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On Castelnuovo's regularity and Hilbert functions

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Abstract. New bounds for Castelnuovo's regularity are established. As a consequence, we show that a property of Hilbert functions stated by J. Harris and D. Eisenbud in [7], p. 82 is only true for curves and false for higher-dimensional subschemes. The letter of W. Vogel [25] gives rise to study this property again.

1. Introduction

Castelnuovo's regularity was first defined by D. Mumford [10], who attributes the idea to G. Castelnuovo, for coherent sheaves on projective spaces. In a more algebraic setting it was defined by D. Eisenbud and S. Goto [3] and A. Ooishi [13] (see Section 2). It comes out that Castelnuovo's regularity gives an upper bound for the maximal degrees of the syzygies in a minimal free resolution [3]. D. Bayer and M. Stillman [2] showed that an estimate of the regularity of an ideal gives a bound on the complexity of algorithms for computing syzygies. In [3], p. 93 D. Eisenbud and S. Goto stated the following well-known conjecture on such an estimation.

Let $X \subseteq \mathbb{P}^n_K$ (K an algebraically closed field) be a nondegenerate, that is, X is not contained in a hyperplane of $\mathbb{P}^n$, irreducible, reduced subvariety then holds

$$\operatorname{reg}(X) \leqslant \operatorname{degree}(X) - \operatorname{codim}(X) + 1.$$

So far, this conjecture has been proved for curves [6] and, if $\operatorname{char}(K) = 0$, for smooth surfaces [9] (see also [5]), for a large class of smooth threefolds in $\mathbb{P}^5_K$ [18], Theorem 3.3, and if X is arithmetically Buchsbaum or $\operatorname{degree}(X) \leqslant \operatorname{codim}(X) + 2$ [21]. In the other cases only weaker results are known by using certain correction terms (see, e.g., [1], [22], [23], [5], [11], [18]). We also note that [11] and [12] describe applications of Castelnuovo's regularity. The aim of this paper is to describe a new approach for providing Castelnuovo bounds as presented in [23]. This provides new bounds for Castelnuovo's regularity which improve bounds of [23] in some cases. Moreover, we prove special cases of the conjectures of D. Eisenbud and S. Goto and of D. Bayer and M. Stillman [2] (see Theorem 2). In Section 4 we will apply these results in order to show that an assertion of J. Harris and D. Eisenbud [7], p. 82 on the equality of the abstract

Hilbert function and the Hilbert polynomial is not true in general (see Theorem 3). Providing our Theorem 3 we construct our counterexamples.

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2. Notations and preliminary results

We work over an algebraically closed field K. Let $S = K[x_0, \ldots, x_n]$ be a polynomial ring and $\mathfrak{a} \subseteq S$ be a homogeneous ideal. We set $A = S/\mathfrak{a}$., that is, A is a graded K-algebra. We denote by $P_A := \oplus_{n>0} A_n$ the irrelevant ideal of A. When there is no possibility of confusion we will denote P_A simply by P. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded A-module. The i-th local cohomology module of M with support in P, denoted by $H^i_P(M)$, is also a graded A-module. Let $[M]_i$ denote the i-th graded part of M for $i \in \mathbb{Z}$, i.e. $[M]_i = M_i$. Let j be an integer then let $M(j)$ denote the graded A-module whose underlying module is the same as that of M and whose grading is given by $[M(j)]_i = [M]_{i+j}$ for all $i \in \mathbb{Z}$. We set for an arbitrary A-module M: $e(M) := \sup\{t \in \mathbb{Z}: [M]_t \neq 0\}$

$$\operatorname{sgn}(M_n) := \begin{cases} 1, & \text{if } M_n \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$r_k(M) := \sup\{i + e(H^i_P(M)): i \geqslant k\}.$$

For a finitely generated graded A-module M we define Castelnuovo's regularity, denoted by $\operatorname{reg}(M)$, by $\operatorname{reg}(M) := r_0(M) = r_{\operatorname{depth}(M)}(M)$.

Let $\{x_1, \ldots, x_m\}$ be a part of a system of parameters for M. It is said to be a filter-regular sequence if

$$e((x_1, \ldots, x_{i-1})M : {}_M x_i/(x_1, \ldots, x_{i-1})M) < \infty \quad \text{for } i = 1, \ldots, m$$

(see, e.g., [20], appendix for further informations). We set $M^i = M/(x_1, \ldots, x_i)M$ $(i = 0, \ldots, m)$ for a filter-regular sequence $\{x_1, \ldots, x_m\}$ for M. We have the following result.

