

COMPOSITIO MATHEMATICA

A. FAUNTLEROY

**Invariant theory for linear algebraic groups
II (char k arbitrary)**

Compositio Mathematica, tome 68, n° 1 (1988), p. 23-29

http://www.numdam.org/item?id=CM_1988__68_1_23_0

© Foundation Compositio Mathematica, 1988, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

Invariant theory for linear algebraic groups II (char k arbitrary)

A. FAUNTLEROY

*Department of Mathematics, North Carolina State University, Raleigh,
NC 27695-8205, USA*

Received 2 July 1987; accepted 24 March 1988

Given a linear algebraic group G and an action of it on a quasi-projective variety X , all defined over an algebraically closed field k , there will in general be no quasi-projective orbit space. When the group G is reductive, Mumford in GIT gave a reasonable criteria for the existence of a quasi-projective quotient using his notion of stable points. In order to generalize his concepts to arbitrary linear groups it is necessary to treat the case of unipotent group actions. If H is the unipotent radical of G , then one first must construct $Y = X/H$ and assuming Y is well behaved, apply the technique of Mumford to the action of the reductive group G/H on Y .

There are two technical problems involved in this program. The first is to find reasonable conditions which guarantee that Y exists and is quasi-projective. The second is to insure that the pair $(Y, G/H)$ satisfies the hypothesis required in order to apply the methods of GIT.

The results of [1] give a method for handling these problems when the ground field k has characteristic zero. The purpose of this note is to extend the key results on unipotent actions given in [1; Section 1] to the case of arbitrary characteristics. It is then a straightforward matter to extend to arbitrary algebraic groups G over arbitrary fields k the notions of stability given in [1].

Let X be a quasi-factorial variety over k , i.e., $B = \Gamma(X, \mathcal{O}_X)$ is a unique factorization domain and the canonical map: $X \rightarrow \text{Spec } B$ is an open immersion. Let H be a connected unipotent algebraic group defined over k . We assume throughout that H acts on X and that the isotropy group in H of each point of X is finite. We recall here some definitions from [1].

(1) A point $x \in X$ is *semi-stable* if $\dim c^{-1}(cx) = \dim H$ where $c: X \rightarrow \text{Spec } A$, $A = B^H$, is the natural map. If X^{ss} denotes the set of semi-stable points of X then X^{ss} is open and H -stable (c.f. [1]). Moreover there exists a quasi-factorial variety Q and an H -equivariant map $\pi: X \rightarrow Q$ making Q an s -categorical quotient of X by H . This means that for any morphism f :

$X \rightarrow Y$, Y a separated algebraic scheme, with f constant on H -orbits, there is a unique map $g: Q \rightarrow Y$ with $f = g\pi$. Further $Q = c(X^{ss})$ is open in $\text{Spec } A$.

(2) A point $x \in X$ is *stable* if there is an open neighborhood U of x with $HU = U$ and such that U/H exists and is affine. We denote by X^s the open set of stable points. It is evidently invariant under the action of H and a geometric quotient X^s/H exists as an algebraic scheme. A point x is *properly stable* if it is stable and there exists an open H -invariant neighborhood V of x with $V \subseteq X^s$ such that the action of H on V is proper. The set of properly stable points X^{ps} is evidently open and H -stable.

PROPOSITION 1.1. *Let $U \subseteq X^s$ be an open H stable subset and suppose $Y = U/H$. If the quotient morphism $U \rightarrow Y$ is affine then $U \subseteq X^{ss}$. If further, Y is separated then the natural map $Y \rightarrow Q$ is an open immersion so Y is quasi-factorial.*

Proof. First assume that Y and U are affine. Let $A = B^H$. Then A is factorial and $k[U] = B[a^{-1}]$, $k[Y] = A[a^{-1}]$ for some $a \in A$. The triangle below clearly commutes

$$\begin{array}{ccc} U & \xrightarrow{c} & \text{Spec } A \\ & \searrow q & \nearrow \\ & & Y \end{array}$$

The non empty fibres of $c|_U$ are of dimension $l = \dim H$. If $x \in U \subset X$ then $c(x) = q(x) = y$ so $a(y) \neq 0$ and hence $a(x) \neq 0$. Thus each point of the fibre $c^{-1}(c(x))$ lies in U and it follows that the dimension of each fiber is l so $U \subset X^{ss}$. Now in the general case U is covered by H -stable open affine subsets with affine quotients so $U \subseteq X^{ss}$. If Y is separated, then $Y \rightarrow Q$ is a birational quasi-finite map, hence an open immersion by Zariski's Main Theorem.

