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On Kirillov's conjecture for Archimedean fields

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Abstract. Let $G_n = \mathrm{GL}(n, \mathbb{F})$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$ and let P_n be the subgroup consisting of those matrices whose last row is $(0, 0, \ldots, 0, 1)$. A long standing conjecture of A.A. Kirillov asserts that any irreducible unitary representation of G_n remains irreducible upon restriction to P_n. In this paper this conjecture is completely proved for $\mathbb{F} = \mathbb{C}$ and partial results are obtained for $\mathbb{F} = \mathbb{R}$.

Introduction

The general linear groups form a natural class of reductive groups whose representation theory is now fairly well understood [B, T, V]. The foundation of the *non-archimedean* theory is the work of Bernstein and Zelevinsky [BZ], and central to their approach is the notion of the derivatives of an admissible representation of $\mathrm{GL}(n, \mathbb{F})$, where $\mathbb{F}$ is non-archimean. This notion involves restriction of the given admissible representation to a subgroup $P_n(\mathbb{F}) \cong \mathrm{GL}(n-1, \mathbb{F}) \rtimes \mathbb{F}^{n-1}$ consisting of matrices with last row $(0, \ldots, 0, 1)$.

This paper was motivated by three considerations. First, since the group $P_n(\mathbb{F})$ is not reductive, it is problematic to define the derivatives in the case when $\mathbb{F}$ is archimedean; but it seemed that one might still be able to make sense of the *highest* derivative for irreducible unitary representations. Of course, this was intimately involved with Kirillov's Conjecture [K] which states

CONJECTURE 1.1. If $\mathbb{F}$ is a local field and π is an irreducible unitary representation of $\mathrm{GL}(n, \mathbb{F})$ then $\pi \mid P_n(\mathbb{F})$ is irreducible.

So the second motivation was to study this conjecture. This turned out to be inextricably intertwined with the following conjecture which was the third motivation.

CONJECTURE 1.2. If $\mathbb{F}$ is a local field and π_1 and π_2 are irreducible unitary representations of $\mathrm{GL}(n_1, \mathbb{F})$ are $\mathrm{GL}(n_2, \mathbb{F})$. Then $\pi_1 \times \pi_2$ is an irreducible representation of $\mathrm{GL}(n_1 + n_2, \mathbb{F})$. (By $\pi_1 \times \pi_2$ we mean the representation obtained by unitary parabolic induction from the representation $\pi_1 \otimes \pi_2$ of the Levi subgroup $\mathrm{GL}(n_1, \mathbb{F}) \times \mathrm{GL}(n_2, \mathbb{F})$).

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For non-archimedean fields these conjectures are all theorems due to Bernstein. For $\mathbb{R}$, and $\mathbb{C}$. Conjecture 1.2 is implicit in [V].

A word now about our methods and results. From now on $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$; and for $z \in \mathbb{F}$, we will write $\operatorname{Re} z$ for its real part (so $\operatorname{Re} z = z$ if $\mathbb{F} = \mathbb{R}$).

The first ingredient is the classical Mackey theory of unitary representations of semi-direct products. In this connection, we note the following facts about the groups $P_n(\mathbb{F})$.

(i) The group $P_n(\mathbb{F})$ is a semi-direct product $\mathrm{GL}(n-1, \mathbb{F}) \ltimes \mathbb{F}^{n-1}$ with the second factor normal and abelian.
(ii) For $z \in \mathbb{F}$, we will denote by e the unitary character $e(z) = \exp(i \operatorname{Re} z)$, and use it to identify the algebraic and unitary duals of $\mathbb{F}^n$, writing $(\mathbb{F}^n)^*$ for both.
(iii) On $(\mathbb{F}^{n-1})^*$, the group $\mathrm{GL}(n-1, \mathbb{F})$ has exactly two orbits viz., 0 and $(\mathbb{F}^{n-1})^* \backslash 0$. Furthermore, if $\chi \in (\mathbb{F}^{n-1})^* \backslash 0$ is given by $\chi({}^t(x_1, \ldots, x_{n-1})) = x_{n-1}$, then $\mathrm{Stab}_{\mathrm{GL}(n-1,\mathbb{F})}(\chi) \cong P_{n-1}(\mathbb{F})$, imbedded in the top left corner of $\mathrm{GL}(n, \mathbb{F})$. This implies the following result (see [M]).

FACT 1.1. Every irreducible unitary representation of $P_n(\mathbb{F})$ is obtained in one of two ways

either (a) by trivially extending an irreducible unitary representation of $\mathrm{GL}(n-1, \mathbb{F})$
or (b) by extending an irreducible unitary representation of $P_{n-1}(\mathbb{F})$ to $P_{n-1}(\mathbb{F}) \ltimes \mathbb{F}^{n-1}$ by the character χ and unitarily inducing to $P_n(\mathbb{F})$.

NOTATION. We shall write $\hat{G}$ for the set of irreducible unitary representations of G. Also we shall use E and I for the two constructions (really functors) given in parts (a) and (b) of the above fact.

Then the above fact, and a bit more, is summarized by the equality

$$\hat{P}_n(\mathbb{F}) = E(\widehat{\mathrm{GL}}(n-1, \mathbb{F})) \amalg I(\hat{P}_{n-1}(\mathbb{F})). \tag{$*$}$$

We shall use the convention that $P_1(\mathbb{F}) = G_0(\mathbb{F}) =$ trivial group (consisting of the identity element alone). With this in mind and iterating $(*)$ we observe:

FACT 1.2. Every irreducible unitary representation τ of $P_n(\mathbb{F})$ is of the form

$$\tau = I^{k-1} E\sigma \quad \text{for some } k \geqslant 1,$$

where k and $\sigma \in \widehat{\mathrm{GL}}(n-k, \mathbb{F})$ are *uniquely* determined by τ.

DEFINITION 1.1. (i) If τ is a unitary representation of $P_n(\mathbb{F})$ (not necessarily irreducible) we shall say that τ is *homogeneous of depth* k if

$$\tau = I^{k-1}E\sigma$$

for some unitary representation σ of $\mathrm{GL}(n-k, \mathbb{F})$.

(ii) If ρ is a unitary representation of $\mathrm{GL(n, \mathbb{F})}$ we shall say that ρ is *adducible* of depth k if $\rho|p_n(\mathbb{F})$ is homogeneous of depth k; and if

$$\rho|p_n(\mathbb{F}) = I^{k-1}E\sigma,$$

we shall write $\sigma = A\rho$ and term it the *adduced* representation of ρ.