LEMMA 1. *Let M be a Noetherian graded A-module of dimension $d > 0$. Then there is a filter-regular sequence $\{l_1, \ldots, l_d\}$ of forms $\in A_1$.*

Proof. It is sufficient to show that there is a filter-regular element $l \in A_1$ for M.

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of prime ideals $\mathfrak{p}$ of $\mathrm{Ass}_A(M)$ with Krull-dim$(A/\mathfrak{p}) \geqslant d$. Using, for example [16], Theorem 2.3 we can find an element $l \in A_1 \setminus (P \cdot A_1 \cup \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s)$ since $P \not\subset \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_s$. It follows from [20], Theorem 7 of the appendix that l is a filter-regular element for M. Q.E.D.

Let M be a Noetherian graded A-module of dimension $d \geqslant 1$. We denote by $h_0(M)$ the $(d-1)!$-fold of the leading coefficient of the Hilbert polynomial $p_M(t)$. We recall that $p_M(t) = \mathrm{rank}_K[M]_t$ for all $t \gg 0$. If $M = \mathrm{A} = S/\mathfrak{a}$ we define

$$\mathrm{degree}(\mathfrak{a}) := \begin{cases} h_0(S/\mathfrak{a}), & \text{if Krull-dim}(S/\mathfrak{a}) > 0 \\ \text{length}(S/\mathfrak{a}), & \text{if Krull-dim}(S/\mathfrak{a}) = 0. \end{cases}$$

Further, we set $\mathfrak{a}\colon \langle P \rangle = \{x \in S\colon \text{there is an integer } m \geqslant 0 \text{ with } P^m \cdot x \subseteq \mathfrak{a}\}$.

Let X be a subscheme of P^n_K. Then we denote by $I(X)$ the defining ideal of X in $S = K[x_0, \ldots, x_n]$. If $\mathfrak{a} \subseteq S$ is a homogeneous ideal let $V(\mathfrak{a})$ be the corresponding subscheme of $\mathbb{P}^n_K$. The ideal $\mathfrak{a}$ is said to be regular if $V(\mathfrak{a})$ is smooth. Note that degree $(X) =$ degree$(I(X))$. We set $r_k(X) := r_{k+1}(I(X)) = r_k(S/I(X)) + 1$ $(k \geqslant 0)$.

For a set B we write card(B) for its cardinality. Finally, we set for integers a, $b \geqslant 0$: $\{a/b\} := \inf\{t \in \mathbb{Z}\colon a \leqslant tb\}$. If $a > b$ we define a sum $\sum_{i=a}^{b} \ldots$ to be zero and a condition, say B_i, for $i = a, a+1, \ldots, b$ to be empty.

3. Castelnuovo bounds

Studying our integers r_k we will prove a generalization of a theorem of D. Mumford [10], p. 99 and A. Ooishi [13], Theorem 2.

THEOREM 1. *Let M be a finitely generated graded A-module of dimension d and let m and $k \geqslant 1$ be integers. Suppose that $[H^i_P(M)]_{m-i} = 0$ for all $i \geqslant k$. Then $r_k(M) \leqslant m - 1$. Moreover,* reg$(M) \leqslant m - 1$ *provides $A_i M_j = M_{i+j}$ for all $i \geqslant 0$ and $j \geqslant m - 1$.*

Proof. We induct on d. In case of $d = 0$ the assertions are trivial since $H^0_P(M) = M$ and $H^i_P(M) = 0$ for all $i > 0$. Let $d > 0$. According to Lemma 1 we can choose a filter-regular element $l \in A_1$ for M. Then we get $H^i_P(M/0\colon l) \cong H^i_P(M)$ for $i > 0$ from the long exact cohomology sequence of $0 \to 0\colon_M l \to M \to M/0\colon_M l \to 0$. The exact sequence $0 \to M/0\colon l(-1) \xrightarrow{l} M \to M/lM \to 0$ gives rise to the cohomology sequence

$$0 \to 0\colon_M l(-1) \to H^0_P(M)(-1) \xrightarrow{l} H^0_P(M) \to H^0_P(M/lM) \to H^1_P(M)(-1) \to \cdots \qquad (*)$$