In [1] it was shown that when $\text{char } k = 0$, X^{ps} is the set of points in X for which the action of H is locally trivial and that $H \times X^{ps} \rightarrow X^{ps} \times X^{ps}$, $(h, x) \rightarrow (hx, x)$ is proper so the morphism $X^{ps} \rightarrow Y = X^{ps}/H$ is affine and Y is separated hence quasi-factorial. The main purpose of this note is to give the appropriate generalization of this result in arbitrary characteristics. Since H contains a normal series with successive quotients isomorphic to the additive group G_a , one would expect the answer to lie in G_a -actions. This is indeed the case. A first guess might be to replace *locally trivial* by locally

trivial in the finite radical topology. However, the example 3 of [2] gives a counterexample to this conjecture.

It is important to note here that without the hypothesis that X be quasi-factorial, the action of H on X^{ps} need not be proper! (See Example 2, p. 727 in [2].)

A point $x \in X$ will be called *finitely-stable* or *f-stable* if there exists an open affine neighborhood V of x invariant under the action of H and an H -equivariant finite morphism $H \times S \rightarrow V$ for some affine variety S . Let X^{fs} denote the set of finitely stable points of X . In the definition we may assume without loss of generality that S is normal.

LEMMA 1.2. X^{fs} is contained in X^s .

Proof. It suffices to show that if $H \times S \rightarrow V$ is a finite surjective H -morphism with V normal then V/H exists and is affine. By [3; p. 539] we can find a Seshadri cover $Z \rightarrow V$ of V with respect to H such that $k(Z)$ is the normal closure of $k(H \times S)$ in an algebraic closure of $k(V)$. It follows that Z is the normalization of $H \times S$ in $k(Z)$ so in particular is affine. Moreover, $Z \rightarrow H \times S$ is a Seshadri cover of $H \times S$. The action of $H \times S$ is easily seen to be proper so the action of H on Z is proper. Then $W = Z/H \rightarrow S$ is finite so W is affine. By Theorem 7.1 of [3] V/H exists and is affine.

LEMMA 1.3. Let $Z \rightarrow X$ be a Seshadri cover of X . If the action of H on Z is proper, then the action of H on X is proper.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H \times Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ H \times X & \xrightarrow{\Phi} & X \times X \end{array}$$

The vertical and upper horizontal maps are finite hence Φ is finite hence proper.

The following lemma is the key to our description of X^{ps} . It describes the situation locally when $H = G_a$.

LEMMA 1.4. Let V be a factorial affine variety on which G_a acts. Let R denote the coordinate ring of V . Then the following conditions are equivalent:

- (1) There exists a variety S and a finite surjective G_a -equivariant morphism $p: G_a \times S \rightarrow V$

- (2) *There is an element $g \in R$ such that $\tilde{\sigma}(g)$ is monic in $R(\lambda) = k[G_a \times V]$ where $\tilde{\sigma}$ is the comorphism for the action of G_a on V .*

Proof. Suppose first that (1) holds. Let G_a act diagonally on $G_a \times V$. Then $1 \times p: G_a \times S \rightarrow G_a \times V$ is finite and G_a equivariant. Let W be the image of $1 \times p$. Then W is a G_a -stable subvariety of $G_a \times V$ of codimension one. Thus W is defined by a single irreducible invariant polynomial $F(T)$ in $R[T]$. The composition of $1 \times p$ with the second projection $G_a \times V \rightarrow V$ is the original morphism p . Hence the restriction of the second projection to W is a finite morphism. It follows that $F(T)$ can be taken monic in T . Write $F(T) = a_0 + a_1T + \cdots + T^n$ with $a_i \in R$.