Note that if ρ is adducible, the $\rho|P_n(\mathbb{F})$ is irreducible if and only if $A\rho$ is irreducible.

We are now in a positon to state the key result about adducibility.

THEOREM 2.1. *If ρ and σ are adducible representations of* $\mathrm{GL}(r, \mathbb{F})$ *and* $\mathrm{GL}(s, \mathbb{F})$ *of depths k and ℓ, then $\rho \times \sigma$ is adducible of depth $k + \ell$ and*

$$A(\rho \times \sigma) = A\rho \times A\sigma.$$

Using Theorem 2.1 and a semi-classical fact about the discrete series of $\mathrm{GL}(2, \mathbb{R})$ we can deduce Conjectures 1.1 and 1.2 for *tempered* representations of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$. This is the content of Theorem 3.1.

Then, using an extension of some arguments in Stein [St] and results of Vogan on the unitary dual of $\mathrm{GL}(n, \mathbb{C})$, we establish both Conjectures 1.1 and 1.2 for $\mathbb{F} = \mathbb{C}$. This is Theorem 3.2.

In some remarks at the end of Section 3 we indicate the additional effort necessary to prove Kirillov's Conjecture for $\mathbb{F} = \mathbb{R}$ and also suggest some connections with the theory of rank of representations as developed by Howe and Scaramuzzi.

Section 2. The considerations in this section are motivated by Bernstein and Zelevinsky [BZ].

We will be concerned with three instances of the following situation:

A is a group and B and C are "disjoint" commuting subgroups of A, so that $B \times C$ is imbedded in A. D is an extension of $B \times C$ by a normal, abelian factor N such that A/D is compact.

For a group G, let Rep G be the category of unitary representations of G, and then we define a bifunctor "$\times$" from Rep $B \times$ Rep C to Rep A as follows.

For $\pi_1 \in$ Rep B and $\pi_2 \in$ Rep C, we shall denote by $\pi_1 \times \pi_2$ the representation of A obtained by extending $\pi_1 \otimes \pi_2$ from $B \times C$ to D (trivially on N) and then unitarily inducing from D to A.

The three cases are

(1) $A = G_{p+q}, \quad B = G_p, \quad C = G_q, \quad N = M_{p \times q}$

$$\begin{array}{|c|c|} \hline G_p & N \\ \hline & G_q \\ \hline \end{array}$$

(2) $A = P_{p+q}, \quad B = G_p, \quad C = P_q, \quad N = M_{p \times q}$

$$\begin{array}{|c|c|} \hline G_p & N \\ \hline & P_q \\ \hline \end{array}$$

(3) $A = P_{p+q}, \quad B = P_p, \quad C = G_q, \quad N = M_{p-1 \times q}$

$$\begin{array}{|c|c|c|} \hline P_p & N & P_p \\ \hline & G_q & \\ \hline \end{array}$$

Here by G_n, P_n and $M_{m \times n}$ we mean the groups $\mathrm{GL}(n, \mathbb{F})$, $P_n(\mathbb{F})$ and $M_{m \times n}(\mathbb{F})$ – the *additive* group of $m \times n$ matrixes with entries in $\mathbb{F}$, where $\mathbb{F}$ is some fixed local field.

We will also consider the following functors:

(a) E: Rep $G_{n-1} \to$ Rep P_n is the trivial extension functor.
(b) I: Rep $P_{n-1} \to$ Rep P_n is the "Mackey induction" functor.
(c) R: Rep $G_n \to$ Rep P_n is the restriction functor.

The above functors enter into the definition of "adduction" . The following lemma shows how they "distribute" over the " $\times$ -functors".

LEMMA 2.1. *If* $\rho \in \operatorname{Rep} G_r, \sigma \in \operatorname{Rep} G_s, \tau \in \operatorname{Rep} P_t$, *then*

(i) $R(\rho \times \sigma) = R\rho \times \sigma$,
(ii) $I(\rho \times \tau) = \rho \times I\tau$,
(iii) $E(\rho \times \sigma) = \rho \times E\sigma$,
(iv) $I(\tau \times \sigma) = I\tau \times \sigma$,
(v) $I(\rho \times R\sigma) = E\rho \times \sigma$.

The rest of the section is devoted to the proof of this Lemma. The ideas are closely related to [*BZ*] and depend on classical results of Mackey.

Proof.

(i) Let $Q = \begin{array}{cc|c} r & s & \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{*} & r \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{*} & s \\ \hline \end{array}$. Then $\rho \times \sigma = \mathrm{Ind}_Q^{G_{r+s}}(\rho \otimes \sigma \otimes 1)$.

Now on $G = G_{r+s}$, there are two Q, P double cosets viz. $\bar{1} = Q.1.P$ and $\bar{w} = QwP$ where w is the matrix of the cyclic permutation $(r, r+s, r+s-1, \ldots, r-1)$. Since $\bar{1}$ has strictly smaller dimension than $G, \bar{w}$ is the *only* double coset with nonzero Haar measure.

So by Mackey's subgroup Theorem ([Wr; 5.3.4.1])

$$\begin{aligned} R(\rho \times \sigma) &= \mathrm{Ind}_P^G(\rho \otimes \sigma \otimes 1) \mid P, \\ &= \mathrm{Ind}_{P \cap Q^w}^P((\rho \otimes \sigma \otimes 1)^w \mid P \cap Q^w, \\ &= R\rho \times \sigma \quad \text{(by definition).} \end{aligned}$$

Parts (ii), (iii), (iv) are applications of unitary induction by stages. We shall be considering various subgroups of G_n and P_n and their representations. It is convenient to adopt the following convention:

