to S$ is the obvious restriction of Γh; by Proposition B, $\phi=s\otimes t$ for allowable

$$s:([Q,M]\otimes P_A)\otimes N_D\to S \text{ and } t:P_D\otimes N_A\to I\otimes I$$

Also by Proposition B, $t=t_1\otimes t_2$ for allowable $t_1:P_D\to I$ and $t_2:N_A\to I$. $r(P_D\otimes N_A)>0$ implies $r(s)<r(h)$; $\sigma=\eta'(<\xi'>\otimes 1)$, where $\xi':P_A\to Q$ and $\eta':M\otimes N_D\to S$ are the obvious restrictions of ξ and η, and $\sigma\notin\mathfrak{M}$; so, by induction, $s=q'(<p'>\otimes 1)$ for allowable $p':P_A\to Q$ and $q':M\otimes N_D\to S$. Take

$$p=b(p'\otimes t_1)z:P\to Q \quad\text{and}\quad q=b(q'\otimes t_2)a^{-1}(1\otimes cw):M\otimes N\to S.$$

Then

$$h=b\,\boldsymbol{b}\,a^{-1}\phi u((1\otimes z)\otimes w)$$

$$= b b a^{-1}(s\otimes 1)(1\otimes(t_1\otimes 1))(1\otimes(1\otimes t_2))u((1\otimes z)\otimes w)$$

$$= b b a^{-1}(q'\otimes 1)((<p'>\otimes 1)\otimes 1)(1\otimes(t_1\otimes 1))(1\otimes(1\otimes t_2))u((1\otimes z)\otimes w)$$

$$= b b a^{-1}(q'\otimes(1\otimes t_2))((<p'>\otimes 1)\otimes 1)(1\otimes(t_1\otimes 1))u((1\otimes z)\otimes w)$$

(using Theorem 4.9 of [3])

$$= b(q'\otimes t_2)(1\otimes bc)((<p'>\otimes 1)\otimes 1)(1\otimes(t_1\otimes 1))u((1\otimes z)\otimes w)$$

$$= b(q'\otimes t_2)(1\otimes bc)((<p'>\otimes 1)\otimes(t_1\otimes 1))u((1\otimes z)\otimes w)$$

(using Theorem 4.9 of [3])

$$= b(q'\otimes t_2)(1\otimes bc)(1\otimes cb^{-1})a^{-1}(1\otimes c)(<b(p'\otimes t_1)>\otimes 1)((1\otimes z)\otimes w)$$

$$= b(q'\otimes t_2)a^{-1}(1\otimes cw)(<b(p'\otimes t_1)z>\otimes 1) = q(<p>\otimes 1)$$

and $\Gamma p = \xi, \Gamma q = \eta$.

Subcase III: $r(P_D) > 0$ and $r(N_A) = 0$. Then (since N has no constant prime factors) $N_A = I$ and all prime factors of N are associated with prime factors of D via x. Let ϕ be the composite

$$b^{-1}b((1\otimes z^{-1})\otimes 1)\ v^{-1} : (([Q,M]\otimes P_A)\otimes N)\otimes P_D \to S\otimes I$$

where v is the obvious central

$$([Q,M]\otimes(P_A\otimes P_D))\otimes N \to (([Q,M]\otimes P_A)\otimes N)\otimes P_D .$$

$\Gamma\phi$ is of the form $\sigma\otimes\tau$ where $\sigma : ([Q,M]\otimes P_A)\otimes N \to S$ is the obvious restriction of Γb; by Proposition B, $\phi = s\otimes t$ for allowable $t : P_D \to I$ and $s : ([Q,M]\otimes P_A)\otimes N \to S$. $r(P_D) > 0$ implies $r(s) < r(b)$;

$$\sigma = \eta(<\xi'>\otimes 1),$$

where $\xi' : P_A \to Q$ is the obvious restriction of ξ, and $\sigma \notin \mathfrak{M}$; so, by induction, $s = q(<p'>\otimes 1)$ for allowable $p' : P_A \to Q$ and $q : M\otimes N \to S$. Take $p = b(p'\otimes t)z : P \to Q$. Then

$$b = b\phi v((1\otimes z)\otimes 1) = b(1\otimes t)(s\otimes 1)v((1\otimes z)\otimes 1)$$

$$= b(1\otimes t)(q\otimes 1)((<p'>\otimes 1)\otimes 1)v((1\otimes z)\otimes 1)$$

