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Duistermaat-Heckman measures in a non-compact setting

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Abstract. We prove a Duistermaat-Heckman type formula in a suitable non-compact setting. We use this formula to evaluate explicitly the pushforward of the Liouville measure via the moment map of both an abelian and a non-abelian group action. As an application we obtain continuous versions of well-known multiplicity formulas for the holomorphic discrete series representations.

0. Introduction

Let T be a torus with Lie algebra $\mathfrak{t}$. If (M, ω) is a compact symplectic manifold of dimension $2n$ with a Hamiltonian T-action, let $\Phi\colon M \to \mathfrak{t}^*$ be the corresponding moment map, and denote by $\beta = \frac{1}{(2\pi)^n}\frac{\omega^n}{n!}$ the Liouville volume form. Assume initially that M is compact. Consider the integral

$$\int_M e^{\sqrt{-1}\langle \Phi, Z\rangle}\beta, \qquad Z \in \mathfrak{t}^{\mathbb{C}}. \tag{0.1}$$

In their fundamental papers [DH] Duistermaat and Heckman use the method of exact stationary phase to prove a formula that expresses this integral explicitly in terms of local invariants of the T-fixed point set, F, in M.

The Duistermaat-Heckman formula has a number of important applications. For example, consider the measure $\Phi_*|\beta|$ on $\mathfrak{t}^*$, push-forward via Φ of the Liouville measure on M; we will refer to this measure as the *Duistermaat-Heckman* measure. Notice that the integral (0.1) is the Fourier-Laplace transform of $\Phi_*|\beta|$. Guillemin, Lerman, and Sternberg [GLS] use the Duistermaat-Heckman formula to obtain an explicit formula for $\Phi_*|\beta|$ itself under the assumption that F is isolated. This formula is generalized in [GP] to non-abelian group actions. Recently Jeffrey and Kirwan [JK] extended these formulas to allow non-isolated fixed points and arbitrary equivariantly closed forms.

If M is not compact the integral (0.1) may not exist. We study this integral in the case that there exists a component Φ_{X_0} of the moment map that is proper and

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bounded from below. We also assume, for simplicity, that the T-fixed point set is finite; though one could more generally assume that F has finitely many connected components. In this paper, we establish a Duistermaat-Heckman type formula in this setting (Theorem 2.2) and as an application we obtain explicit formulas for the Duistermaat-Heckman measure in both the abelian (Theorem 3.2) and non-abelian (Theorem 3.7) case. From the point of view of physics, the integral (0.1), when Z is purely imaginary, is the partition function of a statistical system with phase space M and energy Φ_{X_0}. Our assumption is natural since usually the phase space is not compact and when this is the case, the energy is bounded only from below (and not from above). However we should remark that in our setting the Hamiltonian flow is periodic; this condition is quite special and mathematically this is what makes the stationary phase exact.

In Section 1, we explore the immediate consequences of our assumption and review some basic facts about tempered distributions (with typically non-compact supports) and their Fourier-Laplace transformations. In Section 2, we prove a Duistermaat-Heckman type formula for torus actions; this is obtained in stages, by initially considering the case of circle actions on manifolds with boundary. Our formula is formally identical to the Duistermaat-Heckman formula, except that it only makes sense for $\mathrm{Im}\,(Z)$ belonging to a special open cone in $\mathfrak{t}$. (This corresponds physically to the positivity of temperature.) In Section 3 we first obtain an explicit formula for the measure $\Phi_*|\beta|$; then we evaluate the measure $\Phi_*^K|\beta|$, where Φ^K is the moment map for the action of a compact connected Lie group K with Cartan subgroup T. In Section 4, we study the regular elliptic orbits of a non-compact semisimple Lie group G that correspond to its holomorphic discrete series representations; we observe that these orbits satisfy our assumption and we evaluate the Duistermaat-Heckman measures associated to the action of a compact Cartan T and to the action of a maximal compact subgroup K of G. The non-abelian measure was first evaluated by Duflo, Heckman and Vergne [DHV] for elliptic orbits corresponding to all the discrete series. Finally the appendix contains a review of basic facts about polyhedral sets and cones that are used throughout the paper.

1. Preliminaries

Let M be a non-compact connected symplectic manifold, T a torus (with Lie algebra $\mathfrak{t}$) acting on M in a Hamiltonian fashion, and $\Phi\colon M \to \mathfrak{t}^*$ the corresponding moment map.

1.1 Some properties of the moment map

Assume for a moment that $\Phi_X = \langle \Phi, X \rangle$, for a certain $X \in \mathfrak{t}$, is a proper function (it may happen that such an X does not exist). Then the moment map Φ itself is a proper mapping. Moreover, Φ_X is a function of Morse-Bott type with critical submanifolds (if any) of even indices; thus the levels of Φ_X either are empty or have a constant number of connected components [A]. This observation leads to strong restrictions on the occurrence of (local) extrema for Φ_X, and on the image set $\Phi_X(M)$:

LEMMA 1.1. *Assume that Φ_X is a proper function. If Φ_X is surjective there are no extrema. If Φ_X is not surjective there is a unique extremal value and Φ_X is an interval of the types:* $[m_X, \infty), (-\infty, n_X]$.

Proof. According to the standard Bott-Morse theory, passing through an extremal value entails adding an additional connected component to the level $\Phi_X^{-1}(a)$; but the number of connected components is constant so all extrema must be global. Now, since Φ_X is proper $\Phi_X(M)$ is an unbounded interval and there can be at most one extremal value, none if Φ_X is surjective. Assume that Φ_X is not surjective. Then we will have, for example, $\Phi_X > c$ strictly for some real number c. Let a be a regular value of Φ_X and consider the manifold $M_- = \Phi_X^{-1}([c, a])$ with boundary $\Phi_X^{-1}(a)$; M_- is compact since Φ_X is proper. Let m_X be the global minimum of Φ_X on M_-. It is easy to see that $m_X < a$; this ensures that m_X corresponds to a (global) minimum on M itself, and that $\Phi_X(M) = [m_X, \infty)$. □

Let us now focus on the case where Φ_X is not surjective.

