COMPOSITIO MATHEMATICA

A. SILVERBERG

YU G. ZARHIN

Connectedness results for ℓ-adic representations associated to abelian varieties

Compositio Mathematica, tome 97, nº 1-2 (1995), p. 273-284

<http://www.numdam.org/item?id=CM_1995__97_1-2_273_0>

Compositio Mathematica **97**: 273–284, 1995.

Connectedness results for ℓ-adic representations associated to abelian varieties

Dedicated to Frans Oort on the occasion of his 60th birthday

A. SILVERBERG[1]* and YU. G. ZARHIN[2]**
[1]*Department of Mathematics, Ohio State University, Columbus, OH 43210-1174, USA*
e-mail address: silver@math.ohio-state.edu
[2]*Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA, Institute for Mathematical Problems in Biology, Russian Academy of Sciences, Pushchino, Moscow Region, 142292, Russia*
e-mail address: zarhin@math.psu.edu

Received 28 September 1994; accepted in final form 4 April 1995

1. Introduction

Suppose X is an abelian variety defined over a field F, ℓ is a prime number, and $\ell \neq \mathrm{char}(F)$. Let F^s denote a separable closure of F, let $T_\ell(X) = \varprojlim X_{\ell^r}$ (the Tate module), let $V_\ell(X) = T_\ell(X) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$, and let $\rho_{X,\ell}$ denote the ℓ-adic representation

$$\rho_{X,\ell}\colon \mathrm{Gal}(F^s/F) \to \mathrm{Aut}(T_\ell(X)) \subseteq \mathrm{Aut}(V_\ell(X)) \cong \mathrm{GL}_{2d}(\mathbf{Q}_\ell),$$

where $d = \dim(X)$. If L is an extension of F in F^s, let $G_{L,X}$ denote the image of $\mathrm{Gal}(F^s/L)$ under $\rho_{X,\ell}$. Let $\mathfrak{G}_\ell(F,X)$ denote the algebraic envelope of the image of $\rho_{X,\ell}$, i.e., the Zariski closure in $\mathrm{GL}_{2d}(\mathbf{Q}_\ell)$ of $G_{F,X}$. Let $F_{\Phi,\ell}(X)$ be the smallest extension F' of F such that $\mathfrak{G}_\ell(F',X)$ is connected. If G is an algebraic group, let G^0 denote the identity connected component. Let Φ denote the group of connected components

$$\Phi = \mathfrak{G}_\ell(F,X)/\mathfrak{G}_\ell(F,X)^0.$$

The algebraic group $\mathfrak{G}_\ell(F,X)$, the finite group Φ, and the field $F_{\Phi,\ell}(X)$ were introduced and studied by Serre (see [16] and [17]). Our goal in this paper is to compare the field $F_{\Phi,\ell}(X)$ with other extensions of F (especially those generated by torsion points on X) and to prove sufficient conditions for the connectedness of $\mathfrak{G}_\ell(F,X)$.

* Partially supported by the National Science Foundation.
** Partially supported by the Ohio State University Mathematics Research Institute and the Deutsche Forschungsgemeinschaft.

Let $F(\mathrm{End}(X))$ denote the smallest extension of F over which all the endomorphisms of X are defined. We have (see Proposition 2.10)

$$F(\mathrm{End}(X)) \subseteq F_{\Phi,\ell}(X).$$

In Theorem 3.7 (see also Theorem 3.8) we show that if $n \geq 5$, n is not divisible by $\mathrm{char}(F)$, and λ and $\widetilde{X}_n$ are as above, then

$$F(\mathrm{End}(X)) \subseteq F(\widetilde{X}_n, \boldsymbol{\mu}_n, \lambda).$$

Suppose now that F is a finitely generated extension of $\mathbf{Q}$. Serre proved that $F_{\Phi,\ell}(X)$ is independent of ℓ (see Theorem 2.11), so we will denote the field $F_{\Phi,\ell}(X)$ by $F_\Phi(X)$. If n is an integer greater than 2, then (see Remark 3.1)

$$F_\Phi(X) \subseteq F(X_n).$$

A consequence of our main result of Section 3 (see Theorem 3.2) is that if X is an abelian variety defined over a finitely generated extension F of $\mathbf{Q}$, n is an integer greater than 4, λ is a polarization on X, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to the Weil pairing induced by λ, then

$$F_\Phi(X) \subseteq F(\widetilde{X}_n, \boldsymbol{\mu}_n, \lambda).$$

In other words, if F is a field of definition for the polarization λ, the points of $\widetilde{X}_n$, and the nth roots of unity, then $\mathfrak{G}_\ell(F, X)$ is connected. (See also Theorems 3.4 and 3.6 for results for global fields and arbitrary fields, respectively.) This gives a new criterion, in terms of torsion points of X, for the connectedness of $\mathfrak{G}_\ell(F, X)$.

In conversations with Silverberg in 1990, Serre asked whether it is true that $F_\Phi(X) = \bigcap_{p \geq n_0} {}_{p'} F(X_p)$ for every integer $n_0 \geq 3$. We discuss this question further elsewhere.