$$= b(q\otimes 1)((<p'>\otimes 1)\otimes t)v((1\otimes z)\otimes 1)$$

$$= qb((e\otimes 1)\otimes 1)(((1\otimes p')\otimes 1)\otimes t)v((1\otimes z)\otimes 1)$$

$$=qb((e\otimes 1)\otimes 1)v((1\otimes(p'\otimes t))\otimes 1)((1\otimes z)\otimes 1)$$

$$=q(e\otimes 1)bv((1\otimes(p'\otimes t))\otimes 1)((1\otimes z)\otimes 1)$$

(by Theorem 4.9 of [3])

$$=q(e\otimes 1)((1\otimes b)\otimes 1)((1\otimes(p'\otimes t))\otimes 1)((1\otimes z)\otimes 1)=q(<p>\otimes 1)$$

and $\Gamma p=\xi$, $\Gamma q=\eta$.

Subcase IV: $r(P_D)=0$ and $r(N_A)>0$. Then (since P has no constant prime factors) $P_D=I$ and all prime factors of P are associated with prime factors of A via x. Let ϕ be the composite

$$b^{-1}b(1\otimes w^{-1}c)a:(([Q,M]\otimes P)\otimes N_D)\otimes N_A\to S\otimes I.$$

$\Gamma\phi$ is of the form $\sigma\otimes\tau$ where $\sigma:([Q,M]\otimes P)\otimes N_D\to S$ is the obvious restriction of Γh; by Proposition B, $\phi=s\otimes t$ for allowable $t:N_A\to I$ and $s:([Q,M]\otimes P)\otimes N_D\to S$. $r(N_A)>0$ implies $r(s)<r(h)$; $\sigma=\eta'(<\xi>\otimes 1)$, where $\eta':M\otimes N_D\to S$ is the obvious restriction of η, and $\sigma\notin\mathfrak{M}$; so, by induction, $s=q'(<p>\otimes 1)$ for allowable $p:P\to Q$ and q': $M\otimes N_D\to S$. Take $q=b(q'\otimes t)a^{-1}(1\otimes cw):M\otimes N\to S$. Then

$$h=b\phi a^{-1}(1\otimes cw)=b(1\otimes t)(s\otimes 1)a^{-1}(1\otimes cw)$$

$$=b(1\otimes t)(q'\otimes 1)((<p>\otimes 1)\otimes 1)a^{-1}(1\otimes cw)$$

$$=b(q'\otimes 1)(1\otimes t)((<p>\otimes 1)\otimes 1)a^{-1}(1\otimes cw)$$

$$=b(q'\otimes t)a^{-1}(1\otimes cw)(<p>\otimes 1)=q(<p>\otimes 1)$$

and $\Gamma p=\xi$, $\Gamma q=\eta$.

Case (ii). $[Q,M]$ is associated with a prime factor of A via x.

Subcase I: $[B,C]$ associated with a prime factor of P via x. Then we may assume that f is not of type $<>$ and $[B,C]$ is not constant (see the proof of Theorem 2.4 in [3]). Suppose that Γf can be written in the form $<>$ as $\Gamma f=\zeta(<\lambda>\otimes 1)\tau$ for a central graph

$$\tau:A\to([X,Y]\otimes Z)\otimes W$$

and allowable graphs

$$\lambda:Z\to X \quad\text{and}\quad \zeta:Y\otimes W\to B.$$

We may assume $[X,Y]$ is non-constant and λ cannot be written in the form $<>$ as above. Let t be the central morphism in $\underline{H}$ with $\Gamma t = \tau$. Then $\Gamma(ft^{-1}) \notin \mathfrak{M}$ since $\Gamma f \notin \mathfrak{M}$,

$$r(ft^{-1}) = r(f) < r(h)$$

and

$$\Gamma(ft^{-1}) : ([X, Y] \otimes Z) \otimes W \to B$$

is of the form $\zeta(<\lambda> \otimes 1)$: by induction, there are allowable morphisms $l : Z \to X$, $z : Y \otimes W \to B$ such that

$$ft^{-1} = z(<l> \otimes 1), \quad \Gamma l = \lambda, \quad \Gamma z = \zeta .$$