because we have $0\colon_M l \subseteq H^0_P(M)$ by the choice of l. Considering the following

sequences of (∗)

$$[H^i_P(M)]_{m-i} \to [H^i_P(M/lM)]_{m-i} \to [H^{i+1}(M)]_{m-i-1} \quad (i \geqslant 0)$$

we get $[H^i_P(M/lM)]_{m-1} = 0$ for all $i \geqslant k$ by assumption. Therefore the induction hypothesis provides $[H^i_P(M/lM)]_j = 0$ for all i and j with $i \geqslant k$ and $i + j \geqslant m$. The following sequences of (∗)

$$[H^i_P(M)]_{m-i} \to [H^i_P(M)]_{m-i+1} \to [H^i_P(M/lM)]_{m+1-i} \quad (i \geqslant 0)$$

gives us $[H^i_P(M)]_{m+1-i} = 0$ for all $i \geqslant k$. By induction we therefore have the first assertion. Proving the second assertion we first note that $\operatorname{reg}(M) \leqslant m - 1$ and (∗) yield $\operatorname{reg}(M/lM) \leqslant m - 1$. If we set $M' = M/lM$ and $A' = A/lA$ we have $A'_i M'_j = M'_{i+j}$ for $i \geqslant 0$ and $j \geqslant m - 1$ by induction hypothesis. Hence $A_i M_j + lM_{i+j-1} = M_{i+j}$. It follows from this by induction on i that $M_{i+j-1} = A_{i-1} M_j$. We obtain $M_{i+j} = A_i M_j + lM_{i+j-1} = A_i M_j$. Q.E.D.

Moreover we get from the proof of Theorem 1. (see also [24], Lemma 2.3):

LEMMA 2. *Let M be a finitely generated graded A-module and let $l \in A_1$ be a filter-regular element for M then*

$$e(H^{i+1}_P(M)) < e(H^i_P(M/lM)) \leqslant \max\{e(H^i_P(M)),\ 1 + e(H^{i+1}_P(M))\} \quad (i \geqslant 0).$$

Proof. The assertions follows from the exact sequence (∗) of the proof of Theorem 1. Q.E.D.

LEMMA 3. *Let $k \geqslant 1$ and c be integers. Then we have for all finitely generated graded A-modules M of dimension d*

$$r_k(M) \leqslant c - 1 + \sum_{i=k}^{d} \operatorname{card}\{t \in \mathbb{Z} : t \geqslant c - i \text{ and } [H^i_P(M)]_t \neq 0\}.$$

Proof. The assertion is trivial for $k > d$. Let $1 \leqslant k \leqslant d$. We have $\sum_{i=k}^{d} \operatorname{sgn}[H^i_P(M)]_{m-i} \geqslant 1$ for all $m \leqslant r_k(M)$ by Theorem 1. Thus it follows from all $c \in \mathbb{Z}$:

$$\begin{aligned} r_k(M) &\leqslant c - 1 + \sum_{m=c}^{r_k(M)} \sum_{i=k}^{d} \operatorname{sgn}[H^i_P(M)]_{m-i} \\ &= c - 1 + \sum_{i=k}^{d} \sum_{t=c-i}^{r_k(M)-i} \operatorname{sgn}[H^i_P(M)]_t \\ &= c - 1 + \sum_{i=k}^{d} \operatorname{card}\{t \in \mathbb{Z} : t \geqslant c - i \text{ and } [H^i_P(M)]_t \neq 0\} \end{aligned}$$

where the last equality follows from the definition of $r_k(M)$. Q.E.D.

In case of $M = A$ we obtain something more.

MAIN LEMMA: *Let $k \geqslant 0$ and c be integers. Let A be a graded K-algebra of Krulldimension d. Then:*

$$r_k(A) \leqslant c - 1 + \sum_{i=k}^{d} \operatorname{card}\{t \in \mathbb{Z} : t \geqslant c - i \text{ and } [H_P^i(A)]_t \neq 0\}.$$