The first author would like to thank J.-P. Serre for introducing her to some of the ideas in this paper. The authors would like to thank the Mathematische Institut der Universität Erlangen–Nürnberg for its hospitality.

2. Definitions, notation, and lemmas

Let $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote respectively the integers, rational numbers, real numbers, and complex numbers. If F is a field, let $\bar{F}$ denote an algebraic closure and let F^s denote a separable closure. Suppose X is an abelian variety defined over F. Write $\mathrm{End}_F(X)$ for the set of endomorphisms of X which are defined over F, let $\mathrm{End}(X) = \mathrm{End}_{F^s}(X)$, and let $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. If λ is a polarization on X, n is a positive integer not divisible by $\mathrm{char}(F)$, and $\boldsymbol{\mu}_n$ is the $\mathrm{Gal}(F^s/F)$-module of nth roots of unity in F^s, then the e_n-pairing induced by the polarization λ

$$e_{\lambda,n}\colon X_n \times X_n \to \boldsymbol{\mu}_n$$

(see Section 75 of [23]), is a skew-symmetric bilinear map which satisfies:

$$\sigma(e_{\lambda,n}(x_1, x_2)) = e_{\sigma(\lambda),n}(\sigma(x_1), \sigma(x_2))$$

for every $\sigma \in \mathrm{Gal}(F^s/F)$ and $x_1, x_2 \in X_n$. If n is relatively prime to the degree of the polarization λ, then the pairing $e_{\lambda,n}$ is nondegenerate. If $\tilde{X}$ is a subset of X_n, then

$$F(\tilde{X}, e_{\lambda,n}(X_n, \tilde{X}), \lambda)$$

denotes the smallest extension of F in F^s which contains the roots of unity in $e_{\lambda,n}(X_n, \tilde{X})$ and which is a field of definition for the polarization λ and the elements of $\tilde{X}$.

We recall some results from [21] and [22], which we extend and apply.

LEMMA 2.1 (Lemma 5.2 of [22]). *Suppose that d and n are positive integers, and for each prime ℓ which divides n we have a matrix $A_\ell \in M_{2d}(\mathbf{Z}_\ell)$ such that the characteristic polynomials of the A_ℓ have integral coefficients independent of ℓ, and such that $(A_\ell - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$. Then for every eigenvalue α of A_ℓ, $(\alpha - 1)/\sqrt{n}$ satisfies a monic polynomial with integer coefficients.*

If k is a positive integer, define a finite set $N(k)$ by

$$N(k) = \{\text{prime powers } \ell^m : 0 \leq m(\ell - 1) \leq k\}.$$

If n is a positive integer which is not in $N(k)$, let $R(k, n) = 1$. Let $R(k, 1) = 0$. If $1 \neq n = \ell^m \in N(k)$ with ℓ a prime, let

$$R(k, n) = \ell^{r(k,n)} \quad \text{where} \quad r(k, n) = \max\{r \in \mathbf{Z}^+ : m(\ell - 1)\ell^{r-1} \leq k\}.$$

THEOREM 2.2 (Corollary 3.3 of [21]). *Suppose n and k are positive integers, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime which divides n is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, α has finite multiplicative order, and $(\alpha - 1)^k \in n\mathcal{O}$. Then $\alpha^{R(k,n)} = 1$.*

In the case $k = 2$ we have the following corollary.

COROLLARY 2.3. *Suppose n is an integer greater than 4, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime divisor of n is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, α has finite multiplicative order, and $(\alpha - 1)^2 \in n\mathcal{O}$. Then $\alpha = 1$.*

LEMMA 2.4. *Suppose $\mathcal{O}$ is an integral domain of characteristic zero, n and k are positive integers such that no rational prime which divides n is a unit in $\mathcal{O}$, $A \in \mathrm{GL}_g(\mathcal{O})$ satisfies $(A - I)^k \in nM_g(\mathcal{O})$, and α is a root of unity in the multiplicative group generated by the eigenvalues of A. Then $\alpha^{R(k,n)} = 1$.*

Proof. View the eigenvalues of A as lying in the integral closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ in an algebraically closed field containing $\mathcal{O}$. As shown in Lemma 6.6 of [21], no rational prime divisor of n is a unit in $\bar{\mathcal{O}}$. If μ is an eigenvalue of A, then $\mu \in \bar{\mathcal{O}}$ and $(\mu-1)^k \in n\bar{\mathcal{O}}$. Therefore, the multiplicative group $G = \{\beta \in \bar{\mathcal{O}} : (\beta-1)^k \in n\bar{\mathcal{O}}\}$ contains the multiplicative group generated by the eigenvalues of A. By Theorem 2.2, every root of unity α in G satisfies $\alpha^{R(k,n)} = 1$. □

The following proposition gives a means of verifying the connectedness or disconnectedness of a linear algebraic group. See also [2], especially Section 8 in Chapter III, or [10], especially Chapter VI.

