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On the mean square value of Dirichlet's L-functions*

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Abstract. The main purpose of this paper is to give a sharper asymptotic formula for the mean square value

$$\sum_{\chi \bmod q} L(\sigma + it, \chi)L(1 - \sigma - it, \bar{\chi})$$

where $0 < \sigma < 1$. This will be derived from the functional equation of Hurwitz's zeta-function and the analytic methods.

1. Introduction

For integer $q > 2$, let χ denote a typical Dirichlet character mod q, and $L(s, \chi)$ be the corresponding Dirichlet L-function. We define the function $T(q, s)$ as follows:

$$T(q, s) = \sum_{\chi \bmod q} L(s, \chi)L(1 - s, \bar{\chi})$$

where the summation is over all Dirichlet characters mod q, and $s = \sigma + it$, $0 < \sigma < 1$.

The main purpose of this paper is to study the asymptotic property of $T(q, s)$. We know very little at present about this problem. Although D. R. Heath-Brown [1] first introduced the function $T(q, s)$, he obtained an asymptotic series only for $T(q, 1/2)$. Enlightened by the idea in [2], this paper, using the functional equation of Hurwitz's zeta-function and the analytic method, studies the asymptotic property of $T(q, s)$ for all $0 < \sigma < 1$ and proves the following three theorems:

THEOREM 1. *Let integer $q > 2$ and real $t > 3$, $0 < \sigma < 1$, $c(\sigma) = \mathrm{Max}(\sigma, 1 - \sigma)$, $s = \sigma + it$, then we have*

$$T(q, s) = \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)} + \sum_{p/q}\frac{\ln p}{p - 1} + \frac{1}{2}\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \frac{\Gamma(s/2)}{\Gamma(s/2)}\right]$$
$$+ 0\left[(qt)^{c(\sigma)}\exp\left(\frac{\ln(qt)}{\ln\ln(qt)}\right)\right]$$

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where γ is the Euler constant, $\sum_{p|q}$ denote the summation over all distinct prime divisors of q, $\Gamma(s)$ is Gamma function and $\exp(y) = e^y$.

THEOREM 2. *Let* $\bmod q > 2$, *then we have asymptotic formula*

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2}, \chi)|^2 = \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{8\pi}\right) + \gamma + \sum_{p|q}\frac{\ln p}{p-1}\right] + 0\left[q^{1/2}\exp\left(\frac{\ln q}{\ln\ln q}\right)\right]$$

THEOREM 3. *The asymptotic formula*

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2} + it, \chi)|^2 = \frac{\phi^2(q)}{q}\left[\ln\left(\frac{qt}{2\pi}\right) + 2\gamma + \sum_{p|q}\frac{\ln p}{p-1}\right] + 0(qt^{-1}) + 0\left[(qt)^{1/2}\exp\left(\frac{\ln(qt)}{\ln\ln(qt)}\right)\right]$$

holds for all $\bmod q$ *and real* $t > 2$.

From the theorems we may immediately deduce the following:

COROLLARY 1. *Let* $0 < \sigma < 1$, $s = \sigma + it$, $c = \mathrm{Min}\left(\frac{\sigma}{1-\sigma}, \frac{1-\sigma}{\sigma}\right)$, *if* $1 < |t| < q^{c-\varepsilon}$, *then we have*

$$T(q, s) \sim \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)} + \sum_{p/q}\frac{\ln p}{p-1} + \frac{1}{2}\,\frac{\Gamma'\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right]$$

where ε *is any fixed positive number.*

COROLLARY 2. *If* $|t| < q^{1-\varepsilon}$, *then*

$$\sum_{\chi \bmod q} |L(\tfrac{1}{2} + it, \chi)|^2 \sim \frac{\phi^2(q)}{q}\ln|qt| \quad \text{for all } t > 3.$$

Corollary 2 is an improvement of result of Balasubramanian [5], who gave the asymptotic formula in the range $|t| < q^{3/4-\varepsilon}$.

2. Some lemmas

In this section, we shall give some basic lemmas which are necessary in the course of proving the theorems.

