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The discriminants of curves of genus 2

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In our previous paper [S1], we studied the relation between the discriminant and the conductor of curves of arbitrary genus. In the case of genus 2, there is another discriminant slightly different from that used there. The former is defined by using the canonical isomorphism of Mumford and the latter is classical and related to the Siegel modular form of weight 10. The purpose of this paper is to study the difference between these two discriminants. Our main result asserts that it is determined by the base locus of the canonical divisor when the curves degenerate.

Let g: $Y \to T$ be a proper and smooth morphism with geometrically connected fibers and of relative dimension 1. Then there exists a unique canonical isomorphism [D]

$$\Delta\colon \det Rg_*\omega_{Y/T}^{\otimes 2} \to (\det Rg_*\omega_{Y/T})^{\otimes 13}.$$

Here $\omega_{Y/T}$ is the relative canonical sheaf. In this case, it is isomorphic to $\Omega^1_{Y/T}$. The notation det denotes the determinant invertible sheaf of perfect complex ([K–M] Chapter 1). In this case $\det Rg_*\omega_{Y/T}$ (resp. $\det Rg_*\omega_{Y/T}^{\otimes 2}$) is simply isomorphic to $\wedge^{\gamma} g_*\omega_{Y/T}$ (resp. $\wedge^{3\gamma-3} g_*\omega_{Y/T}^{\otimes 2}$) where γ is the genus of Y. When the genus is 2, there is a canonical section $\Delta' \in \Gamma(T, (\det Rg_*\omega_{Y/T})^{\otimes 10})$. This is the well known Siegel modular form of weight 10 ([M] p. 317). It is defined using Δ as follows. It is well known that the canonical morphism

$$S^2 g_*\omega_{Y/T} \to g_*\omega_{Y/T}^{\otimes 2}$$

is an isomorphism where S^2 denotes the second symmetric power. From this we have a canonical isomorphism

$$\Delta_1\colon (\det Rg_*\omega_{Y/T})^{\otimes 3} \to \det Rg_*\omega_{Y/T}^{\otimes 2}.$$

Then Δ' is defined by

$$\Delta' = (\Delta \circ \Delta_1) \otimes id\colon \mathcal{O}_T \to (\det Rg_*\omega_{Y/T})^{\otimes 10}.$$

Therefore to know the difference between these two discriminants Δ and Δ', it is sufficient to study Δ_1.

Now let us study it in the situation of [S1]. Let S be the spectrum of a discrete valuation ring with algebraically closed residue field. Let s (resp. η) be the closed (resp. generic) point of S. Let $f\colon X \to S$ be a proper flat geometrically connected and regular S-scheme with smooth generic fiber and of relative dimension 1. Assume that the genus of X is 2. Then we have the canonical non-zero rational sections Δ, Δ' and Δ_1 of the invertible $\mathcal{O}_S$-modules $\mathrm{Hom}_{\mathcal{O}_S}(\det Rf_*\omega_{X/S}^{\otimes 2}, (\det Rf_*\omega_{X/S})^{\otimes 13})$, $(\det Rf_*\omega_{X/S})^{\otimes 10}$ and $\mathrm{Hom}_{\mathcal{O}_S}((\det Rf_*\omega_{X/S})^{\otimes 3}, \det Rf_*\omega_{X/S}^{\otimes 2})$ respectively. For a non-zero rational section l of an invertible $\mathcal{O}_S$-module L, let ord l denote the unique integer such that $\mathcal{O}_S \cdot l = \mathfrak{p}^n L$ where $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}_S$. We called ord Δ the discriminant of X over S in [S1] Section 1. Our purpose is to know ord Δ_1.

We will show that ord Δ_1 is determined by the base locus of the relative canonical sheaf $\omega_{X/S}$. Let $\kappa\colon f^*f_*\omega_{X/S} \to \omega_{X/S}$ be the canonical morphism. The support of the cokernel of κ, i.e., the base locus of $\omega_{X/S}$, is in X_s since the canonical divisor of a smooth curve of genus $\geqslant 1$ is base point free. Let I be the annihilater ideal of the cokernel of κ, let D be the unique divisor of X such that $I_D \supset I$ and I_D/I is of finite length and let R be the 0-cycle $\Sigma_{x \in X}$ $\mathrm{length}_{\mathcal{O}_{X,x}}(I_D/I)_x \cdot [x]$. Then the support of D and R are in X_s. Our main result is the following.

