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Representations of $GSp(4)$ over a p-adic field: part 1

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Introduction

In this paper, we classify the irreducible admissible representations of the group $G = GSp(4)$ over a p-adic field F of odd residual characteristic by adapting the techniques used for the group $U(2, 1)$ in [M1]. The classification is thus based on two concepts: nondegenerate representations and Hecke algebra isomorphisms.

We define a nondegenerate representation as a representation σ of a compact open subgroup $L \subset G$ satisfying a certain cuspidality or semisimplicity condition (see Section 1). The importance of these representations for p-adic groups is that they play a role analogous to the role of minimal or lowest K-types in real Lie groups [V]. In particular, we prove every admissible representation of G contains a nondegenerate representation. Furthermore, we show how they provide a means for partitioning the set of equivalence classes of irreducible admissible representations. Nondegenerate representations thus provide an anchor for investigating the representations of G. This is analogous to the role played by minimal K-types in real groups.

In real groups, the representations of a reductive group G which contain a given minimal K-type σ can be classified by relating them to the representations of some smaller reductive group G' which contain a minimal K-type σ' derived from σ. In the p-adic case, a similar relation between representations of G containing a nondegenerate representation and representations of some smaller group G', is effected by Hecke algebra isomorphisms. As in [HM1, HM2] and [M1], the transfer of representations between G and G', arising from the Hecke algebra isomorphisms established here, preserve certain important properties such as temperedness, square integrability and supercuspidality. This is very much in the spirit of cohomological induction results in the case of real groups [V].

As already mentioned, we define the nondegenerate representations in Section 1. A general theory is beginning to emerge for defining these representations for reductive groups (see [M2], [HM3]), but it is still in a state of flux. As such, we define nondegenerate representations via an exhaustive list. We show, in Section 2, every representation of G contains a nondegenerate representation after tensoring by a one dimensional character of G. Section 3 is devoted to a description of the representations of G which contain nondegenerate representations of level one. These representations consist of a cuspidal representation of a parahoric subgroup of G. Part two of this paper will give a similar description of the representations of G which contain nondegenerate representations of higher level.

1. Nondegenerate representations

Let F be a p-adic field of odd residual characteristic and let R denote the ring of integers in F, $\mathfrak{p}$ the prime ideal in R, ϖ a prime element in $\mathfrak{p}$ and $\mathbb{F}_q = R/\mathfrak{p}$ the residue field with q elements. Let $\mathbf{G}$ be the algebraic group $\mathbf{GSp(4)}$. We begin our study of the representations of $G = \mathbf{G}(F)$ by recalling a convenient realization of G. Let F^4 denote the four dimensional space of column vectors. We define a symplectic form $\langle\, ,\rangle$ on F^4. Given $\mathbf{y} \in F^4$ (resp. $g \in M_4(F)$), let $\mathbf{y}^t$ (resp. g^t) be the transpose matrix. For $\mathbf{x}^t = (x_1, x_2, x_3, x_4)$ and $\mathbf{y}^t = (y_1, y_2, y_3, y_4)$ in F^4, set

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_4 + x_2 y_3 - x_3 y_2 - x_4 y_1 .$$

Then

$$G = \{g \in GL(F^4) | \langle g\mathbf{v}, g\mathbf{w} \rangle = \mu(g)\langle \mathbf{v}, \mathbf{w} \rangle, \mathbf{v}, \mathbf{w} \in F^4\}.$$

Here, $\mu(g)$ is a scalar. Let e_1, e_2, e_3 and e_4 denote the standard basis vectors of F^4, and let $E_{i,j}$ be the 4×4 matrix whose (r, s) entry is $\delta_{r,i}\delta_{s,j}$. Set

$$H = E_{1,4} + E_{2,3} - E_{3,2} - E_{4,1} .$$

Then $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^t H \mathbf{y}$ and

$$G = \{g \in GL_4(F) | g^t H g = \mu(g) H\}.$$

In this setup, the F-rational points of the Lie algebra of $\mathbf{G}$ is the set

$$\mathfrak{g}(F) = \{g \in M_4(F) | g^t H + Hg = \Lambda(g) H\}.$$

The Lie algebra $\mathfrak{g}(F)$ is the direct sum of

$$\mathfrak{sp}(4)(F) = \{g \in M_4(F) | gH + Hg^t = 0\}$$

$$= \left\{ \begin{bmatrix} a & M & X & Z \\ m & d & N & X \\ x & n & -d & -M \\ z & x & -m & -a \end{bmatrix} \right\}$$

and the set of scalar matrices.

