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Representations of $GSp(4)$ over a p-adic field: part 2

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In Part 1 of this paper, we defined a nondegenerate representation of $G = GSp(4)$ as a pair (L, Ω) consisting of a compact open subgroup L and a representation Ω of L satisfying a certain cuspidality or semisimplicity condition (see Section 1). Theorem 2.1 showed that every irreducible representation of G must obtain a nondegenerate representation (see also [HM2], [M]). In Section 3, we analyzed those irreducible representations of G which contained a nondegenerate representation (L, Ω) with L a parahoric subgroup.

In Part 2, we give an analysis of those representations of G which contain the remaining nondegenerate representations. These are the unramified and ramified representations. The key point again is to reduce the classification of representations of G to the same question for a smaller group. This is done by establishing Hecke algebra isomorphisms (see Section 3).

4. Unramified representations

Recall from Section 1 that a nondegenerate unramified representation $(L, \Omega_{\mathbf{s}})$ is described by two parameters. One parameter is a positive integer which is a measure of the level of the nondegenerate representation. The other parameter is a nonscalar semisimple element $\mathbf{s} \in \mathfrak{g}(\mathbb{F}_q)$, the $\mathbb{F}_q$-rational points of the algebraic Lie algebra $\mathfrak{g} = \mathcal{G}\mathcal{s}\mathcal{p}(4)$. We make a few remarks about the semisimple parameter $\mathbf{s}$.

A semisimple element is of course contained in a Cartan subalgebra. Therefore, it is natural to give a rough classification of semisimple elements based on Cartan subalgebras. The Cartan subalgebras of $\mathfrak{g}(\mathbb{F}_q)$ are parametrized, up to conjugacy, by the conjugacy classes of elements in the Weyl group (e.g. see [C]). In the case of $GSp(4)$, the Weyl group is the dihedral group of order eight. In particular, there are five classes of Cartan subalgebras. In order to give representatives for the five classes, we recall some notation from Section 1. The 4×4 matrix whose (r, s) entry is $\delta_{r,i}\delta_{s,j}$ is denote $E_{i,j}$. Consider the five Cartan subalgebras in $\mathfrak{sp}(4)(\mathbb{F}_q)$ given by

$$\{a(E_{1,1} - E_{4,4}) + b(E_{2,2} - E_{3,3})\} \tag{4.1a}$$

$$\{a(E_{1,1} - E_{4,4}) + b(E_{2,3} + \varepsilon E_{3,2})\} \tag{4.1b}$$

$$\{a(E_{1,1} + E_{2,2} - E_{3,3} - E_{4,4}) + b(E_{1,2} + \varepsilon E_{2,1} - E_{3,4} - \varepsilon E_{4,3})\} \tag{4.1c}$$

$$\{a(E_{1,4} + \varepsilon E_{4,1}) + b(E_{2,3} + \varepsilon E_{3,2})\} \tag{4.1d}$$

$$\left\{ \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & \varepsilon b & a \\ u & v & 0 & 0 \\ \varepsilon v & u & 0 & 0 \end{bmatrix} \middle| \, u + \sqrt{\varepsilon} v = (a + \sqrt{\varepsilon} b)(A + \sqrt{\varepsilon} B) \right\}. \tag{4.1e}$$

In the above sets, a and b run over $\mathbb{F}_q$. In (4.1e), $A + B\sqrt{\varepsilon}$ is a nonsquare in $\mathbb{F}_q[\sqrt{\varepsilon}]$. The five conjugacy classes of Cartan subalgebras of $\mathfrak{g}(\mathbb{F}_q)$ are the subalgebras in (4.1) direct sum with the scalar matrices.

The element $\mathbf{s} \neq 0$ can be taken to be in the form (4.1). If $\mathbf{s}$ is type (4.1b, c, d, e), we can further assume that $b \neq 0$. If $\mathbf{s}$ is type (4.1d), we assume $a \notin \{0, \pm b\}$.

As stated above, the other parameter needed to specify an unramified nondegenerate representation is a level parameter $i \in \mathbb{N}$. To describe how to construct $(L, \Omega_{\mathbf{s}})$ from i and $\mathbf{s}$, we recall some notation established in Section 1. The group $\tilde{G}$ (resp. $\tilde{K}$) is the group $GL_4(F)$ (resp. $GL_4(R)$). For $u \in \mathbb{N}$, the group $\tilde{K}_u$ is the u-th principal congruence subgroup of $\tilde{K}$. If $\tilde{L}$ is a subgroup of $\tilde{G}$, then L is the intersection of $\tilde{L}$ with G. To establish some more notation, let

$$\begin{aligned} \tilde{\mathscr{K}} &= M_4(R) \\ \tilde{\mathscr{B}} &= \{x \in \tilde{\mathscr{K}} \mid x \text{ is upper triangular mod } \mathfrak{p}\} \\ \tilde{\mathscr{Q}} &= \tilde{\mathscr{B}} + \tilde{\mathscr{B}} s_0 \tilde{\mathscr{B}} \\ \tilde{\mathscr{M}} &= \tilde{\mathscr{B}} + \tilde{\mathscr{B}} s_1 \tilde{\mathscr{B}}. \end{aligned} \tag{4.2}$$

Each set in (4.2) is a R-order in $M_4(F)$, i.e., an R-submodule which is also a ring. If $\tilde{\mathscr{R}}$ is an R-order in (4.2), let $\tilde{G}_{\tilde{\mathscr{R}}}$ denote the units in $\tilde{\mathscr{R}}$. The group

$\tilde{G}_{\tilde{\mathscr{R}}}$ is an open compact subgroup of $\tilde{G} = GL_4(F)$. As $\tilde{\mathscr{R}}$ varies over the four orders in (4.2), we use the obvious notation $\tilde{K}$ (resp. $\tilde{B}$, $\tilde{Q}$ and $\tilde{M}$) for $\tilde{G}_{\tilde{\mathscr{R}}}$. The groups K, B, Q and M are the parahoric subgroups of G encountered in Section 1. In the notation of Section 1, we have

$$\begin{aligned} K &= P_{\{s_0, s_1\}} \\ B &= P_{\varnothing} \\ Q &= P_{\{s_0\}} \\ M &= P_{\{s_1\}}. \end{aligned} \tag{4.3}$$

Each parahoric subgroup possesses a natural filtration (see Section 1). The filtration subgroups can be easily interpreted in terms of the R-orders in (4.2). To do this let $\mathscr{J}(\tilde{\mathscr{R}})$ to be the topological Jacobson radical of $\tilde{\mathscr{R}}$. The description of $\mathscr{J}(\tilde{\mathscr{R}})$ for each of the four orders in (4.2) is given by

$$\begin{aligned} \mathscr{J}(\tilde{\mathscr{K}}) &= \{x \in \tilde{\mathscr{K}} \mid x = 0 \bmod \mathfrak{p}\} \\ \mathscr{J}(\tilde{\mathscr{B}}) &= \{x \in \tilde{\mathscr{B}} \mid \text{the diagonal entries of } x \text{ lie in } \mathfrak{p}\} \\ \mathscr{J}(\tilde{\mathscr{Q}}) &= \{x \in \mathscr{J}(\tilde{\mathscr{B}}) \mid \text{entry } (2, 3) \text{ of } x \text{ lies in } \mathfrak{p}\} \\ \mathscr{J}(\tilde{\mathscr{M}}) &= \{x \in \mathscr{J}(\tilde{\mathscr{B}}) \mid \text{entries } (1, 2) \text{ and } (3, 4) \text{ of } x \text{ lie in } \mathfrak{p}\}. \end{aligned} \tag{4.4}$$

For $u \in \mathbb{N}$, consider the ideal $\mathscr{J}(\tilde{\mathscr{R}})^u$. A trivial calculation shows

$$\begin{aligned} \mathscr{J}(\tilde{\mathscr{K}})^{u+1} &= \varpi \mathscr{J}(\tilde{\mathscr{K}})^u \\ \mathscr{J}(\tilde{\mathscr{B}})^{u+4} &= \varpi \mathscr{J}(\tilde{\mathscr{B}})^u \\ \mathscr{J}(\tilde{\mathscr{Q}})^{u+3} &= \varpi \mathscr{J}(\tilde{\mathscr{Q}})^u \\ \mathscr{J}(\tilde{\mathscr{M}})^{u+2} &= \varpi \mathscr{J}(\tilde{\mathscr{M}})^u. \end{aligned} \tag{4.5}$$

This periodicity relation permits us to define $\mathscr{J}(\tilde{\mathscr{R}})^u$ for all integers u. For $u \in \mathbb{N}$, let

$$\tilde{G}_{\tilde{\mathscr{R}},u} = 1 + \mathscr{J}(\tilde{\mathscr{R}})^u = \{1 + x \mid x \in \mathscr{J}(\tilde{\mathscr{R}})^u\}$$

and

$$G_{\tilde{\mathscr{R}},u} = G \cap \tilde{G}_{\tilde{\mathscr{R}},u}. \tag{4.6}$$

The $G_{\tilde{\mathscr{R}},u}$'s are the filtration subgroups of $G_{\tilde{\mathscr{R}}}$ defined in Section 1 (cf. (1.15)). As $\tilde{\mathscr{R}}$ varies over the four orders in (4.2), we denote the groups $G_{\tilde{\mathscr{R}},u}$ by K_u, B_u, Q_u and M_u respectively. The Cayley transform $c(x) = (1 - x)(1 + x)^{-1}$ maps $\mathscr{J}(\tilde{\mathscr{R}})^u$ to $\tilde{G}_{\tilde{\mathscr{R}},u}$ and takes

$$\mathfrak{g}_{\tilde{\mathscr{R}},u} = \{\mathfrak{g} \cap \mathscr{J}(\tilde{\mathscr{R}})^u\} \tag{4.7}$$

to $G_{\tilde{\mathscr{R}},u}$. If $u \geqslant v$, the induced maps from

$$\begin{aligned} \mathscr{J}(\tilde{\mathscr{R}})^u/\mathscr{J}(\tilde{\mathscr{R}})^{u+v} &\to \tilde{G}_{\tilde{\mathscr{R}},u}/\tilde{G}_{\tilde{\mathscr{R}},u+v} \\ \mathfrak{g}_{\tilde{\mathscr{R}},u}/\mathfrak{g}_{\tilde{\mathscr{R}},u+v} &\to G_{\tilde{\mathscr{R}},u}/G_{\tilde{\mathscr{R}},u+v} \end{aligned} \tag{4.8}$$

are isomorphisms.

An unramified nondegenerate representation $(L\ \Omega_{\mathrm{s}})$ has L equal to some filtration subgroup $G_{\tilde{\mathscr{R}},u}$. Indeed, if i is the level of the nondegenerate representation (cf. (1.24)), then

$$\begin{aligned} L &= K_i \quad \mathrm{s} \neq 0 \text{ of type (4.1d) or (4.1e)} \\ L &= B_{4i} \quad \mathrm{s} \neq 0 \text{ of type (4.1a)} \\ L &= Q_{3i} \quad \mathrm{s} \neq 0 \text{ of type (4.1b)} \\ L &= M_{2i}\ \mathrm{s} \neq 0 \text{ of type (4.1c).} \end{aligned} \tag{4.9}$$

Let n be the index of the filtration group L in (4.9), i.e., $L = G_{\tilde{\mathscr{R}},n}$. The representation Ω_{s} is a character of $G_{\tilde{\mathscr{R}},n}/G_{\tilde{\mathscr{R}},n+1}$. To describe Ω_{s}, we first recall more notation from Section 1.

The rings $\tilde{\mathfrak{g}} = M_4(F)$ and $\tilde{\mathfrak{g}}(R) = M_4(R)$ are the F and R-rational points of the Lie algebra $\mathscr{gl}(4)$. The Lie rings, $\mathfrak{g}$ (resp. $\mathfrak{g}(R)$) are the analogous sets for the Lie algebra $\mathscr{Gsp}(4)$. Given a lattice $\tilde{\mathfrak{l}} \subset \tilde{\mathfrak{g}}$, the dual lattice

$$\tilde{\mathfrak{l}}^* = \{x \in \tilde{\mathfrak{g}} \mid \langle x, \tilde{\mathfrak{l}} \rangle \subset R\}$$

was defined in (1.17). Recall, $\langle\, ,\rangle$ is the form $\langle x, y \rangle = \operatorname{tr}\, \tilde{\mathfrak{g}}(xy)$. For $\tilde{\mathscr{R}}$ an order in (4.2) let r be the period given in (4.5), i.e., $\mathscr{J}(\tilde{\mathscr{R}})^{l+r} = \varpi \mathscr{J}(\tilde{\mathscr{R}})^l$.

The dual lattice of $\mathscr{J}(\tilde{\mathscr{R}})^j$, $j \in \mathbb{Z}$ is

$$\{\mathscr{J}(\tilde{\mathscr{R}})^j\}^* = \mathscr{J}(\tilde{\mathscr{R}})^{(-r+1)-j}. \tag{4.10}$$

Let Ψ be the additive character of F with conductor R used in (1.21). For integers $j \geqslant k > 0$, identify the character group $\{\tilde{G}_{\tilde{\mathscr{R}},j}/\tilde{G}_{\tilde{\mathscr{R}},j+k}\}^\wedge$ with $\mathscr{J}(\tilde{\mathscr{R}})^{(-r+1)-(j+k)}/\mathscr{J}(\tilde{\mathscr{R}})^{(-r+1)-j}$ by the map $(\alpha + \mathscr{J}(\tilde{\mathscr{R}})^{(-r+1)-j}) \to \Omega_\alpha$

$$\Omega_\alpha(c(x)) = \Psi(\langle x, -\alpha\rangle/2), \quad x \in \mathscr{J}(\tilde{\mathscr{R}})^j. \tag{4.11}$$

This is just (1.21). With the obvious identifications, this realization of characters identifies the character group $\{G_{\tilde{\mathscr{R}},j}/G_{\tilde{\mathscr{R}},j+k}\}^\wedge$ with

$$\{\mathfrak{g}_{\tilde{\mathscr{R}},(-r+1)-(j+k)}\}/\{\mathfrak{g}_{\tilde{\mathscr{R}},(-r+1)-j}\}. \tag{4.12}$$

We now explain how the character $\Omega_{\mathbf{s}}$ of $G_{\tilde{\mathscr{R}},n}/G_{\tilde{\mathscr{R}},n+1}$ is realized as a coset (4.12) in terms of the parameter $\mathbf{s}$. For each of the subalgebras $\mathfrak{h}(\mathbb{F}_q)$ in (4.1), let $\mathfrak{h} \subset \mathfrak{g}$ be the obvious subalgebra so that

$$\mathfrak{h}(\mathbb{F}_q) = \{\mathfrak{h} \cap M_4(R)\}/\{\mathfrak{h} \cap \varpi M_4(R)\}. \tag{4.13}$$

The character $\Omega_{\mathbf{s}}$ is realized by choosing an $\varpi^{-(i+1)}\alpha \in \{\mathfrak{h} \cap \mathscr{J}(\tilde{\mathscr{R}})^{(-r-n)}\}$ satisfying $\mathbf{s} = \alpha \bmod \not{p}$. The integer i, is the integer in (4.9).