Proof. We consider only the case $k = 0$ according to Lemma 3. Since $r_0(A) = \max\{e(H_P^0(A)), r_1(A)\}$ the assertion follows from Lemma 3 with $k = 1$ in assuming $e(H_P^0(A)) \leqslant r_1(A)$. We therefore suppose that $e(H_P^0(A)) > r_1(A)$. We set $A = S/\mathfrak{a} \cap \mathfrak{q}$ where $S = K[x_0, \ldots, x_n]$ and $\mathfrak{a}, \mathfrak{q} \subseteq S$ are homogeneous ideals such that $\mathfrak{a}$: $\langle P \rangle = \mathfrak{a}$, $\mathfrak{a} \nsubseteq \mathfrak{q}$ and $\mathfrak{q}$ is a primary ideal belonging to P. If we set $A' = S/\mathfrak{a}$ we get $\operatorname{depth}(A') \geqslant 1$. Hence $\operatorname{reg}(A') = r_1(A') = r_1(A)$, consequently $\operatorname{reg}(\mathfrak{a}) = 1 + r_1(A)$. It follows from the second assertion of Theorem 1. that $\mathfrak{a}$ is generated by forms of degree $\leqslant 1 + r_1(A)$. Since $H_P^0(A) = \mathfrak{a}/\mathfrak{a} \cap \mathfrak{q}$ we can deduce $[H_P^0(A)]_t \neq 0$ for all t with $r_1(A) < t \leqslant e(H_P^0(A))$. This gives us

$$r_0(A) = e(H_P^0(A)) = r_1(A) + \operatorname{card}\{t \in \mathbb{Z} : t > r_1(A) \text{ and } [H_P^0(A)]_t \neq 0\}.$$

Therefore the assertion follows again from Lemma 3 with $k = 1$. Q.E.D.

COROLLARY 1. *Let A be a graded K-algebra of Krulldimension d. Let j and k be integers such that $j + k \leqslant d$ and $\{l_1, \ldots, l_j\}$ be a filter-regular sequence for A. We set $A^j := A/(l_1, \ldots, l_j)A$. Then:*

$$r_k(A) \leqslant c - 1 + \sum_{i=k}^{k+j-1} \operatorname{card}\{t \in \mathbb{Z} : t \geqslant c - i \text{ and } [H_P^i(A)]_t \neq 0\}$$

for all $c > r_k(A^j)$.

Proof. Lemma 2 gives us for $i = k, \ldots, d - j$

$$i + j + e(H_P^{i+j}(A)) \leqslant i + e(H_P^i(A^j)) \leqslant r_k(A^j).$$

Therefore the assertion follows from the Main Lemma. Q.E.D.

REMARKS (i) If we suppose $k \geqslant 1$ in Corollary 1 the above result remains true even for finitely generated graded A-modules because we can apply Lemma 3.

(ii) If we set $k = 1$ and $j = d - 1$ in Corollary 1 we obtain the main lemma of [23]. Hence we could deduce the Castelnuovo bounds of [23]. Here we want to state some new bounds.

THEOREM 2. *Let* $\mathfrak{a} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m \subset S = K[x_0, \ldots, x_n]$ *be an intersection of* m *equidimensional (homogeneous) prime ideals. Let* d *be the Krulldimension of* $A = S/\mathfrak{a}$. *Then we have:*

(i) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-2} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i \text{ and } [H^i_P(A)]_t \neq 0\}$ *for all* $c \geqslant \operatorname{degree}(\mathfrak{a})$.

(ii) *In case of* $m = 1$ *we get:*

(ii.1) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-1} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i \text{ and } [H^i_P(A)]_t \neq 0\}$ *for all* $c > \{\operatorname{degree}(\mathfrak{a}) - 1/\operatorname{rank}_K[A]_1 - d\}$ *if* $\operatorname{char}(K) = 0$

(ii.2) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-2} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i \text{ and } [H^i_P(A)]_t \neq 0\}$ *for all* $c > \operatorname{degree}(\mathfrak{a}) + d - \operatorname{rank}_K[A]_1$

(ii.3) $\operatorname{reg}(\mathfrak{a}) \leqslant c + \sum_{i=1}^{d-3} \operatorname{card}\{t \in \mathbb{Z}: t \geqslant c - i \text{ and } [H^i_P(A)]_t \neq 0\}$ *for all* $c > \operatorname{degree}(\mathfrak{a}) + d - \operatorname{rank}_K[A]_1$ *if* $\mathfrak{a}$ *is regular and* $\operatorname{char}(K) = 0$.