A diagram such as

$$\begin{array}{c|c|c|c|} & r & t & 1 \\ \hline r & \rho & 1 & \\ \cline{1-3} t-1 & & \tau & \chi \\ \cline{1-3} 1 & & & \\ \hline \end{array}$$

means a representation of the subgroup

$$\begin{array}{c|c|c|c|} & r & t & 1 \\ \hline r & * & * & \\ \cline{1-3} t-1 & 0 & * & * \\ \cline{1-3} 1 & \multicolumn{2}{c|}{0 \ldots\ldots\ldots\ldots 01} & \\ \hline 1 & \multicolumn{3}{c|}{0 \ldots\ldots\ldots\ldots\ldots\ldots 01} \\ \hline \end{array}$$

of P_{r+t+1} which is ρ on

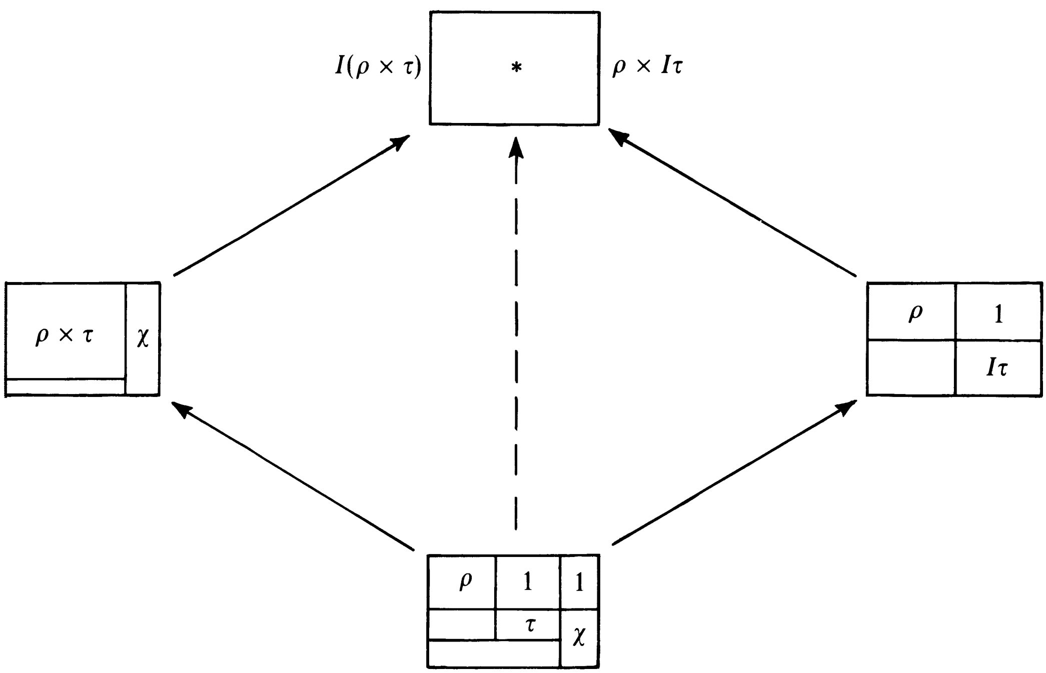
$$\begin{array}{c|c|c|} & r & t+1 \\ \hline r & * & 0 \\ \hline t+1 & 0 & I \\ \hline \end{array}$$

1 on

$$\begin{array}{c|c|c|c|} & r & t & 1 \\ \hline r & I & * & 0 \\ \hline t+1 & 0 & \multicolumn{2}{c|}{I} \\ \hline \end{array}$$

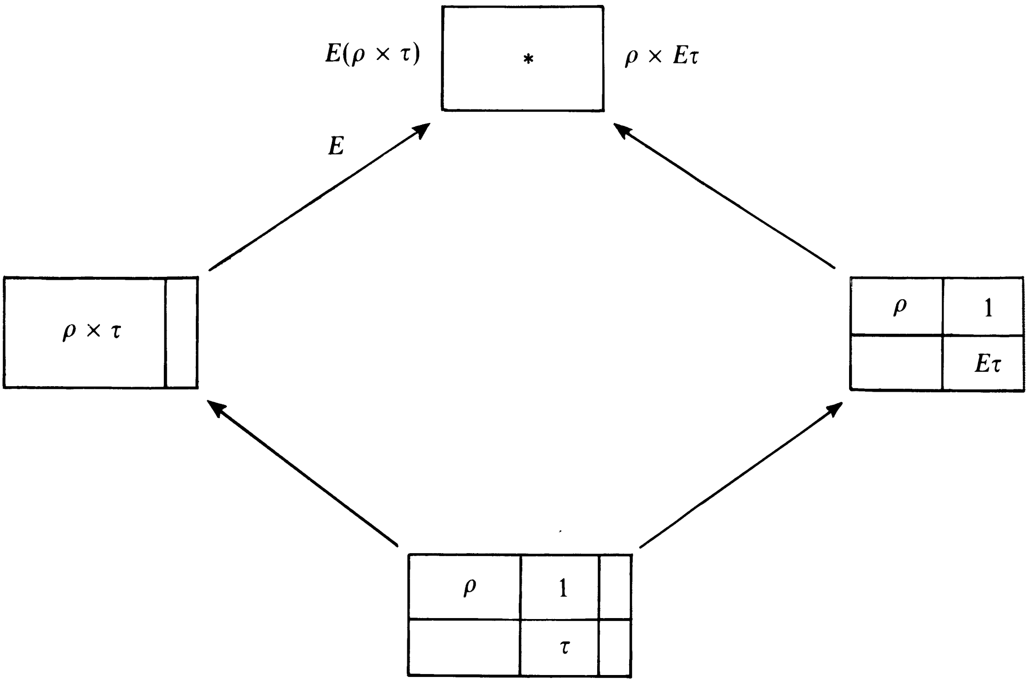
etc.

(ii) Consider the following diagram of subgroups of P_{r+t+1} and their representations.



Starting from the indicated representation at the bottom, the induced representation at the top ($-\ -\rightarrow$) may be computed via the left and right paths (shown by $\rightarrow$). The two methods yield $I(\rho \times \tau)$ and $\rho \times I\tau$ respectively.

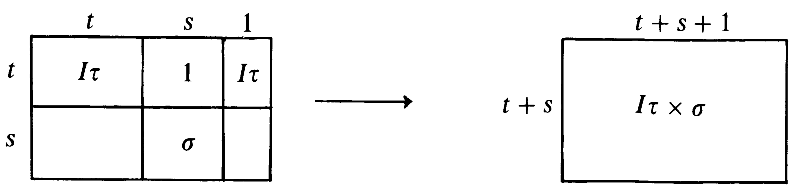
(iii) Consider the following diagram of subgroups of P_{r+t} and their representations



The arrows marked "E" correspond to the functor of trivial extension to the last column. The other two arrows correspond to unitary induction and from the definition of induction the diagram obviously commutes. The left and right paths yield $E(\rho \times \tau)$ and $\rho \times E\tau$ respectively.

(iv) The proof of this identity is a little long but quite straightforward.

(a) The representation $I\tau \times \sigma$ is obtained by induction in the following diagram



Let us write S for the group on the left and μ for the indicated representation.