With this description of L and $\Omega_{\mathbf{s}}$, we proceed to formulate the key result on Hecke algebras. Let $\mathfrak{g}'(\mathbb{F}_q)$ be the centralizer $C_{\mathfrak{g}(\mathbb{F}_q)}(\mathbf{s})$ of $\mathbf{s} \in \mathfrak{g}(\mathbb{F}_q)$. We can pick α so that if $\mathfrak{g}' = C_{\mathfrak{g}}(\alpha)$, then

$$\mathfrak{g}'(\mathbb{F}_q) = \{\mathfrak{g}' \cap M_4(R)\}/\{\mathfrak{g}' \cap \varpi M_4(R)\}. \tag{4.14}$$

Let $G' = C_G(\alpha)$ be the centralizer of α in G. Set $J' = G' \cap J$. Our goal is to classify the irreducible representations of G which contain $(L, \Omega_{\mathbf{s}}')$ via the Hecke algebra $\mathscr{H}(G'//J', 1)$. To prepare for the precise statement, we need some constructions based on the Cartan subalgebra $\mathfrak{h}$ and its orthogonal complement in $\mathfrak{g}$.

If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, let $\mathfrak{h}^\perp$ (resp. $\mathfrak{h}^*$) be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$). We have $\mathfrak{h}^* = \mathfrak{h}^\perp \oplus \mathfrak{g}^*$. Let

$$\mathfrak{h}_{\tilde{\mathscr{R}},j} = \mathfrak{h} \cap \mathscr{J}(\tilde{\mathscr{R}})^j$$

$$\mathfrak{h}^\perp_{\tilde{\mathscr{R}},j} = \mathfrak{h}^\perp \cap \mathscr{J}(\tilde{\mathscr{R}})^j$$

$$\mathfrak{h}^{\#}_{\tilde{\mathscr{R}},j} = \mathfrak{h}^{\#} \cap \mathscr{J}(\tilde{\mathscr{R}})^{j}$$

$$\mathfrak{g}^{\#}_{\tilde{\mathscr{R}},j} = \mathfrak{g}^{\#} \cap \mathscr{J}(\tilde{\mathscr{R}})^{j}. \tag{4.15}$$

With the obvious identifications via the Cayley transform, we have

$$\tilde{G}_{\tilde{\mathscr{R}},j}/\tilde{G}_{\tilde{\mathscr{R}},j+k} = G_{\tilde{\mathscr{R}},j}/G_{\tilde{\mathscr{R}},j+k} \oplus \mathfrak{g}^{\#}_{\tilde{\mathscr{R}},j}/\mathfrak{g}^{\#}_{\tilde{\mathscr{R}},j+k}, \quad j \geqslant k > 0.$$

We are now ready to state the main result on Hecke algebras. Let j (resp. j') be the greatest integer in $(n + 1)/2$ (resp. $(n + 1)/2$). Define lattices

$$\begin{aligned} \mathfrak{l} &= \mathfrak{g}_{\tilde{\mathscr{R}},n} + \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},j} \\ \mathfrak{l}_{+} &= \mathfrak{g}_{\tilde{\mathscr{R}},n} + \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},j'} \end{aligned} \tag{4.16}$$

and let J (resp. J_{+}) be the group $J = c(\mathfrak{l})$ (resp. $J_{+} = c(\mathfrak{l}_{+})$). Clearly $L \subset J_{+} \subset J$.

If $n + 1 = 2j$, then $J_{+} = J$ and Ω_s can be extended to a character Ω of J by setting σ equal to 1 on $c(\mathfrak{g}'^{\perp}_{\tilde{\mathscr{R}},j})$.

If $n = 2j$, then $J_{+} \neq J$. The character Ω_s extends trivially on $c(\mathfrak{g}'^{\perp}_{\tilde{\mathscr{R}},j+1})$ to give a character on J_{+}.

In both cases, let Ω denote the extended character. There is a unique representation σ of J whose restriction to J_{+} is a multiple of Ω. In the case $J \neq J_{+}$, σ is the Heisenberg representation. We state our main results as

THEOREM 4.1. *Any irreducible representation of G which contains (L, Ω_s) must contain (J, σ).*

THEOREM 4.2. *There is a *-isomorphism of algebras*

$$\eta : \mathscr{H}(G'//J', 1) \to \mathscr{H}(G//J, \sigma)$$

so that $\mathrm{supp}(\eta(f)) = J\{\mathrm{supp}(f)\}J$ *for* $f \in \mathscr{H}(G'//J', 1)$.

The isomorphism η shall be described explicitly in the proof of Theorem 4.2.

In the case $n + 1 = 2j$, η is easily described. Fix $g \in G'$. There is an unique element $f_g \in \mathscr{H}(G//J, \sigma)$ with $\mathrm{supp}(f_g) = JgJ$ and $f_g(g) = 1$ (cf. Proposition 4.9). Let $e_g \in \mathscr{H}(G'//J', 1)$ be the characteristic function of $J'gJ'$. Then, the map η is given by

$$\eta(e_g) = \{\mathrm{vol}(J'gJ')/\mathrm{vol}(JgJ)\}^{1/2} f_g. \tag{4.17}$$

We defer the description of η when $n + 1$ is odd.

We begin the proof of Theorems 4.1 and 4.2 with a few preliminary results on the support of Hecke algebras. Let α realize Ω_s as in (4.11). The decomposition

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'^{\perp}$$

yields a decomposition of $\mathfrak{g}_{\tilde{\mathscr{R}},u}$ and $\mathfrak{g}_{\tilde{\mathscr{R}},u}/\mathfrak{g}_{\tilde{\mathscr{R}},u+v}$ as

$$\begin{aligned} \mathfrak{g}_{\tilde{\mathscr{R}},u} &= \mathfrak{g}'_{\tilde{\mathscr{R}},u} \oplus \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u} \\ \mathfrak{g}_{\tilde{\mathscr{R}},u}/\mathfrak{g}_{\tilde{\mathscr{R}},u+v} &= \mathfrak{g}'_{\tilde{\mathscr{R}},u}/\mathfrak{g}'_{\tilde{\mathscr{R}},u+v} \oplus \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+v}. \end{aligned} \tag{4.18}$$

For $x \in \mathfrak{g}$, let ad(x) be the adjoint map

$$\mathrm{ad}(x)(y) = xy - yx \quad y \in \mathfrak{g}. \tag{4.19}$$

Let $d = -r - n$, so that $\alpha + \mathfrak{g}_{\tilde{\mathscr{R}},d+1}$ is coset of $\mathfrak{g}_{\tilde{\mathscr{R}},d}/\mathfrak{g}_{\tilde{\mathscr{R}},d+1}$ representing Ω_s. An element $x \in \alpha + \mathfrak{g}_{\tilde{\mathscr{R}},d+1}$, determines a map ad$(x)$: $\mathfrak{g}_{\tilde{\mathscr{R}},u} \to \mathfrak{g}_{\tilde{\mathscr{R}},u+d}$. Furthermore, the map ad(x) induces a quotient map

$$\mathrm{ad}(x)\colon \mathfrak{g}_{\tilde{\mathscr{R}},u}/\mathfrak{g}_{\tilde{\mathscr{R}},u+1} \to \mathfrak{g}_{\tilde{\mathscr{R}},u+d}/\mathfrak{g}_{\tilde{\mathscr{R}},u+d+1} \tag{4.20}$$

which is independent of x. The map ad(x) maps $\mathfrak{g}'_{\tilde{\mathscr{R}},u}/\mathfrak{g}'_{\tilde{\mathscr{R}},u+1}$ to zero and $\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+1}$ to $\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+d}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+d+1}$.

LEMMA 4.3. *For all* u,

$$\mathrm{ad}(x)\colon \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+1} \to \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+d}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+d+1}$$

is an isomorphism.

Proof. For each of the Cartan subalgebras in (4.1), let $\mathfrak{h} \subset \mathfrak{g}$ be the Cartan subalgebra in (4.13). In all cases, except

$$\text{(4.1a) } a = \pm b, \quad \text{(4.1a) } b = 0, \quad \text{(4.1b) } a = 0 \quad \text{and} \quad \text{(4.1c) } a = 0, \tag{4.21}$$

$\mathfrak{g}' = \mathfrak{h}$. In these regular cases, $\mathfrak{g}^{\perp}$ is the direct sum of those root spaces of $\mathfrak{g}$ with respect to $\mathfrak{h}$ on which ad acts nontrivially. These root space decompositions respect the groups $\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+1}$. Since ad$(x)$ is nonzero on each root space, we have

$$\operatorname{ad}(x)\colon \mathfrak{g}^{\perp}_{\mathscr{R},u}/\mathfrak{g}^{\perp}_{\mathscr{R},u+1} \rightarrow \mathfrak{g}^{\perp}_{\mathscr{R},u+d}/\mathfrak{g}^{\perp}_{\mathscr{R},u+d+1} \tag{4.22}$$

is an isomorphism.

Consider the four "singular" cases of $\mathbf{s}$ in (4.21). Type (4.1a) with $a = \pm b$. We can assume $a = b$. Then, in the notation of (1.2) and (1.7), $\mathfrak{g}'$ is the subalgebra

$$\mathfrak{g}' = \mathfrak{m}_{\{s_1\}} \tag{4.23}$$

and

$$\mathfrak{g}^{\perp} = \{\theta E_N + \gamma E_X + \delta E_Z + \kappa E_n + \lambda E_x + \varphi E_z\} \tag{4.24}$$

It is obvious from (4.24) that $\mathfrak{g}^{\perp} = \mathfrak{g}^{\perp}_{+} \oplus \mathfrak{g}^{\perp}_{-}$, where

$$\mathfrak{g}^{\perp}_{+} = \{x \in \mathfrak{g}^{\perp} \mid \lambda, \kappa, \varphi = 0\}$$

$$\mathfrak{g}^{\perp}_{-} = \{x \in \mathfrak{g}^{\perp} \mid \theta, \delta, \gamma = 0\}$$

are $\operatorname{ad}(x)$ invariant. On $\mathfrak{g}^{\perp}_{+}$ (resp. $\mathfrak{g}^{\perp}_{-}$), $\operatorname{ad}(x)$ is multiplication by $\varpi^{-\prime}2a$ (resp. $-\varpi^{-i}2a$). It is obvious from this that (4.22) is an isomorphism. □

The other three singular cases are similar.

Type (4.1a) with $b = 0$. Then $\mathfrak{g}'$ is the subalgebra

$$\mathfrak{g}' = \mathfrak{m}_{\{s_0\}} \tag{4.25}$$

and

$$\mathfrak{g}^{\perp} = \{\gamma E_M + \theta E_X + \delta E_Z + \kappa E_m + \lambda E_x + \varphi E_z\} \tag{4.26}$$

For the type (4.1b) and $a = 0$, we have

$$\mathfrak{g}' = \{\upsilon(E_{1,1} - E_{4,4}) + \nu E_{1,4} + \mu E_{4,1} + \gamma(E_{2,3} + \varepsilon E_{3,2})\} \tag{4.27}$$

and

$$\mathfrak{g}^{\perp} = \left\{\begin{bmatrix} 0 & \theta & \delta & 0 \\ \varphi & \lambda & \omega & \delta \\ \mu & -\varepsilon\omega & -\lambda & -\theta \\ 0 & \mu & -\varphi & 0 \end{bmatrix}\right\} \tag{4.28}$$

For type (4.1c) and $a = 0$, we have

$$\mathfrak{g}' = \left\{ \begin{bmatrix} \upsilon & \gamma & 0 & \nu \\ \varepsilon\gamma & \upsilon & -\varepsilon\nu & 0 \\ 0 & \mu & -\upsilon & -\gamma \\ -\varepsilon\mu & 0 & -\varepsilon\gamma & -\upsilon \end{bmatrix} \right\} \tag{4.29}$$

and

$$\mathfrak{g}^{\perp} = \left\{ \begin{bmatrix} -\lambda & \theta & \delta & \omega \\ -\varepsilon\theta & \lambda & \varepsilon\omega & \delta \\ \mu & \varphi & -\lambda & -\theta \\ \varepsilon\varphi & \mu & \varepsilon\theta & \lambda \end{bmatrix} \right\} \tag{4.30}$$

In all cases, (4.22) is easily verified to be an isomorphism.

The next lemma is analogous to Lemma 3.2 in [HM3].

LEMMA 4.4. *Suppose* $\alpha + \mathfrak{g}_{\tilde{\mathscr{X}},d+1}$ *realizes* Ω_s *and* $x \in \alpha + \mathfrak{g}_{\tilde{\mathscr{X}},d+1}$ *lies in* $\mathfrak{g}'$ mod $\mathfrak{g}_{\tilde{\mathscr{X}},d+u}$ *for some integer* $u \geqslant 1$. *Then* x *is conjugate by* $G_{\tilde{\mathscr{X}},u}$ *to an element in* $\mathfrak{g}'$.

Proof. The proof is essentially that of Lemma 3.2 in [HM3]. We show x is $G_{\tilde{\mathscr{X}},u}$-conjugate to an element of $\mathfrak{g}'$ mod $\mathfrak{g}_{\tilde{\mathscr{X}},d+u+1}$. Write x as

$$x = x_1 + x^{\perp} \quad \text{with} \quad x_1 \in \mathfrak{g}'_{\tilde{\mathscr{X}},d} \quad \text{and} \quad x^{\perp} \in \mathfrak{g}^{\perp}_{\tilde{\mathscr{X}},d+u}.$$

By Lemma 4.3, there is $z^{\perp} \in \mathfrak{g}^{\perp}_{\tilde{\mathscr{X}},u}$ so that

$$\mathrm{ad}(-2x)(z^{\perp}) = x^{\perp} \bmod \mathfrak{g}_{\tilde{\mathscr{X}},d+u+1}.$$

For $g \in G$ and $y \in \mathfrak{g}$, let $\mathrm{Ad}(g)(y) = gyg^{-1}$. Then

$$\begin{aligned} \mathrm{Ad}(c(z^{\perp}))(x) &= x + 2\mathrm{ad}(x)(z^{\perp}) \bmod \mathfrak{g}_{\tilde{\mathscr{X}},d+u+1} \\ &= x_1 \bmod \mathfrak{g}_{\tilde{\mathscr{X}},d+u+1}. \end{aligned}$$

By induction, there is a convergent sequence $x_v \to x'$ of elements in $\mathfrak{g}'$ and a convergent sequence $g_v \to g$ of elements in $G_{\tilde{\mathscr{R}},u}$ so that

$$\mathrm{Ad}(g_w)(x) = x_v \bmod \mathfrak{g}_{\tilde{\mathscr{R}},d+u+v} \text{ for } w \geqslant v.$$

Hence, $\mathrm{Ad}(g)(x) = x'$. □

As an immediate corollary we have

COROLLARY 4.5. $\mathrm{Ad}(G_{\tilde{\mathscr{R}},1})(\alpha + \mathfrak{g}'_{\tilde{\mathscr{R}},d+1}) = \alpha + \mathfrak{g}_{\tilde{\mathscr{R}},d+1}$.