THEOREM: *Under the above notation,*

$$\text{ord } \Delta_1 = 2c_1(\omega_{X/S}) \cap D - D^2 + \deg R.$$

COROLLARY 1: *If $\omega_{X/S}$ is base point free, then Δ_1 defines an isomorphism of invertible $\mathcal{O}_S$-modules*

$$(\det Rf_*\omega_{X/S})^{\otimes 3} \xrightarrow{\sim} \det Rf_*\omega_{X/S}^{\otimes 2}.$$

Assume X is minimal. Then Δ_1 always defines a homomorphism of $\mathcal{O}_S$-modules and it is an isomorphism if and only if $\omega_{X/S}$ is base point free.

COROLLARY 2: *Under the assumption of the theorem,*

$$\text{ord } \Delta' = -\mathrm{Art}(X/S) + 2c_1(\omega_{X/S}) \cap D - D^2 + \deg R,$$

where Art(X/S) *is the Artin conductor of* X *over* S *defined in* [*S1*] *Section 1. Always* Δ' *defines a section of* $\mathcal{O}_S$*-module and it is a generator if and only if the minimal model of* X *is smooth.*

First we give the proof of the corollaries admitting the theorem.

Proof of Corollary 1: First, we remark that Δ_1 defines a homomorphism (resp. an isomorphism) of $\mathcal{O}_S$-modules if and only if ord $\Delta_1 \geqslant 0$ (resp. ord $\Delta_1 = 0$). If $\omega_{X/S}$ is base point free, it is clearly follows from the theorem that ord $\Delta_1 = 0$. Assume X is minimal. Then $c_1(\omega_{X/S}) \cap D \geqslant 0$. On the other hand, always we have $-D^2 \geqslant 0$ and deg $R \geqslant 0$. Hence we have the first assertion. For the second, ord $\Delta_1 = 0$ is now equivalent to $c_1(\omega_{X/S}) \cap D = D^2 = \deg R = 0$. It is easy to see that this holds if and only if $D = 0$ and $R = 0$. This condition is evidently equivalent to that $\omega_{X/S}$ is base point free.

Proof of Corollary 2: The equality immediately follows from the theorem above, Theorem 1 of [S1] and the equality

$$\operatorname{ord} \Delta' = \operatorname{ord} \Delta + \operatorname{ord} \Delta_1 .$$

To see the rest, we may assume X is minimal since ord Δ' is invariant under blowing-up. Now the assertion follows from Corollary 1 above and Corollary 1 to Theorem 1 of [S1].

Proof of Theorem: We consider the perfect complex of $\mathcal{O}_X$-modules $M = [f^*f_*\omega_{X/S} \to \omega_{X/S}]$. Here $f^*f_*\omega_{X/S}$ (resp. $\omega_{X/S}$) is put on degree 0 (resp. 1). This complex is considered by Mumford ([M] Section 8). It is easy to see that $H^0(M)$ is an invertible $\mathcal{O}_X$-module. The support of $H^1(M)$ i.e., the base locus of $\omega_{X/S}$ is in X_s and other cohomology sheaves vanish. Hence the construction of the localized chern class of Bloch–MacPherson ([B] Section 1, [F] Section 18) can be naturally applied to this case and we obtain $c^X_{2X_s}(M)$ in $\mathrm{CH}_0(X_s)$.

Next we consider the derived 2nd exterior power complex $L\wedge^2 M$. The complex $L\wedge^2 M$ is defined to be

$$[\wedge^2 f^*f_*\omega_{X/S} \xrightarrow{\kappa\otimes 1|_{\wedge 2}} \omega_{X/S} \otimes f^*f_*\omega_{X/S} \xrightarrow{1\otimes\kappa} \omega_{X/S}^{\otimes 2}],$$

where $\wedge^2 f^*f_*\omega_{X/S}$ is put on degree 0. Although the general definition of the derived exterior power complex is given by the method of Dold–Puppe