PROPOSITION 2.5. *Suppose φ is an invertible linear operator on a finite-dimensional vector space V over a field of characteristic zero. Then the multiplicative group generated by the eigenvalues of φ contains no non-trivial roots of unity if and only if the smallest algebraic subgroup of* $\mathrm{GL}(V)$ *containing φ is connected.*

Proof. The connectedness or disconnectness of an algebraic group is invariant under extensions of the ground field, so we may assume the ground field k is algebraically closed. The Jordan decomposition (see Section 4 in Chapter I of [2]) gives a unipotent operator u and a semisimple operator s such that $\varphi = su = us$. If $f \in \mathrm{GL}(V)$, let G_f denote the smallest algebraic subgroup of $\mathrm{GL}(V)$ containing f. Let $x = \log(u)$. Then $G_u(k) = \{\exp(tx) : t \in k\}$, a (zero- or one-dimensional) connected algebraic group. Let $\alpha_1, \dots, \alpha_n$ denote the eigenvalues of s, with multiplicity. Then

$$G_s \cong \left\{ \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix} : \text{ if } \prod \alpha_i^{b_i} = 1 \text{ with } b_i \in \mathbf{Z} \text{ then } \prod \beta_i^{b_i} = 1 \right\}.$$

The multiplication map $G_s \times G_u \to G_\varphi$ is an isomorphism (by the definition of G_φ and the above characterizations of the groups G_s and G_u). Since G_u is connected and the eigenvalues of u are all 1, we can reduce to the case $\varphi = s$. Let $X(G_s) = \mathrm{Hom}(G_s, \mathbf{G}_m)$, the group of characters of G_s. Then $X(G_s) \cong \mathbf{Z}^n/B$, where

$$B = \left\{(b_1, \dots, b_n) \in \mathbf{Z}^n : \prod \alpha_i^{b_i} = 1\right\}.$$

We next show that G_s is connected if and only if $X(G_s)$ has no non-trivial torsion. If G_s is connected then it is a connected commutative algebraic group with no nilpotent radical, so $G_s \cong \mathbf{G}_m^r$ for some r, and so $X(G_s) \cong \mathbf{Z}^r$. Conversely, if G_s is not connected then there is a non-trivial homomorphism $G_s/G_s^0 \to \mathbf{G}_m$, which induces a homomorphism $G_s \to \mathbf{G}_m$ which is a non-trivial torsion element of $X(G_s)$.

Non-trivial torsion elements of $X(G_s)$ correspond to elements $(c_1, \ldots, c_n) \in \mathbf{Z}^n$ for which $\prod \alpha_i^{c_i}$ is a non-trivial root of unity in the multiplicative group generated by the eigenvalues of s. We therefore obtain the desired result. □

PROPOSITION 2.6. *Suppose $\mathcal{O}$ is an integral domain of characteristic zero, F is its fraction field, and n and k are positive integers such that no rational prime which divides n is a unit in $\mathcal{O}$. Suppose G is a subgroup of $\mathrm{GL}_g(F)$ generated by elements $A \in \mathrm{GL}_g(\mathcal{O})$ such that $(A-I)^k \in nM_g(\mathcal{O})$. If $n \notin N(k)$, then the Zariski closure of G in $\mathrm{GL}_g(F)$ is connected.*

Proof. By the Corollary on p. 56 of [10], an algebraic group which is generated (as an abstract group) by closed connected subgroups is connected. The Proposition therefore follows from Lemma 2.4 and Proposition 2.5. □

LEMMA 2.7. *If X is an abelian variety over a field F, and L is a finite extension of F in F^s, then $\mathfrak{G}_\ell(L,X) \subseteq \mathfrak{G}_\ell(F,X)$ and $\mathfrak{G}_\ell(L,X)^0 = \mathfrak{G}_\ell(F,X)^0$. In particular, if $\mathfrak{G}_\ell(F,X)$ is connected, then $\mathfrak{G}_\ell(F,X) = \mathfrak{G}_\ell(L,X)$.*

Proof. Since $G_{L,X}$ is a subgroup of finite index in $G_{F,X}$, the group $G_{F,X}$ is a finite disjoint union of cosets of $G_{L,X}$. Therefore $\mathfrak{G}_\ell(F,X)$ is a finite disjoint union of cosets of $\mathfrak{G}_\ell(L,X)$. Thus $\mathfrak{G}_\ell(L,X)$ is a closed subgroup of finite index in $\mathfrak{G}_\ell(F,X)$. By the Proposition on p. 53 of [10], $\mathfrak{G}_\ell(F,X)^0 \subseteq \mathfrak{G}_\ell(L,X)$. Therefore, $\mathfrak{G}_\ell(F,X)^0 = \mathfrak{G}_\ell(L,X)^0$. □

REMARK 2.8. If X is an abelian variety over a finitely generated extension F of the prime field, and $\ell \neq \mathrm{char}(F)$, then the algebraic group $\mathfrak{G}_\ell(F,X)^0$ is reductive, since the representation $\rho_{X,\ell}$ is semisimple (by Faltings ([7], [8]) in the characteristic zero case, by Zarhin ([25], [26]) in the case of characteristic greater than 2, and by Mori ([11], especially Section 5 of Chapter VI and Section 2 of Chapter XII) in the characteristic 2 case. See also [28].). Note also (see [1]) that if F is a finitely generated extension of $\mathbf{Q}$ then $G_{F,X}$ is an open subgroup of $\mathfrak{G}_\ell(F,X)(\mathbf{Q}_\ell)$.