LEMMA 1. *Let integer $q > 2$, then for any $0 < \sigma < 1$ and $s = \sigma + it$ we have*

$$T(q, s) = \frac{\phi(q)}{q} \sum_{d|q} \mu(d) \sum_{n=1}^{q/d} \zeta\left(s, \frac{n}{q/d}\right) \zeta\left(1 - s, \frac{n}{q/d}\right)$$

where $\zeta(s, \alpha)$ is Hurwitz zeta-function, $\phi(q)$ is Euler function and $\mu(n)$ is Möbius function.

Proof. From the orthogonality of Dirichlet characters and

$$L(s, \chi) = \frac{1}{q^s} \sum_{1 \leqslant a \leqslant q} \chi(a)\zeta(s, a/q)$$

we may get

$$\begin{aligned} T(q, s) &= \frac{1}{q}\sum_{\chi_q} \left(\sum_{a=1}^{q} \chi(a)\zeta\left(s, \frac{a}{q}\right)\right)\left(\sum_{a=1}^{q} \bar{\chi}(a)\zeta\left(1 - s, \frac{a}{q}\right)\right) \\ &= \frac{\phi(q)}{q} \sum_{1 \leqslant a \leqslant q, (a,q)=1} \zeta\left(s, \frac{a}{q}\right)\zeta\left(1 - s, \frac{a}{q}\right) \\ &= \frac{\phi(q)}{q} \sum_{d/q} \mu(d) \sum_{n=1}^{q/d} \zeta\left(s, \frac{n}{q/d}\right)\zeta\left(1 - s, \frac{n}{q/d}\right) \end{aligned}$$

□

Let

$$F(w, s, k) = \frac{\Gamma\left(\frac{1 - s + w}{2}\right)\Gamma\left(\frac{s + w}{2}\right)}{(\pi k)^w} \cdot \frac{\sin(\pi s) + \sin(\pi w)}{2} \times \sum_{h=1}^{k} \zeta\left(s + w, \frac{h}{k}\right) \zeta\left(1 - s + w, \frac{h}{k}\right).$$

We then have the following.

LEMMA 2. *If integer $k > 2$, $\mathrm{Re}(w) \geqslant 1$, then*

$$\begin{aligned} F(-w, s, k) = F(w, s, k) &- \frac{\Gamma\left(\frac{1 - s + w}{2}\right)\Gamma\left(\frac{s + w}{2}\right)}{(\pi k)^w} \times \frac{\sin(\pi w)}{2} \\ &\times \left[\sum_{h=1}^{k} \zeta\left(s + w, \frac{h}{k}\right)\zeta\left(1 - s + w, \frac{h}{k}\right) + \zeta(s + w)\zeta(1 - s + w)\right. \\ &\left. + \sum_{\gamma=1}^{k-1} \zeta\left(s + w, \frac{\gamma}{k}\right) \zeta\left(1 - s + w, \frac{k - \gamma}{k}\right)\right] \end{aligned}$$

Proof. From the functional equation of Hurwitz zeta-function (See [3], theorem 12.8) and the property of Gamma function we know that

$$\zeta\left(1-s,\frac{h}{k}\right)\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\sin\frac{\pi}{2}(1-s)$$
$$=\frac{\Gamma(s/2)\pi^{-(s/2)}}{2k^s}\sum_{\gamma=1}^{k}\left(\exp\left(\frac{2\pi i\gamma h}{k}-\frac{\pi i s}{2}\right)+\exp\left(\frac{\pi i s}{2}-\frac{2\pi i\gamma h}{k}\right)\right)\zeta\left(s,\frac{\gamma}{k}\right)$$

holds for all integers $1 \leqslant h \leqslant k$.