Proof. First we show (i) and (ii.2). If $d = 0$ or $d = 1$ then A is Cohen-Macaulay and we get $\operatorname{reg}(\mathfrak{a}) \leqslant \operatorname{degree}(\mathfrak{a}) + d - \operatorname{rank}_K[A]_1 + 1$ by [13], Proposition 13. This proves (i) and even (ii.2) since $\operatorname{rank}_K[A]_1 \geqslant d$, where equality holds if and only if A is isomorphic to a polynomial ring over K. But in this case we have $\operatorname{degree}(\mathfrak{a}) = \operatorname{reg}(\mathfrak{a}) = 1$. Let $d \geqslant 2$. According to H. Flenner [4] there are generic linear forms $l_1, \ldots, l_{d-2} \in S$ such that $\mathfrak{a} + (l_1, \ldots, l_i)S$ is an intersection of m prime ideals of dimension $d - i$ up to a primary component belonging to P for $i = 0, \ldots, d - 2$. Therefore $\{l_1, \ldots, l_{d-2}\}$ is a filter-regular sequence for A. If we set $A^{d-2} = A/(l_1, \ldots, l_{d-2})A$ and $\mathfrak{a}' = \mathfrak{a} + (l_1, \ldots, l_{d-2})S: \langle P \rangle$ we obtain $r_1(A^{d-2}) = r_1(S/\mathfrak{a}') = \operatorname{reg}(S/\mathfrak{a}')$. Thus we get $r_1(A^{d-2} \leqslant \operatorname{degree}(\mathfrak{a}') + 2 - \operatorname{rank}_K[S/\mathfrak{a}']_1$ by Theorem 1.1. of [6] for $m = 1$ and $r_1(A^{d-2}) \leqslant \operatorname{degree}(\mathfrak{a}') - 1$ according to the remark after the proof of Theorem 1.1. in [6] for $m \geqslant 1$. Therefore Corollary 1 with $k = 1$ and $j = d - 2$ proves (i) since $\operatorname{degree}(\mathfrak{a}) = \operatorname{degree}(\mathfrak{a}')$ by Bezout's theorem.

Proving (ii.2) we will show that $2 - \operatorname{rank}_K[S/\mathfrak{a}']_1 \leqslant d - \operatorname{rank}_K[A]_1$. Then we can apply Corollary 1. It follows from [22], Lemma 3 that

$$\begin{aligned}
\operatorname{rank}_K[\mathfrak{a}']_1 &= \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-3})S]_1 + 1 \\
&\leqslant \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-3})S: \langle P \rangle]_1 + 1 \\
&= \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-4})S]_1 + 2 \\
&\leqslant \cdots \leqslant \operatorname{rank}_K[\mathfrak{a}]_1 + d - 2.
\end{aligned}$$

The proof of (ii.3) is analogous to the proof of (ii.2). For this we note that we can choose the linear forms l_i according to [4] such that $\mathfrak{a}' := \mathfrak{a} + (l_1, \ldots, l_{d-3})S: \langle P \rangle$ is regular if $\mathfrak{a}$ is regular. Therefore the assertion follows from Corollary 1 with

$k = 1$ and $j = d - 3$ by using

$$\begin{aligned} r_1(A/(l_1, \ldots, l_{d-3})A) &= \operatorname{reg}(S/\mathfrak{a}') \\ &\leqslant \operatorname{degree}(\mathfrak{a}') + 3 - \operatorname{rank}_K[S/\mathfrak{a}']_1 \quad \text{by [9] (or [5])} \\ &\leqslant \operatorname{degree}(\mathfrak{a}') + d - \operatorname{rank}_K[A]_1 \quad \text{by [22],} \end{aligned}$$

Lemma 3 and Bezout's theorem.

Now we show (ii.1). The assertion is trivial for $d = 0$ and $d = 1$. Let $d \geqslant 2$. Take the linear forms $l_1, \ldots, l_{d-2}$ constructed in our proof of (ii.2). Consider a general linear form l_{d-1} and set $\mathfrak{a}' = \mathfrak{a} + (l_1, \ldots, l_{d-1})S : \langle P \rangle$. Then we get

$$r_1(A/(l_1, \ldots, l_{d-1})A) = \operatorname{reg}(S/\mathfrak{a}') \leqslant \left\{ \frac{\operatorname{degree}(\mathfrak{a}') - 1}{\operatorname{rank}_K[S/\mathfrak{a}']_1 - 1} \right\}$$