In particular, Theorem 4.1 follows directly from Corollary 4.5.

Let $\mathfrak{l}_+^*$ be the dual lattice of $\mathfrak{l}_+$ so

$$\mathfrak{l}_+^* = \mathfrak{g}'_{\tilde{\mathscr{R}},-r-n+1} + \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},-r-j'+1}. \tag{4.31}$$

Inspection of the proof of Lemma 4.4. gives

COROLLARY 4.6. $\mathrm{Ad}(J)(\alpha + \mathfrak{g}'_{\tilde{\mathscr{R}},d+1}) = \alpha + \mathfrak{l}_+^*$

The next two lemmas on Hecke algebra support are analogues of Lemmas 3.4 and 3.5 in [HM3].

LEMMA 4.7. *An element $g \in G$ lies in* supp $\mathscr{H}(G//J_+, \Omega_s)$ *if and only if the intersection*

$$\mathrm{Ad}(g)(\alpha + \mathfrak{l}_+^*) \cap (\alpha + \mathfrak{l}_+^*)$$

is nonempty.

Proof. This is exactly as in Lemma 3.4 in [HM3]. □

LEMMA 4.8. supp $\mathscr{H}(G//J_+, \Omega_s) \subset JG'J$

Proof. By Lemma 4.7, if $g \in \mathscr{H}(G//J_+, \Omega_s)$, then there are x_1, $x_2 \in \alpha + \mathfrak{l}_+^*$, so that $\mathrm{Ad}(g)(x_1) = x_2$. By Corollary 4.6, we can find x_1', $x_2' \in \alpha + \mathfrak{g}'_{\tilde{\mathscr{R}},d+1}$ and k_1, $k_2 \in J$ so that $x_s = \mathrm{Ad}(k_s)(x_s')$ for $s = 1$ or 2. This means the element $g' = k_2^{-1}gk_1$ conjugates x_1' to x_2'. Inspection of the various cases for $\mathfrak{g}'$, i.e., s regular or singular, shows $g' \in G'$, so $g \in JG'J$. □

As a consequence of Lemmas 4.7 and 4.8, we have

PROPOSITION 4.9. supp $\mathscr{H}(G//J, \sigma) = JG'J$.

Proof. By Lemma 4.7, $G' \subset \operatorname{supp} \mathscr{H}(G//J, \sigma)$. Also, by Lemma 4.8 $\operatorname{supp} \mathscr{H}(G//J, \sigma) \subset JG'J$. Whence, $\operatorname{supp} \mathscr{H}(G//J, \sigma) = JG'J$. □

Our next goal is to show

PROPOSITION 4.10. *Let $g' \in G'$. Then*

$$Jg'J \cap G' = J'g'J'.$$

Proposition 4.10 is trivial in cases (4.1d, e) since G' is compact and J is normalized by G'. To show Proposition 4.10 in cases (4.1a, b, c) we use the Bruhat decomposition [BT]. Let

$$\begin{aligned} N &= \text{monomial matrices in } G \\ N(R) &= N \cap \tilde{B} \\ B' &= G_{\tilde{\mathscr{B}}} \cap G' = \text{Iwahori subgroup of } G' \\ N' &= N \cap G' \\ N'(R) &= N' \cap B'. \end{aligned} \tag{4.32}$$

The Bruhat decomposition for G' is

$$G' = B'N'B'. \tag{4.33}$$

For $w_1, w_2 \in N'$, we have $B'w_1B' = B'w_2B'$ if and only if $w_1w_2^{-1} \in N'(R)$. Let

$$I = \begin{cases} \varnothing & \text{case (4.1a)} \\ \{s_0\} & \text{case (4.1b)}. \\ \{s_1\} & \text{case (4.1c)} \end{cases} \tag{4.34}$$

In the notation of (1.24), set

$$\mathfrak{J} = M_I \cap G_{\tilde{\mathfrak{R}}}. \tag{4.35}$$

The group $\mathfrak{J}$ is normalized by N'. For c a root of the root system C_2, and $v \in \mathbb{N}$, let $U_{c,v}$ be the v-th filtration root group in (1.4). Let C be the set of roots c so that $U_c \cap \mathfrak{J} = \{1\}$. For $w \in N'$, $wU_{c,v}w^{-1}$ (resp. $w^{-1}U_{c,v}w$) is equal to some $U_{c',v'}$(resp. $U_{c'',v''}$). For $c \in \mathrm{C}$, let $J_c = U_c \cap J$. Define

$$\begin{aligned} S(-) &= \{c \in \mathrm{C} \mid J_{c''} \not\subset w^{-1}J_c w\} \\ S(0) &= \{c \in \mathrm{C} \mid J_{c''} = wJ_c w^{-1}\} \\ S(+) &= \{c \in \mathrm{C} \mid J_{c'} \not\subset wJ_c w^{-1}\}. \end{aligned} \tag{4.36}$$

Fix some ordering on the sets in (4.36) and define sets

$$\begin{aligned} J^s &= \prod_{c \in S(s)} J_c \quad s \in \{+, -\} \\ J^0 &= \{\mathfrak{J} \cap J\} \prod_{c \in S(0)} J_c. \end{aligned} \tag{4.37}$$

Let J'^+ (resp. J'^-, J'^0) be the intersection of J^+ (resp. J^-, J^0) with G'. Then

Lemma 4.11. *Suppose* $\mathbf{s}$ *is regular or of type (4.1a). If* $w \in N'$, *then the double cosets* JwJ *and* $J'wJ'$ *have the unique decompositions*

$$\begin{aligned} JwJ &= J^- wJ^0 J^+ \\ J'wJ' &= J'^- wJ'^0 J'^+. \end{aligned}$$

There is an analogue of Lemma 4.11 in the remaining cases: (4.1b, c) with $a = 0$. Let H denote the Cartan subgroup of G corresponding to $\mathfrak{h}$ in (4.13). In the notation of (1.2), define subspaces of $\mathfrak{g}$ for case (4.1b) by

$$\begin{aligned} \mathfrak{u}_{M,X} &= \text{span of } E_M \text{ and } E_X \\ \mathfrak{u}_{m,x} &= \text{span of } E_m \text{ and } E_x \\ \mathfrak{u}_Z &= \text{span of } E_Z \\ \mathfrak{u}_z &= \text{span of } E_z, \end{aligned} \tag{4.38b}$$

and case (4.1c) by

$$\begin{aligned}
\mathfrak{u}_{Z-\varepsilon N} &= \text{span of } E_Z - \varepsilon E_N \\
\mathfrak{u}_{n-\varepsilon z} &= \text{span of } E_n - \varepsilon E_z \\
\mathfrak{u}_{X,Z+N} &= \text{span of } E_X \text{ and } E_Z + E_N \\
\mathfrak{u}_{x,z+n} &= \text{span of } E_x \text{ and } E_z + E_n .
\end{aligned} \tag{4.38c}$$

These subspaces can be exponentiated into the group G. Denote these sets by

$$U_{M,X}, \quad U_{m,x}, \quad U_Z, \quad U_z \tag{4.39b}$$

and

$$U_{Z-\varepsilon N}, \quad U_{n-\varepsilon z}, \quad U_{X,Z+N}, \quad U_{x,z+n}. \tag{4.39c}$$

The individual sets are $\mathrm{Ad}(H)$-invariant. In either case b or c, the collection of sets is $\mathrm{Ad}(N')$-invariant. Let $\mathtt{C}$ be the set of indices in (4.39b, c). For $c \in \mathtt{C}$, let $J_c = U_c \cap J$. Define sets as in (4.36) and (4.37). With these definitions, one readily verifies

LEMMA 4.12. *Suppose* $\mathbf{s}$ *is of type* (*4.1b, c*) *and* $a = 0$. *If* $w \in N'$, *then the double cosets* JwJ *and* $J'wJ'$ *have the unique decompositions*

$$\begin{aligned}
JwJ &= J^- w J^0 J^+ \\
J'wJ' &= J'^- w J'^0 J'^+ .
\end{aligned}$$

Proposition 4.10 for cases (4.1a, b, c) follows from Lemmas 4.11 and 4.12.

Proof of Theorem 4.2 in cases (*4.1d, e*)

Here, $JG'J = G'J$ is a group. In particular, when σ is one-dimensional, the map η in (4.17) reduces to $\eta(e_g) = f_g$ and is clearly an isomorphism. The other case we need to consider is when n is even. Consider the restriction of Ω to J'. Fix an extension φ of Ω on J' to G'. There is an extension of σ to $G'J$ provided by φ and the oscillator representation [H]. For $g \in G'$, let $f_g \in \mathscr{H}(G//J, \sigma)$ be the element with support JgJ and $f_g(g) = \sigma(g)$. Then, the map η: $\mathscr{H}(G'//J') \to \mathscr{H}(G//J, \sigma)$ defined by

$$\eta(e_g) = \varphi(g)^{-1} f_g$$

is an isomorphism. □

Modulo the group T in (3.8), $G'J$ is compact. Hence, a representation of G which contains σ' must be supercuspidal. These representations are obtained by taking the different extensions of σ to $G'J/T$ and inducing up to G. There are $\#\{G'/J'T\}$ such representations.

Proof of Theorem 4.2 in the regular cases of (4.1a, b, c)

The proof in all three cases is very similar to the proofs of Proposition 3.7, 3.15 and 3.20. As such, we merely outline the proofs and omit the rather tedious details.

Case (4.1a). Here, $G' = A$ is the subgroup of diagonal elements. Let i be as in (4.9), and q be the order of the residue field $\mathbb{F}_q = R/\wp$. The representation σ is one dimensional when i is even and q dimensional when i is odd. In the latter case, it is convenient to set $k = (i - 1)/2$ and define, using the root group notation of (1.4),

$$\begin{aligned} J^\# &= U_{a+b,k} \\ J_* &= J_+ J^\# . \end{aligned} \tag{4.40}$$

The group J_* is normalized by B', the character Ω on J_+ extends to a character ϱ on J_* trivial on $J^\#$, and $\sigma = \operatorname{Ind} \varrho$. This means $\mathscr{H}(G//J, \sigma) = \mathscr{H}(G//J_*, \varrho)$. Furthermore, Propositions 4.9 and 4.10 hold with (J, σ) replaced by (J_*, ϱ). For notational convenience, we shall use the notation (J, σ) to refer to (J, σ) when i is even and (J_*, ϱ) when i is odd. Regardless of the parity of i, we can therefore assume σ is one dimensional. For each $g \in A$, let e_g and f_g be as in (4.17). Take d_+, h_+ as in (3.23). The proof of Proposition 3.5 is easily modified to show $Jg'JdJ \cap JG'J = Jg'dJ$, when $g' \in G'$ and $d \in \{\varpi I, d_+, h_+\}$. So, $f_{g'} * f_d$ is a multiple of $f_{g'd}$. Since $\{\varpi I, d_+, h_+\} \cup B'$ generate G', we conclude that $f_g * f_{g'}$, is multiple of $f_{gg'}$ for arbitrary $g,g' \in G'$. The proof of Proposition 3.7 is now easily adapted to the present situation to show

$$\{\mathrm{vol}(J'gJ')/\mathrm{vol}(JgJ)\}^{1/2} f_d * \{\mathrm{vol}(J'g'J')/\mathrm{vol}(Jg'J)\}^{1/2} f_{d'}$$

$$= \{\mathrm{vol}(J'gg'J')/\mathrm{vol}(Jgg'J)\}^{1/2} f_{gg'}, \tag{4.41}$$

i.e., η is an isomorphism.

Case (4.1b). Regardless of the dimension of σ, let V_σ denote the space of σ. Set $G'(R) = G' \cap G_{\tilde{\varsigma}}$. A character φ of $G'(R)$ which extends Ω on J' again determines an extension of σ to $G'(R)$. Let T be the group (3.8), and define h_+, h_- as in (3.23). For x an element of the group generated by $T \cup \{h_+, h_-\}$, define $\sigma(x)$ to be the identity operator, and define $f_x \in \mathscr{H}(G//J, \sigma)$, $x \in G'(R) \cup T \cup \{h_+, h_-\}$, to be the element with support JxJ and $f_x(x) = \sigma(x)$. Each element $g \in G'$ has a unique decomposition $g = th^m k$, where $t \in T$, $h \in \{h_+, h_-\}$, $m \geqslant 0$, and $k \in G'(R)$. Define $f_g \in \mathscr{H}(G//J, \sigma)$ by

$$f_g = \varphi(k)^{-1} f_t * \{f_h\}^m * f_k. \tag{4.42}$$

Observe that

$$JgJ = \{JtJ\}\{JhJ\}^m\{JkJ\}. \tag{4.43}$$

The terms on the right hand side can be taken in any order. This means the terms on the right hand side of (4.42) can be taken in any order, i.e., they commute. To show the map η defined by

$$\eta(e_g) = \{\mathrm{vol}(J'gJ')/\mathrm{vol}(JgJ)\}^{1/2} f_g. \tag{4.44}$$

is an isomorphism, we need only show

$$f_{h_+} * f_{h_-} = q^4 f_I. \tag{4.45}$$

The proof of Proposition 3.5 again easily adapts to show $Jh_+Jh_-J \cap JG'J = J$; whence, $f_{h_+} * f_{h_-}$ is a multiple of f_I. Since

$$f_{h_+} * f_{h_-}(1) = \int_G f_{h_+}(g) f_{h_-}(g^{-1})\, dg = q^4 I, \tag{4.46}$$

(4.45) follows immediately.