We now briefly recall some of the structure theory of G with respect to its parahoric subgroups. For a general discussion of such topics, we refer to [BT]. Let **A** be the maximal torus in **G** consisting of diagonal elements. For $a, b, c \in F^\times$ let

$$d(a, b, c) = aE_{1,1} + bE_{2,2} + cE_{3,3} + (bc/a) E_{4,4}. \tag{1.1}$$

Then, $A = \mathbf{A}(F)$ is the set of matrices $\{d(a, b, c)\}$. Let $\mathfrak{a}(F)$ be the F-rational points of the Lie algebra of **A**. A basis for the Cartan subalgebra $\mathfrak{a}(F)$ is given by the three vectors

$$H_1 = E_{1,1} - E_{4,4}$$

$$H_2 = E_{2,2} - E_{3,3}$$

$$H_3 = E_{1,1} + E_{2,2} + E_{3,3} + E_{4,4} = \text{identity matrix.}$$

Define two linear functionals a and b on $\mathfrak{a}(F)$ by

$$a(rH_1 + sH_2 + tH_3) = r - s$$

$$b(rH_1 + sH_2 + tH_3) = 2s.$$

The set of eight linear functionals

$$\Phi = \{\pm a, \pm b, \pm(a + b), \pm(2a + b)\}$$

form a root system of type C_2. The Borel subgroup **B** of upper triangular matrices in **G** determines an ordering of the roots in which a and b are the simple roots. Set

$$
\begin{aligned}
E_M &= E_a &&= E_{1,2} - E_{3,4} \\
E_N &= E_b &&= E_{2,3} \\
E_X &= E_{a+b} &&= E_{1,3} + E_{2,4} \\
E_Z &= E_{2a+b} &&= E_{1,4} \\
E_m &= E_{-a} &&= E_{2,1} - E_{4,3} \\
E_n &= E_{-b} &&= E_{3,2} \\
E_x &= E_{-a-b} &&= E_{3,1} + E_{4,2} \\
E_z &= E_{-2a-b} &&= E_{4,1}.
\end{aligned}
\tag{1.2}
$$

These vectors are root vectors, i.e.

$$[H, E_c] = c(H)E_c \text{ for } c \text{ a root and } H \in \mathfrak{a}(F).$$

The one-dimensional root space $\mathscr{u}_c(F) = FE_c$ of $\mathfrak{g}(F)$ can be exponentiated into root groups U_c inside G. Denote by $\mathbf{U}_c$ the corresponding algebraic subgroup of $\mathbf{G}$ so that $U_c = \mathbf{U}_c(F)$. The exponential map

$$u_c\colon \mathscr{u}_c(F) \to U_c \tag{1.3}$$

is given by $u_c(xE_c) = 1 + xE_c$. Abbreviate $u_c(xE_c)$ to $u_c(x)$. The F-rational points of $\mathbf{B}$ is equal to the subgroup of G generated by A and U_c $(c > 0)$. We write this as

$$\mathbf{B}(F) = A \prod_{c>0} U_c.$$

For i an integer, let

$$
\begin{aligned}
U_{c,i} &= u_c(\mathscr{p}^i), \quad c \in \Phi \\
U_{0,i} &= \{d(a, b, c) \mid a, b, c \equiv 1 \bmod \mathscr{p}^i\}, \quad i \geqslant 0.
\end{aligned}
\tag{1.4}
$$

The subgroup of G generated by $\mathbf{A}(R)$, $U_{c,0}$ $(c > 0)$ and $U_{c,1}$ $(c < 0)$, which we write as

$$B = \mathbf{A}(R) \prod_{c>0} U_{c,0} \prod_{c<0} U_{c,1},$$

is an *Iwahori* subgroup of G. It consists of those elements of $K = \mathbf{G}(R)$ which are upper triangular mod $\mathfrak{p}$.