LEMMA 2.9. *Suppose X is an abelian variety defined over a field F, λ is a polarization of X, n is a positive integer not divisible by the characteristic of F, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to the pairing $e_{\lambda,n}$. Suppose the polarization λ, the points of $\widetilde{X}_n$, and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ are all defined over F. Then $(\sigma-1)^2 X_n = 0$ for every $\sigma \in \mathrm{Gal}(F^s/F)$.*

Proof. The pairing $e_{\lambda,n}$ induces a natural homomorphism

$$X_n \to \mathrm{Hom}(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n)),$$

which is $\mathrm{Gal}(F^s/F)$-equivariant since the polarization λ is defined over the field F. Since $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n, $\widetilde{X}_n$ is the kernel of the map, and we can view $X_n/\widetilde{X}_n$ as a $\mathrm{Gal}(F^s/F)$-submodule of $\mathrm{Hom}(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n))$. If $\sigma \in \mathrm{Gal}(F^s/F)$, then $\sigma = 1$ on $\widetilde{X}_n$ and on $e_{\lambda,n}(X_n, \widetilde{X}_n)$. Therefore, $\sigma = 1$

If $\sigma \in \mathrm{Gal}(F^s/F)$, then $\sigma = 1$ on $\widetilde{X}_n$ and on $e_{\lambda,n}(X_n, \widetilde{X}_n)$. Therefore, $\sigma = 1$ on $X_n/\widetilde{X}_n$, i.e., $(\sigma-1)X_n \subseteq \widetilde{X}_n$. Since $(\sigma-1)\widetilde{X}_n = 0$ we have $(\sigma-1)^2 X_n = 0$.□

PROPOSITION 2.10. *If X is an abelian variety over a field F, ℓ is a prime, and $\ell \neq \mathrm{char}(F)$, then*

$$F(\mathrm{End}(X)) \subseteq F_{\Phi,\ell}(X).$$

Proof. Without loss of generality we may assume $F = F_{\Phi,\ell}(X)$. It then suffices to show that all the endomorphisms of X are defined over F. Let $V = V_\ell(X)$. If L is a finite extension of F in F^s, we have

$$\mathrm{End}_L(X) \subseteq (\mathrm{End}(V))^{\mathrm{Gal}(F^s/L)} = (\mathrm{End}(V))^{\mathfrak{G}_\ell(L,X)}.$$

Since $\mathfrak{G}_\ell(F,X)$ is connected, by Lemma 2.7 we have $\mathfrak{G}_\ell(F,X) = \mathfrak{G}_\ell(L,X)$. Therefore,

$$\mathrm{End}_L(X) \subseteq (\mathrm{End}(V))^{\mathfrak{G}_\ell(F,X)} = (\mathrm{End}(V))^{\mathrm{Gal}(F^s/F)}.$$

But

$$\mathrm{End}_L(X) \cap (\mathrm{End}(V))^{\mathrm{Gal}(F^s/F)} = \mathrm{End}_F(X).$$

Therefore, $\mathrm{End}_L(X) = \mathrm{End}_F(X)$. Now taking L to be a finite separable extension of F over which all the endomorphisms of X are defined, we have $\mathrm{End}(X) = \mathrm{End}_F(X)$. □

Although we do not make use of the following result in our proofs, we include it because of its importance to the subject of this paper.

THEOREM 2.11 (Serre). *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then the field $F_{\Phi,\ell}(X)$ is independent of the prime ℓ.*

Proof. See [16] (see also Corollary 3.8 of [5], [15], and [18]). □

The following result is an immediate corollary.

COROLLARY 2.12 (Serre). *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then*

(i) *if the algebraic group $\mathfrak{G}_\ell(F,X)$ is connected for one prime ℓ then it is connected for every prime ℓ,*
(ii) *the group Φ of connected components is independent of the prime ℓ.*

3. Field inclusions

REMARK 3.1. If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, and n is an integer greater than 2, then

$$F_\Phi(X) \subseteq F(X_n)$$

(see [4], [3], and Proposition 3.6 of [5]).

In the result below we replace the n-torsion by a maximal isotropic subgroup.

THEOREM 3.2. *Suppose X is an abelian variety defined over a finitely generated extension F of $\mathbf{Q}$, λ is a polarization on X, n is an integer, $n \geq 5$, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Then*

$$F_\Phi(X) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda).$$

Proof. Suppose ℓ is a prime number. Without loss of generality, we may assume

$$F = F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda).$$

It then suffices to show that $\mathfrak{G}_\ell(F, X)$ is connected. Let R be a finitely generated smooth sub-$\mathbf{Z}$-algebra of F whose fraction field is F, and such that X is the generic fiber of an abelian scheme over Spec(R). Let $S = \mathrm{Spec}(R[\frac{1}{n\ell}])$, and let $\pi_1(S)$ denote the étale fundamental group of S with respect to the geometric point Spec$(\bar{F})$. Then $\pi_1(S)$ is a quotient of Gal$(\bar{F}/F)$, and the action of Gal$(\bar{F}/F)$ on $V_\ell(X)$ factors through $\pi_1(S)$. To each closed point $y \in S$ we can associate a conjugacy class Fr_y of a Frobenius element in $\pi_1(S)$ (see p. 206 of [8]). By the Chebotarev density theorem (see Theorem 12 on p. 289 of [24] in the number field case, and see the Theorem on p. 206 of [8] for the Chebotarev density theorem in the generality of finitely generated extensions of $\mathbf{Q}$), the Fr_y are dense in $\pi_1(S)$. Let $\sigma \in \mathrm{Gal}(\bar{F}/F)$ be an element which maps to an element of a Frobenius conjugacy class associated to a closed point $y \in S$. By Lemma 2.9, we have $(\sigma - 1)X_n = 0$, and therefore for all prime numbers q we have