([I] Chapter 1 Section 4), for our purpose, it is more convenient to refer [S2] Section 1. For the perfect complex K of amplitude in [0, 1], we can define $L \wedge^q K$ to be $D(L \wedge^q DK)$. Here $DK = R\,\mathrm{Hom}_{\mathcal{O}_X}(K, \mathcal{O}_X)$ and the definition given there applies to it. More directly, for the compex $K = [E \to L]$, where E is put on degree 0, $L \wedge^q K$ is defined by

$$L \wedge^q K = [\wedge^q E \to \wedge^{q-1} E \otimes L \to \wedge^{q-2} E \otimes S^2 L \to \dots \\ \to E \otimes S^{q-1} L \to S^q L],$$

where $\wedge^q E$ is put on degree 0.

Since $L \wedge^2 M$ is acyclic outside the base locus of $\omega_{X/S}$, the localized chern class $c^X_{2X_s}(L \wedge^2 M) \in CH_0(X_s)$ is also defined. We will prove the following three equalities.

(a) $\deg c^X_{2X_s}(M) = 2c_1(\omega_{X/S}) \cap D - D^2 + \deg R$.

(b) $\deg c^X_{2X_s}(L \wedge^2 M) = -\mathrm{ord}\,\Delta_1$.

(c) $c^X_{2X_s}(M) = -c^X_{2X_s}(L \wedge^2 M)$.

The theorem follows from these equalities at once.

The equality (c) and the fact that $c^X_{1X_s}(L \wedge^2 M) = 0$ is proved in quite the same way as Proposition (2.1) of [S2]. Or, we may directly apply it to the dual DK. We show the equality (b). By applying Proposition (2.3) of [S2], we have $-\deg c^X_{2X_s}(L \wedge^2 M) = \chi(S, Rf_* L \wedge^2 M)$. Hence we are reduced to show

(b′) $\mathrm{ord}\,\Delta_1 = \chi(S, Rf_* L \wedge^2 M)$.

First, we compute $\mathrm{ord}\,\Delta_1$. Since $f\colon X \to S$ is cohomologically flat ([R] Proposition (9.5.1)), we have $\det Rf_*\omega_{X/S} = \det f_*\omega_{X/S}$. Now it is easy to see, and follows from Lemma 2 of [S1] for example, that

$$\mathrm{ord}\,\Delta_1 = \chi(S, [S^2 f_*\omega_{X/S} \to Rf_*\omega^{\otimes 2}_{X/S}]).$$

Here $S^2 f_*\omega_{X/S}$ is put on degree -1. Next we compute $Rf_* L \wedge^2 M$. We consider the spectral sequence

$$E_1^{p,q} = R^q f_*(L \wedge^2 M)^p \Rightarrow R^{p+q} f_* L \wedge^2 M.$$

The complex of E_1-terms is

$$E_1^{\cdot,q} = \begin{cases} [\wedge^2 f_*\omega_{X/S} \to \otimes^2 f_*\omega_{X/S} \to f_*\omega^{\otimes 2}_{X/S}] & q = 0 \\ [\wedge^2 f_*\omega_{X/S} \otimes R^1 f_*\mathcal{O}_X \to f_*\omega_{X/S} \to R^1 f_*\omega^{\otimes 2}_{X/S}] & q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since the sequence

$$0 \to \wedge^2 f_*\omega_{X/S} \to \otimes^2 f_*\omega_{X/S} \to S^2 f_*\omega_{X/S} \to 0$$

is exact, $E_1^{\cdot,0}$ is quasi-isomorphic to $[S^2 f_*\omega_{X/S} \to f_*\omega_{X/S}^{\otimes 2}]$, where $S^2 f_*\omega_{X/S}$ is put on degree 1. For $q = 1$, by the Serre duality $R^1 f_*\mathcal{O}_X \simeq \mathrm{Hom}_{\mathcal{O}_S}(f_*\omega_{X/S}, \mathcal{O}_S)$,

$$\wedge^2 f_*\omega_{X/S} \otimes R^1 f_*\mathcal{O}_X \to f_*\omega_{X/S}$$

is the canonical isomorphism. Hence $E_1^{\cdot,1}$ is quasi-isomorphic to $R^1 f_*\omega_{X/S}[-2]$. Now it is easy to see that

$$\chi(S, L \wedge^2 M) = \chi(S, [S^2 f_*\omega_{X/S} \to Rf_*\omega_{X/S}^{\otimes 2}]).$$

Hence we have proved the equalities (b′) and (b).