(see, e.g., [22], Lemma 1). We have again $\operatorname{degree}(\mathfrak{a}) = \operatorname{degree}(\mathfrak{a}')$ by Bezout's theorem and $\operatorname{rank}_K[\mathfrak{a}']_1 \leqslant \operatorname{rank}_K[\mathfrak{a}]_1 + d - 1$. On the other hand Lemma 3 of [22] gives us

$$\begin{aligned} \operatorname{rank}_K[\mathfrak{a}']_1 &= \operatorname{rank}_K[\mathfrak{a} + (l_1, \ldots, l_{d-2})S]_1 + \\ &\quad + 1 \geqslant \operatorname{rank}_K[\mathfrak{a}]_1 + d - 1. \end{aligned}$$

Putting all together we obtain

$$r_1(A^{d-1}) \leqslant \left\{ \frac{\operatorname{degree}(\mathfrak{a}) - 1}{\operatorname{rank}_K[A]_1 - d} \right\}. \qquad (+)$$

Consequently Corollary 1 with $k = 1$ and $j = d - 1$ proves the assertion (ii.1). This completes the proof of Theorem 2. Q.E.D.

COROLLARY 2. *Let X be a nondegenerate, irreducible and reduced subscheme of $\mathbb{P}^n$ of dimension d. Then we have:*

$$d + 1 + e(H_P^{d+1}(S/I(X))) \leqslant \left\{ \frac{\operatorname{degree}(X) - 1}{\operatorname{codim}(X)} \right\}.$$

Proof. The assertion follows from (+) of the proof of Theorem 2(ii.1) and Lemma 2. Q.E.D.

REMARKS. Theorem 2(ii.1) is Theorem 2(ii) of [23]. Theorem 2(i), (ii.2) and (ii.3) improve Theorem 2(i) of [23] in some special cases.

The assumption $\operatorname{char}(K) = 0$ is necessary in Theorem 2(ii.1) because the general position lemma does not remain true if $\operatorname{char}(K) > 0$ (see [17], Example 1.2).

The conjecture of D. Bayer and M. Stillman [2] gives $\operatorname{reg}(\mathfrak{a}) \leqslant \operatorname{degree}(\mathfrak{a})$. Therefore Theorem 2(i) and (ii.2) (see also [22], corollary) prove the conjectures of Bayer and Stillman and of Eisenbud and Goto in case of $\operatorname{depth}(S/\mathfrak{a}) \geqslant d - 1$. This means, for example, that the latter conjecture is true for surfaces in $\mathbb{P}^4$ if the homogeneous coordinate ring has depth $\geqslant 2$. Note that (see the introduction) singular surfaces in $\mathbb{P}^4$ are the simplest varieties such that the conjecture of D. Eisenbud and S. Goto is open.

4. Counterexamples to an assertion of J. Harris and D. Eisenbud

In this section we will apply Theorem 2 in order to study the equality between Hilbert functions and Hilbert polynomials. Let X be a subscheme of P^n and $A = S/I(X)$ be its homogeneous coordinate ring. We recall that the Hilbert function of X is defined by $h_X(t) := \operatorname{rank}_K[A]_t$ for $t \geqslant 0$. The so-called Hilbert polynomial, denoted by $p_X(t)$, is given by $h_X(t)$ for $t \gg 0$. It is well-known that $p_X(t) = \sum_{i \geqslant 0}(-1)^i h^i(X, \mathcal{O}_X(t))$ where $h^i(X, O_X(t))$ is the dimension of $H^i(X, \mathcal{O}_X(t))$. Following [7] the function $h'_X(t) := h^0(X, \mathcal{O}_X(t))$ is said to be the abstract Hilbert function of X. The index of regularity of X, denoted by $r(X)$, is defined as $r(X) := \min\{t \in \mathbb{N}: h_X(i) = p_X(i) \text{ for all } i \geqslant t\}$. Moreover, we set

$$r'(X) := \min\{t \in \mathbb{N}: h'_X(i) = p_X(i) \text{ for all } i \geqslant t\},\ \operatorname{reg}(X) = \operatorname{reg}(I(X)).$$

LEMMA 4. (i) ([14], *Corollary* 2.2) $r(X) \leqslant \operatorname{reg}(X) - \operatorname{depth}(S/I(X))$,
(ii) $r'(X) \leqslant r_2(X) - \max\{2, \operatorname{depth}(S/I(X))\}$.