$$(\rho_{X,q}(\sigma) - I)^2 \in n\,\mathrm{End}(T_q(X)) \cong nM_{2d}(\mathbf{Z}_q),$$

where d is the dimension of X. If q is a prime not equal to the residue characteristic of y, then the characteristic polynomial of $\rho_{X,q}(\sigma)$ has integer coefficients which are independent of q. Note that the residue characteristic p of y does not divide ℓn. Let $\bar{\mathbf{Z}}$ denote the ring of algebraic integers. The eigenvalues of $\rho_{X,\ell}(\sigma)$ are in $1 + \sqrt{n}\bar{\mathbf{Z}}$ by Lemma 2.1, and are in $(\bar{\mathbf{Z}}[\frac{1}{p}])^\times$ by Weil's theorem. The multiplicative group generated by the eigenvalues of $\rho_{X,\ell}(\sigma)$ is a subset of the multiplicative semi-group $1 + \sqrt{n}\bar{\mathbf{Z}}[\frac{1}{p}]$, and therefore by Corollary 2.3 contains no non-trivial root of unity. By Proposition 2.5 and the Chebotarev density theorem, $\mathfrak{G}_\ell(F, X)$ is connected. (We again use that an algebraic group which is generated by closed connected subgroups is connected.) □

The following result is an immediate corollary.

COROLLARY 3.3. *Suppose X is an abelian variety defined over a finitely generated extension F of $\mathbf{Q}$, λ is a polarization on X, n is an integer, $n \geq 5$, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Then*

$$F_\Phi(X) \subseteq F(\widetilde{X}_n, \mu_n, \lambda).$$

THEOREM 3.4. *Suppose X is an abelian variety defined over a global field F of positive characteristic p, ℓ is a prime number different from p, λ is a polarization on X, n is an integer not divisible by p, $n \geq 5$, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Then*

$$F_{\Phi,\ell}(X) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda) \subseteq F(\widetilde{X}_n, \mu_n, \lambda).$$

Proof. The proof is the same as the proof of Theorem 3.2. For the Chebotarev density theorem for global fields, see Theorem 12 on p. 289 of [24]. □

REMARK 3.5. Theorem 3.2 and the result stated in Remark 3.1 should also hold for F a finitely generated extension of a finite field, using Theorem 3.4 and Mori's technique (see [12]) for inducting on the transcendence degree of F.

THEOREM 3.6. *Suppose X is an abelian variety defined over an arbitrary field F, λ is a polarization on X, n is a positive integer relatively prime to* $\mathrm{char}(F)$, *and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Suppose ℓ is a prime divisor of n, and either*

(i) $\ell \geq 5$, *or*
(ii) $\ell = 3$ *and n is divisible by* 9, *or*
(iii) $\ell = 2$ *and n is divisible by* 8.

Then

$$F_{\Phi,\ell}(X) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda) \subseteq F(\widetilde{X}_n, \mu_n, \lambda).$$

Proof. Without loss of generality, we may assume

$$F = F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda).$$

It then suffices to show that $\mathfrak{G}_\ell(F, X)$ is connected. Let ℓ^w be the highest power of ℓ which divides n. By Lemma 2.9, if $\sigma \in \mathrm{Gal}(F^s/F)$ then

$$(\rho_{X,\ell}(\sigma) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell) = \ell^w M_{2d}(\mathbf{Z}_\ell).$$

By Proposition 2.6, $\mathfrak{G}_\ell(F, X)$ is connected. □

We now give a direct proof, valid over an arbitrary field F, that $F(\mathrm{End}(X)) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda)$. Theorems 3.7 and 3.8 extend earlier results in [19]; see also [20].