From above and notice that

$$\sin\frac{\pi}{2}(1-s+w)\sin\frac{\pi}{2}(s+w)=\frac{\sin(\pi s)+sin(\pi w)}{2},$$

$$\sum_{\gamma=1}^{k}\exp(2\pi i n\gamma/k)=\begin{cases}k & \text{if } k/n\\ 0 & \text{if } k\nmid n.\end{cases}$$

we may immediately get

$$F(-w,s,k)=\sum_{h=1}^{k}\frac{\Gamma\left(\frac{1-s-w}{2}\right)\Gamma\left(\frac{s-w}{2}\right)\pi^{1/2}}{k^{-w}\pi^{\frac{1-s-w}{2}}\pi^{\frac{s-w}{2}}}\times\frac{\sin(\pi s)-\sin(\pi w)}{2}$$

$$\times\zeta\left(1-s-w,\frac{h}{k}\right)\zeta\left(s-w,\frac{h}{k}\right)$$

$$=\frac{\Gamma\left(\frac{1-s+w}{2}\right)\Gamma\left(\frac{s+w}{2}\right)}{4k(\pi k)^w}\sum_{\gamma=1}^{k}\sum_{\gamma_1=1}^{k}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{\gamma_1}{k}\right)$$
$$\times\sum_{h=1}^{k}\left(\exp\left(-\frac{\pi i}{2}(s+w)+\frac{2\pi i\gamma h}{k}\right)+\exp\left(\frac{\pi i}{2}(s+w)-\frac{2\pi i\gamma h}{k}\right)\right)$$
$$\times\left(\exp\left(-\frac{\pi i}{2}(1-s+w)+\frac{2\pi i\gamma_1 h}{k}\right)+\exp\left(\frac{\pi i}{2}(1-s+w)-\frac{2\pi i\gamma_1 h}{k}\right)\right)$$
$$=\frac{\Gamma\left(\frac{1-s+w}{2}\right)\Gamma\left(\frac{s+w}{2}\right)}{(\pi k)^w}$$
$$\times\left[\frac{\sin(\pi s)}{2}\sum_{\gamma=1}^{k}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{\gamma}{k}\right)\right.$$
$$-\frac{\sin(\pi w)}{2}\left(\zeta(s+w)\zeta(1-s+w)+\sum_{\gamma=1}^{k-1}\zeta\left(s+w,\frac{\gamma}{k}\right)\right.$$

$$\times\zeta\left(1-s+w,\frac{k-\gamma}{k}\right)\Bigg)\Bigg]$$

$$= F(w,s,k) - \frac{\Gamma\left(\frac{1-s+w}{2}\right)\Gamma\left(\frac{s+w}{2}\right)}{(k\pi)^w}\cdot\frac{\sin(\pi w)}{2}$$

$$\times\left[\sum_{\gamma=1}^{k}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{\gamma}{k}\right)+\zeta(s+w)\zeta(1-s+w)\right.$$

$$\left.+\sum_{\gamma=1}^{k-1}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{k-\gamma}{k}\right)\right]$$

This completes the proof of the lemma 2. □

LEMMA 3. *Let integer* $k>2$, $s=\sigma+it$, $0<\sigma<1$, *then*

$$\sum_{\gamma=1}^{k}\zeta\left(s,\frac{\gamma}{k}\right)\zeta\left(1-s,\frac{\gamma}{k}\right)=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{g(s,w)}{g(s,0)}\left(2+\frac{\sin(\pi w)}{\sin(\pi s)}\right)$$

$$\times\sum_{\gamma=1}^{k}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{\gamma}{k}\right)\frac{e^{w^2}}{w}\,dw$$

$$-\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{g(s,w)}{g(s,0)}\frac{\sin(\pi w)}{\sin(\pi s)}\left(\sum_{\gamma=1}^{k-1}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{k-\gamma}{k}\right)\right.$$

$$\left.+\zeta(s+w)\zeta(1-s+w)\right)\frac{e^{w^2}}{w}\,dw+0\left(\frac{k^\sigma+k^{1-\sigma}}{|s|}\right)$$

where

$$g(s,w)=\Gamma\left(\frac{1-s+w}{2}\right)\Gamma\left(\frac{s+w}{2}\right)\Big/(\pi k)^w$$