Consider also the group R

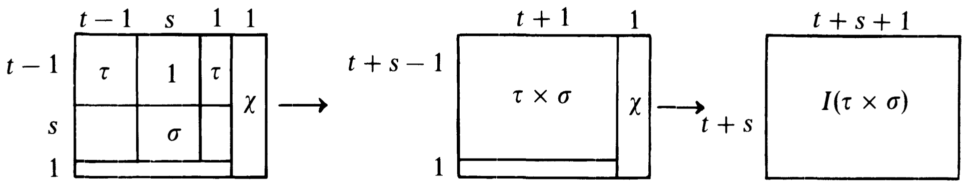
$$R = \begin{array}{c|c|c|} & t & s+1 \\ \hline t & * & * \\ \hline s & & * \\ \hline \end{array}$$

and let $\delta = \operatorname{Ind}_S^R(\mu)$.

Then, appealing to induction by stages, we have

$$I\tau \times \sigma = \operatorname{Ind}_R^{P_{t+s+1}}(\delta).$$

(b) On the other hand, the representation $I(\tau \times \sigma)$ is obtained by induction in the diagram shown below.



Where χ is the character of the normal subgroup $N \approx \mathbb{F}^{t+s}$ corresponding to the last column given by $\chi({}^t(x_1, \ldots, x_{t+s})) = e(x_{t+s})$.

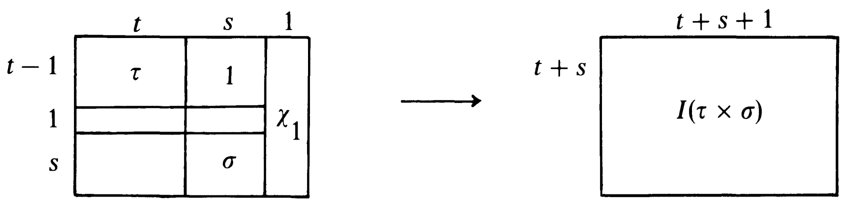
Let us write S_0 for the group on the left and μ_0 for the indicated representation, then

$$I(\tau \times \sigma) = \operatorname{Ind}_{S_0}^{P_{t+s+1}}(\mu_0).$$

Now, the group S_0 is not a subgroup of R. However, let w be the matrix of the cyclic permutation $(t, t+1, \ldots, t+s)$; and write $S_1 = w \cdot S_0$ and $\mu_1 = w \cdot \mu_0$ for the conjugates of S_0 and μ_0 by w. Then S_1 *is* a subgroup of R and, clearly,

$$I(\tau \times \sigma) = \operatorname{Ind}_{S_1}^{P_{t+s+1}}(\mu_1)$$

This identity corresponds to induction in the following diagram



Where $\chi_1 = w \cdot \chi_0$ so that $\chi_1({}^t(x_1, \dots, x_{s+t})) = e(x_t)$.

Let us write $\delta_1 = \mathrm{Ind}_{S_1}^{R}(\mu_1)$. Then it suffices to prove that δ and δ_1 are equivalent representations of R.

(c) Let $\mathscr{H}_\sigma$ and $\mathscr{H}_\tau$ be the representation spaces of σ and τ and write $\mathscr{H}$ for the Hilbert space tensor product of $\mathscr{H}_\sigma$ and $\mathscr{H}_\tau$. Our method of proof is to give explicit realizations of δ and δ_1 on the space $L^2(\mathbb{F}^{t+s}; \mathscr{H})$ and show that the desired equivalence is implemented by a partial Fourier transform.

(d) Note first that the group R is a semidirect product $Q \ltimes N$ where $N = \mathbb{F}^{s+t}$ and Q is a maximal parabolic subgroup of G_{t+s}.

$$Q = \begin{array}{c|c|c|} & t & s \\ \hline t & * & * \\ \hline s & & * \\ \hline \end{array}$$

R acts on N by *conjugation* and so also on its unitary dual V. V is again isomorphic to $\mathbb{F}^{t+s}$ and under the action of R, there is a unique orbit $\mathcal{O}$ which is open, dense and of full Lebesgue measure. Furthermore χ_1 belongs to $\mathcal{O}$ and

$$\mathrm{Stab}_R(\chi_1) = S_1.$$

so δ_1 may be realized explicitly on $L^2(\mathcal{O}; \mathscr{H}) \simeq L^2(V; \mathscr{H})$.

Carrying out the necessary computations, one obtains the following expressions.

$$\text{Suppose } r = \begin{array}{cc} & \begin{matrix} t-1 & 1 & s & 1 \end{matrix} \\ \begin{matrix} t-1 \\ 1 \\ s \\ 1 \end{matrix} & \begin{bmatrix} a & b & c & x \\ \alpha & \beta & \gamma & y \\ 0 & 0 & h & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \in R < P_{t+s+1}$$

and $v = (\overset{t-1}{\xi}, \overset{1}{\eta}, \overset{s}{\zeta}) \in V$.

$$\text{write } q = \begin{bmatrix} a & b & c \\ \alpha & \beta & \gamma \\ 0 & 0 & h \end{bmatrix} \quad \text{and } n = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let $p \in P_t$ be uniquely determined by solving

$$p \cdot \begin{array}{cc} & \begin{array}{cc} t-1 & 1 \end{array} \\ \begin{array}{c} t-1 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 \\ * & * \end{bmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cc} t-1 & 1 \end{array} \\ \begin{array}{c} t-1 \\ 1 \end{array} & \begin{bmatrix} 1 & 0 \\ \xi & \eta \end{bmatrix} \end{array} \cdot \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix}$$

Finally, let $|q| = |\det q|$ if $\mathbb{F} = \mathbb{R}$ and $|q| = |\det q|^2$ if $\mathbb{F} = \mathbb{C}$. Then

$$(\delta_1(r)f)(v) = (|q|^{1/2}\, \mathrm{e}(vn))[\tau(p) \otimes \sigma(h)]\, f(vq).$$

(e) Let N_t, N_s be the subgroups of N corresponding to the top t and lower s entries in the last column.

$$N_t = \begin{array}{cc} & \begin{array}{ccc} t & s & 1 \end{array} \\ \begin{array}{c} t \\ s \end{array} & \begin{array}{|c|c|c|} \hline & & * \\ \hline & & \\ \hline \end{array} \end{array} \qquad N_s = \begin{array}{cc} & \begin{array}{ccc} t & s & 1 \end{array} \\ \begin{array}{c} t \\ s \end{array} & \begin{array}{|c|c|c|} \hline & & \\ \hline & & * \\ \hline \end{array} \end{array}$$

Write V_t and V_s for the unitary duals of N_t and N_s. Then the representation $I\tau$ is realized on the space $L^2(V_t; \mathscr{H}_\tau)$.