Lastly we prove (a). This is merely a computation of the chern class. Let $A = H^0(M) = \mathrm{Ker}\, \kappa$ and $B = H^1(M) = \mathrm{Coker}\, \kappa$. We recall that A is an invertible $\mathcal{O}_X$-module. Then we have

$$c_1(M) = -c_1(\omega_{X/S})$$

and

$$c_{2X,}^{X}(M) = -c_{2X,}^{X}(B) - c_1(M) \cap c_{1X,}^{X}(B) \qquad (*)$$

$$= -c_{2X,}^{X}(B) + c_1(\omega_{X/S}) \cap c_{1X,}^{X}(B).$$

Let $B' = B \otimes_{\mathcal{O}_X} \mathcal{O}_D$ and $B'' = \mathrm{Ker}(B \to B')$. Here $B' = \omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ and $B'' \simeq I_D/I$. Hence

$$c_{1X,}^{X}(B') = D, \; c_{2X,}^{X}(B') = -(c_1(\omega_{X/S}) - D) \cap D$$

and

$$c_{1X,}^{X}(B'') = 0, \; c_{2X,}^{X}(B'') = -R.$$

From the exact sequence $0 \to B'' \to B \to B' \to 0$, we have

$$c_{1X,}^{X}(B) = D, \; c_{2X,}^{X}(B) = -(c_1(\omega_{X/S}) - D) \cap D - R.$$

By substituting this to $(*)$, we obtain the equality (a). Thus we have completed the proof of the theorem.

We compute ord Δ_1 in the semi-stable case. Let $f\colon X \to S$ be a regular curve of genus 2 as above. When X_s is a reduced normal crossing divisor, we say X is semi-stable.

PROPOSITION: *Assume X is minimal and semi-stable. Let P be the set of singular points of X_s such that $X_s - \{p\}$ is disconnected. Then*

$$\text{ord } \Delta_1 = \text{Card } P.$$

Proof: When X satisfies the following condition, we say X is of type 1 and otherwise X is of type 0.
$(*)$ By suitably choosing the numbering of the irreducible components C_i $0 \leqslant i \leqslant m$ of X_s, the following equalities are satisfied

$$(C_i, C_j) = \begin{cases} 1 & i - j = \pm 1 \\ -2 & i = j \neq 0, m \\ -1 & i = j = 0, m \\ 0 & \text{otherwise.} \end{cases}$$

$$(C_i, c_1(\omega_{X/S})) = \begin{cases} 1 & i = 0, m \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 1: *If X is of type 0, $\omega_{X/S}$ is base point free and $P = \varnothing$. Assume X is of type 1. Let p_i be the intersection point of C_i and C_{i+1} for $0 \leqslant i < m$. Then*

$$D = \begin{cases} \sum_{k=1}^{r-1} k(C_k + C_{m-k}) + rC_r & \text{if } m = 2r \\ \sum_{k=1}^{r} k(C_k + C_{m-k}) & \text{if } m = 2r + 1 \end{cases}$$

$$R = \begin{cases} 0 & \text{if } m = 2r \\ [p_r] & \text{if } m = 2r + 1 \end{cases}$$

and $P = \{p_i; 0 \leqslant i < m\}$.

We show Lemma 1 implies Proposition. If X is of type 0, ord $\Delta_1 =$ Card $P = 0$ by Lemma 1 and Corollary 1 of Theorem. Assume X is of type 1. Then by some easy calculation, we see ord $\Delta_1 = m$ follows from Lemma 1 and Theorem. On the other hand, it is clear that Card $P = m$.

Proof of Lemma 1: The assertions on P is easy to check. To study the base locus of $\omega_{X/S}$, we use the following fact which is stated in [M] Footnote 5.

LEMMA 2: *Let Z be a stable curve over a field k. Let B be the base locus of $\omega_{Z/k}$. Let W be the set of singular points p of Z such that $Z - \{p\}$ is disconnected and let V be the set of the smooth and rational irreducible components C of Z such that every singular point of Z in C is contained in W. Then*

$$B = W \cup \bigcup_{C \in V} C.$$

We will give a proof of Lemma 2 later in Appendix, because the author cannot find a proper reference of the proof.