Case (*4.1c*). The representation σ has dimension q. An extension Ω of Ω_s to $G'(R) = G' \cap G_{\tilde{\mathscr{U}}}$, again determines an extension of σ to $G'(R)J$. An element $g \in G'$ has a unique decomposition $g = td^m k$, with $t \in T$, $h \in \{d_+, d_-\}$, $m \geqslant 0$, and $k \in G'(R)$. Define $f_g \in \mathscr{H}(G//J, \sigma)$ by the analogue of (4.43). The map η defined by (4.44) is an isomorphism. □

REMARK. In all three cases, G'/T has a noncompact center and so no square integrable representations. By implication, there are no square integrable representations of G/T containing (J, σ').

Proof of Theorem 4.2 in the singular cases of (*4.1a, b, c*)

Here, as in Section 3, we shall give a presentation of $\mathscr{H}(G'//J', 1)$ in terms of generators and relations and show that $\mathscr{H}(G//J, \sigma)$ possesses parallel generators and relations. We then define a *-isomorphism via these generators.

Case (*4.1a*), $a = b \neq 0$. Recall some notation from Section 1, e.g.,

$$d(x, y, z) = xE_{1,1} + yE_{2,2} + zE_{3,3} + (yz/x)E_{4,4}. \tag{4.47}$$

Let $u_a(x)$ and $u_{-a}(x)$ be the root elements (1.3). Let

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.48}$$

In the notation of (1.7),

$$\begin{aligned} G' &= M_{\{s_1\}}(F) \\ &= \left\{ \begin{bmatrix} A & \\ & B \end{bmatrix} \middle| A \in GL_2(F),\ \lambda \in F^\times \text{ and } B = \lambda S(A^t)^{-1}S \right\} \end{aligned} \tag{4.49}$$

(cf. (3.80)). In particular, G' is isomorphic to $GL_2(F) \times F^\times$. Let

$$\begin{aligned} N' &= G' \cap N \\ B' &= G' \cap B. \end{aligned} \tag{4.50}$$

The group B' is an Iwahori subgroup of G' and a Bruhat decomposition of G' is given by

$$G' = B'N'B'. \tag{4.51}$$

The affine Weyl group $W'^{\mathrm{aff}} = N'/\{B' \cap N'\}$ is generated by the images of

$$\begin{aligned} r_0 &= \varpi^{-1}E_{1,2} + \varpi E_{2,1} + \varpi^{-1}E_{3,4} + \varpi E_{4,3} = s_2 s_0 s_1 s_0 s_2 \\ r_1 &= E_{1,2} + E_{2,1} + E_{3,4} + E_{4,3} = s_1 \\ d_+ &= d(\varpi^{-1}, \varpi^{-1}, 1), \quad d_- = d_+^{-1} \\ t' &= E_{1,2} + \varpi E_{2,1} + E_{3,4} + \varpi E_{4,3} = s_2 s_0 t d(1, -1, -1) \end{aligned} \tag{4.52}$$

(cf. (3.40)).

PROPOSITION 4.13. *The Hecke algebra $\mathscr{H}(G'//J', 1)$ is generated by the elements e_g, $g \in \{r_0, r_1, d_+, d_-, t', t'^{-1}\} \cup B'$. These elements satisfy the relations*

(a) $e_k * e_{k'} = e_{kk'}$ $k, k' \in B'$

(b) $e_s * e_k = e_{\mathrm{Ad}(s)(k)} * e_s$ $s \in \{r_0, r_1, t'\}$ *and* $k \in sB's \cap B$

(c) (i) $e_{r_1} * e_{r_1} = q \sum_x e_{u_{-a}(\varpi' x)}$ *(x runs over R* mod $\not{p}$)

(ii) $e_{t'} * e_{r_1} = e_{r_0} * e_{t'}$

(iii) $e_{t'} * e_{t'^{-1}} = e_I$

(d) $e_{r_1} * e_{u_a(x)} * e_{r_1} = q\, e_{u_a(x^{-1})} * e_{r_1} * e_{d(x, -x^{-1}, -x)} * e_{u_a(x^{-1})}$, $x \in R^\times$

(e) (i) $e_{d_+} * e_{d_-} = e_I = e_{d_-} * e_{d_+}$

(ii) $e_{d_+} * e_g = e_g * e_{d_+}$ $g \in \{r_0, r_1, t'\} \cup B'$.

The above relations are a defining set of relations for the algebra.

Proof. This is proved exactly as Theorem 2.1 in Chapter 3 [HM1]. □

To exhibit parallel relations in $\mathscr{H}(G//J, \sigma)$, we need to make a preliminary reduction. The representation σ is one-dimensional when i is even and q dimensional when i is odd. In the odd case define (J_*, ϱ) by (4.40), then $\mathscr{H}(G//J, \sigma) = \mathscr{H}(G//J_*, \varrho)$; whence, if we replace (J, σ) by (J_*, ϱ) when i is odd, we can assume σ is one-dimensional in all cases. We have $\mathrm{supp}(\mathscr{H}(G//J, \sigma)) = JG'J$. If $g \in G'$, there is a unique element $f_g \in \mathscr{H}(G//J, \sigma)$ with $\mathrm{supp}(f_g) = JgJ$ and $f_g(g) = 1$. Define $\eta\colon \mathscr{H}(G'//J', 1) \to \mathscr{H}(G//J, \sigma)$ by

$$\eta(e_g) = \{\mathrm{vol}(J'gJ')/\mathrm{vol}(JgJ)\}^{1/2} f_g. \tag{4.53}$$

The map η is clearly a linear isomorphism. That η is a isomorphism of algebras, is equivalent to

PROPOSITION 4.14. *The Hecke algebra $\mathscr{H}(G//J, \sigma)$ is generated by the elements f_g, $g \in \{r_0, r_1, d_+, d_-, t'\} \cup B'$. The elements f_g satisfy the relations*

$$(a)\ f_k * f_{k'} = f_{kk'} \quad k, k' \in B'$$

$$(b)\ f_s * f_k = f_{\mathrm{Ad}(s)(k)} * f_s \quad s \in \{r_0, r_1, t'\} \text{ and } k \in sB's \cap B'$$

$$(c)\ (i)\ f_{r_1} * f_{r_1} = \{\mathrm{vol}(Jr_1J)/\mathrm{vol}(J'r_1J')\} q \sum_x f_{u_{-a}(\varpi' x)}$$

$$(x \text{ runs over } R \bmod \mathfrak{p})$$

$$(ii)\ f_{r_1} * f_{t'} = f_{r_1 t'} = f_{t' r_0}$$

$$= \{\mathrm{vol}(Jt'r_0J)/(\mathrm{vol}(Jt'J)\ \mathrm{vol}(Jr_0J))\}^{1/2} f_{t'} * f_{r_0}$$

$$(iii)\ f_{t'} * f_{t'} = \mathrm{vol}(Jt'J) f_{\varpi' I}$$

$$(d)\ f_{r_1} * f_{u_a(x)} * f_{r_1} = \{\mathrm{vol}(Jr_1J)/\mathrm{vol}(J'r_1J')\}^{1/2}$$

$$\times\ q f_{u_a(x^{-1})} * f_{r_1} * f_{d(x, -x^{-1}, -x)} * f_{u_a(x^{-1})}, \quad x \in R^\times$$

$$(e)\ (i)\ f_{d_+} * f_{d_-} = q^3 f_I = f_{d_-} * f_{d_+}$$

$$(ii)\ f_{d_+} * f_g = \{\mathrm{vol}(Jd_+J)\ \mathrm{vol}(JgJ)/\mathrm{vol}(Jd_+gJ)\}^{1/2} f_{gd_+}$$

$$= f_g * f_{d_+}, \quad g \in GL_2(F).$$

Proof. If g, $g' \in G'$ and $JgJg'J = Jgg'J$, i.e., $\mathrm{vol}(JgJ)\mathrm{vol}(Jg'J) = \mathrm{vol}(Jgg'J)$, then $f_g * f_{g'}$ is a multiple of $f_{gg'}$. If either of JgJ, $Jg'J$ has volume one, we in fact have equality. Since J is normalized by B', this immediately implies relations (a), (b) and (e(ii)) when $g \in B'$. Let N_0' (resp. N_1') be the subgroup of N' generated by r_0, r_1, and $A'(R) = N' \cap B'$ (resp. N_0', t'). The group $W_0' = N_0'/A'(R)$ is an infinite dihedral group, while the group $W_1' = N_1'/A(R)$ is the affine Weyl group of the $GL_2(F)$ component of G'. Let ℓ denote the length function on N_1'. We have

$$\mathrm{vol}(Jr_0J) = \begin{cases} q^5 & i \text{ even} \\ q^3 & i \text{ odd} \end{cases}$$

$$\mathrm{vol}(Jr_1J) = \begin{cases} q^1 & i \text{ even} \\ q^3 & i \text{ odd} \end{cases} \tag{4.54}$$

$$\mathrm{vol}(J(r_1r_0)^uJ) = q^{6|u|}.$$

It follows that $JwJw'J = Jww'J$ whenever w, $w' \in N_0'$, and $\ell(ww') = \ell(w) + \ell(w')$; whence, under these hypotheses

$$f_w * f_{w'} = f_{ww'}. \tag{4.55}$$

When i is odd the element t' normalizes J and so (4.55) is also valid for w, $w' \in N_1'$. This in particular implies relations (c(ii)) and (c(iii)). When i is even t' does not normalize J. We have

$$\begin{aligned} \mathrm{vol}(Jt'J) &= q^2 \\ \mathrm{vol}(Jt'(r_0r_1)^uJ) &= q^{6u-2} \quad u \geqslant 1 \\ \mathrm{vol}(Jt'(r_0r_1)^ur_0J) &= q^{6u+3} \quad u \geqslant 0 \\ \mathrm{vol}(Jt'(r_1r_0)^uJ) &= q^{6u+2} \quad u \geqslant 0 \\ \mathrm{vol}(Jt'(r_1r_0)^ur_1J) &= q^{6u+3} \quad u \geqslant 0. \end{aligned} \tag{4.56}$$

Thus, $f_{t'} * f_w = f_{t'w}$ and $f_{t'r_0} * f_w = f_{t'r_0w}$ provided w is of the form $(r_1r_0)^u$ or $(r_1r_0)^ur_1$, $u \geqslant 0$. We show $f_{t'} * f_{r_0} = qf_{t'r_0}$. Regardless of the parity of i, let k be the greatest integer in $i/2$. For $h \in G$, denote by δ_h, the Dirac point mass measure at h. When i is even, write $f_{t'}$ as

$$f_{t'} = \sum_{z} f_I * \delta_{t'} * \delta_z, \tag{4.57}$$

where z runs over the q^2 elements of J mod $\{J \cap t'Jt'^{-1}\}$. In the notation of (1.4), we can take representatives for z in the form $z = u_1 u_2$ where u_1 (resp. u_2) runs over $U_{b,k}/U_{b,k+1}$ (resp. $U_{-2a-b,k+1}/U_{-2a-b,k+2}$). Similarly, we have

$$f_{r_0} = \sum_{x} \delta_x * \delta_{r_0} * f_I \tag{4.58}$$

with $x = v_1 v_2 v_3$ and $v_1 \in U_{b,k}/U_{b,k+2}$, $v_2 \in U_{-2a-b,k+1}/U_{-2a-b,k+3}$, $v_3 \in U_{-a,k+1}/U_{-a,k+2}$. Thus,

$$f_{t'} * f_{r_0} = \sum_{z,x} f_I * \delta_{t'} * \delta_z * \delta_x * \delta_{r_0} * f_I. \tag{4.59}$$

But, $Jt'zxr_0J \cap G' = \varnothing$ unless $u_1 v_1 \in U_{b,k+1}$ and $u_2 v_2 \in U_{-2a-b,k+2}$. Under these conditions, the nonzero summands have the form

$$f_I * \delta_{t'} * \delta_{v_3} * \delta_{r_0} * f_I = f_I * \delta_{t'} * \delta_{r_0} * f_I.$$

Therefore,

$$\begin{aligned} f_{t'} * f_{r_0} &= q^5 f_I * \delta_{t'r_0} * f_I = q^3 f_{t'r_0} \\ &= q^3 f_{r_1 t'} = q^3 f_{r_1} * f_{t'}. \end{aligned} \tag{4.60}$$

An analogous calculation shows

$$f_{t'} * f_{t'} = q^2 f_{\varpi I}. \tag{4.61}$$

We conclude relations (c(ii)) and (c(iii)) from (4.60) and (4.61). We can further conclude from the above that regardless of the parity of i,

$$\begin{aligned} &\{\mathrm{vol}(J'wJ')/\mathrm{vol}(JwJ)\}^{1/2} f_w * \{\mathrm{vol}(J'w'J')/\mathrm{vol}(Jw'J)\}^{1/2} f_{w'} \\ &\quad = \{\mathrm{vol}(J'ww'J')/\mathrm{vol}(Jww'J)\}^{1/2} f_{ww'} \end{aligned} \tag{4.62}$$

provided w, $w' \in N_1'$, and $\ell(ww') = \ell(w) + \ell(w')$. Consequently, the elements f_g, $g \in GL_2(F)$ lie in the subalgebra of $\mathscr{H}(G//J, \sigma)$ generated by f_g, $g \in \{r_0, r_1, t'\} \cup B'$. Once we show relations (e(i)) and (e(ii)) it will follow that f_g, $g \in \{r_0, r_1, d_+, d_-, t'\} \cup B'$ generate $\mathscr{H}(G//J, \sigma)$. To show the first relation in (e(i)), write

$$f_{d_+} = \sum_x f_I * \delta_{d_+} * \delta_x \quad \text{and} \quad f_{d_-} = \sum_x \delta_x * \delta_{d_-} * f_I, \tag{4.63}$$

where $x = x_0 x_1 x_2$ and $x_\alpha \in U_{\alpha a+b,k}/U_{\alpha a+b,k+1}$, $\alpha = 1, 2$; $x_0 \in U_{b,k}/U_{b,k+1}$ (resp. $U_{b,k+1}/U_{b,k+2}$) when i is even (resp. odd). Then,

$$f_{d_+} * f_{d_-} = q^3 \sum_x f_I * \delta_{d_+} * \delta_x * \delta_{d_-} * f_I.$$