THEOREM 3.7. *Suppose (X, λ) is a polarized abelian variety defined over a field F, n is a positive integer which is greater than 4 and is not divisible by the characteristic of F, $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to the pairing $e_{\lambda,n}$, and the points of $\widetilde{X}_n$ and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ are all defined over F. Then every endomorphism of X is defined over F.*

Proof. The action of $\mathrm{Gal}(F^s/F)$ on X induces a representation

$$\rho\colon \mathrm{Gal}(F^s/F) \to \mathrm{Aut}(\mathrm{End}(X)).$$

Suppose $\sigma \in \mathrm{Gal}(F^s/F)$ and α is an eigenvalue of $\rho(\sigma)$. Then α is an algebraic integer. Since the endomorphisms of X are defined over a finite separable extension of F, $\rho(\sigma)$ has finite order and α is a root of unity. Let $p = \mathrm{char}(F)$ and let ℓ be a prime number different from p. Using the injections

$$\mathrm{End}(X) \hookrightarrow \mathrm{End}(X) \otimes \mathbf{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell(X)),$$

we can view ρ as a map from $\mathrm{Gal}(F^s/F)$ to $\mathrm{Aut}(\mathrm{End}(T_\ell(X)))$. Then ρ is the adjoint representation of $\rho_{X,\ell}$. Let $\bar{\mathbf{Z}}$ and $\bar{\mathbf{Z}}_\ell$ denote integral closures of $\mathbf{Z}$ and $\mathbf{Z}_\ell$, respectively. For every embedding of $\bar{\mathbf{Z}}$ into $\bar{\mathbf{Z}}_\ell$, we can write $\alpha = a/b$ with a and b eigenvalues of $\rho_{X,\ell}(\sigma)$. By Lemma 2.9, we have $(\rho_{X,\ell}(\sigma) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$. Therefore, $(a-1)/\sqrt{n}$ and $(b-1)/\sqrt{n}$ satisfy monic polynomials over $\mathbf{Z}_\ell$, i.e., $a, b \in 1 + \sqrt{n}\bar{\mathbf{Z}}_\ell$. Thus, $\alpha \in 1 + \sqrt{n}\bar{\mathbf{Z}}_\ell$, i.e., every embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_\ell$ sends $(\alpha - 1)/\sqrt{n}$ into $\bar{\mathbf{Z}}_\ell$, for every prime $\ell \neq p$. Therefore $(\alpha-1)/\sqrt{n} \in \bar{\mathbf{Z}}[\frac{1}{p}]$, so $(\alpha - 1)^2 \in n\bar{\mathbf{Z}}[\frac{1}{p}]$. By Corollary 2.3, if $n \geq 5$ then $\alpha = 1$. Therefore $\rho(\sigma) = 1$ and all the endomorphisms of X are defined over F. □

THEOREM 3.8. *Suppose (X, λ) and (Y, μ) are polarized abelian varieties defined over a field F, and n is a positive integer which is greater than 4 and is not divisible by the characteristic of F. Suppose $\widetilde{X}_n$, respectively $\widetilde{Y}_n$, is a maximal isotropic subgroup of X_n, respectively Y_n, with respect to the pairing $e_{\lambda,n}$, respectively $e_{\mu,n}$. Suppose the points of $\widetilde{X}_n$ and $\widetilde{Y}_n$ and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ and $e_{\mu,n}(Y_n, \widetilde{Y}_n)$ are all defined over F. Then every homomorphism between X and Y is defined over F.*

Proof. Apply Theorem 3.7 to the polarized abelian variety $(X \times Y, \lambda \times \mu)$ with maximal isotropic subgroup $\widetilde{X}_n \times \widetilde{Y}_n \subseteq (X \times Y)_n$. □

4. Mumford–Tate groups

Next we define the Mumford–Tate group of a complex abelian variety X (see Section 2 of [14] or Section 6 of [27]). If X is a complex abelian variety, let $V = H_1(X(\mathbf{C}), \mathbf{Q})$ and consider the Hodge decomposition $V \otimes \mathbf{C} = H_1(X(\mathbf{C}), \mathbf{C}) = H^{-1,0} \oplus H^{0,-1}$. Define a homomorphism $\mu\colon \mathbf{G}_m \to \mathrm{GL}(V)$ as follows. For $z \in \mathbf{C}$, let $\mu(z)$ be the automorphism of $V \otimes \mathbf{C}$ which is multiplication by z on $H^{-1,0}$ and is the identity on $H^{0,-1}$.

DEFINITION 4.1. The *Mumford–Tate group* MT_X of X is the smallest algebraic subgroup of $\mathrm{GL}(V)$, defined over $\mathbf{Q}$, which after extension of scalars to $\mathbf{C}$ contains the image of μ.

It follows from the definition that MT_X is connected.

REMARK 4.2. Define a homomorphism $\varphi\colon \mathbf{G}_m \times \mathbf{G}_m \to \mathrm{GL}(V)$ as follows. For $z, w \in \mathbf{C}$, let $\varphi(z, w)$ be the automorphism of $V \otimes \mathbf{C}$ which is multiplication by z on $H^{-1,0}$ and is multiplication by w on $H^{0,-1}$. Then MT_X can also be defined as the smallest algebraic subgroup of $\mathrm{GL}(V)$, defined over $\mathbf{Q}$, which after extension of scalars to $\mathbf{C}$ contains the image of φ. The equivalence of the definitions follows easily from the fact that $H^{-1,0}$ is the complex conjugate of $H^{0,-1}$. (See Section 3 of [15], where MT_X is called the Hodge group. See also Section 6 of [27].)