We continue the proof of Lemma 1 admitting Lemma 2. Let $g\colon Y \to S$ be the stable model of X i.e., the contraction $\pi\colon X \to Y$ of the (-2)-curves in X. Then the base locus of $\omega_{X/S}$ is the inverse image of $\omega_{Y/S}$ since $\pi^*\omega_{Y/S} = \omega_{X/S}$ and $\pi_*\omega_{X/S} = \omega_{Y/S}$. Since the formation of $g_*\omega_{Y/S}$ commutes with the base change, the base locus of $\omega_{Y/S}$ has the same support as that of $\omega_{Y_s/s}$. Assume X is of type 0. Then we can easily check that W and V for Y_s is empty. Hence by Lemma 2, the base locus of $\omega_{Y_s/s}$ is empty. Therefore that of $\omega_{Y/S}$ and that of $\omega_{X/S}$ are also empty.

Now we assume X is of type 1. Let D_k be the divisor $\Sigma_{i=0}^{k} (k - i) \cdot C_i$ of X for $1 \leqslant k \leqslant m$. We will show the following, later.

CLAIM: $\omega_{X/S}(D_k)|_{D_k} \simeq \mathcal{O}_{D_k}$ for $1 \leqslant k \leqslant m$.

We continue the proof admitting Claim. Let E_k be the divisor $k \cdot X_s$. Since the formation of $f_*\omega_{X/S}$ commutes with the base change, we have

$$H^0(X, \omega_{X/S}) \otimes_{\mathcal{O}_S} \mathcal{O}_S/\mathfrak{p}^k = H^0(E_k, \omega_{X/S}|_{E_k}).$$

We define an injection $\alpha\colon \mathcal{O}_{D_k} \to \omega_{X/S}|_{E_k}$ as the composite of the following morphisms. The isomorphism in the claim $\mathcal{O}_{D_k} \simeq \omega_{X/S}(D_k)|_{D_k}$, an isomorphism $\omega_{X/S}(D_k)|_{D_k} \simeq \omega_{X/S}(D_k - E_k)|_{D_k}$ defined by an element of $\mathcal{O}_S$ of order k, and the canonical injection $\omega_{X/S}(D_k - E_k)|_{D_k} \to \omega_{X/S}|_{E_k}$. Let $a \in H^0(X, \omega_{X/S})$ be a section of $\omega_{X/S}$ such that the reduction $\bar{a} \in H^0(E_m, \omega_{X/S}|_{E_m})$ comes from

$1 \in H^0(D_m, \mathcal{O}_{D_m})$ by α. By inverting the numbering of the irreducible components, similarly we define a section b of $H^0(X, \omega_{X/S})$. We prove that (a, b) is a basis of the free $\mathcal{O}_S$-module $H^0(X, \omega_{X/S})$ of rank 2. For this, it is sufficient to show that $(\bar{a}, \bar{b})$ is a basis the $\kappa(s)$-vector space $H^0(X_s, \omega_{X_s/s})$. Since $\omega_{X/S}(C_0 - X_s)|_{C_0} \oplus \omega_{X/S}(C_m - X_s)|_{C_m}$ naturally defines the subsheaf of $\omega_{X/S}|_{X_s}$, we have an injection

$$H^0(C_0, \omega_{X/S}(C_0 - X_s)|_{C_0}) \oplus H^0(C_m, \omega_{X/S}(C_m - X_s)|_{C_m}) \to H^0(X_s, \omega_{X/S}|_{X_s}).$$

By Claim, $(\bar{a}, \bar{b})$ is a basis of the vector space on the left hand side. Hence this injection is an isomorphism and $(\bar{a}, \bar{b})$ is a basis of $H^0(X_s, \omega_{X/S}|_{X_s})$.

