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# Simple constructions of algebraic curves with nodes

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In this paper we shall give a simple proof of the following result:

THEOREM. *There exists an integral non-degenerate (i.e. lying in no hyperplane) curve of degree $d \geqslant n$ in $\mathbb{P}_n$, with $\delta$ real nodes and no other singular point for all $\delta$ less than or equal to the Castelnuovo bound.*

Over the complex field, the case $n = 2$ was solved by Severi and the case $n \geqslant 3$ by Tannenbaum. Tannenbaum used deformation theory to generalize Severi's result (*c.f.* [$T_1$], [$T_2$]).

Our method is entirely different: we simplify the nodes of some very simple Lissajous's curves with many real nodes, according to the following "elementary" rule.

SIMPLIFICATION OF NODES. Let $\mathscr{L}$ be an affine plane curve of degree $d$ having only $k$ real nodes $Q_i$ in the affine plane. Let $E$ be a vector space of polynomials of degrees $\leqslant d$. If the conditions $P(Q_i) = 0$ are independent on $E$, then there is a curve $\mathscr{L} + G = 0$ with $G \in E$, having real nodes near $Q_1, \dots, Q_\delta$ ($\delta \leqslant k$) and no other singular points in the affine plane.

A very readable proof of this principle is in [BR] p. 270–273; it uses only the implicit function theorem.

Let us now define our Lissajous's curves. Let $T_h$ denote the Tchébycheff polynomial: $\cos(hu) = T_h(\cos u)$.

PROPOSITION 1. *If $a$ and $b$ are coprime integers, the affine curve parametrized by $x = T_b(t)$, $y = T_a(t)$ is an irreducible curve having $(a-1)(b-1)/2$ real nodes. Its equation is $T_a(x) = T_b(y)$.*

*Proof.* Easy (c.f. [P]). □

If we take $a = d$, $b = d - 1$, we get an irreducible curve of degree $d$ with $(d-1)(d-2)/2$ real nodes. As an introduction to our method, let us show how the result follows for $n = 2$.

COROLLARY. *For any $\delta \leqslant (d-1)(d-2)/2$ there exists an irreducible curve of degree $d$ with $\delta$ real nodes, and no other singular point in $\mathbb{P}_2(\mathbb{C})$.*

*Proof.* Let $\mathscr{L}$ be an irreducible curve of degree $d$ with $(d-1)(d-2)/2$ real nodes in the affine plane. Let $E$ be the set of real polynomials of degrees $\leqslant d-3$, and $F$ be the set of real functions defined on the nodes of $\mathscr{L}$. We have a linear mapping $E \to F$ between spaces of the same dimension. Let $P$ be in the kernel of this mapping. If $P$ is not the zero polynomial, the curves $P(x, y) = 0$ and $\mathscr{L}$ have at least $2((d-1)(d-2)/2)$ intersections, which is absurd by Bézout's theorem since $d(d-3) < (d-1)(d-2)$. Consequently the mapping $E \to F$ is an isomorphism, which means that the simplifications of the nodes are independent. We can then find a polynomial $G \in E$ such that the curve $\mathscr{L}(x, y) + G(x, y) = 0$ has $\delta$ nodes in the affine plane. Moreover, it has no singular point at infinity. □

For the general case our construction is based on the following:

PROPOSITION 2. *Let $a > e$ and $b$ be integers such that $(a-e, b) = 1$. There exists polynomials $A(t)$, $B(t)$, $E(t)$ of degrees $a, b$ and $e$, such that the curve $(B(t), A(t)/E(t))$ has $(b-1)(a+e-1)/2$ real nodes, and no other singular point in the affine plane.*

*Proof.* Let $t_1, \dots, t_e$ be such that the vertical lines $x = T_b(t_i)$ are distinct and each intersects the Lissajous's curve $(T_b(t), T_{a-e}(t))$ in $b$ real regular points. Then it is easy to see that the curve $(T_b(t), T_{a-e}(t) + \eta/(t-t_1)\cdots(t-t_e))$ has the required properties if $\eta$ is sufficiently small (c.f. [P]). □

We shall also need the fact that the equation of this curve is of degree $b$ in $y$ and $a$ in $x$.

We shall now give the proof of the theorem for $n \geqslant 3$.

First some notations. If $d \geqslant n \geqslant 3$ are integers:

$$d-1 = m(n-1)+\varepsilon \quad \text{with } 0 \leqslant \varepsilon < n-1$$

$$d-1 = (m+1)(n-k)+e \quad \text{with } 0 \leqslant e < m+1$$

The Castelnuovo bound $C(d, n)$ is:

$$C(d, n) = m((n-1)(m-1)+2\varepsilon)/2 = m((n-2k+1)(m+1)+2e)/2$$

We have $C(d, n) \geqslant 0$ and $C(d, n) = 0$ iff $d = n$. We also have from the last formula $\lambda = (n-2k+1) \geqslant -1$, if $\lambda = -1$ then $k = (n+2)/2$, $n$ is even and $n \geqslant 4$. Since $(m+1)(n-k+1) > d-1 \geqslant m(n-1)$ we get $m < n/(n-2) \leqslant 2$, and then $m = 1$. The last relation shows then that $C(d, n) = 0$, in which case the theorem is trivial. Thus we can always suppose that $\lambda \geqslant 0$.