If X is an abelian variety over a subfield F of $\mathbf{C}$, we fix an embedding of $\bar{F}$ in $\mathbf{C}$. This gives an identification of $V_\ell(X)$ with $H_1(X, \mathbf{Q}) \otimes \mathbf{Q}_\ell$, and allows us to view $MT_X \times \mathbf{Q}_\ell$ as a linear $\mathbf{Q}_\ell$-algebraic subgroup of $\mathrm{GL}(V_\ell(X))$. Let $MT_{X,\ell} = MT_X \times_{\mathbf{Q}} \mathbf{Q}_\ell$. Then $MT_X(\mathbf{Q}_\ell) = MT_{X,\ell}(\mathbf{Q}_\ell)$.

REMARK 4.3. The Mumford–Tate conjecture for abelian varieties (see [15]) may be reformulated as the equality of $\mathbf{Q}_\ell$-algebraic groups, $\mathfrak{G}_\ell(F, X)^0 = MT_{X,\ell}$.

THEOREM 4.4 (Piatetski-Shapiro [13], Deligne [6], Borovoi [3]). *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then $MT_{X,\ell}(\mathbf{Q}_\ell)$ contains an open subgroup of finite index in $G_{F,X}$.*

COROLLARY 4.5. *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then $\mathfrak{G}_\ell(F, X)^0 \subseteq MT_{X,\ell}$.*

Proof. By Theorem 4.4, we can find a finite algebraic extension L of F such that $G_{L,X} \subseteq MT_{X,\ell}(\mathbf{Q}_\ell)$. Then $\mathfrak{G}_\ell(L, X) \subseteq MT_{X,\ell}$. By Lemma 2.7, $\mathfrak{G}_\ell(F, X)^0 = \mathfrak{G}_\ell(L, X)^0 \subseteq \mathfrak{G}_\ell(L, X)$. □

In [4] (see also [3]) Borovoi showed that if X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, n is an integer greater than 2, and $F = F(X_n)$, then $G_{F,X}$ is contained in $MT_{X,\ell}(\mathbf{Q}_\ell)$, i.e., $\mathfrak{G}_\ell(F, X) \subseteq MT_{X,\ell}$. We have the following strengthening of Borovoi's result.

THEOREM 4.6. *Suppose (X, λ) is a polarized abelian variety over a finitely generated extension F of $\mathbf{Q}$, n is an integer greater than 4, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Suppose the points of $\widetilde{X}_n$ and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ are all defined over F. Then $\mathfrak{G}_\ell(F, X) \subseteq MT_{X,\ell}$.*

Proof. By Theorem 3.2, we have $\mathfrak{G}_\ell(F, X) = \mathfrak{G}_\ell(F, X)^0$. The result now follows from Corollary 4.5. □

5. Semistable reduction and connectedness

Suppose X is an abelian variety over a field F and v is a discrete valuation on F. Let $\bar{v}$ be an extension of v to F^s, and let I_v denote the corresponding inertia subgroup of $\mathrm{Gal}(F^s/F)$. For a definition of semistable reduction, see p. 349 of [9] or Section 3 of [22] (or define it from the following theorem).

THEOREM 5.1 (Grothendieck, Proposition 3.5 and Corollaire 3.8 of [9]). *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, and ℓ is a prime number different from the residue characteristic of v. Let $V = V_\ell(X)$. Then the following statements are equivalent:*

(i) *X has semistable reduction at v,*
(ii) *there is a subspace W of V such that I_v acts as the identity on W and on V/W,*
(iii) *I_v acts by unipotent operators on V.*

The definition of *motif semi-stable* on p. 396 of [18] suggests that the following result is already known. Since it follows easily from the techniques used in this paper, we have included it here.

THEOREM 5.2. *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, and ℓ is a prime number different from the residue characteristic of v. Then X has semistable reduction at v if and only if the Zariski closure of $\rho_{X,\ell}(I_v)$ is connected.*

Proof. Let $\mathfrak{G}$ denote the Zariski closure of $\rho_{X,\ell}(I_v)$ in $\mathrm{GL}(V_\ell(X))$. If X has semistable reduction at v, then I_v acts on V by unipotent operators by Theorem 5.1, so 1 is the only eigenvalue of elements of $\rho_{X,\ell}(I_v)$. By Proposition 2.5, $\mathfrak{G}$ is connected.

Conversely, suppose $\mathfrak{G}$ is connected. Let L be a finite Galois extension of F over which X has semistable reduction above v, let w denote the restriction of $\bar{v}$ to L, and let I_w be the inertia subgroup for $\bar{v}$ over w. Let $W = V^{I_w}$, the subspace of V on which I_w acts as the identity. Then I_w is the identity on V/W, by Theorem 5.1. Let $\mathfrak{G}_w$ denote the Zariski closure of $\rho_{X,\ell}(I_w)$. Then $\mathfrak{G}_w$ acts as the identity on W and on V/W. Since I_w is an open subgroup of finite index in I_v, $\rho_{X,\ell}(I_w)$ is an open subgroup of finite index in $\rho_{X,\ell}(I_v)$. Therefore $\mathfrak{G}_w \subseteq \mathfrak{G}$, and $\mathfrak{G}$ is a finite disjoint union of cosets of $\mathfrak{G}_w$. Since $\mathfrak{G}$ is connected, $\mathfrak{G} = \mathfrak{G}_w$. Therefore, the subgroup $\rho_{X,\ell}(I_v)$ of $\mathfrak{G}(\mathbf{Q}_\ell)$ acts as the identity on W and on V/W. By Theorem 5.1, X has semistable reduction at v. □