Then $f = z(<l> \otimes 1)t$, contrary to our assumption. So Γf cannot be written in the form $<>$. This implies that Q is non-constant. Also, the hypothesis that ξ cannot be written in the form $<>$ implies that B is non-constant. Then $\Gamma h \notin \mathfrak{M}$ implies that Γh is (also) of one of the types π, $\otimes$ and «central». Γh cannot be central because, by the form of h and Γh, the mates of variables in B and Q (B and Q being non-constant) cannot be in $v(S)$. If Γh is of type $\otimes$, then, by Proposition B, h is also of type $\otimes$ and this case is done. If Γh is of type π, then $S = [K, L]$ and

$$\pi^{-1}(\Gamma h) = \Gamma(\pi^{-1}(h)) : (([Q, M] \otimes P) \otimes N) \otimes K \to L$$

is of type $<>$, not in $\mathfrak{M}$ and with rank lower than $r(h)$; so, by induction, $\pi^{-1}(h) = \hat{q}(<p> \otimes 1)$ for allowable $p : P \to Q$ and $q : (M \otimes N) \otimes K \to L$. Then

$$h = \pi(\hat{q}(<p> \otimes 1)) = \pi(\hat{q})(<p> \otimes 1) = q(<p> \otimes 1)$$

for $q = \pi(\hat{q}) : M \otimes N \to S$, and $\Gamma p = \xi$, $\Gamma q = \eta$.

Subcase II: $[B, C]$ associated with a prime factor of N via x. Let P_A (resp. P_D) be an iterated $\otimes$-product of the prime factors of P associated with prime factors of A (resp. D) via x. There is a central $z : P \to P_A \otimes P_D$. If $r(P_D) = 0$, then (since P has no constant prime factors) $P_D = I$ and every prime factor of P is associated with a prime factor of A via x; continue as in [3] for this case, observing that $\Gamma(ft) \notin \mathfrak{M}$ (in [3]). If $r(P_D) > 0$, let ϕ be the composite

$$h^{-1} h((1 \otimes z^{-1}) \otimes 1)u : (([Q, M] \otimes P_A) \otimes N) \otimes P_D \to S \otimes I,$$

where u is the obvious central

$$(([Q,M]\otimes P_A)\otimes N)\otimes P_D \rightarrow ([Q,M]\otimes(P_A\otimes P_D))\otimes N.$$

The mate of an element of $v(P_D)$ under $\Gamma\phi$ is in $v(P)+v(Q)$ by hypothesis and in $v(C)+v(D)+v(S)$ by the form of h, so it must be in $v(P_D)$. So $\Gamma\phi$ is of the form $\sigma\otimes\tau$ where $\sigma:([Q,M]\otimes P_A)\otimes N\rightarrow S$ is the obvious restriction of Γh; by Proposition B, $\phi=s\otimes t$ for allowable $s:([Q,M]\otimes P_A)\otimes N\rightarrow S$ and $t:P_D\rightarrow I$. Then

$$h=b\phi u^{-1}((1\otimes z)\otimes 1)=y'(s\otimes t)x'$$

with x' and y' central, i.e. h is of type $\otimes$ and this case is done.

Case (iii). $[Q,M]$ is associated with a prime factor of D via x.

Subcase I: $[B,C]$ associated with a prime factor of P via x. Let N_A (resp. N_D) be an iterated $\otimes$-product of the prime factors of N associated with prime factors of A (resp. D) via x. There is a central $w: N\rightarrow N_A\otimes N_D$. If $r(N_A)=0$, then (since N has no constant prime factors) $N_A=I$ and every prime factor of N is associated via x with a prime factor of D, i.e. every prime factor of A is associated with a prime factor of P via x; as in [3], this case is excluded, since it implies that ξ is of the form $\rho(\langle\sigma\rangle\otimes 1)\omega$ with ω central. If $r(N_A)>0$, let ϕ be the composite

$$b^{-1}h(1\otimes w^{-1}c)a:(([Q,M]\otimes P)\otimes N_D)\otimes N_A\rightarrow S\otimes I.$$

The mate of an element of $v(N_A)$ under $\Gamma\phi$ is in $v(M)+v(N)+v(S)$ by hypothesis and in $v(A)+v(B)$ by the form of h, so it must be in $v(N_A)$. So $\Gamma\phi$ is of the form $\sigma\otimes\tau$ where $\sigma:([Q,M]\otimes P)\otimes N_D\rightarrow S$ is the obvious restriction of Γh; by Proposition B, $\phi=s\otimes t$ for allowable $s:([Q,M]\otimes P)\otimes N_D\rightarrow S$ and $t:N_A\rightarrow I$. Then

$$h=b\phi a^{-1}(1\otimes cw)=y'(s\otimes t)x'$$

with x' and y' central, i.e. h is of the type $\otimes$ and this case is done.