Let T be the following subgroup of R:

$$T = \begin{array}{cc} & \begin{array}{ccc} t & s & 1 \end{array} \\ \begin{array}{c} t \\ s \end{array} & \begin{array}{|c|c|c|} \hline & & \\ \hline & * & * \\ \hline \end{array} \end{array}$$

Then the natural projection from R to T is a homomorphism.

Now T acts transitively on N_s by *left multiplication* and we may lift this action to R. Let 0 denote the identity element of N_s, then it is easily checked that

$$\mathrm{Stab}_R(0) = S.$$

so δ may be realized on the space:

$$\begin{aligned} &L^2(N_s; L^2(V_t; \mathscr{H}_\tau) \otimes \mathscr{H}_\sigma) \\ &\quad \approx L^2(V_t \oplus N_s; \mathscr{H}_\tau \otimes \mathscr{H}_\sigma) \\ &\quad \simeq L^2(V_t \oplus N_s; \mathscr{H}). \end{aligned}$$

Again, after some computation, one obtains the following explicit description. Let r, q, n, p be as before, and let

$$\mu = (\xi, \eta) \quad \text{with} \quad \xi \in \mathbb{F}^{t-1}, \eta \in \mathbb{F}, \quad \text{and} \quad u \in N_s \simeq \mathbb{F}^s.$$

Write

$$g = \begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix}, \; w = \begin{bmatrix} c \\ \gamma \end{bmatrix}, \; m = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } u' = h^{-1}(u - z).$$

Then, for $\varphi \in L^2(V_t \oplus N_s; \mathscr{H})$.

$$\delta(r)\varphi(\mu, u) = |h|^{-1/2}|g|^{1/2}\,\mathrm{e}(\mu(wu' + m))[\tau(p) \otimes (h)]\varphi(\mu g, u').$$

(f) Now corresponding to the decomposition $N = N_t \oplus N_s$, we have the dual decomposition $V = V_t \oplus V_s$.
Define

$$\mathscr{F}: L^2(V; \mathscr{H}) \to L^2(V_t \oplus N_s; \mathscr{H})$$

by

$$\mathscr{F}f(\mu, u) = \int \mathrm{e}(-\zeta u) f(\mu, \zeta)\,\mathrm{d}\zeta.$$

We shall now show that $\mathscr{F}$ intertwines δ and δ_1.

$$\mathscr{F}(\delta_1(r)f)(\mu, u) = \int \mathrm{e}(-\zeta\mu)\delta_1(r)f(\mu, \zeta)\,\mathrm{d}\zeta.$$

Now using the formula for $\delta_1(r)$ from (d) and writing $v = (\mu, \zeta)$, $n = [\begin{smallmatrix} m \\ z \end{smallmatrix}]$, the integrand above becomes

$$\begin{aligned} &|q|^{1/2}\,\mathrm{e}(-\zeta u + vn)[\tau(p) \otimes \sigma(h)]f(vq) \\ &\quad = |q|^{1/2}\,\mathrm{e}(\mu m - \zeta(u - z))[\tau(p) \otimes \sigma(h)]f(vq) \end{aligned}$$

so

$$\mathscr{F} \circ \delta_1(r)f(\mu, u) = |q|^{1/2}\,\mathrm{e}(\mu m)[\tau(p) \otimes \sigma(h)] \cdot \int (*)\,\mathrm{d}\zeta. \tag{$\dagger$}$$

with

$$\int (*)\,\mathrm{d}\zeta = \int \mathrm{e}(-\zeta(u-z))\,f(\overbrace{\mu g, \mu w + \zeta h}^{vq})\,\mathrm{d}\zeta$$

Substituting $\zeta' = \mu w + \zeta h$, this becomes

$$|h|^{-1}\int \mathrm{e}(-\zeta' - \mu w)h^{-1}(u-z))\,f(\mu g, \zeta')\,\mathrm{d}\zeta'$$

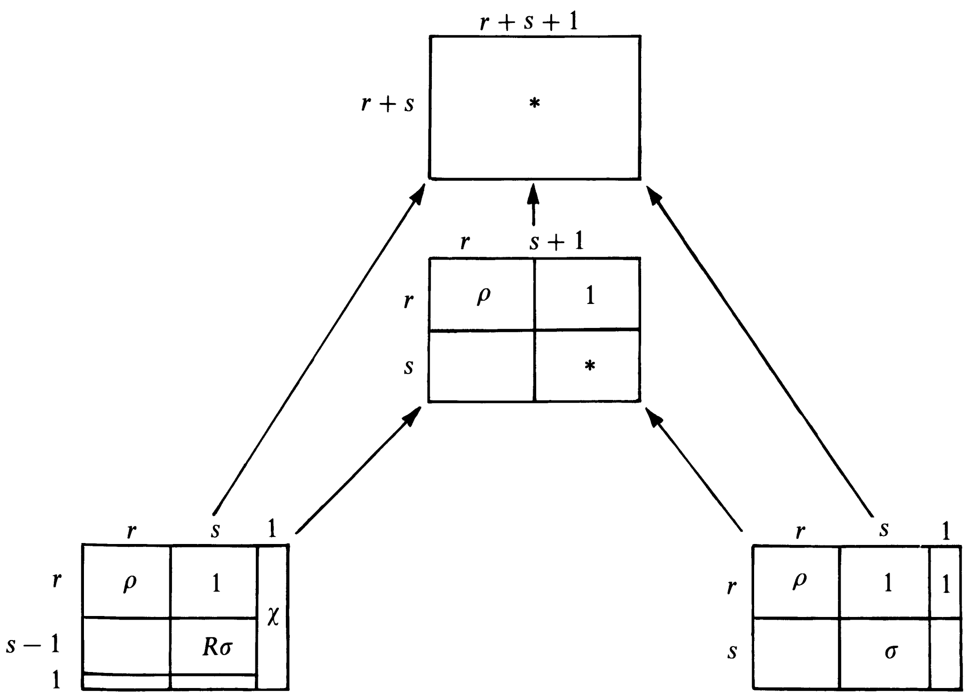
$$= |h|^{-1}\,\mathrm{e}(\mu w u')\int \mathrm{e}(-\zeta' u')\,f(\mu g, \zeta')\,\mathrm{d}\zeta'$$

$$= |h|^{-1}\,\mathrm{e}(\mu w u')\mathscr{F}f(\mu g, u').$$

Substituting in (†), and noting that $|\det q| = |\det g||\det h|$ we have

$$\begin{aligned}
&\mathscr{F}\circ\delta_1(r)f(\mu, u)\\
&\quad = |h|^{-1/2}|g|^{1/2}\,\mathrm{e}(\mu(\mathrm{w}u' + m))[\tau(p)\otimes\sigma(h)]\mathscr{F}(\mu g, u')\\
&\quad = \delta_2(\gamma)\circ\mathscr{F}f(\mu, u).
\end{aligned}$$