References

1. F. A. Bogomolov, *Sur l'algébricité des représentations ℓ-adiques*, C.R.A.S. Paris 290 (1980) 701–703.

2. A. Borel, Linear Algebraic Groups (Second Enlarged Edition), Springer-Verlag, New York, 1991.
3. M. Borovoi, *The action of the Galois group on the rational cohomology classes of type* (p,p) *of abelian varieties* (Russian), Mat. Sbornik (N. S.) 94 (136) (1974) 649–652 = Math. USSR Sbornik 23 (1974) 613–616.
4. M. Borovoi, *The Shimura-Deligne schemes* $M_{\mathbf{C}}(G,h)$ *and the rational cohomology classes of type* (p,p) *of abelian varieties* (Russian), Problems of group theory and homological algebra, No. 1 (Russian), pp. 3–53, Yaroslav. Gos. Univ., Yaroslavl', 1977.
5. W. Chi, *ℓ-adic and λ-adic representations associated to abelian varieties defined over number fields*, Amer. J. Math. 114 (1992) 315–353.
6. . Deligne (notes by J. Milne), *Hodge cycles on abelian varieties*, in Hodge cycles, motives, and Shimura varieties (P. Deligne, J. Milne, A. Ogus, K.-y. Shih), Lecture Notes in Mathematics 900, Springer-Verlag, Berlin–Heidelberg–New York, 1982, pp. 9–100.
7. G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73 (1983) 349–366.
8. G. Faltings, *Complements to Mordell*, Chapter VI of Rational Points (G. Faltings, G. Wüstholz, et al.) (Third Enlarged Edition), Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 1992.
9. A. Grothendieck, *Modèles de Néron et monodromie*, in Groupes de monodromie en géometrie algébrique, SGA7 I (A. Grothendieck, ed.), Lecture Notes in Math. 288, Springer, Berlin–Heidelberg–New York, 1972, pp. 313–523.
10. J. E. Humphreys, Linear Algebraic Groups, Graduate Texts in Mathematics 21, Springer-Verlag, New York-Heidelberg-Berlin, 1975.
11. L. Moret-Bailly, Pinceaux de Variétés Abéliennes, Astérisque 129 (1985).
12. S. Mori, *On Tate conjecture concerning endomorphisms of abelian varieties*, in Proceedings of the International Symposium on Algebraic Geometry, Kyoto 1977 (M. Nagata, ed.), Kinokuniya Book-store Co., Ltd., Tokyo, 1978.
13. I. I. Piatetski-Shapiro, *Interrelations between the Tate and Hodge conjectures for abelian varieties* (Russian), Mat. Sbornik 85 (1971) 610–620 = Math. USSR Sbornik 14 (1971) 615–625.
14. K. Ribet, *Hodge classes on certain types of abelian varieties*, Amer. J. Math. 105 (1983) 523–538.
15. J.-P. Serre, *Représentations ℓ-adiques*, in Algebraic Number Theory (Proceedings of the International Taniguchi Symposium, Kyoto, 1976) (S. Iyanaga, ed.), Japan Society for the Promotion of Science, Tokyo, 1977, pp. 177–193.
16. J.-P. Serre, Letters to K. Ribet, Jan. 1, 1981 and Jan. 29, 1981.
17. J.-P. Serre, Résumé des cours de 1985–1986, Collège de France.
18. J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques*, in Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.), Proc. Symp. Pure Math 55 (1994), Part 1, pp. 377–400.
19. A. Silverberg, *Fields of definition for homomorphisms of abelian varieties*, J. Pure and Applied Algebra 77 (1992) 253–262.
20. A. Silverberg, Yu. G. Zarhin, *Isogenies of abelian varieties*, J. Pure and Applied Algebra 90 (1993) 23–37.
21. A. Silverberg, Yu. G. Zarhin, *Variations on a theme of Minkowski and Serre*, to appear in J. Pure and Applied Algebra.
22. A. Silverberg, Yu. G. Zarhin, *Semistable reduction and torsion subgroups of abelian varieties*, to appear in Ann. Inst. Fourier 45, 2 (1995).
23. A. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948.
24. A. Weil, Basic Number Theory, Springer-Verlag, Berlin–Heidelberg–New York, 1967.
25. Yu. G. Zarhin, *Endomorphisms of abelian varieties over fields of finite characteristic* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975) 272–277 = Math. USSR - Izv. 9 (1975) 255–260.
26. Yu. G. Zarhin, *Abelian varieties in characteristic p* (Russian), Mat. Zametki 19 (1976) 393–400 = Math. Notes 19 (1976) 240–244.
27. Yu. G. Zarhin, *Weights of simple Lie algebras in the cohomology of algebraic varieties* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 48 (1984) 264–304 = Math. USSR - Izv. 24 (1985) 245–282.
28. Yu. G. Zarhin, A. N. Parshin, *Finiteness problems in Diophantine Geometry*, Amer. Math. Soc. Transl. 143 (1989) 35–102.