Let $\hat{\sigma}: R[T] \rightarrow R[T][\lambda] = R[T, \lambda]$ denote the comorphism for the action of G_a on $G_a \times V$. If $\Sigma b_i T^i \in R[T]$. Then $\sigma(\Sigma b_i T^i) = \Sigma \tilde{\sigma}(b_i)(T + \lambda)^i$. Using the fact that $\hat{\sigma}(F(T)) = F(T)$ we find

$$\begin{aligned} (T + \lambda)^n + \hat{\sigma}(a_{n-1})(T + \lambda)^{n-1} + \cdots + \hat{\sigma}(a_0) \\ = T^n + a_{n-1} + \cdots + a_1T + a_0 \end{aligned}$$

Taking for T the value $-\lambda$ we see that $\hat{\sigma}(a_0) = (-\lambda)^n + a_{n-1}(-\lambda)^{n-1} + \cdots + a_0$. Thus $g = (-1)^n a_0$ satisfies $\hat{\sigma}(g)$ is monic in λ and (2) holds.

Conversely suppose $g \in R$ with $\hat{\sigma}(g) = g + g_1\lambda + \cdots + g_{n-1}\lambda^{n-1} + \lambda^n$. Let W be the closed subset of $G_a \times V$ defined by

$$G(T) = g + g_1(-T) + \cdots + g_{n-1}(-T)^{n-1} + (-T)^n = 0.$$

Note that if $(\mu, p) \in W$ then

$$\begin{aligned} g(-\mu \cdot p) &= g(p) + g_1(p)(-\mu) + \cdots + g_{n-1}(p)(-\lambda)^{n-1} + (-\mu)^n \\ &= G(T)(\mu, p) = 0. \end{aligned}$$

Conversely if $g(-\mu \cdot p) = 0$ then $(\mu, p) \in W$. Now let G_a act diagonally on $G_a \times V$ then if $(\mu, p) \in W$ and $\lambda \in G_a$ we have

$$\begin{aligned} G(T)(\lambda \cdot (u, p)) &= G(T)(\lambda + u, \lambda \cdot p) \\ &= g((-\lambda - u) \cdot (\lambda p)) \\ &= g(-\mu \cdot p) = 0. \end{aligned}$$

Thus W is G_a -stable. The mapping $W \rightarrow V$ obtained by restricting the second projection $G \times V \rightarrow V$ to W is finite since $G(T)$ is monic. Replacing W by a suitable irreducible component if necessary, we obtain a G_a -stable closed subvariety W of $G_a \times V$ such that the mapping $W \rightarrow V$ is finite and G_a -stable closed subvariety W of $G_a \times V$ such that the mapping $W \rightarrow V$ is finite and G_a -equivariant. Finally, since $G_a \times V$ is trivial as a G_a -space so also is any G_a -stable subvariety so that $W \simeq G_a \times S$ for some variety S . This gives the desired implication (2) implies (1) and completes the proof of the lemma.

THEOREM 1.5. *Let X be a quasi-factorial variety on which the connected unipotent group H acts. Then H acts properly on $X^{fs}(H)$. In particular, $Y = X^{fs}(H)/H$ is quasi-factorial and $q: X^{fs}(H) \rightarrow Y$ is an affine morphism.*

Proof. We argue by induction on $\dim H$. Assume the result holds for connected subgroups $N \subseteq H$ with $0 < \dim N < \dim H$ and let N be such a subgroup which is normal in H . Recall, [1; Sec. 3] that $H \simeq N \times (H/N)$ as an N -space. It follows that $X^{fs}(H) \simeq X^{fs}(N)$ and by the inductive assumption H/N acts properly on $Y_N^{fs}(H/N)$ where $Y_N = X^{fs}(N)/N$.