(v) Consider the following diagram



The representation obtained by induction along the extreme left arrow is $I(\rho \times R\sigma)$ and invoking induction by stages we see that

$$I(\rho \times R\sigma) \approx \rho \times IR\sigma.$$

Similarly $E\rho \times \sigma$ is obtained via the extreme right arrow and we see that

$$E\rho \times \sigma \approx \rho \times \mathrm{Ind}_{G_s}^{P_s-1}(\sigma).$$

So to complete the proof, it suffices to prove the following claim:

CLAIM. If σ is a unitary representation of G_s, then

$$\mathrm{Ind}_{G_s}^{P_s+1}(\sigma) \approx IR\sigma.$$

Proof. Write $P_{s+1} = G_s \ltimes N_s$ with $N_s \approx \mathbb{F}^s$. Then $\mathrm{Ind}_{G_s}^{P_s+1}(\sigma)$ can be realized on $L^2(N_s; \mathscr{H}_\sigma)$ with the action:

$$\gamma\left(\begin{bmatrix} g & x \\ 0 & 1 \end{bmatrix}\right)f(y) = |g|^{-1/2}\sigma(g)f(g^{-1}(y-x)). \tag{1}$$

On the other hand, let V be the dual group of N_s, then $I(R\sigma)$ is realizable on $L^2(V; \mathscr{H}_\sigma)$ with the following action:

For $\xi = (\xi_1, \ldots, \xi_s)$ in V, let $v = v(\xi)$ denote the $s \times s$ matrix

$$v(\xi) = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & & & & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & & & 1 & 0 \\ \xi_1 & \xi_2 & . & . & . & \xi_s \end{bmatrix}. \tag{2}$$

For $g \in G_s$ let $\xi' = \xi g$ and $v' = v(\xi')$. Then v' is invertible if $\xi'_s \neq 0$ (i.e. for almost all ξ), and if

$$p = vg(v')^{-1} \tag{3}$$

then p is an element of P_s.

With the above notation $\gamma_1 = I(R\sigma)$ is given by:

$$\gamma_1\left(\begin{bmatrix} g & x \\ 0 & 1 \end{bmatrix}\right)\varphi(\xi) = |g|^{1/2}(\mathrm{e}(\xi x))\sigma(p)\varphi(\xi') \tag{4}$$

Let $\mathscr{F}: L^2(N_s; \mathscr{H}_\sigma) \to L^2(V; \mathscr{H}_\sigma)$ be the (inverse) Fourier transform given by

$$\mathscr{F}f(\xi) = \int \mathrm{e}(\xi y) f(h)\,\mathrm{d}y.$$

Then

$$\mathscr{F} \circ \gamma\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} f(\xi) = \int \mathrm{e}(\xi y)|g|^{-1/2}\sigma(g) f(g^{-1}(y - x))\,\mathrm{d}y \qquad \text{by (1).}$$

After the substitution $z = g^{-1}(y - x)$, this becomes

$$\begin{aligned} &\int \mathrm{e}(\xi(gz + x))|g|^{-1/2}\sigma(g) f(z)\,\mathrm{d}z \\ &\quad = |g|^{-1/2}\,\mathrm{e}(\xi x)\sigma(g)\mathscr{F}f(\xi'). \end{aligned} \tag{5}$$

Now let $T: L^2(V; \mathscr{H}_\sigma) \to L^2(V; \mathscr{H}_\sigma)$ be the (unitary) "multiplication" operator

$$Tf(\xi) = \sigma(v) f(\xi)$$

where $v = v(\xi)$ as in (2). Then

$$\begin{aligned} &T \circ \mathscr{F} \circ \gamma\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} f(\xi) \\ &\quad = \sigma(v)|g|^{1/2}\,\mathrm{e}(\xi x)\sigma(g)\mathscr{F}f(\xi') \qquad \text{by (5),} \\ &\quad = |g|^{1/2}\,\mathrm{e}(\xi x)\sigma(v) \circ \sigma(g) \circ \sigma(v')^{-1}(T \circ \mathscr{F}) f(\xi'), \\ &\quad = |g|^{1/2}\,\mathrm{e}(\xi x)\sigma(p) T \circ \mathscr{F}) f(\xi') \qquad \text{by (2),} \\ &\quad = \gamma_1\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} \circ (T \circ \mathscr{F}) f(\xi) \qquad \text{by (4).} \end{aligned}$$

This shows that $T \circ \mathscr{F}$ is a (unitary) intertwining operator for γ and γ_1.

Q.E.D.

Here is the crucial

THEOREM 2.1. *If ρ and σ are adducible representations of* $\mathrm{GL}(r, \mathbb{F})$ *and* $\mathrm{GL}(s, \mathbb{F})$

of depths k and ℓ, then $\rho \times \sigma$ is adducible of depth $k + \ell$, and

$$A(\rho \times \sigma) = A\rho \times A\sigma.$$

Proof. By assumption, $R\rho \cong I^{k-1}E(A\rho)$ and $R\sigma \cong I^{\ell-1}(A\sigma)$. Now

$$\begin{aligned} R(\rho \times \sigma) &= (R\rho) \times \sigma && \text{by Lemma 2.1 (i)} \\ &= (I^{k-1}EA\rho) \times \sigma \\ &= I^{k-1}((EA\rho) \times \sigma) && \text{by (iv)} \\ &= I^{k-1}(I(A\rho \times R\sigma)) && \text{by (v)} \\ &= I^{k}(A\rho \times I^{\ell-1}EA\sigma) \\ &= I^{k+\ell-1}(A\rho \times EA\sigma) && \text{by (ii)} \\ &= I^{k+\ell-1}E(A\rho \times A\sigma) && \text{by (iii)} \end{aligned}$$

Q.E.D.