Now we calculate the base locus of $\omega_{X/S}$ using the basis (a, b) of $H^0(X, \omega_{X/S})$. Let A be the effective divisor div $a - \Sigma_{k=0}^m k \cdot C_k$. Then by the definition of a, we have $(C_k, A) = 0$ for $k \neq m$. By definition, the divisor A is a linear combination of C_m and a flat divisor. Hence A is a flat divisor. In the same way, the effective divisor $B =$ div $b - \Sigma_{k=0}^m (m - k) \cdot C_k$ is a flat divisor and $(C_k, B) = 0$ for $k \neq 0$. Using this, we can easily check that D and R are as in the statement of Lemma 1.

Our remaining task is to prove Claim.

Proof of Claim: For a divisor D of X, the invertible $\mathcal{O}_D$-module $\omega_{X/S}(D)|_D$ is isomorphic to the canonical sheaf ω_D of D over S. Let π be a prime element of $\mathcal{O}_S$ and D'_k be the divisor $\Sigma_{i=0}^{k-1} C_i$ of X. It is easy to see that there are exact sequences

(a) $\quad 0 \to \omega_{D_{k-1}} \to \omega_{D_k} \to \omega_{D_k}|_{D'_k} \to 0$

(b) $\quad 0 \to \omega_{D'_k} \to \omega_{D_k} \to \omega_{D_{k-1}} \to 0$

such that the composite $\omega_{D_k} \to \omega_{D_{k-1}} \to \omega_{D_k}$ coincides with the multiplication by π.

First we prove $H^0(D_k, \omega_{D_k}) \simeq \mathcal{O}_S/\mathfrak{p}^k$ for $1 \leqslant k \leqslant m$, by induction on k. For simplicity, when the base space is clear, we omit to write it as $H^0(\omega_{D_k}) = H^0(D_k, \omega_{D_k})$ and by small h, we denote the length of the cohomology H as in $h^0(\omega_{D_k}) = \text{length}_{\mathcal{O}_S} H^0(\omega_{D_k})$. For the proof of the above fact, it is sufficient to prove that $\text{Ker}(\pi\colon H^0(\omega_{D_k}) \to H^0(\omega_{D_k})) \simeq \kappa(s)$ i.e., $h^0(\omega_{D'_k}) = 1$ and that $h^0(\omega_{D_k}) = k$. First we show $h^0(\omega_{D'_k}) = 1$. Since $(D'_k, D'_k) = -1$ and $(D'_k, c_1(\omega_{X/S})) = 1$, by Riemann–Roch, we have $\chi(\mathcal{O}_{D'_k}) = 0$. Since D'_k is reduced and connected, $H^0(\mathcal{O}_{D'_k}) \simeq \kappa(s)$. Hence we have $h^1(\mathcal{O}_{D'_k}) = 1$. By the Serre duality $H^0(\omega_{D'_k}) \simeq \text{Hom}_{\mathcal{O}_S}(H^1(\mathcal{O}_{D'_k}), \kappa(\eta)/\kappa(s))$, we have $h^0(\omega_{D'_k}) = 1$. Next we show $h^0(\omega_{D_k}) = k$. For $k = 1$, since

$D_k = D'_k$, we have seen this. Since $(D_k, D_k) = -k$ and $(D_k, c_1(\omega_{X/S})) = k$, by Riemann–Roch, $\chi(\mathcal{O}_{D_k}) = 0$. Since $\mathcal{O}_S/\mathfrak{p}^k \subset H^0(\mathcal{O}_{D_k})$, we have $h^0(\mathcal{O}_{D_k}) \geqslant k$. Hence $h^1(\mathcal{O}_{D_k}) \geqslant k$. Also by the Serre duality, we have $h^0(\omega_{D_k}) \geqslant k$. On the other hand, by the exact sequence (b), we have $h^0(\omega_{D_k}) \leqslant h^0(\omega_{D_{k-1}}) + h^0(\omega_{D'_k})$. Here $h^0(\omega_{D_{k-1}}) = k - 1$ by the assumption of induction and $h^0(\omega_{D'_k}) = 1$ as we have seen before. Hence we have $h^0(\omega_{D_k}) \leqslant k$. Thus we have proved $h^0(\omega_{D_k}) = k$. This argument also proves $H^0(\mathcal{O}_{D_k}) \simeq \mathcal{O}_S/\mathfrak{p}^k$ and the global section of the exact sequence (b)

$$\text{(B)} \qquad 0 \to H^0(\omega_{D'_k}) \to H^0(\omega_{D_k}) \to H^0(\omega_{D_{k-1}}) \to 0$$

is exact.