Subcase II: $[B,C]$ associated with a prime factor of N via x. Let P_A and P_D be as in case (ii). If $r(P_A)=0$, then (since P has no constant prime factors) every prime factor of P is associated via x with a prime factor of D; continue as in [3] for this case, observing that $\zeta \notin \mathfrak{M}$ (in [3]). If $r(P_A) > 0$, let ϕ be the composite

$$(([Q,M]\otimes P_D)\otimes N)\otimes P_A \xrightarrow{v} ([Q,M]\otimes P)\otimes N \xrightarrow{h} S \xrightarrow{b^{-1}} S\otimes I$$

where v is the obvious central. The mate of an element of $v(P_A)$ under $\Gamma\phi$ is in $v(P)+v(Q)$ by hypothesis and in $v(A)+v(B)$ by the form of h, so it must be in $v(P_A)$. So $\Gamma\phi$ is of the form $\sigma\otimes\tau$, where $\sigma:([Q,M]\otimes P_D)\otimes N \to S$ is the obvious restriction of Γh; by Proposition B, $\phi = s\otimes t$ for allowable

$$s:([Q,M]\otimes P_D)\otimes N \to S \quad \text{and} \quad t:P_A \to I.$$

Then $h = b\phi v^{-1} = y'(s\otimes t)x'$ with x' and y' central, i.e. h is of type $\otimes$ and this case is done. ■

If we replace graphs by N-graphs in defining $\mathfrak{M}_0$, we define a class $\mathfrak{M}_{0\,N}$ of N-allowable N-graphs. (Note that the fact that ξ, in this definition of $\mathfrak{M}_{0\,N}$, is of the form <>, excludes the possibility of its being of type «unblocked string».) Then, by modifying definitions and propositions above in the way the analogous part of [3] is modified in [4], we obtain the definition of a class $\mathfrak{M}_N$ of N-allowable N-graphs (which is essentially $\mathfrak{M}$) and the following extension of Theorem 2.4_N of [4]:

THEOREM 1_N. *There is a finite test for deciding whether a N-allowable N-graph is in* $\mathfrak{M}_N$.

THEOREM 2_N. *If* W *is any naturality and* $f, f' : |T|_W \to |S|_W$ *are N-allowable natural transformations with* $\Gamma_N f = \Gamma_N f' \notin \mathfrak{M}_N$, *then* $f = f'$.

2. Conjectures, based on the idea behind the class $\mathfrak{M}$ of § 1.

a) The result of [5] (with the notation of [5] and with $\mathfrak{M}'$ being the obvious analog of $\mathfrak{M}$ for D-graphs) may be improved to: *Let* $h, h' : T \to S$ *be two morphisms in* N, *such that* $\Gamma h = \Gamma h' \notin \mathfrak{M}$ *and* $\Delta h = \Delta h' \notin \mathfrak{M}'$. *Then* $h = h'$.

b) The result of [1] (with the terminology of [1]) may be improved to: *We have for* L *the coherence result that* $f, g : P \to Q$ *are equal if* $\Gamma f = \Gamma g \notin \mathfrak{M}$.

c) In the case of biclosed categories (i.e. monoidal categories in which $-\otimes B$ has a right adjoint $[B, -]$ with d and e the unit and counit of the adjunction, and $B\otimes -$ has a right adjoint $<B, ->$ with d' and e' the unit and counit of the adjunction), write

$\pi(f)$ for the composite $A \xrightarrow{d} [B, A\otimes B] \xrightarrow{[1,f]} [B, C]$,

$\pi'(g)$ for the composite $A \xrightarrow{d'} <B, B\otimes A> \xrightarrow{<1,g>} <B, C>$,

$\bar{f}$ for the composite $[B, C]\otimes A \xrightarrow{1\otimes f} [B, C]\otimes B \xrightarrow{e} C$,

$\hat{g}$ for the composite $A\otimes <B, C> \xrightarrow{g\otimes 1} B\otimes <B, C> \xrightarrow{e'} C$,

Proceeding as in [3] (see also [2]) we define allowable natural transformations and graphs and we find that, for each allowable $h : T \to S$, at least one of the following is true:

(**1) h is central,

(**2) h is $T \xrightarrow{x} A\otimes B \xrightarrow{f\otimes g} C\otimes D \xrightarrow{y} S$ with f, g allowable and nontrivial and x, y central,

(**3) h is $T \xrightarrow{\pi(f)} [B, C] \xrightarrow{y} S$ with f allowable and y central,

(**4) h is $T \xrightarrow{\pi'(f)} <B, C> \xrightarrow{y} S$ with f allowable and y central,

(**5) h is

$$T \xrightarrow{x} F\otimes(([B, C]\otimes A)\otimes D) \xrightarrow{1\otimes(\bar{f}\otimes 1)} F\otimes(C\otimes D) \xrightarrow{g} S$$

with f, g allowable and x central.