Let

$$\mathbf{N} = \text{normalizer of } \mathbf{A} \text{ in } \mathbf{G}. \tag{1.5}$$

The group $N = \mathbf{N}(F)$ is the normalizer of A in G. For c a root, set

$$w_c(t) = u_c(t)u_{-c}(-t^{-1})u_c(t), \quad t \in F^\times.$$

The $w_c(t)$'s and A generate N. The Weyl group is of course defined to be $W = N/A$. It is generated by the images in W of the two elements

$$s_0 = E_{1,1} + E_{2,3} - E_{3,2} + E_{4,4} = w_b(1) \quad \text{and}$$

$$s_1 = E_{1,2} + E_{2,1} + E_{3,4} + E_{4,3} = w_a(1)d(1, -1, 1). \tag{1.6}$$

If I is a subset of $\{s_0, s_1\}$, the *standard parabolic subgroup* $P(I)$ is the subgroup of G generated by $\mathbf{B}(F)$ and I. Denote by $\mathbf{P}_I$, the parabolic subgroup of $\mathbf{G}$ such that $\mathbf{P}_I(F) = P(I)$. The algebraic group $\mathbf{P}_I$ has a Levi decomposition

$$\mathbf{P}_I = \mathbf{M}_I\mathbf{U}_I. \tag{1.7}$$

The group $\mathbf{M}_I$ may be chosen so that $\mathbf{M}_I(F)$ is the subgroup of G generated by A, I and

$$\{U_c | c \text{ a linear combination of elements in } I\}.$$

Here, we have of course identified s_0 with b and s_1 with a. The group $\mathbf{M}_I(F)$ is invariant under the Cartan involution of inverse transpose.

The *affine Weyl group* is defined to be the group

$$W^{\text{aff}} = N/\mathbf{A}(R). \tag{1.8}$$

It is generated by the images in W^{aff} of s_0, s_1 and the two elements

$$s_2 = -\varpi^{-1}E_{1,4} + E_{2,2} + E_{3,3} + \varpi E_{4,1} = w_{-2a-b}(\varpi) \tag{1.9}$$

and

$$t = E_{1,3} + E_{2,4} + \varpi E_{3,1} + \varpi E_{4,2}.$$

Let

$$S = \{s_0, s_1, s_2\}. \tag{1.10}$$

The element t normalizes the Iwahori subgroup B. We also have $ts_1 = s_1 t$ and $s_0 t = ts_2$. The Dynkin diagrams for W and W^{aff} are

$$o_{s_0} = o_{s_1} \text{ for } W$$

$$o_{s_0} = o_{s_1} = o_{s_2} \text{ for } W^{\mathrm{aff}}$$

If I is a subset of S, the *standard parahoric subgroup* P_I is the subgroup of G generated by B and I. A *parahoric subgroup* is a G-conjugate of a standard parahoric subgroup. The parahoric subgroup P_I generated by B and I is compact if and only if I is a proper subset of S. In particular, there are seven standard compact parahoric subgroups. They are

$$K = P_{\{s_0,s_1\}} = \mathbf{G}(R) \text{ (conjugate to } P_{\{s_2,s_1\}}) \tag{1.11}$$

$$J = P_{\{s_0,s_2\}}$$

$$P_{\{s_0\}} \text{ (conjugate to } P_{\{s_2\}})$$

$$P_{\{s_1\}}$$

$$B = P_{\varnothing}.$$

The two groups $K = P_{\{s_0,s_1\}} = \mathbf{G}(R)$ and $J = P_{\{s_0,s_2\}}$ are representatives for the two conjugacy classes of maximal compact subgroups of G.

Given a parahoric subgroup P, let P_1 be the maximal normal pro-p-subgroup of P. To describe P_1, and more generally a filtration of P which we use in Sections 4 and 5, we review the litany of affine roots and heights.