Section 3.

In this section we examine some consequences of Theorem 2.1. As a first step, we need to prove adducibility for some representations which serve as building blocks.

LEMMA 3.1.

(i) *If π is a unitary character of* GL(n, $\mathbb{F}$), *then π is adducible of depth* 1 *and $A\pi = \pi \,|\, \mathrm{GL}(n-1, \mathbb{F})$ where* GL($n-1, \mathbb{F}$) *is imbedded in the top left corner of* GL(n, $\mathbb{F}$).

(ii) *If π is a discrete series representation of* GL(2, $\mathbb{R}$), *then π is adducible of depth* 2 *and $A\pi$ is the trivial representation of the trivial group.*

Proof. (i) This is trivial.

(ii) This result is well-known, but for the sake of completeness we include the following sketch which uses some pretty ideas from classical complex analysis.

Recall [L] that the discrete series $\{\delta_m\}$ of GL(2, $\mathbb{R}$) is parametrized (up to the action of the center) by the positive integers $m \geqslant 2$, and the representation δ_m may be explicitly realized as follows.

Let $X = C\backslash\mathbb{R}$ be the complex plane minus the real axis. Let

$$D_m(X) = L^2_{\text{hol}}\left(X; y^{\mathrm{m}}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\right), \quad \mathrm{m} \geqslant 2,$$

$$H^2(X) = \left\{ f \in \mathrm{Hol}(X) \colon \lim_{y\to 0+} \int |f(x+iy)|^2\,\mathrm{d}x < \infty \quad \text{and} \right.$$

$$\left. \lim_{y\to 0-} \int |f(x+iy)|^2\,\mathrm{d}x < \infty \right\}.$$

Then the $D_m(X)$ are the representation spaces for the discrete series representation δ_m and $H^2(X)$ is the usual Hardy space (on which the limit of discrete series may be realized).

Now it is a consequence of the Payley-Wiener theorem that

$$S\colon L^2(\mathbb{R}) \to H^2(X).$$

$S\varphi(z) = \int_{-\infty}^{\infty} e^{izt}\,\varphi(t)\,dt$ is an isometry up to a constant; and it is an easy matter to show that

$$S_m\colon L^2(\mathbb{R}) \to D_m(X) \qquad \text{given by}$$

$$S_m\varphi(z) = \int_{-\infty}^{\infty} e^{izt}(it)^{(m+1)/2}\,\varphi(t)\,dt$$

is also an isometry up to a constant.

Finally, let $\mathscr{F}$ be the Fourier transform from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, and let T_m be the operator $S_m \circ \mathscr{F}^{-1}\colon L^2(\mathbb{R}) \to D_m(X)$. Then T_m is an isometry up to a constant and it is completely straightforward to check that

$$(T_m^{-1} \circ \delta_m \circ T_m)\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} f(\xi) = |a|^{-1/2} f(a^{-1}\xi)$$

$$(T_m^{-1} \circ \delta_m \circ T_m)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(\xi) = e^{ib\xi} f(\xi).$$

These formulas show that $(T_m^{-1} \circ \delta_m \circ T_m) \simeq IE(1_0)$ where 1_0 is the trivial representation of the trivial group G_0. Q.E.D.

Before continuing, let us note the following immediate corollary, part of which is well known and may be proved by other methods. (See [W], for (i) and (ii) for $\mathrm{GL}(n, \mathbb{C})$ and [KS] for $\mathrm{GL}(n, \mathbb{R})$).

THEOREM 3.1.

(i) *Every irreducible tempered representation of* $\mathrm{GL}(n, \mathbb{C})$ *or* $\mathrm{GL}(n, \mathbb{R})$ *is fully induced from a relative discrete series representation of some Levi subgroup.*

(ii) *If* $\sigma_1, \dots, \sigma_k$ *are irreducible and tempered so is* $\sigma_1 \times \cdots \times \sigma_k$.

(iii) *Every tempered representation of* $\mathrm{GL}(n, \mathbb{C})$ *and* $\mathrm{GL}(n, \mathbb{R})$ *is adducible of (maximal) depth n, so that the adduced representation is the trivial representation of the trivial group.*

Proof. Let σ be a tempered representation of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$. Then, by a well known result of Harish-Chandra, there is a cuspidal parabolic subgroup $M \cdot N$ and a relative discrete series representation δ of M such that σ is a subrepresentation of $\mathrm{Ind}_{MN}^{G}(\delta \otimes 1)$.

In the case of $\mathrm{GL}(n, \mathbb{C})$, M is necessarily of the form $\mathrm{GL}(1, \mathbb{C}) \times \cdots \times \mathrm{GL}(1, \mathbb{C})$ and $\delta = \pi_1 \otimes \cdots \otimes \pi_n$ where π_i are unitary characters. For $\mathrm{GL}(n, \mathbb{R})$, M has to be of the form

$$\mathrm{GL}_1(\mathbb{R}) \times \cdots \times \mathrm{GL}_1(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R}) \times \cdots \times \mathrm{GL}_2(\mathbb{R}) \text{ and } \delta = \pi_i \otimes \cdots \otimes \pi_m$$

where the π_i's are unitary characters for the $\mathrm{GL}(1, \mathbb{R})$'s and discrete series representations for the $\mathrm{GL}(2, \mathbb{R})$'s.

So $\mathrm{Ind}_{M \cdot N}^{G}(\delta) = \pi_1 \times \cdots \times \pi_k$ where each π_i is either a character of G, or a discrete series representation. Now, by Lemma 3.2 all the π_i's are adducible and $A\pi_i$ is the trivial representation of the trivial group in each case.

So, by Theorem 2.1, $\pi_1 \times \cdots \times \pi_k$ is adducible and

$$\begin{aligned} A(\pi_1 \times \cdots \times \pi_k) &= A\pi_1 \times \cdots \times A\pi_k \\ &= 1_0 = \text{trivial representation of the trivial group.} \end{aligned}$$

This shows first of all that $\pi_1 \times \cdots \times \pi_k) \mid P_n = I^{n-1}E1$ and is therefore irreducible; second since, by assumption σ is contained in $\pi_1 \times \cdots \times \pi_k$, σ must actually be equal to $\pi_1 \times \cdots \times \pi_k$. This proves (i) and (iii). To prove (ii) we note that we can first write each σ_i as $\pi_{il} \times \cdots \times \pi_{in}$ where π_{ij} is either a unitary character of G, or a discrete series representation. So $\sigma_1 \times \cdots \times \sigma_k = \pi_{11} \times \pi_{12} \times \cdots \times \pi_{kn}$ which is irreducible by the argument as before. Q.E.D.