Now we prove $\omega_{D_k} \simeq \mathcal{O}_{D_k}$. Let $\alpha = \alpha_k \colon \mathcal{O}_{D_k} \to \omega_{D_k}$ be the morphism defined by a generator of $H^0(\omega_{D_k}) \simeq \mathcal{O}_S/\mathfrak{p}^k$. We prove this is an isomorphism. The exact sequence (a) defines the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{D_{k-1}} & \to & \mathcal{O}_{D_k} & \to & \mathcal{O}_{D'_k} & \to & 0 \\
 & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\
0 & \to & \omega_{D_{k-1}} & \to & \omega_{D_k} & \to & \omega_{D_k}|_{D'_k} & \to & 0.
\end{array}$$

We prove β and γ are isomorphisms. This proves Claim. First we prove β is an isomorphism. Let $H^0(\beta)\colon H^0(\mathcal{O}_{D_{k-1}}) \to H^0(\omega_{D_{k-1}})$ be the morphism induced by β. Then by the exact sequence (B), $H^0(\beta)$ coincides with $H^0(\alpha) \otimes id_{\mathcal{O}_S/\mathfrak{p}^{k-1}}$ and is an isomorphism. Since $\omega_{D_{k-1}} \simeq \mathcal{O}_{D_{k-1}}$, the fact that $H^0(\beta)$ is an isomorphism implies β is an isomorphism. Lastly we prove γ is an isomorphism. It is easy to check that $\deg \omega_{D_k}|_{C_i} = 0$ for $0 \leqslant i < k$. On the other hand, since $h^0(\omega_{D_j}) = j$ for $j = k, k-1$, we have $h^0(\omega_{D_k}|_{D'_k}) \geqslant 1$. Since D'_k is reduced and connected, these facts show that $\omega_{D_k}|_{D'_k} \simeq \mathcal{O}_{D'_k}$ and the global section of the exact sequence (a)

$$0 \to H^0(\omega_{D_{k-1}}) \to H^0(\omega_{D_k}) \to H^0(\omega_{D_k}|_{D'_k}) \to 0$$

is exact. From this, it follows that $H^0(\gamma)$ is an isomorphism and γ is an isomorphism. Thus we have completed the proof of Claim. Hence we have proved Proposition.

We consider a moduli theoretic meaning of ord Δ_1, which explains this notation. Let M_2 (resp. $\bar{M}_2$) be the fine moduli stack of smooth (resp. stable) curves of genus 2. Let Δ be the normal crossing divisor $\bar{M}_2 - M_2$. Then Δ has two irreducible components Δ_0 and Δ_1. An elliptic curve with a node corresponds to Δ_0 and two elliptic curves transversally meeting at one point corresponds to Δ_1. Let $f\colon X \to S$ be as in Proposition so that X defines a morphism $S \to \bar{M}_2$. Then the meaning of Proposition is that the degree of the inverse image of the divisor by this morphism is equal to ord Δ_1 (cf. [D–M] Proposition 1.5). In other words, ord Δ_1 is the intersection number of S with Δ_1. Thus even for not semi-stable curves, we might consider ord Δ_1 as if it was the intersection number of S with Δ_1.

Lastly, we mention the relation with a work on ord Δ' of K. Ueno [U]. First we briefly review it.

THEOREM (Ueno [U]): *Let $f\colon X \to S$ be as in Theorem above. Assume that char $\kappa(s) \neq 2, 3, 5$ and that X is minimal. Then*

$$\operatorname{ord} \Delta' \;=\; -\operatorname{Art}(X/S) + (\text{correcting term}).$$

The correcting term is determined by the configuration of the closed fiber and given explicitly by a table.

We consider the relation between this and ours. By Corollary 2 of Theorem, this correcting term should be computed by our description of ord Δ_1. However, it is directly checked only for the semi-stable case in Proposition. It remains open to check it generally.