PROPOSITION 1.2. *Each connected component of the critical set of a proper component of the moment map Φ_X contains at least a T-fixed point.*

Proof. Let T' be the closure of the one-parameter subgroup $\{e^{tX}\}$ in T. Then the critical set of Φ_X coincides with

$$M_{T'} = \{p \in M \mid \mathrm{stab}(p) \supseteq T'\}.$$

It is proven in [GS, Theorem 27.2], that $M_{T'}$ is a union of closed T-invariant connected symplectic submanifolds, W_i, and that Φ_X maps each of these to a point. The main observation here is that since Φ_X is proper W_i is compact. But T acts on W_i in a Hamiltonian fashion so that each W_i must contain a point which is fixed by T. □

Denote by F the T-fixed point set. From now on, we make the following:

ASSUMPTION 1.3. Assume that there exists an $X_0 \in \mathfrak{t}$ such that Φ_{X_0} is proper and not surjective. By the above proposition F is non-empty; assume that it is finite.

In general, if the moment map Φ is proper and if the fixed point set has a finite number of connected components, the image $\Phi(M)$ is a polyhedral set [CDM-HNP]. (We will be using a number of properties of *polyhedral sets*; we refer to the Appendix for details.)

PROPOSITION 1.4. *Under Assumption 1.3, the polyhedral set $\Phi(M)$ is proper. Let $\mathcal{C} \subset \mathfrak{t}^*$ be its asymptotic cone, then Φ_X is proper if and only if $X \in \pm\mathrm{Int}(\mathcal{C}')$. If $X \in \mathrm{Int}(\mathcal{C}')$, then $\Phi_X(M) = [m_X, \infty)$ for a suitable $m_X \in \mathbb{R}$.*

Proof. Notice that $\Phi_X = \pi_X \circ \Phi$, where $\pi_X : \mathfrak{t}^* \to \mathbb{R}$ is defined by $\pi_X = \langle \cdot, X \rangle$. It follows from Lemma A.10 and Lemma A.7 that Φ_X is proper if and only if $X \in \pm\mathrm{Int}(\mathcal{C}')$. By Assumption 1.3, $\mathrm{Int}(\mathcal{C}')$ is not empty, hence $\Phi(M)$ is proper. □

1.2 Distributions with non-compact support and the Laplace transform

This subsection is devoted to a brief overview of the elements of the theory of Laplace transforms that will be needed throughout the paper. We refer to [Hö] for all proofs. Let E be a finite-dimensional vector space, and let E^* be its dual. Let $\mathcal{D}'(E)$ be the space of distributions on E, and $\mathcal{S}'(E)$, that of tempered ones. For any $T \in \mathcal{D}'(E)$, the set $\Gamma(T) = \{Y \in E^* \,|\, e^{-\langle Y,x\rangle}T(x) \in \mathcal{S}'(E)\}$ (which may be empty) is convex. Since the Fourier transform $\mathcal{F}$ is a linear isomorphism from $\mathcal{S}'(E)$ to $\mathcal{S}'(E^*)$, if Γ is non-empty one can define the Laplace transform of $T \in \mathcal{D}'(E^*)$ by

$$\mathcal{L}(T)(Z) = \mathcal{F}(e^{-\langle Y,x\rangle}T(x))(X), \qquad Z = X + \sqrt{-1}\,Y, \quad Y \in \Gamma(T).$$

$\mathcal{L}(T)$ is an analytic function on the interior of the above domain. For $T \in \mathcal{S}'$, Γ contains 0 and, by the continuity of the Fourier transform, when $Y \to 0$ inside any closed cone in $\Gamma(T)$, $\mathcal{L}(T)(Z) \to \mathcal{F}(T)(X)$ as tempered distributions. If the distribution is tempered and compactly supported the region $\Gamma(T)$ is all of E, but we will be interested in distributions that have non-compact support. Let's concentrate for a moment on a simple example that will be of fundamental importance.

EXAMPLE 1.5. Let $\alpha_1, \dots, \alpha_n$ ($n \geqslant \dim E$) be a spanning set of vectors in E that generates a proper polyhedral cone, $\mathcal{C}_\alpha$. Let H_{α_i} be the Heaviside distribution defined by

$$H_{\alpha_i}(f) = \int_0^\infty f(t\alpha_i)dt, \qquad f \in C_0^\infty(E).$$

Then the convolution $H_{\alpha_1} * \cdots * H_{\alpha_n}$ defines a tempered distribution on E supported on $\mathcal{C}_\alpha$. There is another description for this measure. Let L_α be the map from the positive n-tant in $\mathbb{R}^n$ to E defined by

$$L_\alpha(s_1, \dots, s_n) = \sum_{i=1}^n s_i\alpha_i, \quad \text{where } s_i \geqslant 0. \tag{1.1}$$

L_α is proper since $\mathcal{C}_\alpha$ is, and the pushforward via L_α of Lebesgue measure, ds, on E, is well defined and given by $(L_\alpha)_* ds = H_{\alpha_1} * \cdots * H_{\alpha_n}$ [GP]. It is quite easy to see that the set $\Gamma(L_\alpha)$ is the dual cone $\mathcal{C}'_\alpha$ and that, for all $Z \in (E^*)^{\mathbb{C}}$ with $\mathrm{Im}\,(Z) \in \mathrm{Int}(\mathcal{C}'_\alpha)$, the Laplace transform is given by

$$\mathcal{L}(H_{\alpha_1} * \cdots * H_{\alpha_n})(Z) = \frac{(\sqrt{-1}\,)^n}{\prod_{i=1}^n \langle \alpha_i, Z\rangle}. \tag{1.2}$$

Returning to the Hamiltonian T-action on (M^{2n}, ω), we take $E = \mathfrak{t}^*$ and hence $E^* = \mathfrak{t}$. Let $\beta = \frac{1}{(2\pi)^n}\frac{\omega^n}{n!}$ be the Liouville volume form and $\Phi_*|\beta|$, the push-forward of the corresponding measure. Under Assumption 1.3, $\Phi_*|\beta|$, supported on $\Phi(M)$, is piecewise polynomial and therefore defines a tempered distribution. It will be shown in Sect. 3 that $\Phi_*|\beta|$ can be written as a sum of distributions of the form considered in Example 1.5.