(**6) h is

$$T \xrightarrow{x} (D\otimes(A\otimes <B, C>))\otimes F \xrightarrow{(1\otimes\hat{f})\otimes 1} (D\otimes C)\otimes F \xrightarrow{g} S$$

with f, g allowable and x central.

Take $\mathfrak{M}_0^*$ to be the class of those allowable graphs which can be written in the forms (**5) and (**6):

$$\xi : T \overset{x}{\rightarrow} F \otimes (([B,C] \otimes A) \otimes D) \xrightarrow{1 \otimes (\bar{f} \otimes 1)} F \otimes (C \otimes D) \overset{g}{\rightarrow} S,$$

$$\xi : T \overset{x'}{\rightarrow} (D' \otimes (A' \otimes < B', C' >)) \otimes F' \xrightarrow{(1 \otimes \hat{f}') \otimes 1} (D' \otimes C') \otimes F' \overset{g'}{\rightarrow} S,$$

with

1) B and B' non-constant,

2) f not of the form (**5) and f' not of the form (**6),

3) $[B, C]$ associated with a prime factor of A' via $x'x^{-1}$,

4) $< B', C' >$ associated with a prime factor of A via $x(x')^{-1}$,

but cannot be written in any of the forms (**1) - (**4). If $\mathfrak{M}^*$ is the class of allowable graphs generated by $\mathfrak{M}_0^*$ (and $\otimes$, $[\,,\,]$, $<,>$ and composition), we may have the following theorem: *If* $h, h' : S \rightarrow T$ *are allowable natural transformations and* $\Gamma h = \Gamma h' \notin \mathfrak{M}^*$, *then* $h = h'$.

3. On the possible relationship between coherence and extensions of the notion of graph of a natural transformation.

There are evidences that, in a general theory of coherence, the notion of graph between shapes (in the sense of linkages between variables) may be replaced by an appropriate use of something broader (call it «superextended graph» here, informally) involving linkages between variables (as in graphs), between constants and between names of functors, with the possibility of certain things being linked with themselves. Examples:

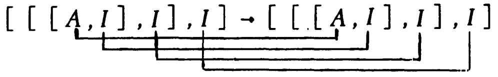
I. In the case of closed categories, graphs do not suffice for deciding equality of natural transformations, as the example of [3] shows: if k_A is the composite (the familiar map to double dual)

$$\begin{tikzcd}
A \arrow[r, "d"] & {[\,[A,I], A \otimes [A,I]\,]} \arrow[r, "{[1,c]}"] & {[\,[A,I], [A,I] \otimes A\,]} \arrow[d, "{[1,e]}"] \\
 & & {[\,[A,I], I\,]},
\end{tikzcd}$$

then

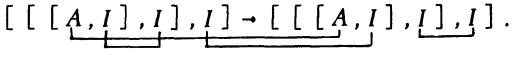
$$k_{[A,I]}\,[k_A, I] \neq 1 : [\,[\,[A,I], I\,], I\,] \rightarrow [\,[\,[A,I], I\,], I\,]$$

although $\Gamma 1 = \Gamma(k_{[A,I]}\,[k_A, I])$. In this case, taking extended graphs* ([6]) (i.e. graphs together with appropriate linkages between I's) we have $G 1 \neq G(k_{[A,I]}\,[k_A, I])$, since



$$G1 \text{ is } [[[A,I],I],I] \to [[[A,I],I],I]$$

and



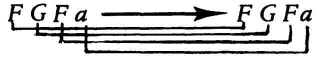
$$G(k_{[A,I]}\,[k_A,I]) \text{ is } [[[A,I],I],I] \to [[[A,I],I],I].$$

Proper use of extended graphs can give additional information (which was inaccessible with graphs) on coherence (see [6] and § 4 below).

II. In the case of an adjunction $F \dashv G$, the triangle

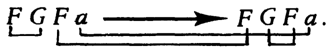
$$\begin{tikzcd}
FGFa \arrow[r, "\varepsilon_{Fa}"] \arrow[dr, "1"'] & Fa \arrow[d, "F\eta_a"] \\
 & FGFa
\end{tikzcd}$$

is not commutative, although $\Gamma 1 = \Gamma(F\eta_a\,\varepsilon_{Fa})$; taking the obvious superextended graphs, we have $G^* 1 \neq G^*(F\eta_a\,\varepsilon_{Fa})$, since



$$G^*1 \text{ is } FGFa \longrightarrow FGFa$$

and



$$G^*(F\eta_a\,\varepsilon_{Fa}) \text{ is } FGFa \longrightarrow FGFa.$$

III. Taking the functor Δ together with Γ in [5] can be considered as an application of the idea of superextended graphs (with all I's linked with themselves).