But $Jd_+ x d_- J \cap G' = \varnothing$ unless $x = 1$, so the summands are all zero except when x is the identity representative. This is the desired result. Similarly $f_{d_-} * f_{d_+} = q^3 f_I$. Consider the relation (e(ii)). We have already shown (e(ii)) when $g \in B'$. By the Bruhat decomposition therefore, it is enough to consider $g = w \in N_1'$. Let $S(-)$ be the set of roots defined in (4.36). Write f_w as

$$f_w = \sum_z \delta_z * \delta_w * f_I, \tag{4.64}$$

where $z = \Pi_{\iota \in S(-)} z_\iota$, $z_\iota \in J_\iota/\{wJ_{\iota''}w^{-1}\}$. The intersection $Jd_+ xzwJ \cap G'$ is empty precisely when one can pick x so that x_α lies in some J_ι and $x_\alpha z_\iota \notin U_{\alpha a+b,k+1}$. Set $\beta = \#\{\{b, a+b, 2a+b\} \cap S(-)\}$. Then,

$$\begin{aligned} f_{d_+} * f_w &= q^{3-\beta} \operatorname{vol}(JwJ) f_I * \delta_{d_+} * \delta_w * f_I \\ &= q^{3-\beta} \{\operatorname{vol}(JwJ)/\operatorname{vol}(Jd_+ wJ)\} f_{d_+ w} \\ &= \{\operatorname{vol}(Jd_+ J) \operatorname{vol}(JwJ)/\operatorname{vol}(Jd_+ wJ)\}^{1/2} f_{d_+ w}. \end{aligned}$$

The calculation for $f_w * f_{d_-}$ is analogous and omitted. The above methods are easily adapted to prove relations (c(i)) and (d). We give the details for (d) when i is odd, and omit those for the easier case of i even as well as relation (c(i)) for all i. For notational convenience, abbreviate $u_a(x)$ to $u(x)$. Write $f_{r_1} * f_{u(x)} = f_{r_1 u(x)}$ and f_{r_1} as

$$f_{r_1 u(x)} = \sum_{z,v} f_I * \delta_{r_1} * \delta_{u(x+\varpi' z)v},$$

$$f_{r_1} = \sum_{z,v} \delta_{vu(x+\varpi' z)} * \delta_{r_1} * f_I,$$

with $z \in R/\mathfrak{p}$, and $v = v_1 v_2$, $v_1 \in U_{2a+b,k}/U_{2a+b,k+1}$, $v_2 \in U_{-b,k+1}/U_{-b,k+2}$. Let $y = x + \varpi' z \in R^\times$. Then

$$f_{r_1} * f_{u(x)} * f_{r_1} = q^3 \sum_{z,v} f_I * \delta_{r_1 u(x+\varpi' z) v r_1} * f_I. \tag{4.65}$$

Since $r_1 u(y) r_1 = u(y^{-1}) r_1 d(y, -y^{-1}, -y) u(y^{-1})$, the intersection

$$J r_1 u(x + \varpi' z) v r_1 J \cap G'$$

is nonempty precisely when v_2 is the identity representative. The summation in (4.65) collapses to

$$f_{r_1} * f_{u(x)} * f_{r_1} = q^4 \sum_{z} f_I * \delta_{r_1 u(y) r_1} * f_I. \tag{4.66}$$

Rewrite the summand as

$$\begin{aligned} f_I * \delta_{r_1 u(y) r_1} * f_I &= f_{u(y^{-1}) r_1 d(y, -y^{-1}, -y) u(y^{-1})}/\mathrm{vol}(J r_1 J) \\ &= f_{u(x^{-1}) r_1 d(x, -x^{-1}, -x) u(x^{-1})}/\mathrm{vol}(J r_1 J). \end{aligned} \tag{4.67}$$

Combining (4.66) and (4.67) we obtain relation (d) when i is odd. This completes the proof of Proposition 4.14.

For the remaining three cases, the techniques of the above case adapt readily. We formulate the results and give details only when they differ significantly from the above arguments.

Case (*4.1a*), $b = 0$. Here,

$$G' = \mathbf{M}_{\{s_0\}}(F) \tag{4.68}$$

As in the above case, we can assume (J, σ) is one dimensional by replacing (J, σ) by (J_*, ϱ) in (4.40), when i is odd. Then the map defined by (4.53) is a $*$-isomorphism of algebras.

Case (*4.1b*), $a = 0$. Here,

$$G' = \left\{ \begin{bmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & \varepsilon y & x & 0 \\ c & 0 & 0 & d \end{bmatrix} \in G \right\}. \tag{4.69}$$

The representation σ is one-dimensional when i is odd and q dimensional when i is even. When σ is one-dimensional, the map η: $\mathscr{H}(G'//J', 1) \to \mathscr{H}(G//J, \sigma)$ defined by (4.53) is a *-isomorphism. When σ is q dimensional, we cannot replace (J, σ) by some (J_*, ϱ) since there are no proper subgroups between J and J_+ normalized by B'. Here, we work directly with σ. The situation is analogous to Case II in Chapter 3 of [HM1]. The oscillator representation yields a representation ω of B' trivial on $G' \cap Q_1$ with the property

$$\omega(g)\sigma(x)\omega(g)^{-1} = \sigma(\mathrm{Ad}(g)(x)), \quad g \in B', x \in J. \tag{4.70}$$

Let Y (resp. Y_+) denote the intersection of J (resp. J_+) with the group (4.68). The group Y has two pertinent properties:

(i) restriction of the representation σ to Y remains irreducible.

(ii) the group $N' = G' \cap N$ normalizes Y.

We deduce an action ω of N' on the space V_σ of σ so that (4.70) holds with $g \in N'$ and $x \in Y$. The ω action is in fact the trivial representation. The Bruhat decomposition of G' yields a mapping

$$\omega: G' \to \mathrm{End}\, V_\sigma$$

satisfying

$$\omega(xgy) = \omega(x)\omega(g)\omega(y). \quad x, y \in B', g \in G'. \tag{4.71}$$

For $g \in G'$, define e_g (resp. f_g) in $\mathscr{H} = \mathscr{H}(G'//J', 1)$ (resp. $\mathscr{H}' = \mathscr{H}(G//J, \sigma)$) by

$$\begin{aligned} \mathrm{supp}\,(e_g) &= J'gJ', \quad e_g(g) = 1 \\ \mathrm{supp}\,(f_g) &= JgJ, \quad f_g(g) = \omega(g). \end{aligned} \tag{4.72}$$

Up to scalar multiples e_g (resp. f_g) are the only elements of $\mathscr{H}$ (resp. $\mathscr{H}'$) with support $J'gJ'$ (resp. JgJ). Define η: $\mathscr{H}' \to \mathscr{H}$ by

$$\eta(e_g) = \{\mathrm{vol}(J'gJ')/\mathrm{vol}(JgJ)\}^{1/2} f_g. \tag{4.73}$$

PROPOSITION 4.15. *The map η: $\mathscr{H}' \to \mathscr{H}$ is a *-isomorphism of algebras.*

Proposition 4.15 is proved by establishing analogues of Propositions 4.13 and 4.14. Let

$$r_0 = E_{1,4} + E_{2,2} + E_{3,3} - E_{4,1}$$

$$r_1 = \varpi^{-1}E_{1,4} + E_{2,2} + E_{3,3} - \varpi E_{4,1}$$

Abbreviate $u_{2a+b}(x)$ (resp. $u_{-2a-b}(x)$) to $u(x)$ (resp. $\underline{u}(x)$).

PROPOSITION 4.16. *The Hecke algebra $\mathscr{H}(G'//J', 1)$ is generated by the elements e_g, $g \in \{r_0, r_1, \varpi I, \varpi^{-1}I\} \cup B'$. The elements e_g satisfy the relations*

(a) $e_k * e_{k'} = e_{kk'} \quad k, k' \in B'$

(b) $e_s * e_k = e_{\mathrm{Ad}(s)(k)} * e_s \quad s \in \{r_0, r_1\}$ *and* $k \in sB's \cap B'$

(c) (i) $e_{r_0} * e_{r_0} = q \sum_x e_{\underline{u}(\varpi' x)} \quad$ (*x runs over* $R \bmod \mathscr{p}$)

$e_{r_1} * e_{r_1} = q \sum_x e_{u(\varpi'^{-1} x)} \quad$ (*x runs over* $R \bmod \mathscr{p}$)

(ii) $e_{\varpi I} * e_{\varpi^{-1} I} = e_I$

(d) *for* $x \in R^\times$

$$e_{r_0} * e_{u(x)} * e_{r_0} = q\, e_{u(x^{-1})} * e_{r_0} * e_{d(x,1,1)} * e_{u(x^{-1})},$$

$$e_{r_1} * e_{\underline{u}(\varpi x)} * e_{r_1} = q\, e_{\underline{u}(\varpi x^{-1})} * f_{r_1} * e_{d(x,1,1)} * e_{\underline{u}(\varpi x^{-1})}.$$

PROPOSITION 4.17. *The Hecke algebra $\mathscr{H}(G//J, \sigma)$ is generated by the elements f_g, $g \in \{r_0, r_1, \varpi I\} \cup B'$. The elements f_g satisfy the relations*

(a) $f_k * f_{k'} = f_{kk'} \quad k, k' \in B'$

(b) $f_s * f_k = f_{\mathrm{Ad}(s)(k)} * f_s \quad s \in \{r_0, r_1, \varpi I\}$ *and* $k \in sB's \cap B'$

(c) (i) $f_{r_0} * f_{r_0} = \mathrm{vol}(Jr_0J)/\mathrm{vol}(Jr_0J)q \sum_x f_{\underline{u}(\varpi' x)}$

(*x runs over R* mod $\mathfrak{p}$)

$f_{r_1} * f_{r_1} = \mathrm{vol}(Jr_1J)/\mathrm{vol}(Jr_1J)q \sum_x f_{u(\varpi'^{-1} x)}$

(*x runs over R* mod $\mathfrak{p}$)

(ii) $f_{\varpi I} * f_{\varpi^{-1} I} = f_I$

(d) *for* $x \in R^\times$

$$f_{r_0} * f_{u(x)} * f_{r_0} = q f_{u(x^{-1})} * f_{r_0} * f_{d(x,1,1)} * f_{u(x^{-1})} \{\mathrm{vol}(Jr_0J)/\mathrm{vol}(J'r_0J')\}^{1/2},$$

$$f_{r_1} * f_{\underline{u}(\varpi x)} * f_{r_1} = q f_{\underline{u}(\varpi x^{-1})} * f_{r_1} * f_{d(x,1,1)} * f_{\underline{u}(\varpi x^{-1})} \{\mathrm{vol}(Jr_1J)/\mathrm{vol}(J'r_1J')\}^{1/2}.$$

Proof. Relations (a), (b) and (c(ii)) are obvious since ω satisfies (4.71). The proof of the four relations in (c(i)) and (d) are similar. We give the details for the first relation in (d) and omit those for the other three easier cases. Let $v(z, z') = u_a(z)u_{a+b}(z')$. Then

$$f_{r_0} * f_{u(x)} = \sum_{x',z,z'} f_I * \delta_{r_0} * \delta_{u(x+\varpi' x')} * \delta_{v(\varpi' z, \varpi' z')} I$$

$$f_{r_0} = \sum_{x',z,z'} I \delta_{v(\varpi' z, \varpi' z')} * \delta_{u(\varpi' x')} * \delta_{r_0} * f_I,$$

where I is the identity operator and x', z, z' run over R mod $\mathfrak{p}$. Let $y = x + \varpi' x'$ and $v = v(\varpi' z, \varpi' z')$. The convolution $f_{r_0} * f_{u(x)} * f_{r_0}$ is therefore

$$f_{r_0} * f_{u(x)} * f_{r_0} = q^3 \sum_{x',v} f_I * \delta_{r_0} * \delta_{u(y)} * \delta_v I * \delta_{r_0} * f_I$$

$$= q^3 \sum_{x',v} f_I * \delta_{r_0} * \delta_{u(y)} * \delta_{r_0} I * \delta_{r_0 v r_0} * f_I.$$

The relation $r_0 u(y) r_0 = u(-y^{-1}) r_0\, d(-y, 1, 1)\, u(-y^{-1})$ allows us to bring the factor $\delta_{r_0 v r_0}$ to the left where, after appropriate transformations, it can be absorded into f_I. When $\delta_{r_0 v r_0}$ is commuted pass $\delta_{u(-y^{-1})}$, the term δ_γ is introduced, where γ is the commutator

$$\gamma = u(y^{-1})(r_0 v r_0)^{-1} u(-y^{-1})(r_0 v r_0) \in Q_{3l}.$$

The factor δ_γ can absorbed into f_I on the right with the introduction of the factor $\sigma(\gamma) = \Psi(\varpi^{-1}(z^2 - \varepsilon z'^2))$. It follows that

$$f_{r_0} * f_{u(x)} * f_{r_0} = q^3 \sum_{x',v} \Psi(\varpi^{-1}(z^2 - \varepsilon z'^2)) f_I * \delta_{r_0 u(x) r_0} I * f_I$$

$$= q^3 \left\{ \sum_{z,z'} \Psi(\varpi^{-1}(z^2 - \varepsilon z'^2)) \right\} q f_I * \delta_{r_0 u(x) r_0} I * f_I$$

The Gauss sum $\{\Sigma_{z,z'} \Psi(\varpi^{-1}(z^2 - \varepsilon z'^2))\}$ is trivially calculated to be q. We conclude

$$f_{r_0} * f_{u(x)} * f_{r_0} = q^3 q^2 f_{r_0 u(x) r_0} / \mathrm{vol}(J r_0 J)$$

$$= q f_{u(x^{-1})} * f_{r_0} * f_{d(x,1,1)} * f_{u(x^{-1})} \{\mathrm{vol}(J r_0 J)/\mathrm{vol}(J' r_0 J')\}^{1/2}.$$

To see that f_g, $g \in \{r_0, r_1, \varpi I\} \cup B'$ do indeed generate the algebra, observe that $\{r_0, r_1, \varpi I\}$ generate N' mod $N' \cap B$. Since $\mathrm{vol}(J r_0 J) = q^3$, $\mathrm{vol}(J r_1 J) = q$ and $\mathrm{vol}(J (r_0 r_1)^u J) = q^{4|u|}$; we have $f_w * f_{w'} = f_{ww'}$ whenever, $w, w' \in N'$ and $\ell(ww') = \ell(w) + \ell(w')$. Here, $\ell(w)$ is of course the number of $r \in \{r_0, r_1\}$ in a reduced expression for w. The algebra is thus, by the Bruhat decomposition, generated by the indicated elements. This completes the proof of Theorem 4.17. □

Case (*4.1c*), $a \doteq 0$. Let

$$r_0 = E_{1,4} - \varepsilon E_{2,3} + E_{3,2} - \varepsilon E_{4,1}$$

$$r_1 = \varpi^{-1} E_{1,4} - \varpi^{-1} \varepsilon E_{2,3} + \varpi E_{3,2} - \varpi \varepsilon E_{4,1}$$

$$t' = E_{1,4} - \varepsilon E_{2,3} + \varpi E_{3,2} - \varpi \varepsilon E_{4,1} \tag{4.74}$$

$$u(x) = u_b(-\varepsilon x) u_{2a+b}(x)$$

$$\underline{u}(x) = u_{-b}(x) u_{-2a-b}(-\varepsilon x).$$

Define e_g, $g \in G'$ as before.