PROPOSITION 1.6. $\Gamma(\Phi_*|\beta|) = \mathcal{C}'$ *(the dual of the asymptotic cone, $\mathcal{C}$, of (M)).*

Proof. Let f_Φ be the Radon-Nikodym derivative of $\Phi_*|\beta|$ with respect to the Lesbegue measure. For any $Y \in \mathcal{C}'$, $e^{\sqrt{-1}\,\langle \mu, X+\sqrt{-1}\,Y\rangle}$ is bounded as μ runs through $\Phi(M)$ (Lemma A.7). So $e^{\sqrt{-1}\,\langle \mu, X+\sqrt{-1}\,Y\rangle} f_\Phi(\mu) \in \mathcal{S}'$, i.e., $Y \in \Gamma$. Conversely, if $Y \notin \mathcal{C}'$, then there is an element $\alpha \in \mathcal{C}$ such that $\langle \alpha, Y\rangle < 0$. Moreover, from the proof of Lemma A.3, one can choose α^0 such that the ray $\alpha^0 + t\alpha \in \Phi(M)$ is contained in $\Phi(M)$ for sufficiently large $t > 0$. In fact, one can choose α^0 such that the ray is in the interior of $\Phi(M)$, considered as a top dimensional subset of its affine hull. The function $f_\Phi(\alpha^0 + t\alpha)$ is a non-zero piecewise polynomial in t. Therefore, $e^{\sqrt{-1}\,\langle \alpha^0 + t\alpha, X+\sqrt{-1}\,Y\rangle} f_\Phi(\alpha^0 + t\alpha)$ increases at least exponentially as $t \to +\infty$. So $e^{\sqrt{-1}\,\langle \mu, X+\sqrt{-1}\,Y\rangle} f_\Phi(\mu) \notin \mathcal{S}'$ and $Y \notin \Gamma$. □

2. A Duistermaat-Heckman type formula

In this section, (M, ω) is a $2n$ dimensional non-compact symplectic manifold with a Hamiltonian torus action satisfying Assumption 1.3. For $p \in F$ let α_i^p, $i = 1, \ldots, n$, be the weights of the isotropy representation of T on the tangent space T_pM.

DEFINITION 2.1. We will say that $Z \in \mathfrak{t}^{\mathbb{C}}$ is *regular* if

$$\alpha_i^p(Z) \neq 0 \quad \text{for} \quad p \in F, \quad i = 1, \ldots, n.$$

THEOREM 2.2. *Under Assumption 1.3, for each regular $Z \in \mathfrak{t}^{\mathbb{C}}$ with* $\mathrm{Im}\,(Z) \in \mathrm{Int}(\mathcal{C}')$, *the following improper integral exists and*

$$\int_M e^{\sqrt{-1}\,\langle \Phi, Z\rangle}\beta = (\sqrt{-1}\,)^n \sum_{p\in F} \frac{e^{\sqrt{-1}\,\langle \Phi(p), Z\rangle}}{\prod_{i=1}^n \alpha_i^p(Z)}. \tag{2.1}$$

Proof. Notice first that it is enough to prove the theorem when Z is purely imaginary, i.e., $Z \in \sqrt{-1}\,\mathfrak{t}$. The general case will follow by analytic continuation since both sides are analytic functions in the domain $\{Z \in \mathfrak{t}^{\mathbb{C}}$ regular $|\, \mathrm{Im}\,(Z) \in \mathrm{Int}(\mathcal{C}')\}$. Consider now the lattice $\mathcal{L} = \{Y \in \mathfrak{t} \,|\, e^{2\pi Y} = 1\}$ in $\mathfrak{t}$ and notice that the set

$$\mathcal{A}_0 = \{Z = \sqrt{-1}\,sY \mid s \geqslant 0, Y \in \mathrm{Int}(\mathcal{C}') \cap \mathcal{L} \text{ is regular}\}$$

is dense in the set of regular elements $Z \in \sqrt{-1}\,\mathfrak{t}$ with $\mathrm{Im}\,(Z) \in \mathrm{Int}(\mathcal{C}')$. Therefore, by continuity, it will be enough to prove the formula for $Z \in \mathcal{A}_0$, $\langle \Phi, Z\rangle = \sqrt{-1}\,s\Phi_Y$. However, since $Y \in \mathcal{L}$, $H = \Phi_Y$ is a moment map for the induced action of $S^1 = \{e^{tY}\}$ on M. Moreover, since Y is regular, the critical set of Φ_Y is isolated and by Proposition 1.2 it coincides with F. We have thus reduced this case to the special case $T = S^1$, which follows from Lemma 2.3 below (the M_- case), with $z = \sqrt{-1}\,s$ and after taking the limit $a \to +\infty$. In fact for $s \geqslant 0$, e^{-sa} decays exponentially as $a \to +\infty$. When a is sufficiently large, the cohomological class of ω_a depends linearly on a [DH], while that of Ω_a remains fixed, since the topology of the bundle $H^{-1}(a) \to M_a$ does not change as a runs through a set of regular values. So the integral over M_a is a polynomial in a,* and consequently, the second sum on the right hand side of (2.2) converges to 0 as $a \to +\infty$. □

LEMMA 2.3. *Let (M^{2n}, ω) be a symplectic manifold on which there is a Hamiltonian S^1-action with an isolated fixed point set F. Let $\alpha_1^p, \dots, \alpha_n^p$ be the weights of the isotropy representation of S^1 on T_pM, $p \in F$. Assume that the moment map $H\colon M \to \mathbb{R}$ is proper and not surjective. Let $a \in \mathbb{R}$ be a regular value of H and let $M_a = H^{-1}(a)/S^1$ be the symplectic quotient with the canonical symplectic form ω_a. Choose a connection of the V-bundle $H^{-1}(a) \to M_a$ with curvature 2-form Ω_a. If H is bounded from above (below, respectively), let $M_+ = H^{-1}([a,\infty))$ ($M_- = H^{-1}((-\infty, a])$, respectively) and $F_\pm = F \cap M_\pm$. Then for any $z \in \mathbb{C}$,*

$$\int_{M_\pm} e^{\sqrt{-1}\,zH}\beta = \left(\frac{\sqrt{-1}}{z}\right)^n \sum_{p\in F_\pm} \frac{e^{\sqrt{-1}\,zH(p)}}{\prod_{i=1}^n \alpha_i^p} \mp \frac{1}{(2\pi)^{n-1}} \sum_{k=0}^{n-1} \frac{e^{\sqrt{-1}\,za}}{(\sqrt{-1}\,z)^{k+1}} \int_{M_a} \frac{\omega_a^{n-1-k}}{(n-1-k)!} \wedge \Omega_a^k. \tag{2.2}$$

Proof. Since a is a regular value of H, there is a small number $\delta > 0$ such that $H^{-1}((-\delta, \delta))$ is diffeomorphic to $H^{-1}(a) \times (-\delta, \delta)$ and the induced symplectic form on the latter is, up to an exact form, $\alpha \wedge dH - (H - a)\Omega_a + \omega_a$ for one (hence any) connection 1-form α of $H^{-1}(a) \to M_a$ [DH,W]. One can find an S^1-invariant Riemannian metric g on M and can choose a connection whose horizontal spaces are induced by g. Denote the vector of the S^1-action by X_M, let

* For an explicit formula of this polynomial, see [W, Theorem 5.2], which can also be deduced by collecting the coefficients of $z^{-(k+1)}$ in (2.2).