Let $b = m+1$, $a = \lambda b + 1 + e = d-(k-1)(m+1)$, we have $a > e$. By Proposition 2 we can find an affine curve $\mathscr{L}$ parametrized by $x = B(t)$, $y = A(t)/E(t)$, where the degrees of the polynomials $B(t)$, $A(t)$, $E(t)$ are $b$, $a$ and $e$, having exactly $(b-1)(a+e-1)/2 = C(d, n)$ real nodes.

Consider the image of $\mathscr{L}$ in $\mathbb{R}^n \subset \mathbb{P}_n(\mathbb{C})$ by the mapping $\psi: \psi(x, y) = (x, \ldots, x^{n-k},\ y, yx, \ldots, yx^{k-1})$. The affine curve $\psi(\mathscr{L})$ is non-degenerate, of degree $d$ and has $C(d, n)$ real nodes. The places at infinity of $\psi(\mathscr{L})$ are of order 1 and do not intersect each other; thus $\psi(\mathscr{L})$ has no singular point at infinity.

Furthermore, the intersection of $\psi(\mathscr{L})$ with a hyperplane is given by substituting $y = P_{n-k}(x)/Q_{k-1}(x)$ in the equation of $\mathscr{L}$ which gives an equation of degree generically exactly $d$ in $x$.

Now we shall see that the simplifications of the nodes of $\psi(\mathscr{L})$ are independent. Let $E$ be the set of real polynomials generated by the monomials $x^\alpha y^\beta$ with:

$$\begin{cases} \beta < b-1 & (*) \\ \beta(a-e) + \alpha b + (b-2)e < (b-1)(a+e-1) & (**) \end{cases}$$

The dimension of $E$ is the number of solutions of this system. Let $h = b - \beta$; $(**)$ is equivalent to: $\alpha < e - 1 + \lambda(h-1) + h/b$, $2 \leqslant h \leqslant b$. For a given $h$, there are $e + \lambda(h-1)$ solutions, and finally:

$$\dim(E) = \sum_{h=2}^{b} (e + \lambda(h-1)) = (2e + \lambda b)(b-1)/2 = (a+e-1)(b-1)/2 = C(d, n)$$

Consider the linear mapping $E \to F$, where $F$ is the set of real functions defined on the nodes of $\mathscr{L}$. Let $P$ be in the kernel of this mapping, and define $p(t)$ by: $p(t) = (E(t))^{b-2} P(B(t), A(t)/E(t))$. By $(*)$ $p(t)$ is a polynomial, by $(**)$ its degree is $< 2C(d, n)$. But this polynomial has $2C(d, n)$ distinct roots, which are the values of the parameter corresponding to the nodes of $\mathscr{L}$. Therefore $P(x, y)$ is zero on the whole irreducible curve $\mathscr{L}$, and $P(x, y) = K(x, y)\mathscr{L}(x, y)$. If we look at the degrees in $y$, we see that $P$ must be the zero polynomial. Our linear mapping is therefore an isomorphism, and the simplifications of the nodes of $\mathscr{L}$ are independent. So, we can find a polynomial $G \in E$ such that the affine curve $\mathscr{C}(x, y) = \mathscr{L}(x, y) + G(x, y) = 0$ has $\delta$ real nodes, $\psi(\mathscr{C})$ has also $\delta$ nodes. Moreover, we see that $\mathscr{L}$ and $\mathscr{C}$ have the same point at infinity and the same tangents at this point: the infinity line and the asymptotes $x = T_b(t_i)$. Therefore the places at infinity of $\psi(\mathscr{C})$ as well as those of $\psi(\mathscr{L})$ are of order one and do not intersect each other. Thus $\psi(\mathscr{C})$ does not have a singular point at infinity.

Let us determine the degree of the space curve $\psi(\mathscr{C})$. We find the intersection with a hyperplane by performing the substitution $y = P_{n-k}(x)/Q_{k-1}(x)$ in the equation $\mathscr{C}(x, y) = 0$. After reduction to the common denominator $(Q_{k-1}(x))^b$, the monomials of $\mathscr{L}$ given an equation of degree generically exactly $d$ in $x$, and the monomials of $G(x, y)$ give polynomials in $x$ of degrees $\alpha + \lambda\beta + b(k-1)$ which is $< d$ (by $(**)$).

This finishes the proof of the theorem. $\square$

REMARKS. The geometric genus of this space curve is $C(d,n)-\delta$. So, we get space curves of arbitrary geometric genus $g \leqslant C(d,n)$. In particular, if $G$ is taken to be a constant polynomial we obtain *constructions* of irreducible smooth curves of degree $d$ and maximal genus $C(d,n)$. We can also get simpler equations for such curves. Let $\mathscr{Y}$ be the plane curve: $(x-x_1)\cdots(x-x_e)(T_{a-e}(x)-T_b(y))=\eta$, then the space curve $\psi(\mathscr{Y})$ is smooth of degree $d$, and genus $C(d,n)$.

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