Let Z be a Seshadri cover of $X^{fs}(H)$. We have a commutative diagram:

$$\begin{array}{ccc}
 Z & \longrightarrow & X^{fs}(H) \\
 \downarrow & & \downarrow \\
 W_1 = Z/N & \longrightarrow & Y_N^{fs}(H/N) \\
 \downarrow & & \downarrow \\
 W & \longrightarrow & Y
 \end{array}$$

where W and Y are quotients under the action of H/N . Since H/N acts properly on $Y_N^{fs}(H/N)$ it also acts properly on W_1 . Thus W is quasi-affine. But $W = Z/H$ and Z is locally trivial. By [1, 1.9] H acts properly on Z . By Lemma 1.3, H acts properly on $X^{fs}(H)$.

To complete the proof we need only establish the result in the case $H = G_a$. By Lemma 1.4 we can find an affine open cover $\{X_x\}$ of $X^{fs}(H)$ consisting of H -stable open affines and an element $g_x = R_x = k[X_x]$ with $\tilde{\sigma}(g_x)$ monic in $R_x[\lambda]$. The map Φ will be proper if it's finite. We consider the cover $\{X_x \times X_\beta\}$ of $X^{fs}(H) \times X^{fs}(H)$. Then $\Phi^{-1}(X_x \times X_\beta) = H \times X_x \cap X_\beta$ so Φ is affine. Let $B = \Gamma(X^{fs}(H), O_X)$ so that $R_x = B[f_x^{-1}]$ with $f_x \in A = B^H$. Then $k[X_x \cap X_\beta] = B[f_x^{-1} \cdot f_\beta^{-1}]$. I claim the map

$$B[f_x^{-1}] \otimes B[f_\beta^{-1}] \xrightarrow{1 \otimes \tilde{\sigma}} B[f_x^{-1} \cdot f_\beta^{-1}][\lambda]$$

is finite. If $b \in B[f_\alpha^{-1} \cdot f_\beta^{-1}]$ and $b = s/f_\alpha^n f_\beta^m$ then $b = (1 \otimes \tilde{\sigma})(s/f_\alpha^n \otimes 1/f_\beta^m)$ so $B[f_\alpha^{-1} f_\beta^{-1}]$ is in the image of $1 \otimes \tilde{\sigma}$. Since $(1 \otimes \tilde{\sigma})(1 \otimes g_\beta) = \tilde{\sigma}(g_\beta)$ is monic in λ the ring $B[f_\alpha^{-1} \cdot f_\beta^{-1}][\lambda]$ is integral over the image of $1 \otimes \tilde{\sigma}$. It follows that Φ is finite and the theorem is proved.

COROLLARY 1.6. *$X^{fs}(H)$ contains every H -stable open subset of X on which H acts properly stably. In particular $X^{fs}(H) = X^{ps}(H)$.*

Proof. Let $U \subseteq X$ be H -stable open and assume H acts properly stably on U . It follows that we can replace U by an affine open subset and assume $Y = U/H$ is affine. If Z is a Seshadri cover of U then Z and W are affine and hence $Z \simeq H \times W$. Since $Z \rightarrow U$ is finite, $U \subset X^{fs}(H)$. The theorem asserts that the action of H on X^{fs} is properly stable hence $X^{fs}(H) \subset X^{ps}(H)$ and equality follows.

The extension of the results of [1] to arbitrary characteristics depends on the invariance under G of the properly stable points of $R_u G$ for actions of arbitrary connected algebraic groups G on quasi-factorial varieties. The following lemma is a key technical tool for this.

LEMMA 1.7 *Let G be a linear algebraic group, N a closed normal subgroup of G and X a quasi-factorial variety on which G acts. If $U \subseteq X$ is an N -stable open subset on which N acts properly then N acts properly on gU for all g in G .*

Proof. $NgU = gNU = gU$ so gU is N -stable. Now $\Phi: N \times U \rightarrow U \times U$ is proper so finite. Let $ad(g)$ denote conjugation by g in G so $ad(g)(n) = gng^{-1}$ and denote by λ_g left multiplication by g . The following diagram is commutative with vertical arrows representing isomorphisms.

$$\begin{array}{ccc}
 N \times U & \xrightarrow{\Phi} & U \times U \\
 \downarrow ad(g) \times \lambda_g & & \downarrow \lambda_g \times \lambda_g \\
 N \times gU & \xrightarrow{\Phi_g} & gU \times gU
 \end{array}$$

Thus Φ_g is finite hence proper.