$\theta = i_{X_M} g / g(X_M, X_M)$ and $\nu = \frac{1}{(2\pi)^n} \theta \wedge (\tilde{d}\theta)^{-1} \wedge e^{\tilde{\omega}}$. Here $\tilde{d} = d - \sqrt{-1}\, z\, i_{X_M}$ is the equivariant derivative and $\tilde{\omega} = \omega + \sqrt{-1}\, zH$ is the closed equivariant extension of ω. Both θ and ν are well-defined on $M \backslash F$ and $\frac{1}{(2\pi)^n} e^{\tilde{\omega}} = \tilde{d}\nu$. For any fixed point $p \in F$, let B_p^ϵ be the ball centered at p and of radius ϵ. Since the top form in $\frac{1}{(2\pi)^n} e^{\tilde{\omega}}$ is $e^{\sqrt{-1}\, zH} \beta$, Stokes' theorem implies

$$\int_{M_\pm} e^{\sqrt{-1}\, zH} \beta = \sum_{p \in F_\pm} \left(\int_{B_p^\epsilon} e^{\sqrt{-1}\, zH} \beta - \int_{\partial B_p^\epsilon} \nu \right) + \int_{\partial M_\pm} \nu. \tag{2.3}$$

A standard argument [BV,GS] shows that as $\epsilon \to +0$,

$$\int_{B_p^\epsilon} e^{\sqrt{-1}\, zH} \beta - \int_{\partial B_p^\epsilon} \nu = \left(\frac{\sqrt{-1}}{z} \right)^n \frac{e^{\sqrt{-1}\, zH(p)}}{\prod_{i=1}^n \alpha_i^p}.$$

The boundary $\partial M_\pm$ is $\pm H^{-1}(a)$. (Here the minus sign refers to the reversed orientation.) One can show easily that, when restricted to $H^{-1}(a)$, $\theta = \alpha$, $\tilde{d}\theta = \Omega_a - \sqrt{-1}\, z$ and $\tilde{\omega} = \omega_a + \sqrt{-1}\, za$. Therefore

$$\begin{aligned} \int_{\partial M_\pm} \nu &= \pm \int_{H^{-1}(a)} \alpha \wedge e^{\omega_a + \sqrt{-1}\, za} \wedge (\Omega_a - \sqrt{-1}\, z)^{-1} \\ &= \pm \frac{1}{(2\pi)^{n-1}} \left(-\frac{1}{\sqrt{-1}\, z} \right) \int_{M_a} e^{\omega_a + \sqrt{-1}\, za} \wedge \left(1 - \frac{\Omega_a}{\sqrt{-1}\, z} \right)^{-1} \\ &= \mp \frac{1}{(2\pi)^{n-1}} \sum_{k=0}^{n-1} \frac{e^{\sqrt{-1}\, za}}{(\sqrt{-1}\, z)^{k+1}} \int_{M_a} \frac{\omega_a^{n-1-k}}{(n-1-k)!} \wedge \Omega_a^k. \end{aligned} \tag{2.4}$$

□

Notice that if the fixed point set F is no longer isolated, but has finitely many connected components, similar arguments show that Lemma 2.3, hence Theorem 2.2, remains valid, after replacing the point-like contribution by an integral over the fixed (symplectic) submanifold, and the products of weights by the equivariant Euler class of the normal bundle. It should also be noted that the proof of Lemma 2.3 above could be adapted to allow, instead of $e^{\tilde{\omega}}$, more general equivariantly closed forms; this could be useful in applications to the ring structure of the ordinary De Rham cohomology of the reduced phase space (cfr. [JK]).

3. Formulas for the Duistermaat-Heckman measure

3.1 THE ABELIAN CASE

Now consider the Hamiltonian T-action on M. The hyperplanes perpendicular to the weights α_i^p ($p \in F, i = 1, \ldots, n$) divide the cone C' into finitely many subcones, each of which we will call a *chamber* for the action of T near F, briefly an *action chamber*. Any regular vector $X \in \mathcal{C}'$ sits in the interior of such a chamber. We fix such an X and call the corresponding chamber, $\mathcal{C}^+$, the *positive* action chamber. Define, for $p \in F$, $i = 1, \ldots, n$: $\beta_i^p = \operatorname{sign}(\alpha_i^p(X))\alpha_i^p$.

DEFINITION 3.1. The set $\beta_i^p, p \in F, i = 1, \ldots, n$ is called a *renormalization* of the set of weights.

Let $\epsilon(p) = \prod_{i=1}^n \operatorname{sign}(\alpha_i^p(X))$, and let δ_μ be the delta distribution supported at $\mu \in \mathfrak{t}^*$.

THEOREM 3.2. *Under Assumption 1.3*

$$\Phi = \sum_{p \in F} \epsilon(p)\, \delta_{\Phi(p)} * H_{\beta_1^p} * \cdots * H_{\beta_n^p}. \tag{3.1}$$

Proof. Both sides of (3.1) are tempered distributions on $\mathfrak{t}^*$. By Theorem 2.2 and Example 1.5 the Laplace transformations of the two sides are equal for all Z with $Y = \operatorname{Im}(Z)$ in the interior of the positive action chamber $\mathcal{C}^+$. Letting $Y \to 0$ inside $\mathcal{C}^+$, we conclude that the Fourier transform of the two sides of (3.1) are equal (as tempered distributions) [Hö]. (3.1) follows from taking the inverse Fourier transformation. □

Notice that (3.1) is actually a collection of formulas, one for each choice of a positive action chamber in $\mathcal{C}'$. What is also remarkable is that if we compute the right-hand side at given point $\mu \in \mathfrak{t}^*$, the terms (and even the number of terms) that contribute will also depend on the choice of the action chamber. More specifically, the summation set above could be replaced, at $\mu \in \mathfrak{t}^*$, by the subset $F_\mu = \{p \in F \mid \Phi(p) + \sum s_i \beta_i^p = \mu,\ \ s_i \geqslant 0\}$. Similar remarks hold for the formulas that will appear in the following.