4. Non-commutative diagrams in closed categories – or, why Theorem 2 of § 1 is the best coherence result for closed categories obtainable by using graphs only.

This paragraph is a preliminary note on the subject and, as such,

---

* (G means «extended graph of»)

mostly descriptive; missing technical details and proofs will appear in [7].

Let $\mathfrak{W}_0$ be the class of those pairs (h, h') of allowable natural transformations which satisfy the following conditions :

h and h' have the same domain, the same codomain and the same graph, and they can be written in the form <> as

$$h : S \xrightarrow{x} ([B, C] \otimes A) \otimes D \xrightarrow{<f> \otimes 1} C \otimes D \xrightarrow{g} T,$$

$$h' : S \xrightarrow{x'} ([B', C'] \otimes A') \otimes D' \xrightarrow{<f'> \otimes 1} C' \otimes D' \xrightarrow{g'} T,$$

with 1) B and B' non-constant,

2) Γf and $\Gamma f'$ not of the form <>,

3) $[B, C]$ associated with a prime factor of A' via $x' x^{-1}$,

4) $[B', C']$ associated with a prime factor of A via $x(x')^{-1}$,

but neither h nor h' can be written in any of the forms «central», $\otimes$ or π.

THEOREM 3. *For every* $(h, h') \in \mathfrak{W}_0$, $h \neq h'$.

To prove Theorem 3, we give a closed category $\mathfrak{K}$ in which, for every $(h, h') \in \mathfrak{W}_0$, h and h' have different components. A brief description of $\mathfrak{K}$ is given in the last part of this paragraph.

THEOREM 4. *For every* $\xi \in \mathfrak{M}$, *there is a pair* (h, h') *of allowable natural transformations such that* $\Gamma h = \Gamma h' = \xi$ *and* $h \neq h'$.

PROOF. We do induction on rank. Suppose the theorem is true for all smaller values, if any, of $r(\xi)$. Using § 1, we distinguish cases according as ξ is of the form (*1), (*2), (*3) or (*4). If ξ is of the form (*1), i.e. $\xi \in \mathfrak{M}_0$, then, by Theorem 2.3 of [3], there is $(h, h') \in \mathfrak{W}_0$ with $\Gamma h = \Gamma h' = \xi$ and, by Theorem 3, $h \neq h'$. If ξ is of the form (*2), $\xi = m(\zeta_1 \otimes \zeta_2)n$ with m and n central, ζ_i allowable non-trivial and at least one of them in $\mathfrak{M}$; by Theorem 4.9 of [3], there are central natural transformations x, y with $\Gamma x = n$, $\Gamma y = m$; using the induction hypothesis or Theorem 2.3 of [3] according as $\zeta_i \in \mathfrak{M}$ or $\zeta_i \notin \mathfrak{M}$, we find pairs of allowable natural transformations (k_i, k'_i) with $\Gamma k_i = \Gamma k'_i = \zeta_i$ and $k_i \neq k'_i$ or $k_i = k'_i$ according as $\zeta_i \in \mathfrak{M}$ or $\zeta_i \notin \mathfrak{M}$; since at least one of ζ_1, ζ_2 is in $\mathfrak{M}$, we have

$$h = y(k_1 \otimes k_2)x \neq y(k'_1 \otimes k'_2)x = h' \quad \text{and} \quad \Gamma h = \Gamma h' = \xi .$$

The remaining cases are done in a similar way, since

$$f \neq f' \text{ implies } y\pi(f) \neq y\pi(f')$$

and

$$f \neq f' \text{ and/ or } g \neq g' \text{ implies } g(<f> \otimes 1)x \neq g'(<f'> \otimes 1)x,$$

for central x, y and the appropriate domains and codomains. ■

Let $\mathfrak{W}$ be the smallest class of pairs of allowable natural transformations satisfying:

$\mathfrak{W}1$. $\mathfrak{W}_0$ is contained in the class,

$\mathfrak{W}2$. If $(f, f' : T \to S)$ is in the class and $u : T' \to T$, $v : S \to S'$ are central, then $(vfu, vf'u : T' \to S')$ is in the class,