We now continue with the main argument.

LEMMA 3.2. *For $s \in (0, \frac{1}{2})$, let $\sigma_{2m}(j, s)$ the Stein complementary series representation with parameter s. (See [V], [Ge], [St]) of $\mathrm{GL}(2m, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $\sigma_{2m}(j, s) \mid p$ is irreducible.*

Proof. As remarked in [V], it suffices to treat the case of $\sigma_{2m}(s) = \sigma_{2m}(1, s)$. Let Q be the parabolic subgroup of GL_{2m}. It follows by [St] that $\sigma_{2m}(s) \mid Q$ are all equivalent, irreducible and isomorphic to the unitarily induced degenerate principal series representation $(1_m \times 1_m) \mid Q$ where 1_m is the trivial representation of GL_m.

$$\begin{array}{c|c|c|} & m & m \\ \hline m & * & * \\ \hline m & & * \\ \hline \end{array}$$

Let $S \subset P \cup Q$ be the "strip" group $\begin{array}{|c|c|} \hline * & * \\ \hline 0 & 1 \\ \hline \end{array}$

Then, it is clearly enough to show that $(1_m \times 1_m)|S$ is irreducible. This is implicit in [St, p. 479] and we sketch the necessary extension of his argument.

First of all, the representation $(1_m \times 1_m)|S$ may be explicitly realized on $L^2(M_m; \mathbb{C})$ with the action

$$R_g \cdot f(x) = |a|^{m/2} f(ax + b),$$

where $g^{-1} = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in S$.

Let T be an intertwining operator for $(1_m \times 1_m)|S$. Since T commutes with translations, its Fourier transform is the operator of multiplication by a bounded measurable function $\mu(x)$. Further, since T commutes with the action of $\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]$, it follows that $\mu(a \cdot x) = \mu(x)$ almost everywhere. But under the action of GL_m there is an orbit of full measure in M_m. So μ must be constant on this orbit, hence T must be a scalar. This proves the irreducibility of $1_m \times 1_m | S$ and the lemma. Q.E.D.

We are now ready to prove Kirillov's conjecture for $\mathbb{C}$. For this we shall use the following result of Vogan.

FACT 3.1 [V]. Every irreducible unitary representation of $\mathrm{GL}_n(\mathbb{C})$ is the $\times$-product of unitary characters and Stein representations.

THEOREM 3.2.
(i) *If π is an irreducible unitary representation of* $\mathrm{GL}(n, \mathbb{C})$ *then $\pi|P$ is irreducible.*
(ii) *If π_i are irreducible unitary representations of* $\mathrm{GL}(n_i, \mathbb{C})$ *and $n = \Sigma n_i$ then $X_i \pi_i$ is an irreducible unitary representation of* $\mathrm{GL}(n, \mathbb{C})$.

Proof. We shall prove (i) and (ii) *simultaneously* by induction on n.

For $n = 0$ or 1 the results are trivially true. Let us assume it for $n \leqslant m$ and let π be an irreducible unitary representation of $\mathrm{GL}(n + 1, \mathbb{C})$. Then, if π is a unitary character or Stein complementary series representation, (i) is implied by Lemmas 3.1 and 3.2. If not then by Fact 3.1, we can write $\pi = \pi_1 \times \pi_2$ with π_1 irreducible unitary representations of $\mathrm{GL}(n; \mathbb{C})$ and each $n_i \leqslant m$. By the inductive hypothesis, each $\pi_i|P$ is irreducible. So by Theorem 2.1 $\pi|P$ is homogeneous and

$$A\pi = A(\pi_1 \times \pi_2) = A\pi_1 \times A\pi_2,$$

where $A\pi_i$ are irreducible representations of $\mathrm{GL}(m_i, \mathbb{C})$ with $m_i = n_i -$ depth$(\pi_i) < n_i$. Thus $m_1 + m_2 < n_1 + n_2 = m + 1$.

Again, by the inductive hypothesis, $A\pi = A\pi_1 \times A\pi_2$ is irreducible. This implies that $\pi|P$ is irreducible, establishing (i) for $n = m + 1$. In fact, the argument just given also proves (ii). Q.E.D.

In conclusion, some remarks are in order.

First, Vogan [V] has shown that every irreducible unitary representation of $\mathrm{GL}(n, \mathbb{R})$ is a $\times$-product of the following:

(i) Unitary characters
(ii) Stein representation
(iii) Speh representations
(iv) Speh complementary series representations.

Our approach shows that in order to prove Conjectures 1.1 and 1.2 for $\mathbb{R}$ it suffices to establish Kirillov's conjecture for representations of types (iii) and (iv).

Secondly, the notion of N-rank of a unitary representation of $\mathrm{GL}(n, \mathbb{F})$ has been considered in [H] and [Sc], and its connection with rates of decay of matrix coefficients has been established in [Sc]. In the existing definition, the rank of a representation is at most $[n/2]$. This is unsatisfactory since a disproportionately large number of representations of $\mathrm{GL}(n, \mathbb{F})$ turn out to have rank $[n/2]$. In this connection it seems that the following alternative definition has some merit:

DEFINITION 3.1. A representation $\pi \in \mathrm{GL}(n, \mathbb{C})$ has rank $n - k$ if k is the smallest value of the index i such at $A^i\pi$ is the trivial representation of the trivial group.

Using results from [Sc] it is not too hard to show that this definition agrees with the earlier definition for rank $< [n/2]$ and further differentiates among representations of rank$[n/2]$. Also, Theorem 3.1 implies that tempered representations have rank $n - 1$.

Once Kirillov's conjecture is proved for $\mathbb{R}$, one can use the same definition for $\mathrm{GL}(n, \mathbb{R})$.

Finally, M. Tadic [T1] has shown that the full unitary dual of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ may be deduced from an a priori knowledge of Kirillov's conjecture (more specifically from Conjecture 1.2). It seems possible that one might be able to give an inductive argument that simultaneously establishes both Kirillov's conjecture and the classification.

Added in proof: It is shown in [S] that $A\sigma_{2m}(j, s) = \sigma_{2m-2}(j, s)$ for all $m \geqslant 1$. Taken together with Theorem 2.1 and Fact 3.1, this enables one to explicitly compute $A\pi$ for all irreducible unitary representations of $\mathrm{GL}(n, \mathbb{C})$.

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