3.2 The non-abelian case

First some notation. Let K be a compact connected (non-abelian) Lie group with Lie algebra $\mathfrak{k}$ and let T be a maximal torus in K with Lie algebra $\mathfrak{t}$. Let $\Delta^+ = \{\sqrt{-1}\,\alpha_1, \ldots, \sqrt{-1}\,\alpha_k\}$ ($\alpha_i \in \mathfrak{t}^*$) be a set of positive roots. For each $i = 1, \ldots, k$ let $X_i \in \mathfrak{t}$ denote the vector dual to α_i with respect to the restriction to $\mathfrak{t}$ of a K-invariant scalar product on $\mathfrak{k}$ and consider $P = \prod_{i=1}^k X_i$, viewed as a polynomial in $\mathfrak{t}^*$. Let $\mathfrak{t}^*_{\mathrm{reg}}$ be the set of elements μ of $\mathfrak{t}^*$ such that $P(\mu) \neq 0$ and let $\mathfrak{k}^*_{\mathrm{reg}}$ be the set $K \cdot \mathfrak{t}^*_{\mathrm{reg}}$ in $\mathfrak{k}^*$.

Assume that K acts on a symplectic manifold M in a Hamiltonian fashion, and denote by Φ^K the corresponding moment map $M \to \mathfrak{k}^*$. The induced action of T on M is also Hamiltonian with moment map, Φ, the composition of the natural projection $\mathfrak{k}^* \to \mathfrak{t}^*$ with Φ^K.

ASSUMPTION 3.3. *Assume that, as a T-space, M satisfies Assumption 1.3. Then the T-fixed point set F is finite; assume in addition that $\Phi^K(p) \in \mathfrak{k}^*_{\text{reg}}$ for each $p \in F$.*

Let us examine for a moment the implications of the first sentence of this assumption. First, the equivariance of the K-moment map implies that the image of the T-moment map, $\Phi(M)$, a polyhedral set in $\mathfrak{t}$, is invariant under the Weyl group W of the pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. So the asymptotic cone $\mathcal{C}$ of $\Phi(M)$ and its dual $\mathcal{C}'$ are also W-invariant. We also know that the open W-invariant convex set $\text{Int}(\mathcal{C}')$ is non-empty since it contains the vector X_0 of Assumption 1.3. By averaging under W, we can assume that X_0 is W-invariant and therefore belongs to the center of $\mathfrak{k}$. In particular, we conclude that the center of K contains a circle.

The assumption that $\Phi^K(p) \in \mathfrak{k}^*_{\text{reg}}$ for each $p \in F$ is not absolutely necessary; if we omit it, most of the argument below goes through, except for the cancellations in the proof of Proposition 3.6. The result would be a more complicated formula (3.6), involving a differential operator.

Notice now that in this setting (Φ thus) Φ^K is a proper mapping; then the measure $\Phi^K_*|\beta|$ is well defined and uniquely determined by its W-invariant restriction, ν, to $\mathfrak{t}^*$, which is defined as follows: if f is a K-invariant smooth compactly supported function on $\mathfrak{k}^*$ and g is its restriction to $\mathfrak{t}^*$, then

$$\int_{\mathfrak{t}^*} g \; \nu = \int_{\mathfrak{k}^*} f \; \Phi^K_*|\beta|.$$

Consider now the symplectic cross-section $M_{\text{reg}} = \left(\Phi^K\right)^{-1}(\mathfrak{t}^*_{\text{reg}})$; M_{reg} is naturally a Hamiltonian T-space. The second sentence of Assumption 3.3 implies that M_{reg} shares its T-fixed point set with M and is therefore non-empty; moreover at each of these fixed points p the set of weights, $\alpha_1^p, \dots, \alpha_n^p$, of the isotropy representation of T on T_pM contains the set of weights of the same representation of T on T_pM_{reg}; the weights that are left are, up to signs, the elements $\alpha_1, \dots, \alpha_k$. After possible relabelings we can assume that at each $p \in F$, $\alpha_i^p = \epsilon_i^p \alpha_i$, $i = 1, \dots, k$, with ϵ_i^p either 1 or -1. Denote by $\epsilon^p = \prod_{i=1}^k \epsilon_i^p$.

The rest of the section will be devoted to write down and prove, under Assumption 3.3, an explicit formula for the measure ν, which is analogous to a formula proven in [GP] for M compact. We start by recalling a useful formula, consequence of a classical formula of Harish-Chandra [HC1]. Consider $P^* = \prod_{i=1}^k \alpha_i$ viewed as a polynomial on $\mathfrak{t}$. Let $\sqrt{-1}\, D_{X_i}$ be differentiation with respect to X_i and let $D_P = \prod_{i=1}^k D_{X_i}$.

PROPOSITION 3.4. *Let f be a K-invariant compactly supported smooth function on $\mathfrak{k}^*$ and let g be its restriction to $\mathfrak{t}^*$; then, for $Z \in \mathfrak{t}^{\mathbb{C}}$,*

$$\int_{\mathfrak{t}^*} g \; e^{\sqrt{-1}\langle -, Z\rangle} \; \nu = c_{\mathfrak{k}}\, D_P \left(P^*(Z) \int_{\mathfrak{k}^*} f \; e^{\sqrt{-1}\langle -, Z\rangle} \Phi^K_*|\beta| \right), \tag{3.2}$$

where $c_{\mathfrak{k}} = (D_P(P^*))^{-1}$ is a non-zero constant.

Proof. Let dk denote the Haar measure on K, normalized so that $\mathrm{vol}(K) = 1$. We have:

$$\begin{aligned} P^*(Z) \int_{\mathfrak{k}^*} f(\xi)\, e^{\sqrt{-1}\langle \xi, Z\rangle}\, \Phi_*^K|\beta| &= P^*(Z) \int_{\mathfrak{k}^*} f(\xi) \int_K e^{\sqrt{-1}\langle k\cdot\xi, Z\rangle} dk\ \Phi_*^K|\beta| \\ &= P^*(Z) \int_{\mathfrak{t}^*} g(\eta) \int_K e^{\sqrt{-1}\langle k\cdot\eta, Z\rangle} dk\ \nu \\ &= c_{\mathfrak{k}}^{-1} \int_{\mathfrak{t}^*} \frac{g(\eta)}{P(\eta)}\, e^{\sqrt{-1}\langle \eta, Z\rangle}\ \nu. \end{aligned}$$