Proof. We have $h_X(t) - p_X(t) = \sum_{i \geqslant 0}(-1)^i \operatorname{rank}_K[H^i_P(S/I(X))]_t$ according to [19]. This proves (i). We obtain (ii) from the characterization of $p_X(t)$ as an Euler-Poincaré characteristic and the isomorphisms $H^i(X, \mathcal{O}_X(t)) \cong [H^{i+1}_P(S/I(X))]_t$ for $i > 0$. Q.E.D.

In [7], p. 82 J. Harris and D. Eisenbud assert for reduced and irreducible subschemes X of $\mathbb{P}^n_K$ ($\operatorname{char}(K) = 0$):

$$r'(X) \leqslant \left\{\frac{\operatorname{degree}(X)}{\operatorname{codim}(X)}\right\}. \qquad (**)$$

The letter [25] gives rise to study this claim again. In this connection, we will prove the following theorem.

THEOREM 3. (i) *We have for nondegenerate, irreducible and reduced curves* X:

$$r'(X) \leqslant \left\{\frac{\text{degree}(X)-1}{\text{codim}(X)}\right\} - 1,$$

that is, the assertion (**) *is true for such curves.*

(ii) *There are nondegenerate, irreducible and reduced subschemes* X *of* $\mathbb{P}^n$, $n \geqslant 4$, *of dimension* d *such that the assertion* (**) *is not true for all* $d \geqslant 2$.

Proof. (i) Corollary 2 gives us for $d = 1$

$$r_2(X) - 1 = 2 + e(H^2_P(S/I(X))) \leqslant \left\{\frac{\text{degree}(X)-1}{\text{codim}(X)}\right\}.$$

Hence (i) follows from Lemma 4(ii).

(ii) We consider the following class of examples: Let $m \geqslant 3$ be an integer. Let $X_m \subseteq \mathbb{P}^4$ be the surface given parametrically by $\{u^m, u^{m-1}v, u^{m-2}vw, uw^{m-1}, w^m\}$. It follows from [8], Proposition 3 that $\text{degree}(X_m) = m + 1$. Moreover, Corollary 3.4(ii) of [24] shows $H^1_P(S/\mathfrak{p}_m) = 0$, that is $\text{depth}(S/\mathfrak{p}_m) \geqslant 2$ where $\mathfrak{p}_m \subset S = K[x_0, \ldots, x_4]$ denotes the defining prime ideal of X_m. That is why we can apply Theorem 2(ii.2) and obtain $\text{reg}(X_m) \leqslant \text{degree}(X_m) - \text{codim}(X_m) + 1 = m$. Hence $\mathfrak{p}_m$ is generated by forms of degree $\leqslant m$ according to Theorem 1. Thus we can compute a minimal basis of $\mathfrak{p}_m$ from its parametrization and obtain

$$\mathfrak{p}_m = (x_1x_4 - x_2x_3, x_0x_2^i x_4^{m-1-i} - x_1^i x_3^{m-i}, \quad i = 0, \ldots, m-1).$$

Since $\mathfrak{p}_m$ needs a generator of degree m we get $\text{reg}(X_m) = m$. Since $p_{X_m}(t) - h'_{X_m}(t) = \sum_{i>0}(-1)^i \text{rank}_K[H^{i+1}_P(S/\mathfrak{p}_m)]_t$ we obtain from Corollary 2 that

$$p_{X_m}(t) - h'_{X_m}(t) = -\text{rank}_K[H^2_P(S/\mathfrak{p}_m)]_t$$
$$\text{for } t \geqslant \left\{\frac{\text{degree}(X_m)-1}{\text{codim}(X_m)}\right\} - 2 = \left\{\frac{m}{2}\right\} - 2.$$

Applying Theorem 1 we get from $\text{reg}(X_m) = m$ and Corollary 2 that $[H^2_P(S/\mathfrak{p}_m)]_t \neq 0$ for

$$\left\{\frac{m}{2}\right\} - 2 \leqslant t \leqslant m-3. \text{ Hence } r'(X_m) = m - 2 > \left\{\frac{m+1}{2}\right\} = \left\{\frac{\text{degree}(X_m)}{\text{codim}(X_m)}\right\}$$

for $m \geqslant 7$. This shows (ii) in case $d = 2$.