Note that if G is unipotent and $U \subset X^s$ (for the action of N) then $gU \subset X^s$ for all $g \in G$. For a proof see [1; Proposition 2.4].

THEOREM 1.7. *Let N be a closed connected normal subgroup of the unipotent group G and X a quasi-factorial variety on which G acts. Let $X_0 = X^{ps}(N)$, $Y_0 = X_0/N$ and $q: X_0 \rightarrow Y_0$ the quotient map. Then X_0 is G stable and $X^{ps}(G) = q^{-1}(Y_0^{ps}(G/N))$.*

Proof. The lemma and the preceding note imply $GX_0 = X_0$. We saw in the proof of Theorem 1.5 that $X^{ps}(G) \subseteq X_0$ and it is evidently N -stable. Its

image Y_1 in Y_0 is thus open, G/N stable and easily seen to be contained in $Y_0^{ps}(G/N)$ (cf. [1; 2.4]). But if $X_1 = q^{-1}(Y_0^{ps}(G/N))$, then X_1 is G -stable and clearly $Y_0^{ps}(G/N)/(G/N) \simeq X_1/G$. It remains only to show that G acts properly on X_1 . This can be seen as follows:

Let $T = X_1/G = Y^{ps}(G/N)/(G/N)$. Then $X_1 \rightarrow Y^{ps}(G/N)$ and $Y^{ps}(G/N) \rightarrow T$ are affine maps because $X_1 \subseteq X^{ps}(N)$ and N act properly on X_1 and G/N acts properly on $Y^{ps}(G/N)$. Let $T_2 \subset T$ be affine, $Y_2 = \alpha^{-1}(T_2)$ where $\alpha: Y^{ps}(G/N) \rightarrow T$ is the quotient map and finally let X_2 be the inverse image of Y_2 in X_1 . If Z is a Seshadri cover of X_2 and $W = Z/G$ is its quotient then we have a commutative diagram

$$\begin{array}{ccc}
 Z & \longrightarrow & X_2 \\
 \downarrow & & \downarrow \\
 S & \dashrightarrow & Y_2 \\
 \downarrow & & \downarrow \\
 W & \longrightarrow & T_2
 \end{array}$$

Since $Z \rightarrow X_2$ is finite and N acts properly on X_2 it acts properly on Z . But Z is locally trivial for the action of G hence also N so $S = Z/N$ exists and is separated. But then the canonical map $S \rightarrow Y_2$ is finite so S is actually affine. Since $W = S/(G/N)$ is separated and $W \rightarrow T_2$ is finite, W is affine. But $Z \rightarrow W$ being locally trivial gives $Z \simeq G \times W$. Thus $X_2 \subset X^{fs}(G) \subset X^{ps}(G)$. Since X_1 can be covered by such open affines it follows that $X_1 \subseteq X^{ps}(G)$ and hence $X^{ps}(G) = q^{-1}(Y_0^{ps}(G/N))$.

COROLLARY 1.8. *Let X be a quasi-factorial variety on which the connected unipotent group G acts. If all stability groups for the action of G are finite then $X^{ps}(G)$ is non-empty.*

Proof. It clearly suffices to establish the result when $G = G_a$. If $f \in \Gamma(X, \mathcal{O}_X)$ is a nonconstant non-invariant function then $\sigma(f) = f + f_1 T + \dots + f_k T^k$ with f_k invariant. Let $X_0 = X_{f_k}$. Lemma 2.4 implies $X_0 \subseteq X^{fs}(G_a) = X^{ps}(G_a)$.

REMARK. The results of [1] contained in Sections 3 and 4 now follow essentially from the arguments given there without any assumption on the characteristic of the ground field.

References

1. Fautleroy, A.: Geometric invariant theory for general algebraic groups, *Compositio Mathematica* 55 (1985) 63–87.
2. Fautleroy, A and Magid, A.: Proper G_a -actions, *Duke J. Math.* 43 (1976) 723–729.
3. Seshadri, C.S.: Quotient spaces modulo reductive algebraic groups, *Ann. of Math.* 95 (1972) 511–566.