We have used the K-invariance of $\Phi_*^K|\beta|$ and f, the definition of ν, the W-invariance of ν and g, and the following formula of Harish-Chandra [HC1, Corollary], with $c_{\mathfrak{k}}^{-1} = c'_{\mathfrak{k}}|W|$:

$$P^*(Z) \int_K e^{\sqrt{-1}\langle k\cdot\eta, Z\rangle} dk = c'_{\mathfrak{k}} \sum_{w\in W} \frac{e^{\sqrt{-1}\langle w\cdot\eta, Z\rangle}}{P(w\cdot\eta)}, \qquad \eta \in \mathfrak{t}^*_{\mathrm{reg}}.$$

The proposition now follows after applying the operator D_P on both terms. The value of $c_{\mathfrak{k}}$ is obtained easily by setting $Z = 0$ in (3.2). □

PROPOSITION 3.5. *Under Assumption 3.3* $\Gamma(\nu) = \mathcal{C}'$ *(the dual of the asymptotic cone,* $\mathcal{C}$*, of* $\Phi(M)$*) and for each* $Z \in \mathfrak{t}^{\mathbb{C}}$ *with* $\mathrm{Im}\,(Z) \in \mathcal{C}'$ *we have*

$$\mathcal{L}(\nu)(Z) = c_{\mathfrak{k}}\, D_P(P^*(Z)\mathcal{L}(\Phi_*^K|\beta|)(Z)). \tag{3.3}$$

Proof. Notice that for $Z \in \mathfrak{t}^{\mathbb{C}}$, $\mathcal{L}(\Phi_*^K|\beta|)(Z) = \mathcal{L}(\Phi_*|\beta|)(Z)$. Thus from Proposition 3.4 we get that $\Gamma(\nu) = \Gamma(\Phi_*|\beta|)$ and this last set equals $\mathcal{C}'$ by Proposition 1.6; (3.3) follows from (3.2). □

PROPOSITION 3.6. *Assume that* M *satisfies Assumption 3.3. For each* $Z \in \mathfrak{t}^{\mathbb{C}}$ *with* $\mathrm{Im}\,(Z) \in \mathrm{Int}(\mathcal{C}')$ *and* $\alpha_i^p(Z) \neq 0$, $p \in F$, $i = k+1, \ldots, n$, *we have*

$$\mathcal{L}(\nu)(Z) = (\sqrt{-1}\,)^n c_{\mathfrak{k}}\, D_P \left(\sum_{p\in F} \epsilon^p \frac{e^{\sqrt{-1}\langle \Phi^K(p), Z\rangle}}{\prod_{i=k+1}^n \alpha_i^p(Z)} \right). \tag{3.4}$$

Proof. By Theorem 2.2 we have, for $Z \in \mathfrak{t}^{\mathbb{C}}$ regular and $\mathrm{Im}\,(Z) \in \mathrm{Int}(\mathcal{C}')$,

$$\mathcal{L}(\Phi_*^K|\beta|)(Z) = (\sqrt{-1}\,)^n \sum_{p\in F} \frac{e^{\sqrt{-1}\langle \Phi^K(p), Z\rangle}}{\prod_{i=1}^n \alpha_i^p(Z)}. \tag{3.5}$$

Then by combining (3.3) and (3.5) and after the appropriate cancellations we get formula (3.4) for the Laplace transformation of ν. □

Consider now the hyperplanes perpendicular to the weights $\alpha_{k+1}^p, \ldots, \alpha_n^p, p \in F$ that are left after the cancellations. They divide the cone $\mathcal{C}'$ into finitely many subcones, which we will call chambers for the action of K near F, or *K-action* *chambers*. Notice that each K-action chamber contains finitely many action chambers for the action of T. Choose now a positive K-action chamber $\mathcal{C}_K^+$ containing a regular element $X \in \mathcal{C}'$, and let β_i^p, $p \in F$, $i = k+1, \dots, n$ be the corresponding renormalized weights. Let $\epsilon(p) = \epsilon^p \prod_{i=k+1}^n \operatorname{sign}(\alpha_i^p(X))$.

THEOREM 3.7. *Under Assumption 3.3*

$$\nu = c_{\mathfrak{k}}\, P \sum_{p \in F} \epsilon(p)\, \delta_{\Phi^K(p)} * H_{\beta_{k+1}^p} * \cdots * H_{\beta_n^p}. \tag{3.6}$$

Proof. Repeat step by step the proof of Theorem 3.2 using Proposition 3.6 instead of Theorem 2.2. □

4. An application: T-types and K-types of the holomorphic discrete series

We begin by reviewing certain basic facts about Hermitian symmetric spaces; we refer to [He, K1, K2] for proofs and a more extensive treatment.

Let (G, K) be an *irreducible Hermitian symmetric* pair: G is a non-compact, simple, connected Lie group with Lie algebra $\mathfrak{g}$ and K a maximal compact subgroup with Lie algebra $\mathfrak{k}$; G has trivial center, K is connected and its center is a circle. Let T be a Cartan subgroup of K; in this setting T is automatically a Cartan subgroup of G. Denote by $(\cdot,\cdot)$ the form on $\mathfrak{g}^*$ that is dual to the Killing form and by W the Weyl group for the pair $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. It is always possible to choose a set of positive roots, $\Delta^+ = \{\sqrt{-1}\,\alpha_1, \dots, \sqrt{-1}\,\alpha_n\}$, for the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that the positive non-compact roots, $\Delta_n^+ = \{\sqrt{-1}\,\alpha_{k+1}, \dots, \sqrt{-1}\,\alpha_n\}$, are larger than the compact ones, $\sqrt{-1}\,\alpha_1, \dots, \sqrt{-1}\,\alpha_k$. With this choice, the elements of Δ_n^+ agree on vectors of the one-dimensional center of $\mathfrak{k}$; as a consequence we have that Δ_n^+ is W-invariant and that $(\alpha, \beta) \geqslant 0$ for each $\alpha, \beta \in \Delta_n^+$. Consider now the proper open W-symmetric cone in $\mathfrak{t}^*$

$$\mathcal{C}_n = \{\lambda \in \mathfrak{t}^* \mid (\alpha_i, \lambda) > 0,\ i = k+1, \dots, n\}.$$

An *elliptic orbit* is, by definition, a coadjoint orbit that intersects $\mathfrak{t}^*$.