Let $\mathbf{X}$ be the $\mathbb{Z}$-span of the simple roots a and b. An affine root is an element $\beta = (c, n)$ in the additive group $\mathbf{X} \times \mathbb{Z}$ subject to the condition that $c \in (\Phi \cup \{0\})$ and $n \in \mathbb{Z}$. If $n \geqslant 0$, set $U_\beta = U_{c,n}$. Each affine root β has a unique decomposition

$$\beta = h_0(\beta)(b, 0) + h_1(\beta)(a, 0) + h_2(\beta)(-2a - b, 1).$$

Thus, for example, we have $(0, 1) = 1(b, 0) + 2(a, 0) + 1(-2a - b, 1)$. For $I \subset S$, define a height function ht_I on the affine roots by the sum

$$ht_I(\beta) = \sum{}' h_i(\beta)$$

over those i for which $s_i \notin I$. Then,

$$P_{I,1} = \text{subgroup of } G \text{ generated by } U_\beta \text{ such that } ht_I(t) \geqslant 1. \tag{1.12}$$

More generally, if $t \in \mathbb{N}$, set

$$P_{I,t} = \text{subgroup of } G \text{ generated by } U_\beta \text{ such that } ht_I(t) \geqslant t. \tag{1.13}$$

The $P_{I,t}$'s are a filtration of normal subgroups in P_I. When $I = \{s_0, s_1\}$, so that $P_I = K = \mathbf{G}(R)$, the group $P_{I,t}$ is the t-th principal congruence subgroup of K. We defer a more general discussion of the groups $P_{I,t}$ until Sections 4 and 5.

The quotient group K/K_1 is of course isomorphic to $\mathbf{G}(\mathbb{F}_q)$. For $I \subset \{s_0, s_1\}$, the group $P_I/P_{I,1}$ is $\mathbf{M}_I(\mathbb{F}_q)$. To identify J/J_1, let $\mathbf{G}(\mathbf{Sp(2)} \times \mathbf{Sp(2)})$ be the algebraic subgroup of $\mathbf{G}$ consisting of those elements whose (1, 2), (1, 3), (2, 1), (2, 4), (3, 1), (3, 4), (4, 2) and (4, 3) entries are zero. For $a \in R$, let $\mathbf{a}$ be the image of a in $\mathbb{F}_q$. The group J/J_1 can be identified with the $\mathbb{F}_q$-rational points of $\mathbf{G}(\mathbf{Sp(2)} \times \mathbf{Sp(2)})$ via the map

$$\begin{bmatrix} a & b & c & d\varpi^{-1} \\ e\varpi & f & g & h \\ i\varpi & j & k & m \\ n\varpi & r\varpi & s\varpi & t \end{bmatrix} \bmod J_1 \to \begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{0} & \mathbf{d} \\ \mathbf{0} & \mathbf{f} & \mathbf{g} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} & \mathbf{k} & \mathbf{0} \\ \mathbf{n} & \mathbf{0} & \mathbf{0} & \mathbf{t} \end{bmatrix}. \tag{1.14}$$

The group $\mathbf{G}(\mathbf{Sp(2)} \times \mathbf{Sp(2)})(\mathbb{F}_q)$ is isomorphic to the group

$$\{(g_1, g_2) \in \mathbf{GL}_2(\mathbb{F}_q) \times \mathbf{GL}_2(\mathbb{F}_q) \mid \det g_1 = \det g_2\}.$$

The isomorphism is given by the map

$$\begin{bmatrix} \mathbf{a} & \mathbf{0} & \mathbf{0} & \mathbf{d} \\ \mathbf{0} & \mathbf{f} & \mathbf{g} & \mathbf{0} \\ \mathbf{0} & \mathbf{j} & \mathbf{k} & \mathbf{0} \\ \mathbf{n} & \mathbf{0} & \mathbf{0} & \mathbf{t} \end{bmatrix} \to \begin{bmatrix} \mathbf{a} & \mathbf{d} \\ \mathbf{n} & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{g} \\ \mathbf{j} & \mathbf{k} \end{bmatrix}. \tag{1.15}$$

In all cases, the group $P_I/P_{I,1}$ can be identified with the $\mathbb{F}_q$-rational points of a reductive group with Dynkin diagram I.

A cuspidal representation of a standard compact parahoric subgroup P is a cuspidal representation of P/P_1 inflated to P. We have developed enough notation to make the following definition.

DEFINITION 1.1. A *nondegenerate representation of level one* in standard position is a pair (P, σ) consisting of a standard compact parahoric subgroup P and an irreducible cuspidal representation σ of P.