PROPOSITION 4.18. *The Hecke algebra $\mathscr{H}(G'//J', 1)$ is generated by the elements e_g, $g \in \{r_0, r_1, t', t'^{-1}\} \cup B'$. The elements e_g satisfy the relations*

$$(a)\ e_k * e_{k'} = e_{kk'} \quad k, k' \in B'$$

$$(b)\ e_s * e_k = e_{\mathrm{Ad}(s)(k)} * e_s \quad s \in \{r_0, r_1, t'\} \text{ and } k \in sB's \cap B'$$

$$(c)\ (i)\ e_{t'} * e_{r_0} = e_{r_1} * e_{t'}$$

$$(ii)\ e_{t'} * e_{t'^{-1}} = e_I$$

$$(d)\ (i)\ e_{r_0} * e_{r_0} = q \sum_x e_{d(-\varepsilon, -\varepsilon, -\varepsilon)} * e_{\underline{u}(\varpi^l x)}$$

$$(ii)\ e_{r_0} * e_{u(x)} * e_{r_0} = q e_{u(-1/(\varepsilon x))} * e_{r_0} * e_{d(-\varepsilon x, -\varepsilon x, -1/x)} * e_{u(-1/(\varepsilon x))}, \quad x \in R^\times.$$

The above relations are a defining set of relations for the algebra.

The dimensional of σ is q^2 when i is odd and q when is even. In the odd case, we can reduce to a one dimensional representation ϱ as before. To do this, let $k = (i-1)/2$. Define

$$\begin{aligned} v(x, y) &= u_{a+b}(x)\, u_{2a+b}(y)\, u_b(\varepsilon y) \\ \underline{v}(x, y) &= u_{-a-b}(x)\, u_{-2a-b}(y)\, u_{-b}(\varepsilon y), \end{aligned} \tag{4.75}$$

and set

$$\begin{aligned} J^\# &= \{v(x, y) \mid x, y \in \varpi^k R\} \\ J_* &= J_+ J^\#. \end{aligned} \tag{4.76}$$

Extend the character Ω on J_+ across $J^\#$ trivially to get a character ϱ of J_*. Clearly, $\sigma = \mathrm{Ind}\, \varrho$, so $\mathscr{H}(G//J, \sigma) = \mathscr{H}(G//J_*, \varrho)$. The obvious map η (cf. (4.44)) is a $*$-isomorphism of algebras.

When $i = 2k$ is even, let Y be the intersection of J with the group (4.49). The group Y is normalized by $N' = G' \cap N$, and σ remains irreducible on restriction to Y. As in case (4.1b), the oscillator representation and Bruhat decomposition determine a mapping

$$\omega\colon G' \to \mathrm{End}\, V_\sigma$$

satisfying (4.71). We may assume

$$\omega(d(x, x, y)) = I, \quad x, y \in R^\times \quad \text{and} \quad \operatorname{trace}(\omega(r_0)) = 1.$$

(see the discussion of Case II in Chapter [HM1]). The map η: $\mathscr{H}' \to \mathscr{H}$, defined by (4.72) and (4.73) is a *-isomorphism of algebras. A novel feature in the proof of this isomorphism is showing

$$f_{r_0} * f_{u(x)} * f_{r_0} = q^3 f_{u(-1/(\varepsilon x))} * f_{r_0} * f_{d(-\varepsilon x, -\varepsilon x, -1/x)} * f_{u(-1/(\varepsilon x))}. \tag{4.77}$$

We use

$$f_{r_0} * f_u(x) = \sum_{x', z, z'} f_I * \delta_{r_0} * \delta_{u(x+\varpi' x')} \delta_{v(\varpi^k z, \varpi^k z')} I,$$

$$f_{r_0} = \sum_{x', z, z'} I \delta_{v(\varpi^k z, \varpi^k z')} * \delta_{u(\varpi' x')} * \delta_{r_0} * f_I.$$

The convolution $f_{r_0} * f_{u(x)} * f_{r_0}$ is equal to

$$q^3 \sum_{x', z, z'} f_I * \sigma(r_0^2) \delta_{r_0} * \delta_{u(x+\varpi' x')r_0} \delta_{r_0} - 1_{v(\varpi^k z, \varpi^k z')r_0} * f_I.$$

The relation

$$r_0 u(y) r_0 = u(-1/(\varepsilon y)) r_0 \, d(-\varepsilon y, -\varepsilon y, 1/y) \, u(-1/(\varepsilon y)), \; y \in R^\times,$$

allows us to bring $\delta_{r_0 v(\varpi^k z, \varpi^k z') r_0}$ to the left to be absorbed into f_I. Let

$$A = \begin{bmatrix} 0 & -1/(\varepsilon y) \\ 1/y & 0 \end{bmatrix} \quad B = \varpi^k \begin{bmatrix} z & z' \\ \varepsilon z' & z \end{bmatrix},$$

so that

$$u(-1/(\varepsilon y)) = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \quad r_0^{-1} v(\varpi^k z, \varpi^k z') r_0 = \begin{bmatrix} I & 0 \\ B & I \end{bmatrix}.$$

When $r_0^{-1}v(\varpi^k z, \varpi^k z')r_0$ is commuted by $u(-1/(\varepsilon y))$, the commutator

$$\gamma = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$$

$$= \begin{bmatrix} S((1 - BA)')^{-1}S & 0 \\ 0 & 1 - BA \end{bmatrix} \text{mod ker } (\sigma)$$

is introduced on the right side. When the measure δ_γ is absorbed into f_I on the right, $\sigma(\gamma)$ appears. We obtain on simplification

$$f_I * \sigma(\gamma)\delta_{r_0 u(x) r_0} * f_I.$$

Hence,

$$f_{r_0} * f_{u(x)} * f_{r_0} = q^3 \sum_{x', z, z'} f_I * \sigma(\gamma)\delta_{r_0 u(x) r_0} * f_I$$

$$= q^3 q f_I * \left\{\sum_{z, z'} \sigma(\gamma)\right\} \delta_{r_0 u(x) r_0} * f_I.$$

But

$$\left\{\sum_{z, z'} \sigma(\gamma)\right\} = \text{multiple of } \sigma(r_0) \tag{4.78}$$

since it intertwines $\sigma(x)$ and $\sigma(r_0 x r_0^{-1})$, $x \in J \cap r_0^{-1} J r_0$. Taking the trace of both sides, we find the multiple to be q. This proves (4.77).

REMARK. In both cases of (4.1a) and also case (4.1b), the center of G'/T is noncompact; and hence G'/T has no square integrable representations. We conclude G/T has no square integrable representations containing (J, σ') in these cases. In case (4.1c), G'/T has a compact center and therefore discrete series. Here, η yields a bijection between the discrete series of G'/T which have a contain a J' fixed vector and those of G/T which contain (J, σ').

5. Ramified representations

We complete our classification of the irreducible representations of G by describing those which contain a ramified representation (L, Ω_s). The results in this section are very analogous to those of the level one and unramified representations in Sections 3 and 4. Consequently, most proofs will just be sketched.

The ramified representations were defined in Section 1. We recall the situation in (1.25). For $i \in \mathbb{N}$, considered the sets

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ u\varpi & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}\right\} \tag{5.1a}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & a \\ d\varpi & e\varpi & 0 & 0 \\ f\varpi & d\varpi & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}\right\} a^2 - bc,\ d^2 - ef \in R^\times \tag{5.1b}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u\varpi & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}\right\} \tag{5.1c}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ u\varpi & 0 & 0 & 0 \\ 0 & v\varpi & 0 & 0 \\ 0 & 0 & -u\varpi & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}\right\} \tag{5.1d}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} \varpi c & 0 & 0 & 1 \\ 0 & \varpi c & \varpi u & 0 \\ 0 & \varpi u\varepsilon & \varpi c & 0 \\ \varpi^2 v^2 \varepsilon & 0 & 0 & \varpi c \end{bmatrix} + \begin{bmatrix} \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^3 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 \end{bmatrix}\right\} (i \geqslant 2). \tag{5.1e}$$

Here, $a, b, c, d, e, f \in R$, and $u, v, \varepsilon \in R^\times$, ε a nonsquare. Write a set $\mathbf{s}$ in (5.1) as

$$\mathbf{s} = \varpi^{-i-1}\alpha + \tilde{\mathfrak{l}} \tag{5.2}$$

(cf. (1.26)). Let $\mathfrak{l} = \tilde{\mathfrak{l}} \cap \mathfrak{g}$. A ramified representation $(L, \Omega_{\mathbf{s}})$ is obtained from $\mathbf{s}$ by setting

$\tilde{\mathfrak{l}}^*$ (resp. $\mathfrak{l}^*$) equal to the dual lattice of $\tilde{\mathfrak{l}}$ (resp. $\mathfrak{l}$) in $\tilde{\mathfrak{g}}$ (resp. $\mathfrak{g}$)

$$\tilde{L} = c(\tilde{\mathfrak{l}}^*), \quad L = c(\mathfrak{l}^*) \tag{5.3}$$

$$\Omega_{\mathbf{s}}(c(x)) = \Psi(\langle x, -\alpha\rangle/2) \quad \text{for } x \in \tilde{\mathfrak{l}}^*.$$

The group L is a parahoric filtration subgroup in cases (5.1a, b, d, e). In cases (5.1a) and (5.1d), L is a filtration subgroup of the Iwahori subgroup B. Indeed, we have $L = B_{4i-1}$ (resp. $L = B_{4i-3}$) for case (5.1a) (resp. (5.1d)). In case (5.1b), $L = M_{2i-1}$ (see (4.9)). For the case (5.1e), let

$$\tilde{\mathcal{N}} = \begin{bmatrix} R & R & R & \mathfrak{p}^{-1} \\ \mathfrak{p} & R & R & R \\ \mathfrak{p} & R & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \end{bmatrix} \qquad \mathcal{J}(\tilde{\mathcal{N}}) = \begin{bmatrix} \mathfrak{p} & R & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}. \tag{5.4}$$

It is trivial to check that $\tilde{\mathcal{N}}$ is an R-order in $M_4(F)$ with radical $\mathcal{J}(\tilde{\mathcal{N}})$. Define the parahoric subgroup N by (4.6) and the filtration subgroups N_k of N by (4.7). In this setup, $L = N_{2i-2} (i \geqslant 2)$. For case (5.1c), the group L is not a filtration group of a parahoric. We say more about this case later.

Cases (*5.1a, d, e*). We begin with some normalizations. The set (5.1d) is conjugate under the Iwahori subgroup B to a set (5.1d) with $u = 1$. Similarly, the set (5.1e) is conjugate via the parahoric subgroup N to a set (5.1e) with $u = 1$. Assume these normalizations. Furthermore, in case (5.1e) if $v^2 = u^2 \bmod \mathfrak{p}$, take $v^2 = u^2$. Define α by (5.2). Let, as in the unramified case, $\mathfrak{g}'$ (resp. G') denote the centralizer of α in $\mathfrak{g}$ (resp. G). Let $\mathfrak{g}'^{\perp}$ be the orthogonal complement to $\mathfrak{g}'$. For α of type (5.1a), matrices in $\mathfrak{g}'$, $\mathfrak{g}'^{\perp}$ have the form

$$\begin{bmatrix} \lambda & \beta & 0 & \gamma \\ u\varpi\gamma & \lambda & \beta & 0 \\ 0 & u\varpi\gamma & \lambda & -\beta \\ u\varpi\beta & 0 & -u\varpi\gamma & \lambda \end{bmatrix}$$

and

$$\begin{bmatrix} \beta & v - \varphi & \mu & 2\kappa \\ -\varpi(\theta + u\kappa) & \delta & -2v & \mu \\ \varpi\zeta & 2\varpi\theta & -\delta & -(v - \varphi) \\ 2\varpi u\varphi & \varpi\zeta & \varpi(\theta + u\kappa) & -\beta \end{bmatrix} \tag{5.5a}$$

respectively. Similarly, if **s** is type (5.1d), matrices in $\mathfrak{g}'$ and $\mathfrak{g}'^{\perp}$ have the form

$$\begin{bmatrix} \lambda & v\beta & 0 & \gamma \\ \varpi\gamma & \lambda & \beta & 0 \\ 0 & v\varpi\gamma & \lambda & -v\beta \\ v\varpi\beta & 0 & -\varpi\gamma & \lambda \end{bmatrix}$$

and

$$\begin{bmatrix} \beta & -(\varphi + vv) & \mu & 2\kappa \\ -(\kappa + \theta) & \delta & -2v & \mu \\ \varpi\zeta & 2v\varpi\theta & -\delta & (\varphi + vv) \\ 2\varpi\varphi & \varpi\zeta & (\kappa + \theta) & -\beta \end{bmatrix}. \tag{5.5d}$$

To describe the centralizer in case (5.1e), we need to distinguish whether v^2 and u^2 are equal. If $v^2 \neq u^2$, then

$$\mathfrak{g}' = \left\{ \begin{bmatrix} \lambda & 0 & 0 & \varpi^{-1}v \\ 0 & \lambda & \mu & 0 \\ 0 & \varepsilon\mu & \lambda & 0 \\ \varpi v^2 \varepsilon v & 0 & 0 & \lambda \end{bmatrix} \right\} \tag{5.5e.1}$$

$$\mathfrak{g}^{\perp} = \left\{ \begin{bmatrix} \beta & \theta & \tau & \varpi^{-1}\varphi \\ -\varpi\gamma & \delta & \omega & \tau \\ \varpi\mu & -\varepsilon\omega & -\delta & -\theta \\ -\varpi v^2\varepsilon\varphi & \varpi\mu & \varpi\gamma & -\beta \end{bmatrix} \right\}.$$

If $v^2 = u^2$, then

$$\mathfrak{g}' = \left\{\begin{bmatrix} \lambda & \zeta & \omega & \varpi^{-1}\nu \\ -\varpi u\zeta & \lambda & \mu & \omega \\ \varpi u\varepsilon\omega & \varepsilon\mu & \lambda & -\zeta \\ \varpi v^2\varepsilon\nu & \varpi u\varepsilon\omega & \varpi u\zeta & \lambda \end{bmatrix}\right\} \tag{5.5e.2}$$

$$\mathfrak{g}^{\perp} = \left\{\begin{bmatrix} \beta & \theta & \tau & \varpi^{-1}\varphi \\ \varpi u\theta & \delta & \omega & \tau \\ -\varpi u\varepsilon\tau & -\varepsilon\omega & -\delta & -\theta \\ -\varpi v^2\varepsilon\varphi & -\varpi u\varepsilon\tau & -\varpi u\theta & -\beta \end{bmatrix}\right\}.$$

In all cases G' is anisotropic. In cases (5.1a), (5.1d) and (5.5e.1), G' is an anisotropic torus. In case (5.5e.2), G' is isogenous to an anisotropic unitary group in two variables over the unramified quadratic extension of F.