Proof. Let

$$I=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}F(w,s,k)\frac{e^{w^2}}{w}\,dw,$$

moving the line of integration in I to $\mathrm{Re}(w)=-1$, in this time the integrand has one order pole at the point $w=0$, s and $1-s$ with the residues $F(0,s,k)$ and

$$\frac{\Gamma(\frac12)\Gamma(s)}{(\pi k)^s}\sin(\pi s)\frac{e^{s^2}}{s}\sum_{\gamma=1}^{k}\zeta\left(2s,\frac{\gamma}{k}\right)\ll\frac{k^\sigma}{|s|},$$

$$\frac{\Gamma(\frac12)\Gamma(1-s)}{(\pi k)^{1-s}}\sin(\pi(1-s))\frac{e^{(1-s)^2}}{1-s}\sum_{\gamma=1}^{k}\zeta\left(2-2s.\frac{\gamma}{k}\right)\ll\frac{k^{1-\sigma}}{|s|}.$$

Thus from lemma 2 and the above we may get

$$I = F(0, s, k) - \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} F(-w, s, k)\frac{e^{w^2}}{w}\, dw + 0\left(\frac{k^\sigma + k^{1-\sigma}}{|s|}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)\sin(\pi s)\sum_{\gamma=1}^{k}\zeta\left(s, \frac{\gamma}{k}\right)\zeta\left(1-s, \frac{\gamma}{k}\right)$$

$$- \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} F(w, s, k)\frac{e^{w^2}}{w}\, dw + \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{g(s, w)}{2}\sin(\pi w)$$

$$\times \sum_{\gamma=1}^{k}\zeta\left(s+w, \frac{\gamma}{k}\right)\zeta\left(1-s+w, \frac{\gamma}{k}\right)\frac{e^{w^2}}{w}\, dw$$

$$+ \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{g(s, w)\sin(\pi w)}{2}\left[\sum_{\gamma=1}^{k-1}\zeta\left(s+w, \frac{\gamma}{k}\right)\zeta\left(1-s+w, \frac{k-\gamma}{k}\right)\right.$$

$$\left. + \zeta(s+w)\zeta(1-s+w)\right]\frac{e^{w^2}}{w}\, dw + 0\left(\frac{k^\sigma + k^{1-\sigma}}{|s|}\right)$$

by the definition of I and $F(w, s, k)$, and the above we immediately deduce lemma 3. □

LEMMA 4. *For real number $t > 3$ and $x > 0$, we have*

$$w(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} x^{-w}\frac{g(s, w)}{g(s, 0)}\left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right)\frac{e^{w^2}}{w}\, dw$$

$$\ll \begin{cases} t/kx, & \text{if } x \geqslant (t/2\pi k), \\ 1, & \text{if } x < (t/2\pi k). \end{cases}$$

Proof. From the Stirling Formula we know that

$$|\Gamma(\beta + it)| = |t|^{\beta - 1/2} e^{-(\pi/2)|t|}\sqrt{2\pi}\left\{1 + 0\left(\frac{1}{t}\right)\right\}, \quad t \to \infty. \tag{1}$$

For $w = \gamma + iy$, by (1) we may estimate

$$\left|\frac{g(s, w)}{g(s, 0)}\right| \ll (|t| + |y|)^\gamma e^{\pi|y|} k^{-\gamma} \tag{2}$$

If $x \geqslant t/(2\pi k)$, then by (2) we may get trivial estimate

$$w(x) \ll \int_{-\infty}^{+\infty} (xk)^{-1}(|t| + |y|) e^{\pi|y|}\left(2 + \frac{e^{\pi|y|}}{e^{\pi|t|}}\right)\frac{e^{-y^2}}{|y| + 1}\, dy$$

$$\ll t/kx \int_0^{+\infty} e^{2\pi y - y^2}\, dy \ll t/kx.$$

If $x < t/(2\pi k)$, then we move the line of integration to

$$\mathrm{Re}(w) = -\min\left(\frac{\sigma}{2}, \frac{1-\sigma}{2}\right),$$

in this time the integrand has one order pole at the point $w = 0$ with the residues 2, from this and (2) we can deduce that

$$w(x) \ll 1 + \left(\frac{t}{kx}\right)^{-\min[\sigma/2),(1-\sigma)/2)]} \int_{-\infty}^{+\infty} e^{2\pi|y| - y^2}\,dy$$
$$\ll 1 + (t/kx)^{-\min[(\sigma/2),(1-\sigma)/2)]} \ll 1.$$