Appendix. Proof of Lemma 2

We prove Lemma 2. Let Z be a stable curve over a field k. Let C be a component of Z. First we show that, unless C is smooth and rational, the invertible $\mathcal{O}_C$-module ω_C is base point free. When C is smooth, this fact is well known. Assume C is singular. Let $\pi\colon \bar{C} \to C$ be the normalization of C and Σ_C be the set of singular points of C. Then we have an exact sequence

$$\text{(a)} \qquad 0 \to \pi_*\omega_{\bar{C}} \to \omega_C \to \omega_C|_{\Sigma_C} \to 0.$$

Since $H^1(\bar{C}, \omega_{\bar{C}}) \simeq H^1(C, \omega_C) \simeq k$, the global section of (a)

$$\text{(A)} \qquad 0 \to H^0(\bar{C}, \omega_{\bar{C}}) \to H^0(C, \omega_C) \to \bigoplus_{p \in \Sigma_C} \omega_C(p) \to 0$$

is exact. When C is not rational, this fact shows that ω_C is base point free. Assume C is rational and not smooth. Similarly as above, we have an exact sequence

$$\text{(B)} \qquad 0 \to H^0(C, \omega_C) \to H^0(\bar{C}, \pi^*\omega_C) \to \bigoplus_{p \in \Sigma_C} \omega_C(p)$$

Here $\pi^*\omega_C \simeq \mathcal{O}(-2 + \pi^{-1}(\Sigma_C))$. For $p \in \Sigma_C$, there is a section $\sigma_p \in H^0(\bar{C}, \pi^*\omega_C)$ such that $\sigma_p(q_1) = -\sigma_p(q_2) \neq 0$ in $\omega_C(p)$ for $\{q_1, q_2\} = \pi^{-1}(p)$ and $\sigma_p(q) = 0$ for $q \in \pi^{-1}(\Sigma_C - \{p\})$. By (B), σ_p defines a section $\in H^0(C, \omega_C)$ and does not vanish on $(C - \Sigma_C) \cup \{p\}$. Therefore ω_C is base point free.

Let Z_C be the closure of the complement $Z - C$ and Q_C be the set of intersection points of C and Z_C. Then we have an exact sequence

$$0 \to \omega_C \oplus \omega_{Z_C} \to \omega_Z \to \omega_Z|_{Q_C} \to 0.$$

This sequence and the above fact show that, unless C is smooth and rational, $(C - Q_C) \cap B = \emptyset$. It also shows that

$$H^0(Z, \omega_Z) \to \bigoplus_{p \in Q_C} \omega_Z(p) \to H^1(C, \omega_C) \oplus H^1(Z_C, \omega_{Z_C}) \to H^1(Z, \omega_Z)$$

is exact. Here $H^1(C, \omega_C) \simeq H^1(Z, \omega_Z) \simeq k$ and $H^1(Z_C, \omega_{Z_C}) \simeq k^{\Pi_C}$, where Π_C is the set of connected component of Z_C. Let α_C: $k^{Q_C} \to k^{\Pi_C}$ be the morphism defined by $e_p \mapsto e_A$ where A is the connected component of Z_C containing p. Then $p \in Q_C$ is contained in the base locus of ω_Z if and only if the projection from Ker α_C to the p-component is the 0-map. This condition is satisfied if and only if the connected component of Z_C containing p does not contain other points of Q_C, which means $Z - \{p\}$ is disconnected. This shows that for a singular point p of Z, p is contained in B if and only if p is in W.

Now what remains to study is the smooth and rational components of Z. Let C be such a component. From the exact sequence $0 \to \omega_{Z_C} \to \omega_Z \to \omega_Z|_C \to 0$, we have the exact sequence $H^0(Z, \omega_Z) \xrightarrow{a} H^0(C, \omega_Z|_C) \xrightarrow{b} H^1(Z_C, \omega_{Z_C})$. Here b is the composite $H^0(C, \omega_Z|_C) \xrightarrow{c} \oplus_{p \in Q_C} \omega_Z(p) \xrightarrow{d} H^1(Z_C, \omega_{Z_C})$. Assume $Q_C \subset W$ i.e. to Ker $c = H^0(C, \omega_C) = 0$. This means $C \subset B$. Contrary, assume $Q_C \not\subset W$ i.e. $C \notin V$. Then by the similar argument as that on the rational and non-smooth component, we see that $(C - W \cap Q_C) \cap B = \emptyset$. Thus we have completed the proof of Lemma 2.

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