Let $j \geqslant 0$ be an integer. We denote by Y_m the projective cone over X_m in $\mathbb{P}^{j+4}$. Then we get $d := \dim(Y_m) = \dim(X_m) + j = 2 + j$, $\text{degree}(Y_m) = \text{degree}(X_m)$ and

$\operatorname{depth}(S/I(Y_m)) = j + 2$ where $S = K[x_0, \ldots, x_{j+4}]$. Moreover, Lemma 2 gives us $\operatorname{reg}(Y_m) = \operatorname{reg}(X_m) = m$. Therefore we obtain from Corollary 2 as above:

$$r'(Y_m) = m - 2 - j > \left\{\frac{m+1}{2}\right\} = \left\{\frac{\operatorname{degree}(Y_m)}{\operatorname{codim}(Y_m)}\right\}$$

for $m \geqslant 2j + 7$ and for all $d = 2 + j$. Q.E.D.

REMARKS. (i) Using results of [6] the subschemes Y_m show that the conjecture of D. Eisenbud and S. Goto is sharp in the sense that there are nondegenerate, irreducible and reduced varieties X with $\operatorname{reg}(X) = \operatorname{degree}(X) - \operatorname{codim}(X) + 1$ in any dimension $\geqslant 1$ and of any degree $\geqslant 4$.

(ii) If the assertion (**) were true we could deduce $r_2(X) \leqslant \{\operatorname{degree}(X)/\operatorname{codim}(X)\} + 2$. But this is also not true in general as the varieties Y_m show.

(iii) (**) is true in assuming, for example, that the subschemes X are arithmetically Buchsbaum, i.e., that the homogeneous coordinate ring $S/I(X)$ is a graded Buchsbaum K-algebra. In this case we obtain from [21], Theorem 1. $\operatorname{reg}(X) \leqslant \{\operatorname{degree}(X) - 1/\operatorname{codim}(X)\} + 1$. Therefore Lemma 4(ii) yields

$$r'(X) < \left\{\frac{\operatorname{degree}(X) - 1}{\operatorname{codim}(X)}\right\}.$$

(iv) The varieties Y_m are not arithmetically Buchsbaum for $m \geqslant 3$ due to [24], Lemma 4.11 and Corollary 4.7 and even not locally Buchsbaum for $m \geqslant 4$ because $Y_m \subset \mathbb{P}^{j+4}$ has a singularity in the point $\mathfrak{p} = (x_1, \ldots, x_{j+4})$ which is not Buchsbaum for $m \geqslant 4$. Otherwise $(S/I(Y_m))_{\mathfrak{p}}$ and consequently also $(S/I(Y_m) + x_3 S)_{\mathfrak{p}}$ would be Buchsbaum. Since $(I(Y_m) + x_3 S)_{\mathfrak{p}} = (x_3, x_1 x_4, x_2^{m-1}, x_2^{m-2} x_4, \ldots, x_2 x_4^{m-2}, x_4^{m-1})_{\mathfrak{p}} = (x_2^{m-1}, x_3, x_4)_{\mathfrak{p}} \cap (x_1, x_3, x_2^{m-1}, x_2^{m-2} x_4, \ldots, x_4^{m-1})_{\mathfrak{p}}$ we have $x_2(x_2^{m-1}, x_3, x_4)_{\mathfrak{p}} \notin (I(Y_m) + x_3 S)_{\mathfrak{p}}$ for $m \geqslant 4$. We immediately obtain a contradiction to a Buchsbaum ring property. (For the facts on Buchsbaum rings used here we refer to [20].)

(v) We can not apply the results of [9] or [5] in order to obtain the bound $\operatorname{reg}(X_m) \leqslant \operatorname{degree}(X_m) - \operatorname{codim}(X_m) + 1$ since the varieties X_m are singular.

(vi) Since our counterexamples are singular varieties it is an open problem if (**) is true in the case of smooth varieties, see [26].

(vii) In [15], p. 370 P. Philippon considers the ideal $\mathfrak{a} := (x_1 x_4 - x_2 x_3, x_0 x_2^2 - x_1^2 x_3, x_0 x_4^2 - x_3^3)$. He asserts that $\mathfrak{a}$ is a prime ideal used in his computations. But it follows from $x_4(x_0 x_2 x_4 - x_1 x_3^2) \in \mathfrak{a}$ that $\mathfrak{a}$ is even not a primary ideal because no power of x_4 and $x_0 x_2 x_4 - x_1 x_3^2$ is contained in $\mathfrak{a}$. (Note that $\mathfrak{p}_3 = \mathfrak{a} + (x_0 x_2 x_4 - x_1 x_3^2)S$.)

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