$\mathfrak{W}3$. If at least one of $(f_1, f'_1 : A \to C)$ and $(f_2, f'_2 : B \to D)$ is in the class and $f_i = f'_i$ in case (f_i, f'_i) is not in the class, then

$$(f_1 \otimes f_2 , f'_1 \otimes f'_2 : A \otimes B \to C \otimes D) \text{ is in the class.}$$

$\mathfrak{W}4$. If $(f, f' : A \otimes B \to C)$ is in the class, then

$$(\pi(f),\ \pi(f') : A \to [B, C]) \text{ is in the class.}$$

$\mathfrak{W}5$. If at least one of $(f_1, f'_1 : A \to B)$ and $(f_2, f'_2 : C \otimes D \to E)$ is in the class and $f_i = f'_i$ in case (f_i, f'_i) is not in the class, then

$$(f_2(<f_1> \otimes 1), f'_2(<f'_1> \otimes 1) : ([B, C] \otimes A) \otimes D \to E) \text{ is in the class.}$$

THEOREM 5. *For every* $(h, h') \in \mathfrak{W}$, $h \neq h'$.

We now turn to a description of the category $\mathcal{K}$ which is used in the proof of Theorem 3. The construction and the good properties (with respect to our purpose) of $\mathcal{K}$ are based on facts about «E-graphs» and «simple graphs» (see [6]) and, in order to make the present exposition readable, we start by (non-rigourously) reminding the reader of these notions.

The E-*graphs* (extended graphs) are ordinary graphs together with linkages between I's, with the possibility of I's linked with themselves allowed. The E-graph of any component of a, b, c, d, e has linkages between I's as if the I's were variables, except in the case of b, where the last I of the domain is linked with itself. E-graphs are composed like graphs. The rules AM1-AM5 of [3] (pp. 105-106) applied to E-graphs (with a, a^{-1}, b, c, d, e now being the E-graphs of the natural transformations with these names, and with b^{-1} replaced by $b^{\dot{-}1}$, the obvious E-graph of b^{-1}, since b is not an isomorphism in the category of E-graphs) define the class of allowable E-graphs.

A shape S is *simple* if either $S=I$ or else any I that appears in S is alone in an inside right place of $[\,,\,]$. For any shape S there is a (well defined) natural isomorphism $l_S : S \to S_0$ with S_0 simple; we denote by λ_S the E-graph of l_S. For any E-graph $\xi : T \to S$, $\xi_0 : T_0 \to S_0$ is the composite $\lambda_S \xi \lambda_T^{\dot{-}1}$.

We write «instance» for both ordinary and expanded instance.

If $k : S \to T$ is a natural transformation, k_{const} denotes a component of k in which every variable in S and T has been replaced by some constant shape.

$[\mathit{const}, \mathit{const}]$ denotes a shape $[A, B]$ with A and B constant.

A *string of* E-*graphs* is a finite sequence $\{x_1, x_2, \dots, x_n\}$ of E-graphs such that $\partial_1 x_i = \partial_0 x_{i+1}$ for $i = 1, 2, \dots, n-1$; the string is *allowable* if every x_i is an instance of one of $a, a^{-1}, b, b^{\dot{-}1}, c, d, e, 1$; we say that a *shape* S *is involved in the string* if S appears as a subshape (proper or not) of the domain or codomain of some x_i; we say that *the string involves* y if $x_i = y$ for some i.

For shapes S and T, let $X(S_0, T_0)$ be the class of all allowable E-graphs $\xi : S_0 \to T_0$ such that $\xi = \xi_n \dots \xi_1$, where $\{\xi_1, \dots, \xi_n\}$ is an allowable string of E-graphs not involving any instance of c_{const} nor any shape $[\mathit{const}, \mathit{const}]$. Let $\mathfrak{X}$ be the category whose objects are the simple shapes and in which, for any simple shapes S and T,

$$Hom_{\mathfrak{X}}(S,T)=X(S,T).$$

In $X(S_0,T_0)$, define a relation $R_0(S_0,T_0)$ by: $(\xi,\eta)\in R_0(S_0,T_0)$ iff there are allowable strings of E-graphs

$$\{\xi_1,\dots,\xi_n\},\ \{\eta_1,\dots,\eta_m\},\ \{\xi'_1,\dots,\xi'_k\},\ \{\eta'_1,\dots,\eta'_l\}$$