Consider $\lambda \in \mathfrak{t}^*_{\text{reg}} \cap \mathcal{C}_n$ and let $\mathcal{O}_\lambda$ be the elliptic orbit through λ. The natural action of K on $\mathcal{O}_\lambda$ is Hamiltonian with moment map, $\Phi^K \colon \mathcal{O}_\lambda \to \mathfrak{t}^*$, given by the restriction to $\mathcal{O}_\lambda$ of the K-invariant projection of $\mathfrak{g}^*$ onto $\mathfrak{k}^*$; as usual we denote by Φ the moment map of the induced T action. Let X_0 be the unique vector in the center of $\mathfrak{k}$ that satisfies $\alpha_i(X_0) = 1$ ($i = k+1, \dots, n$; recall that such α_i's agree on the center).

PROPOSITION 4.1. *$\mathcal{O}_\lambda$ satisfies Assumptions 1.3 and 3.3 with respect to Φ_{X_0}.*

Proof. It is shown in [P, Propositions 2.2 and 2.3], that Φ_{X_0} is proper and not surjective. The proposition now follows from Lemma 4.2 below. □

Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form; $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the Cartan decomposition of $\mathfrak{g}$. K acts naturally on $\mathfrak{p}$ via the adjoint action and the operator $\mathrm{ad}(X_0)$ defines a complex structure on $\mathfrak{p}$; from this it follows easily that $S^1 = \{e^{tX_0}\}$ acts freely on $\mathfrak{p} - 0$. The K-equivariant diffeomorphism of $K \times \mathfrak{p}$ onto G, given by $(k, X) \to e^X k$, induces a K-equivariant diffeomorphism:

$$\mathcal{O}_\lambda \simeq K \cdot \lambda \times \mathfrak{p}. \tag{4.1}$$

We then have:

LEMMA 4.2. *The set of points of $\mathcal{O}_\lambda$ that are fixed by T is finite and given by $W \cdot \lambda = \{w \cdot \lambda \mid w \in W\} \subset \mathfrak{t}^*_{\mathrm{reg}}$.*

Proof. The points of $W \cdot \lambda$ are T-fixed. Conversely, let q be a T-fixed point. Since T is maximal abelian in G and since S^1, thus T, acts freely on $\mathfrak{p} - 0$, one can show quite easily using the K-equivariant diffeomorphism (4.1), that $q \in K \cdot \lambda \cap \mathfrak{t}$. Finally, since $\lambda \in \mathfrak{t}^*_{\mathrm{reg}}$, $K \cdot \lambda \cap \mathfrak{t} = W \cdot \lambda \subset \mathfrak{t}^*_{\mathrm{reg}}$. □

Assume without restrictions that $\alpha_i(\lambda) > 0$, $i = 1, \ldots k$ and notice finally that the weights of the isotropy representation of T on the tangent space of $\mathcal{O}_\lambda$ at $w \cdot \lambda$ are given by $w \cdot \alpha_i$, $i = 1, \ldots, n$. The chambers for the action of T near F here match with the classical Weyl chambers of the non-compact pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$; the chambers for the action of K near F are obtained from these by removing the compact walls. In each case we renormalize the weights with respect to the chamber containing the element λ; we then get the following corollaries of Theorems 3.2 and 3.7 with $\epsilon(p) = \epsilon(w)$, the determinant of w as a linear transformation of $\mathfrak{t}^*$.

THEOREM 4.3.

$$\Phi_*|\beta| = \sum_{w \in W} \epsilon(w)\, \delta_{w \cdot \lambda} * H_{\alpha_1} * \cdots * H_{\alpha_n}. \tag{4.2}$$

THEOREM 4.4.

$$\nu = c_{\mathfrak{k}}\, P \sum_{w \in W} \epsilon(w)\, \delta_{w \cdot \lambda} * H_{\alpha_{k+1}} * \cdots * H_{\alpha_n}. \tag{4.3}$$

The two formulas above are related to the study of the holomorphic discrete series representations of the group G. In fact, following the method of Heckman [H], the first formula is a continuous version of a multiplicity formula of Harish-Chandra [HC2] for the T-multiplicities of such representations. The second formula, on the other hand, is a continuous version of the Blattner formula for the multiplicity of the K-types; the Blattner formula gives the multiplicities of the K-types of all discrete series representations.

It is quite easy to see that for the elliptic orbits that correspond to the non-holomorphic discrete series the T-moment map is not proper. This means, for example, that the measure $\Phi_*|\beta|$, in this setting, is not well defined; this is in perfect agreement with the representation-theoretical fact that, in this setting, the T-multiplicities are not finite. On the other hand one should remark that the K-moment map remains proper, that the corresponding pushforward measure is still well defined and that it has been explicitly evaluated for all discrete series by Duflo, Heckman, and Vergne [DHV]. Our symplectic-theoretical approach does not extend to the non-holomorphic case, since in our proof we are relying on the properness of the T-moment map. However we are hoping that a variation of our approach will soon yield a non-abelian formula that holds even in the event that the T-moment map is not proper.

Appendix A: polyhedral sets and polyhedral cones

This appendix provides a self-contained account on polyhedral sets; we refer to [F-L-B] for related matters. Let E be a finite-dimensional vector space with dual E^* and denote by $\langle\cdot,\cdot\rangle$ the evaluation $E^* \times E \to \mathbb{R}$. For any $\xi \in E^*$, $c \in \mathbb{R}$, let $K(\xi, c)$ be the (closed) half space $\{\alpha \in E \mid \langle\xi,\alpha\rangle \geqslant c\}$ in E.

DEFINITION A.1. A *polyhedral set* $\mathcal{P}$ in E is a finite intersection of half spaces, i.e.,

$$\mathcal{P} = \bigcap_{i=1}^{r} K(\xi_i, c_i) \qquad \text{for } \xi_i \in E^*,\ c_i \in \mathbb{R}. \tag{A.1}$$

It is called a *polyhedral cone* if all $c_i = 0$.

DEFINITION A.2. Let $\mathcal{P}$ be a polyhedral set in E. Its *asymptotic cone*, denoted by $\mathcal{C}(\mathcal{P})$, is the set of vectors $\alpha \in E$ with the property that there exists $\alpha^0 \in E$ such that $\alpha^0 + t\alpha \in \mathcal{P}$ for sufficiently large $t > 0$.

LEMMA A.3. *If* $\mathcal{P} = \bigcap_{i=1}^{r} K(\xi_i, c_i)$, *then* $\mathcal{C}(\mathcal{P}) = \bigcap_{i=1}^{r} K(\xi_i, 0)$.