Combining above two cases we immediately deduce the lemma 4. □

LEMMA 5. For integer k and real $t > 2$, we have

$$\bar{M}_1 \equiv \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} k^{1+2w}\frac{g(s,w)}{g(s,0)}\left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right)\zeta(1+2w)\frac{e^{w^2}}{w}\,dw$$
$$= k\left[\ln\left(\frac{k}{\pi}\right) + \frac{1}{2}\left(\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right) + \frac{\pi}{2\sin(\pi s)} + 2\gamma\right]$$
$$+ O(k^{\sigma}) + O(k^{1-\sigma})$$

Proof. Moving the line of integration in $\bar{M}_1$ to $\mathrm{Re}(w) = -1$, this time the integrand has two order poles at point $w = 0$ and the one order pole at the points $w = -s$ and $w = -(1-s)$ with the residues:

$$\operatorname*{Res}_{w=0}\left[k^{1+2w}\frac{g(s,w)}{g(s,0)}\left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right)w\zeta(1+2w)e^{w^2}\right]'$$
$$= k\left[\ln\left(\frac{k}{\pi}\right) + \frac{1}{2}\left(\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)}\right]$$

and the residues:

$$2k^{1-2s}(k\pi)^s\Gamma(\tfrac{1}{2} - s)\zeta(1-2s)e^{s^2}/(-s\cdot g(s,0)) \ll k^{1-\sigma},$$
$$2\cdot k^{1-2(1-s)}(k\pi)^{1-s}\Gamma(s - \tfrac{1}{2})e^{(1-s)^2}\zeta(2s-1)/(-(1-s)\cdot g(s,0))$$
$$\ll k^{\sigma}.$$

For $w = -1 + iy$, $|y| < t/2$, from the estimate (2) we may get

$$\left| k^{1+2w} \frac{g(s, w)}{g(s, 0)} \left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right) \zeta(1 + 2w) \frac{e^{w^2}}{w} \right|$$
$$\ll k^{-1} \cdot \left(\frac{k}{t}\right) \cdot |\zeta(-1 + 2iy)| \cdot (1 + |y|)^{-1} e^{\pi|y| - y^2}$$
$$\ll e^{2\pi|y| - y^2}. \tag{3}$$

It is clear that the estimate (3) also holds for $|y| > t/2$. From the residues and estimates (3), and notice that

$$\int_{-\infty}^{+\infty} e^{2\pi|y| - y^2} \mathrm{d}y \ll 1$$

we may immediately deduce lemma 5. □

LEMMA 6. *For any fixed* $0 < \sigma < 1$ *and real number* $t > 2$, *we have the asymptotic formula*

$$\frac{\Gamma'(\sigma + it)}{\Gamma(\sigma + it)} = \ln t + \frac{\pi}{2} i + 0\left(\frac{1}{t}\right)$$

Proof. (See [4], Lemma 3). □

LEMMA 7. *For integer k and real number* $t > 2$, *let* $0 < \sigma < 1$, $s = \sigma + it$, *then we have the asymptotic formula*

$$\sum_{\gamma=1}^{k} \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1 - s, \frac{\gamma}{k}\right) = k \left[\ln\left(\frac{k}{\pi}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)} + \frac{1}{2}\left(\frac{\Gamma'((1 - s)/2)}{\Gamma((1 - s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right)\right] + 0((kt)^{c(\sigma)} \ln t)$$

where $c(\sigma) = \max(\sigma, 1 - \sigma)$.