Define $\mathfrak{g}_{\tilde{\mathscr{R}},u}$, $\mathfrak{g}'_{\tilde{\mathscr{R}},u}$ and $\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}$ by the obvious analogues of (4.15). The above descriptions of $\mathfrak{g}'$ and $\mathfrak{g}^{\perp}$ yield decompositions

$$\begin{aligned} \mathfrak{g}_{\tilde{\mathscr{R}},u} &= \mathfrak{g}'_{\tilde{\mathscr{R}},u} \oplus \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u} \\ \mathfrak{g}_{\tilde{\mathscr{R}},u}/\mathfrak{g}_{\tilde{\mathscr{R}},u+1} &= \mathfrak{g}'_{\tilde{\mathscr{R}},u}/\mathfrak{g}'_{\tilde{\mathscr{R}},u+1} \oplus \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+1}. \end{aligned} \tag{5.6}$$

Let

$$\begin{aligned} d &= 4i-3, \quad n = 4i-1, \quad \mathbf{s} \text{ of type (5.1a)} \\ d &= 4i-1, \quad n = 4i-3, \quad \mathbf{s} \text{ of type (5.1d)} \\ d &= -2i, \quad\quad n = 2i-2, \quad \mathbf{s} \text{ of type (5.1e).} \end{aligned}$$

In analogy with (4.20)

$$\mathrm{ad}(x)\colon \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+1} \to \mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+d}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{R}},u+d+1} \tag{5.7}$$

is independent of $x \in \mathbf{s}$, and an isomorphism. Let j (resp. j') be the greatest integer in $(n+1)/2$ (resp. $(n+2)/2$) and define lattices $\mathfrak{J}$ (resp. $\mathfrak{J}_+$) analogous to those in (4.16). Take J and J_+ to be the groups

$$J = c(\mathfrak{J}) \quad \text{and} \quad J_+ = c(\mathfrak{J}_+). \tag{5.8}$$

Again, in analogy with the unramified case, the character Ω_s extends to a character Ω of J_+ which is trivial on $c(\mathfrak{g}^{\perp}_{\mathscr{Y},l'})$. Furthermore, there is a unique representation σ of J whose restriction to J_+ is a multiple of Ω. If $J = J_+$, σ is just equal to Ω, otherwise σ is the Heisenberg representation. The restriction of the character Ω to $J' = G' \cap J$ extends to a character φ of G'. In all cases, φ determines an extension of σ to $G'J$. For $g \in G'$, let e_g (resp. f_g) be the element in $\mathscr{H}' = \mathscr{H}(G'//J', 1)$ (resp. $\mathscr{H} = \mathscr{H}(G//J, \sigma)$) with support $J'gJ'$ (resp. JgJ) and $e_g(g) = \varphi(g)$ (resp. $f_g(g) = \sigma(g)$). Define η: $\mathscr{H}' \to \mathscr{H}$ by

$$\eta(e_g) = f_g. \tag{5.9}$$

THEOREM 5.1. *Suppose* $\mathbf{s}$ *is of type* (5.1*a*, *d*, *e*). *Any irreducible representation of* G *which contains* (L, Ω_s) *must contain* (J, σ).

Proof. The proof follows that of Theorem 4.1. We prove analogues of Lemmas 4.3, 4.4, 4.7, 4.8, and Corollaries 4.5, 4.6. The theorem follows from the analogue of Corollary 4.5. We omit the details. □

THEOREM 5.2. *Under the hypothesis of the Theorem 5.1, the map*

$$\eta\colon \mathscr{H}' \to \mathscr{H}$$

*is a *-isomorphism of algebras.*

Proof. Here, we prove analogues of Propositions 4.9 and 4.10, i.e.,

$$\operatorname{supp} \mathscr{H}(G//J, \sigma) = JG'J$$

$$Jg'J \cap G' = J'g'J' \text{ for } g' \in G'.$$

As in case (4.1d, e), $JG'J = G'J$ is in fact a group. The proof of Theorem 4.2 in cases (4.1d, e) applies verbatim. □

The remarks after the proof of Theorem 4.2 in the cases (4.1d, e) apply here. A representation of G which contains σ' must be supercuspidal. These representations π are obtained as

$$\pi = \operatorname{Ind}_{G'J\uparrow G}(\kappa \otimes \tau), \tag{5.10}$$

where τ is an extension of σ to $G'J$, and κ is an irreducible representation of $G'J/J \cong G'/J'$.

Case (5.1b). We begin with some preliminary normalizations. Write α as

$$\alpha = \begin{bmatrix} & X \\ \varpi Z & \end{bmatrix}$$

where

$$X = \begin{bmatrix} a & b \\ c & a \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} d & e \\ f & d \end{bmatrix}.$$

For $A \in GL_2(F)$, let A' denote the transpose of A. Let S be the 2×2 matrix in (4.48), and set

$$g(A, \lambda) = \begin{bmatrix} A & \\ & \lambda S(A')^{-1} S \end{bmatrix}, \quad A \in GL_2(R), \quad \lambda \in F^{\times}. \tag{5.11}$$

The element $g = g(A, \lambda)$ belongs to the parahoric subgroup M (cf. (4.3)). In particular, the lattice $\mathfrak{l}$ in (5.1b) is invariant under $\mathrm{Ad}(g)$. Clearly,

$$\mathrm{Ad}(g)\alpha = \begin{bmatrix} & X' \\ \varpi Z' & \end{bmatrix}$$

where $X' = \lambda^{-1} AXSA'S$ and $Z' = \lambda S(A')^{-1} SZA^{-1}$. Observe that

$$X'Z' = \{\lambda^{-1} AXSA'S\}\{\lambda S(A')^{-1} SZA^{-1}\} = AXZA^{-1},$$

i.e., $X'Z'$ and XZ are conjugate. Hence, it is natural to classify the set $\mathfrak{s}$ according to whether XZ mod $\mathfrak{p}$

(i) is a scalar

(ii) is a nonscalar unipotent element

(iii) has distinct eigenvalues in $\mathbb{F}_q$ (5.12)

(iv) has eigenvalues not in $\mathbb{F}_q$.

We now make our normalizations. In all cases, after conjugation by some element in M, we can assume X is an antidiagonal matrix, i.e., a matrix with zero entries everywhere except the antidiagonal. In certain cases we shall be able to assume X is diagonal. Consider case (5.12i). If X is an antidiagonal element, then Z' must also be an antidiagonal matrix mod $\mathfrak{p}$. However, if the determinant of X is minus a square then $\mathbf{s}$ is M-conjugate to a set $\mathbf{s}'$ with X' the identity and Z' a scalar. Thus we can assume α has the form (5.13i.1) or (5.13i.2). In case (5.12ii), the nonscalar unipotent condition forces X to have determinant minus a square. Again, $\mathbf{s}$ is M-conjugate to a set $\mathbf{s}'$ with X' the identity. Since $X'Z'$ is unipotent, Z' is either an upper or lower triangular nonscalar unipotent element mod $\mathfrak{p}$. We can of course assume Z' is upper triangular. This gives α of the form (5.13ii). In case (5.12iii), take a symplectic basis with respect to the eigenvectors of XZ. By modifying the basis vector by scalars, we can choose the basis so the change of basis matrix is in M, and $\mathbf{s}$ in the new basis has X and Z antidiagonal matrices. To summarize: in cases (5.12i, ii, iii), we can take α to be

$$(\text{i.1})\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\varepsilon & 0 \\ 0 & \varpi a & 0 & 0 \\ -\varpi a\varepsilon & 0 & 0 & 0 \end{bmatrix},\ a\in R^\times \text{ or } (\text{i.2})\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi a & 0 & 0 & 0 \\ 0 & \varpi a & 0 & 0 \end{bmatrix},\ a\in R^\times$$

$$(\text{ii})\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \varpi a & \varpi x & 0 & 0 \\ 0 & \varpi a & 0 & 0 \end{bmatrix},\quad a\in R^\times,\ x\in R \tag{5.13}$$

$$(\text{iii})\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & a & 0 \\ 0 & \varpi b & 0 & 0 \\ \varpi c & 0 & 0 & 0 \end{bmatrix},\quad a, b, c\in R^\times \quad \text{and} \quad ab \neq \bmod \mathfrak{p}.$$

The centralizer $\mathfrak{g}'$ (resp. G') of α in $\mathfrak{g}$ (resp. G) in cases (5.13i.1), (5.13iii) and (5.12iv) is an anisotropic group. In cases (5.13iii) and (5.12iv), G' is an anisotropic torus. In case (5.13i.1), G' is isogenous to an anisotropic unitary group in two variables over a ramified quadratic extension of F.

In cases (5.13i.2) and (5.13ii), it is advantageous to replace $L = M_{2l-1}$ by B_{4l-2}. Observe that

$$M_{2l-1} \supseteq B_{4l-2} \supseteq B_{4l-1} \supseteq M_{2l}. \tag{5.14}$$

The restriction of Ω_s to B_{4l-2} is trivial on B_{4l-1} and is represented by the set

$$\varpi^{-l-1}\left\{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a\varpi & 0 & 0 & 0 \\ 0 & a\varpi & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \end{bmatrix}\right\}. \tag{5.15}$$

Let α be the obvious element in (5.15). The centralizer G' in this case is a quasi-split group in two variables over a ramified quadratic extension of F.

Matrices in $\mathfrak{g}'$ and $\mathfrak{g}'^{\perp}$ have the form

$$\begin{bmatrix} \lambda & \beta & \xi & \gamma \\ \varepsilon\beta & \lambda & -\varepsilon\zeta & \xi \\ \varpi a\xi & \varpi a\zeta & \lambda & -\beta \\ -\varpi a\varepsilon\gamma & \varpi a\xi & -\varepsilon\beta & \lambda \end{bmatrix}, \begin{bmatrix} \beta & \varphi & \mu & \kappa \\ -\varepsilon\varphi & \delta & \varepsilon\omega & \mu \\ -\varpi a\mu & \varpi a\omega & -\delta & -\varphi \\ \varpi a\varepsilon\kappa & -\varpi a\mu & \varepsilon\varphi & -\beta \end{bmatrix} \tag{5.16}$$

respectively in case (5.12i.1) and

$$\begin{bmatrix} \lambda & 0 & 0 & \gamma \\ 0 & \lambda & a\zeta & 0 \\ 0 & \varpi b\zeta & \lambda & 0 \\ \varpi c\gamma & 0 & 0 & \lambda \end{bmatrix}, \begin{bmatrix} \beta & \varphi & \mu & \kappa \\ \theta & \delta & a\omega & \mu \\ \varpi\varrho & -\varpi b\omega & -\delta & -\varphi \\ -\varpi c\kappa & \varpi\varrho & -\theta & -\beta \end{bmatrix} \tag{5.17}$$

respectively in case (5.13iii). Both $\mathfrak{g}'$ and $\mathfrak{g}'^{\perp}$ can be trivially calculated for case (5.12iv), but due to the messy notation required to write these matrices, we omit their explicit expressions. It follows by inspection of (5.15), (5.16) and case (5.12iv) that

$$\begin{aligned} \mathfrak{g}_{\tilde{x},u} &= \mathfrak{g}'_{\tilde{x},u} \oplus \mathfrak{g}^{\perp}_{\tilde{x},u} \\ \mathfrak{g}_{\tilde{x},u}/\mathfrak{g}_{\tilde{x},u+1} &= \mathfrak{g}'_{\tilde{x},u}/\mathfrak{g}'_{\tilde{x},u+1} \oplus \mathfrak{g}^{\perp}_{\tilde{x},u}/\mathfrak{g}^{\perp}_{\tilde{x},u+1}. \end{aligned} \tag{5.18}$$

Set $d = 2i - 1$ and $n = 2i - 1$. The adjoint map

$$\mathrm{ad}(\alpha)\colon \mathfrak{g}^{\perp}_{\tilde{\mathcal{M}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathcal{M}},u+1} \to \mathfrak{g}^{\perp}_{\tilde{\mathcal{M}},u+d}/\mathfrak{g}^{\perp}_{\tilde{\mathcal{M}},u+d+1}$$

is once again an isomorphism. Define J, J', σ and η as in (5.8) and (5.9). In complete analogy with Theorems 5.1 and 5.2, we have

THEOREM 5.3. *Suppose* $\mathbf{s}$ *is of type* (*5.13i.1*), (*5.13iii*) *or* (*5.12iv*). *Then*
(*a*) *Any irreducible representation of* G *which contains* $(L, \Omega_{\mathbf{s}})$ *also contains* (J, σ).
(*b*) *The map* $\eta\colon \mathscr{H}(G'//J', 1) \to \mathscr{H}(G//J, \sigma)$ *is a* *-*isomorphism of algebras.*

The representations of G which contain σ' are supercuspidal. They are obtained as induced representations (cf. (5.10)).