such that

1) $\{\xi_1,\dots,\xi_n\}$ and $\{\eta_1,\dots,\eta_m\}$ do not involve any $[const, const]$,
2) $\{\xi'_1,\dots,\xi'_k\}$ and $\{\eta'_1,\dots,\eta'_l\}$ are gotten from $\{\xi_1,\dots,\xi_n\}$ and $\{\eta_1,\dots,\eta_m\}$, respectively, by the replacement of each instance of c_{const} by a string of instances of elements of $\{a_{const}, a^{-1}_{const}, 1_{const}\}$ between domain and codomain of the replaced instance of c_{const},
3) $\xi_n\cdots\xi_1=\eta_m\cdots\eta_1\in X(S_0,T_0)$,
4) $\xi=\xi'_k\cdots\xi'_1$ and $\eta=\eta'_l\cdots\eta'_1$.

Let $R(S_0,T_0)$ be the equivalence relation in $X(S_0,T_0)$ generated by $R_0(S_0,T_0)$.

In $Arr\mathfrak{X}$, let R be the equivalence relation defined by:

$$\xi R\eta \text{ iff } \partial_0\xi=\partial_0\eta=S(=S_0),\ \partial_1\xi=\partial_1\eta=T(=T_0)$$
$$\text{and } (\xi,\eta)\in R(S,T).$$

R is compatible with composition of E-graphs. Denote the R-equivalence class of z by $(z)_R$.

$\mathcal{K}$ is the category whose objects are shapes and, for any shapes S and T, $Hom_{\mathcal{K}}(S,T)=X(S_0,T_0)/R$. Composition in $\mathcal{K}$ is (well) defined by $(\eta)_R\circ(\xi)_R=(\eta\circ\xi)_R$.

Functors $\tilde{\otimes}:\mathcal{K}\times\mathcal{K}\to\mathcal{K}$ and $\tilde{[}\,,\tilde{]}:\mathcal{K}^{op}\times\mathcal{K}\to\mathcal{K}$ are (well) defined by:

on objects, $S\tilde{\otimes}T=S\otimes T$ and $\tilde{[}S,T\tilde{]}=[S,T]$,

on arrows,

$$(\xi)_R\tilde{\otimes}(\eta)_R=((\xi\otimes\eta)_0)_R \text{ and } \tilde{[}(\xi)_R,(\eta)_R\tilde{]}=(([\xi,\eta])_0)_R.$$

Natural transformations $\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e}$ satisfying the axioms for a closed category exist, with components

$$\tilde{a}_{ABC}=((a_{ABC})_0)_R:(A\tilde{\otimes}B)\tilde{\otimes}C\to A\tilde{\otimes}(B\tilde{\otimes}C),$$
$$\tilde{b}_A=((b_A)_0)_R:A\tilde{\otimes}I\to I,$$

$$\tilde{c}_{AB} = \begin{cases} ((c_{AB})_0)_R : A \tilde{\otimes} B \to B \tilde{\otimes} A, & \text{if at most one of } A \text{ and } B \text{ is constant,} \\ (1_I = 1_{(A\otimes B)_0})_R : A \tilde{\otimes} B \to B \tilde{\otimes} A, & \text{if both } A \text{ and } B \text{ are constant,} \end{cases}$$

$$\tilde{d}_{AB} = ((d_{AB})_0)_R : A \to \widetilde{[B, A \tilde{\otimes} B]},$$

$$\tilde{e}_{AB} = ((e_{AB})_0)_R : \widetilde{[A, B]} \tilde{\otimes} A \to B.$$

REFERENCES

[1] R. BLACKWELL, Coherence for V-natural transformations, *Abstracts of the Sydney Category Theory Seminar* 1972 (the University of New South Wales), 56-57.

[2] G.M. KELLY, A Cut Elimination Theorem, *Lecture Notes in Math.* 281 Springer (1972), 196-213.

[3] G.M. KELLY and S. MACLANE, Coherence in Closed Categories, *J. Pure and Applied Algebra* 1 (1971), 97-140.

[4] G.M. KELLY and S. MACLANE, Closed Coherence for a Natural Transformation, *Lecture Notes in Math.* 281, Springer (1972), 1-28.

[5] G. LEWIS, Coherence for a Closed Functor, *Lecture Notes in Math.* 281 Springer (1972), 148-195.

[6] R. VOREADOU, *A Coherence Theorem for Closed Categories,* Ph. D. Thesis, The University of Chicago, 1972.

[7] R. VOREADOU, *Non-commutative diagrams in closed categories* (to appear).

P. O. Box 1335, Omonia
ATHENES
GRECE