Proof. From the definition of $W(x)$ and $\zeta(s, \alpha)$, and apply lemma 3 we may get

$$\sum_{\gamma=1}^{k} \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1 - s, \frac{\gamma}{k}\right) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{g(s, w)}{g(s, 0)} \left(2 + \frac{\sin(\pi w)}{\sin(\pi s)}\right) \times \sum_{\gamma=1}^{k} \zeta\left(s + w, \frac{\gamma}{k}\right) \zeta\left(1 - s + w, \frac{\gamma}{k}\right) \frac{e^{w^2}}{w} \mathrm{d}w$$

$$-\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{g(s,w)}{g(s,0)}\frac{\sin(\pi w)}{\sin(\pi s)}\Bigg[\zeta(s+w)\zeta(1-s+w)+$$

$$+\sum_{\gamma=1}^{k-1}\zeta\left(s+w,\frac{\gamma}{k}\right)\zeta\left(1-s+w,\frac{k-\gamma}{k}\right)\Bigg]\frac{e^{w^2}}{w}\,dw$$

$$+0(k^{\sigma})+0(k^{1-\sigma})$$

$$\equiv A(k,s)-B(k,s)+0(k^{c(\sigma)}) \qquad (4)$$

Now we estimate $A(k, s)$ and $B(k, s)$ respectively, we have

$$A(k,s)=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}k^{1+2w}\frac{g(s,w)}{g(s,0)}\left(2+\frac{\sin(\pi w)}{\sin(\pi s)}\right)\zeta(1+2w)\frac{e^{w^2}}{w}\,dw$$

$$+\sum_{\substack{m=0\\ m\neq n}}^{\infty}\sum_{n=0}^{\infty}\sum_{\gamma=1}^{k}\frac{1}{\left(m+\frac{\gamma}{k}\right)^{1-s}\left(n+\frac{\gamma}{k}\right)^{s}}w\left(\left(n+\frac{\gamma}{k}\right)\left(m+\frac{\gamma}{k}\right)\right)$$

$$\equiv M_1+M_2 \qquad (5)$$

Let $c(\sigma)=\max(\sigma,1-\sigma)$, by lemma 4 we may get

$$M_2\ll\sum_{\substack{m=0\\ m\neq n}}^{\infty}\sum_{n=0}^{\infty}\sum_{\gamma=1}^{k}\frac{1}{\left(n+\frac{\gamma}{k}\right)^{\sigma}\left(m+\frac{\gamma}{k}\right)^{1-\sigma}}\min\left(1,\frac{t}{k\left(n+\frac{\gamma}{k}\right)\left(m+\frac{\gamma}{k}\right)}\right)$$

$$\ll\sum_{n=1}^{\infty}\sum_{\gamma=1}^{k}\left[\frac{k^{1-\sigma}}{n^{\sigma}\gamma^{1-\sigma}}+\frac{k^{\sigma}}{n^{1-\sigma}\gamma^{\sigma}}\right]\times\min\left(1,\frac{t}{\gamma n}\right)$$

$$+k\sum_{\substack{m=1\\ m\neq n}}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{\sigma}m^{1-\sigma}}\min\left(1,\frac{t}{kmn}\right)$$

$$\ll\sum_{\gamma n\leqslant t}\left(\frac{k^{1-\sigma}}{n^{\sigma}\gamma^{1-\sigma}}+\frac{k^{\sigma}}{n^{1-\sigma}\gamma^{\sigma}}\right)+\sum_{\substack{mn\leqslant t/k\\ m\neq n}}\frac{k}{m^{1-\sigma}n^{\sigma}}$$

$$+\sum_{\substack{n=1\\ \gamma n>t}}^{\infty}\sum_{\gamma=1}^{k}\left(\frac{k^{1-\sigma}}{n^{\sigma}\gamma^{1-\sigma}}+\frac{k^{\sigma}}{n^{1-\sigma}\gamma^{\sigma}}\right)\frac{t}{\gamma n}$$

$$+k\sum_{\substack{mn>t/k\\ m\neq n}}\frac{t}{m^{2-\sigma}n^{1+\sigma}k}\ll(kt)^{c(\sigma)}\ln t \qquad (6)$$

For Re$(w) = 1$, we have trivial estimate

$$\sum_{\gamma=1}^{k-1} \zeta\left(s + w, \frac{k-\gamma}{k}\right) \zeta\left(1 - s + w, \frac{\gamma}{k}\right)$$
$$\ll \sum_{\gamma=1}^{k-1} \left(\frac{k}{k-\gamma}\right)^{1+\sigma} \left(\frac{k}{\gamma}\right)^{1-\sigma+1}$$
$$\ll k^{1+c(\sigma)} \tag{7}$$

from (2), (7) and the definition of $B(k, s)$ we get

$$B(k, s) \ll \int_{-\infty}^{\infty} \frac{t}{k} \frac{e^{\pi|y|}}{e^{\pi|t|}} k^{1+c(\sigma)} \frac{e^{-y^2}}{1+|y|} \, dy \ll k^{c(\sigma)} \tag{8}$$