The last two cases we need to consider are (5.13i.2) and (5.13ii). As noted in (5.14) we replace the group $L = M_{2i-1}$ by B_{4i-2}, and the set $\mathbf{s}$ by (5.15). We can also relabel the element $a\varpi$ to ϖ. The matrices in $\mathfrak{g}'$ and $\mathfrak{g}^{\perp}$ have the form

$$\begin{bmatrix} \lambda & 0 & \beta & \gamma \\ 0 & -\lambda & \mu & \beta \\ \varpi\beta & \varpi\gamma & \lambda & 0 \\ \varpi\mu & \varpi\beta & 0 & -\lambda \end{bmatrix} \text{ and } \begin{bmatrix} \xi & \nu & \zeta & \kappa \\ \varpi\theta & \xi & \varphi & \zeta \\ -\varpi\zeta & \varpi\kappa & -\xi & -\nu \\ \varpi\varphi & -\varpi\zeta & -\varpi\theta & -\xi \end{bmatrix}. \qquad (5.19)$$

Set $d = 4i - 2$, $n = 4i - 2$. It is clear from (5.19) that

$$\mathrm{ad}(x)\colon \mathfrak{g}^{\perp}_{\tilde{\mathscr{B}},u}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{B}},u+1} \to \mathfrak{g}^{\perp}_{\tilde{\mathscr{B}},u+d}/\mathfrak{g}^{\perp}_{\tilde{\mathscr{B}},u+d+1}$$

is an isomorphism. Define J and σ by (5.8). The representation σ is q dimensional. In analogy with the singular cases of (4.1a) we find it convenient to reduce to *a* one dimensional representation. In the root group notation of Section 1, set

$$J^{\#} = \begin{cases} U_{a,i-1} & i \text{ odd} \\ U_{-a,i} & i \text{ even} \end{cases}$$

$$J_{*} = J_{+}J^{\#}.$$

Observe that $J' = G' \cap J_*$. The character Ω on J_+ extends trivially to $J^\#$ to give a character ϱ on J_* such that ϱ induces to σ. As was pointed out several times before, this means $\mathscr{H}(G//J, \sigma) = \mathscr{H}(G//J_*, \varrho)$. Define the map η: $\mathscr{H}(G'//J', 1) \to \mathscr{H}(G//J_*, \varrho)$ by (4.53). The arguments in Propositions 4.13 and 4.13 easily adapt to show

THEOREM 5.4. *The map η is a *-isomorphism of algebras.*

Since the group G'/T has a compact center, it has square integrable representations. Those with a J' fixed vector transfer to a discrete series representation τ of G/T containing σ'.

Case (5.1c). Relabel $u\varpi$ to ϖ. Matrices in $\mathfrak{g}'$ and $\mathfrak{g}^\perp$ have the form

$$\begin{bmatrix} \lambda & 0 & 0 & \gamma \\ 0 & \lambda+\beta & \mu & 0 \\ 0 & \delta & \lambda-\beta & 0 \\ \varpi\gamma & 0 & 0 & \lambda \end{bmatrix} \text{ and } \begin{bmatrix} \xi & \nu & \mu & \kappa \\ \varpi\theta & 0 & 0 & \mu \\ \varpi\xi & 0 & 0 & -\nu \\ -\varpi\kappa & \varpi\zeta & -\varpi\theta & -\xi \end{bmatrix}. \tag{5.20}$$

As mentioned before, the group L is not a filtration subgroup of a parahoric subgroup as defined in Section 1. We can however, define a convenient filtration of the parahoric subgroup Q (cf. (4.3)), so that we can prove a Hecke algebra result in the vein of those already established. Define this filtration by

$$\tilde{\mathscr{G}}_0 = \tilde{\mathscr{Q}}, \quad \tilde{\mathscr{G}}_1 = \mathscr{J}(\tilde{\mathscr{G}})$$

$$\tilde{\mathscr{G}}_2 = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \quad \tilde{\mathscr{G}}_3 = \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \tag{5.21}$$

$$\tilde{\mathscr{G}}_{4u+v} = \varpi^u \tilde{\mathscr{G}}_v, \quad v = 0, 1, 2, 3.$$

Set $\mathscr{G}_u = \mathfrak{g} \cap \tilde{\mathscr{G}}_u$. Define $\tilde{G}_u = 1 + \tilde{\mathscr{G}}_u, u \geqslant 1$ and $G_u = G \cap \tilde{G}_u = c(\mathscr{G}_u)$. The G_u are normal in Q. We have

$$L = G_{4l-2}, \quad \text{and} \quad \mathbf{s} = \varpi^{-l-1}\{\alpha + \mathscr{G}_3\}.$$

Of equal importance, the constructions and results in Lemma 4.3 through Proposition 4.10 hold with $\mathfrak{g}_{\tilde{\mathfrak{R}},u}$ replaced by $\tilde{\mathcal{G}}_u$. Define lattices and groups

$$\mathfrak{D} = \tilde{\mathcal{G}}_{4i-2} + \tilde{\mathcal{G}}_{2i-1}^{\perp}, \quad \mathfrak{D}_+ = \tilde{\mathcal{G}}_{4i-2} + \tilde{\mathcal{G}}_{2i}^{\perp}$$

$$J = c(\mathfrak{D}), \text{ and } J_+ = c(\mathfrak{D}_+)$$

$$J' = G' \cap J.$$

Take σ to be the Heisenberg representation of J determined by Ω_s. The dimension of σ is q. We have

THEOREM 5.5. *An irreducible representation π of G which contains (L, Ω_s) contains (J, σ).*

We show $\mathcal{H}' = \mathcal{H}(G'//J', 1)$ is isomorphic to $\mathcal{H} = \mathcal{H}(G//J, \sigma)$. To do this, we first determine a presentation of $\mathcal{H}'$.

The intersection $K' = G' \cap Q$ is a maximal compact subgroup of G'. Obviously, K' normalizes J' and J. Take $B' \subset K'$ to be the Iwahori subgroup of those elements in K' which are upper triangular mod $K_1' = G' \cap Q_1$. Let

$$r_0 = E_{1,1} + E_{2,3} - E_{3,2} + E_{4,4}$$

$$r_1 = E_{1,4} + E_{2,3} + \varpi E_{3,2} + \varpi E_{4,1}$$

PROPOSITION 5.6. *The Hecke algebra $\mathcal{H}(G'//J', 1)$ is generated by the elements e_g, $g \in \{r_0, r_1, \varpi I, \varpi^{-1} I\} \cup B'$. These elements satisfy the relations*

(a) $e_k * e_{k'} = e_{kk'}, \quad k, k' \in B'$

(b) $e_s * e_k = e_{\mathrm{Ad}(s)k} * e_s, \quad s \in \{r_0, r_1\}, \quad k \in s^{-1}B's \cap B'$

(c) $e_{\varpi I} * e_g = e_g * e_{\varpi I}, \quad g \in \{r_0, r_1\} \cup B'$

$$e_{\varpi I} * e_{\varpi^{-1} I} = e_I$$

$$(d)\ (i)\quad e_{r_0} * e_{r_0} = e_{d(1,-1,-1)}$$

$$(ii)\quad e_{r_1} * e_{r_1} = q\left\{\sum_x e_{u_b(\varpi^{-1}x)}\right\} * e_{\varpi I}$$

(e) *For* $x \in R^\times$,

$$e_{r_0} * e_{u_b(x)} * e_{r_0} = e_{u_b(-x^{-1})} * e_{r_0} * e_{d(1,-x,-x^{-1})} * e_{u_b(-x^{-1})} * e_{u_b(-x^{-1})}$$

The above relations are a defining set for the algebra.

Proof. We refer again to the proof of Theorem 2.1 in Chapter 3 [HM1].

Consider now the algebra $\mathscr{H}$. Let k be the greatest integer in $i/2$, and set

$$J^\# = \begin{cases} U_{a+b,k} & i \text{ odd} \\ U_{-a,k} & i \text{ even} \end{cases}$$

$$J_* = JJ^\# \tag{5.22}$$

The character Ω on J_+ extends trivially on $J^\#$ to a character ϱ on J_*. We have $\sigma = \operatorname{Ind}\varrho$; whence, $\mathscr{H} = \mathscr{H}_\varrho = \mathscr{H}(G//J_*, \varrho)$. We describe the Hecke algebra $\mathscr{H}_\varrho$. The support of $\mathscr{H}_\varrho$ is the set $J_*G'J_*$. Given $g \in G'$, let as before, $f_g \in \mathscr{H}_\varrho$ denote the element with support J_*gJ_* and $f_g(g) = 1$. The Gauss sum

$$\Gamma = \left\{\sum_y \Psi(\varpi^{-1}y^2)\right\} \tag{5.23}$$

will appear in the multiplicative structure of $\mathscr{H}_\varrho$. Recall that Γ satisfies the property

$$\Gamma^2 = \operatorname{sgn}(-1)q, \tag{5.24}$$

where sgn is the quadratic character of $R^\times$.

PROPOSITION 5.7. *The Hecke algebra $\mathscr{H}_\varrho$ is generated by the elements f_g, $g \in \{r_0, r_1, \varpi I, \varpi^{-1}I\} \cup B'$. The elements satisfy the relations*

$$(a)\quad f_k * f_{k'} = f_{kk'}, \quad k, k' \in B'$$

$$(b)\quad f_s * f_k = f_{\mathrm{Ad}(s)k} * f_s, \quad s \in \{r_0, r_1\}, k \in s^{-1}B's \cap B'$$

(c) $f_{\varpi I} * f_g = f_g * f_{\varpi I}, \quad g \in \{r_0, r_1\} \cup B'$

$$f_{\varpi I} * f_{\varpi^{-1} I} = f_I$$

(d) (i) $f_{r_0} * f_{r_0} = q f_{d(1,-1,-1)}$

(ii) $f_{r_1} * f_{r_1} = q \left\{ \sum_x f_{u_b(\varpi^{i-1} x)} \right\} * e_{\varpi I}$

(e) *For* $x \in R^\times$,

$$f_{r_0} * f_{u_b(x)} * f_{r_0} = \operatorname{sgn}(x) \Gamma f_{u_b(-x^{-1})} * f_{r_0} * f_{d(1,-x,-x^{-1})} * f_{u_b(-x^{-1})}$$

Proof. Relations (a), (b), and (c) are obvious. We give the details of the argument for relations (d) and (e) when i is odd, and omit the easy modification when i is even. To abbreviate notation, let $u(x) = u_b(x)$, $\underline{u}(x) = u_{-b}(x)$, $v(x) = u_{a+b}(x)$ and $\underline{v}(x) = u_{-a-b}(x)$. Consider relation (di). Write

$$\begin{aligned} f_{r_0} &= \sum_x f_I * \delta_{r_0} * \delta_{v(\varpi^k x)} \\ &= \sum_x \delta_{v(\varpi^k x)} * \delta_{r_0} * f_I. \end{aligned} \tag{5.25}$$

Thus,

$$f_{r_0} * f_{r_0} = q \sum_y f_I * \delta_{r_0} * \delta_{v(\varpi^k y)} * \delta_{r_0} * f_I. \tag{5.26}$$

The summand has support inside the set $\mathscr{S} = Jr_0 v(\varpi^k y) r_0 J$. The intersection of G' and $\mathscr{S}$ is empty except when $y = 0 \bmod \wp$. Therefore, the sum in (5.26) collapses to one term. Relation (d(i)) follows. Relation (d(ii)) is only slightly more complicated. We have

$$\begin{aligned} f_{r_1} &= \sum_z f_I * \delta_{r_1} * \delta_{\underline{u}(\varpi^i z)} \\ &= \sum_z \delta_{\underline{u}(\varpi^i z)} * \delta_{r_1} * f_I. \end{aligned}$$

So,

$$\begin{aligned} f_{r_1} * f_{r_1} &= q \sum_z f_I * \delta_{r_1} * \delta_{\underline{u}(\varpi' z)} * \delta_{r_1} * f_I \\ &= q \sum_z f_I * \delta_{r_1} * \delta_{\underline{u}(\varpi' z)} * \delta_{r_1} * f_I \\ &= q \sum_z f_{u(\varpi'^{-1} z)}. \end{aligned}$$

To prove relation (e) we use

$$f_{r_0} * f_{u(x)} = \sum_y f_I * \delta_{r_0} * \delta_{u(x)v(\varpi^k y)}.$$

Let $u'(x) = u_a(x)$. The convolution $f_{r_0} * f_{u(x)} * f_{r_0}$ is equal to

$$q \sum_y f_I * \delta_{r_0} * \delta_{u(x)v(\varpi^k y)} * \delta_{r_0} * f_I = q \sum_y f_I * \delta_{r_0 u(x) r_0} * \delta_{u'(\varpi^k y)} * f_I. \tag{5.27}$$

The identity $r_0 u(x) r_0 = u(-x^{-1}) r_0 d(1, -x, -x^{-1}) u(-x^{-1})$ can be used to bring $\delta_{u'(\varpi^k y)}$ to the left where, after conjugation by r_0, it can be absorded into f_I. When $u'(\varpi^k y)$ is commuted by $u(-x^{-1})$ the commutator

$$\gamma = u(x^{-1}) u'(-\varpi^k y) u(-x^{-1}) u'(\varpi^k y) = u_{a+b}(-\varpi^k y/x) u_{2a+b}(\varpi^{2k} y^2/x)$$

is introduced. When δ_γ is absorbed into f_I, the factor of $\varrho(\gamma) = \Psi(\varpi^{-1} y^2/x)$ appears. Thus, (5.27) becomes

$$\begin{aligned} f_{r_0} * f_{u(x)} * f_{r_0} &= \left\{ \sum_y \Psi(\varpi^{-1} y^2/x) \right\} \cdot f_{u(-x^{-1})} * f_{r_0} * f_{d(1,-x,-x^{-1})} * f_{u(-x^{-1})} \\ &= \operatorname{sgn}(x)\, \Gamma f_{u(-x^{-1})} * f_{r_0} * f_{d(1,-x,-x^{-1})} * f_{u(-x^{-1})}. \end{aligned}$$

Finally, $\mathrm{vol}(Jr_0J) = 1$, $\mathrm{vol}(Jr_1J) = q$ and $\mathrm{vol}(J(r_0r_1)^v J) = q^v$; hence, $f_w * f_{w'} = f_{ww'}$ when $w, w' \in N' = G' \cap N$ and the lengths of w and w' in G' add. By the Bruhat decomposition, the indicated elements generate $\mathscr{H}_\varrho$. This completes the proof of Theorem 5.7. □

Let μ be the character of B' whose value on $k \in B'$ is equal to $\operatorname{sgn}(k_{2,2})$, where $k_{2,2}$ is the (2,2) component of k.

COROLLARY 5.8. *The map* $\eta: \mathscr{H}(G'//J', 1) \to \mathscr{H}(G//J_*, \varrho)$ *defined on the generators of* $\mathscr{H}(G'//J', 1)$ *by*

$$\eta(e_k) = \mu(k) f_k, \quad \eta(e_{\varpi^{\pm 1} I}) = f_{\varpi^{\pm 1} I},$$

$$\eta(e_{r_0}) = f_{r_0}, \quad \eta(e_{r_1}) = f_{r_1}/\Gamma$$

*is a *-isomorphism of algebras.*

The group G'/T has a compact center. It possesses square integrable representations. Those with a J' invariant vector will transfer to square integrable representations of G/T containing ϱ'.

This completes our classification of the representations of G. In particular, it follows that any supercuspidal representation of G/T is induced from an open compact subgroup. An enumeration of the supercuspidal representations as well as the nonsupercuspidal discrete series can be obtained by unraveling the various Hecke algebra isomorphisms. We hope to return to this in the future.

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