Proof. If $\alpha \in \mathcal{C}(\mathcal{P})$, then for sufficiently large $t > 0$, $\alpha^0 + t\alpha \in \mathcal{P}$, i.e, $\langle\xi_i, \alpha^0 + t\alpha\rangle \geqslant c_i$. So $\langle\xi, \alpha\rangle \geqslant 0$, $\forall i = 1, \ldots, r$, i.e., $\alpha \in \bigcap_{i=1}^{r} K(\xi_i, 0)$. Conversely, if $\alpha \in \bigcap_{i=1}^{r} K(\xi_i, 0)$, choose $\alpha^0 \in \mathcal{P}$, then $\langle\xi_i, \alpha^0 + t\alpha\rangle \geqslant c_i$ for all $t \geqslant 0$, i.e., $\alpha^0 + t\alpha \in \mathcal{P}$. □

DEFINITION A.4. A polyhedral cone $\mathcal{C}$ is *proper* if there exists a vector $\xi \in E^*$ such that $\langle\xi, \mathcal{C}\backslash\{0\}\rangle > 0$. A polyhedral set $\mathcal{P}$ is *proper* if $\mathcal{C}(\mathcal{P})$ is.

LEMMA A.5. *A polyhedral set* $\mathcal{P}$ *is proper if and only if it does not contain a line.*

Proof. If $\mathcal{P}$ contains a line $\{\alpha^0 + t\alpha \mid t \in \mathbb{R}\}$, then $\{\pm\alpha\} \subset \mathcal{C}(\mathcal{P})$. So $\mathcal{C}(\mathcal{P})$, hence $\mathcal{P}$, is not proper. Conversely, if $\mathcal{C}(\mathcal{P})$ is not proper, then $\exists \alpha \neq 0$, $\{\pm\alpha\} \subset \mathcal{C}(\mathcal{P})$, which means that there exist $\alpha^0, \alpha^1 \in E$ such that $\alpha^0 + t\alpha, \alpha^1 - t\alpha \in \mathcal{P}$ for sufficiently large $t > 0$. Since $\mathcal{P}$ is convex and closed, it contains the set $\{s\alpha^0 + (1-s)\alpha^1 + t\alpha \mid s \in [0,1], t \in \mathbb{R}\}$ and hence at least a line. □

DEFINITION A.6. The *dual* of a polyhedral cone $\mathcal{C}$ is the set $\mathcal{C}' = \{\xi \in E^* \mid \langle \xi, \mathcal{C}\rangle \geqslant 0\}$.

It is easy to see that $\mathcal{C}'$ is a polyhedral cone in E^* and if $\mathcal{C} = \bigcap_{i=1}^r K(\xi_i, 0)$, then $\mathcal{C}' = \{\sum_{i=1}^r s_i\xi_i \mid s_i \geqslant 0\}$. Moreover, a polyhedral set $\mathcal{P}$ is proper if and only if the interior of $\mathcal{C}(\mathcal{P})'$ is not empty.

LEMMA A.7. *For any $\xi \in E^*$, $\xi \in \mathcal{C}(\mathcal{P})'$ if and only if $\langle \xi, \mathcal{P}\rangle$ is bounded from below.*

Proof. Let $\mathcal{P} = \bigcap_{i=1}^r K(\xi_i, c_i)$. If $\xi \in \mathcal{C}(\mathcal{P})'$, then $\xi = \sum_{i=1}^r s_i\xi_i$ for some $s_i \geqslant 0$. So $\langle \xi, \mathcal{P}\rangle = \sum_{i=1}^r s_i\langle \xi_i, \mathcal{P}\rangle \geqslant \sum_{i=1}^r s_i c_i$. Conversely, if $\xi \notin \mathcal{C}(\mathcal{P})'$, then $\exists \alpha \in \mathcal{C}(\mathcal{P})$, $\langle \xi, \alpha\rangle < 0$. By Definition A.2, there exists α^0 such that $\alpha^0 + t\alpha \in \mathcal{P}$ for sufficiently large $t > 0$. But $\langle \xi, \alpha^0 + t\alpha\rangle = \langle \xi, \alpha^0\rangle + t\langle \xi, \alpha\rangle$ is not bounded from below. □

COROLLARY A.8. *For any $\xi \in E^*$, $\langle \xi, \mathcal{P}\rangle$ is compact if and only if $\langle \xi, \mathcal{C}(\mathcal{P})\rangle = 0$.*

Proof. $\langle \xi, \mathcal{C}(\mathcal{P})\rangle = 0$ is equivalent to $\{\pm\xi\} \subset \mathcal{C}(\mathcal{P})'$. Using Lemma A.7, this is equivalent to the statement that $\langle \xi, \mathcal{P}\rangle$ is bounded both from above and from below. □

COROLLARY A.9. *A polyhedral set $\mathcal{P}$ is compact if and only if $\mathcal{C}(\mathcal{P}) = \{0\}$.*

Proof. $\mathcal{P}$ is compact if and only if for any ξ, $\langle \xi, \mathcal{P}\rangle$ is bounded, or equivalently, $\langle \xi, \mathcal{C}(\mathcal{P})\rangle = 0$. □

LEMMA A.10. *For any $\xi \in E^*$, let $\pi_\xi = \langle \xi, \cdot\rangle : E \to \mathbb{R}$. Then $\pi_\xi|_{\mathcal{P}}$ is a proper map if and only if $\xi \in \pm\mathrm{Int}(\mathcal{C}(\mathcal{P})')$.*

Proof. The inverse image

$$(\pi_\xi|_{\mathcal{P}})^{-1}([a,b]) = \mathcal{P} \cap \pi_\xi^{-1}([a,b]) = \mathcal{P} \cap K(\xi, a) \cap K(-\xi, -b) \tag{A.2}$$

is a polyhedral set with asymptotic cone $\mathcal{C}_\xi = \mathcal{C}(\mathcal{P}) \cap K(\xi, 0) \cap K(-\xi, 0)$. π_ξ is proper if and only if $\mathcal{C}_\xi = \{0\}$, i.e., for any $\alpha \in \mathcal{C}(\mathcal{P})\backslash\{0\}$, $\langle \xi, \alpha\rangle \neq 0$. This is equivalent to $\langle \xi, \mathcal{C}(\mathcal{P})\backslash\{0\}\rangle > 0$ or < 0, i.e., $\xi \in \pm\mathrm{Int}(\mathcal{C}(\mathcal{P})')$. □

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