Combining (4), (5), (6), (8) and lemma 5 we may obtain

$$\sum_{\gamma=1}^{k} \zeta\left(s, \frac{\gamma}{k}\right) \zeta\left(1 - s, \frac{\gamma}{k}\right)$$
$$= k\left[\ln\left(\frac{k}{\pi}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)} + \frac{1}{2}\left(\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right)\right]$$
$$+ 0((qt)^{c(\sigma)} \ln t)$$

This completes the proof of the lemma 7. □

3. Proof of the theorems

In this section, we shall give the proof of the theorems. First we prove theorem 1; by lemma 1 and lemma 7 we may get

$$T(q, s) = \frac{\phi(q)}{q} \sum_{d/q} \mu(d) \sum_{n=1}^{q/d} \zeta\left(s, \frac{n}{q/d}\right) \zeta\left(1 - s, \frac{n}{q/d}\right)$$
$$= \frac{\varphi(q)}{q} \sum_{d/q} \mu(d) \left\{\frac{q}{d}\left[\ln\left(\frac{q}{\pi d}\right) + \frac{\pi}{2\sin(\pi s)} + 2\gamma\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right)\right] + 0\left(\left(\frac{qt}{d}\right)^{c(\delta)} \ln t\right)\right\}$$
$$= \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)} + \frac{1}{2}\left(\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)}\right)\right]$$
$$- \frac{\phi(q)}{q} q \sum_{p/q} \frac{\mu(d)\ln d}{d} + 0\left[(qt)^{c(\sigma)} \ln t \sum_{d/q} |\mu(d)|\right]$$

Notice that

$$\sum_{d/q} \frac{\mu(d)\ln d}{d} = -\frac{\phi(q)}{q} \sum_{p/q} \frac{\ln p}{p-1}, \quad \sum_{d/q} |\mu(d)| \ll \exp\left(\frac{\ln q}{\ln\ln q}\right)$$

From (9) we may immediately obtain

$$T(q, s) = \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2\sin(\pi s)} + \sum_{p/q} \frac{\ln p}{p-1} + \frac{1}{2}\left(\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} + \frac{\Gamma'(s/2)}{\Gamma(s/2)}\right)\right] + 0\left[(qt)^{c(\sigma)} \exp\left(\frac{\ln(qt)}{\ln\ln(qt)}\right)\right]$$

This completes the proof of the theorem 1.

Notice that $L(\frac{1}{2} + it, \chi)L(\frac{1}{2} - it, \bar{\chi}) = |L(\frac{1}{2} + it, \chi)|^2$, from theorem 1 and lemma 6 we can easily deduce theorem 3.

From the properties of Gamma function we may get

$$\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \frac{\Gamma'(\frac{3}{4})}{\Gamma(\frac{3}{4})} = -\pi \quad \text{and} \quad \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} + \frac{\Gamma'(\frac{3}{4})}{\Gamma(\frac{3}{4})} = -2\gamma - 2\ln 8$$

Applying the method of proving theorem 1 and above we can deduce

$$\begin{aligned}\sum_{\chi \bmod q} |L(\tfrac{1}{2}, \chi)|^2 &= \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{\pi}\right) + 2\gamma + \frac{\pi}{2} + \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} + \sum_{p/q} \frac{\ln p}{p-1}\right] \\ &\quad + 0\left[q^{1/2} \exp\left(\frac{\ln q}{\ln\ln q}\right)\right] \\ &= \frac{\phi^2(q)}{q}\left[\ln\left(\frac{q}{8\pi}\right) + \gamma + \sum_{p/q} \frac{\ln p}{p-1}\right] + 0\left[q^{1/2} \exp\left(\frac{\ln q}{\ln\ln q}\right)\right]\end{aligned}$$

This completes the proof.

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