Our next goal is to define the nondegenerate representations of unramified and ramified type. To accomplish this we first establish more notation and conventions. Let $\tilde{G}$ denote the group $GL_4(F)$, $\tilde{K}$ denote $GL_4(R)$ and $\tilde{K}_i$ the ith principal congruence subgroup of $\tilde{K}$. If $\tilde{L}$ is a subgroup of $\tilde{G}$, let L denote the intersection of $\tilde{L}$ with G. This is consistent with our previous use of K as $\mathbf{G}(R)$. Let $\tilde{\mathfrak{g}} = M_4(F)$ (resp. $\tilde{\mathfrak{g}}(R) = M_4(R)$) be the F-rational (resp. R-rational) points of the Lie algebra of $\mathbf{GL_4}$. Let $\mathfrak{g}$ (resp. $\mathfrak{g}(R)$) be the analogous sets for the Lie algebra of $\mathbf{G}$. Define a form on $\tilde{\mathfrak{g}}$ by

$$\langle x, y \rangle = tr_{\tilde{\mathfrak{g}}}(xy). \tag{1.16}$$

The restriction of $\langle\ ,\ \rangle$ to $\mathfrak{g}$ is nondegenerate. A lattice $\mathfrak{l}$ is an open compact R-submodule in $\tilde{\mathfrak{g}}$ or $\mathfrak{g}$. If $\mathfrak{l} \subset \tilde{\mathfrak{g}}$, define the dual lattice $\mathfrak{l}^*$ in $\tilde{\mathfrak{g}}$ by

$$\mathfrak{l}^* = \{x \in \tilde{\mathfrak{g}} \mid \langle x, \mathfrak{l} \rangle \subset R\}. \tag{1.17}$$

If $\mathfrak{l} \subset \mathfrak{g}$, the dual lattice $\mathfrak{l}^*$ in $\mathfrak{g}$ is defined by the obvious modification of (1.17), i.e., $\tilde{\mathfrak{g}}$ is replaced by $\mathfrak{g}$. The lattices $\tilde{\mathfrak{g}}(R)$ and $\mathfrak{g}(R)$ are self-dual in $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ respectively. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ let $\mathfrak{h}^\perp$ (resp. $\mathfrak{h}^\#$) be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$). We have

$$\mathfrak{h}^\# = \mathfrak{h}^\perp \oplus \mathfrak{g}^\#. \tag{1.18}$$

Let $i \geqslant j$ be positive integers. The Cayley transform

$$c(x) = (1 - x)(1 + x)^{-1} \tag{1.19}$$

maps $\varpi^i \tilde{\mathfrak{g}}(R)$ and $\varpi^i \mathfrak{g}(R)$ bijectively to $\tilde{K}_i$ and K_i respectively. Taking quotients, the Cayley transform becomes an isomorphism

$$\varpi^i \tilde{\mathfrak{g}}(R)/\varpi^{i+j}\tilde{\mathfrak{g}}(R) \to \tilde{K}_i/\tilde{K}_{i+j}$$

$$\varpi^i \mathfrak{g}(R)/\varpi^{i+j}\mathfrak{g}(R) \to K_i/K_{i+j}.$$

We use this isomorphism to identify the quotient groups. Let $\mathfrak{g}^{\#}(R) = \mathfrak{g}^{\#} \cap \tilde{\mathfrak{g}}(R)$. We have

$$\tilde{K}_i/\tilde{K}_{i+j} = K_i/K_{i+j} \oplus \varpi^i \mathfrak{g}^{\#}(R)/\varpi^{i+j}\mathfrak{g}^{\#}(R). \tag{1.20}$$

In this setting, the characters of K_i/K_{i+j} can be viewed as the characters of $\tilde{K}_i/\tilde{K}_{i+j}$ trivial on $\varpi^i \mathfrak{g}^{\#}(R)/\varpi^{i+j}\mathfrak{g}^{\#}(R)$. We need to describe a very important realization of these characters. Let ψ be an additive character of F with conductor R. We identify the character group $\{\tilde{K}_i/\tilde{K}_{i+j}\}^{\wedge}$ with $\varpi^{-(i+j)}\tilde{\mathfrak{g}}(R)/\varpi^{-i}\tilde{\mathfrak{g}}(R)$ via the map $(\alpha + \varpi^{-i}\tilde{\mathfrak{g}}(R)) \to \Omega_\alpha$,

$$\Omega_\alpha(c(x)) = \psi(\langle x, -\alpha\rangle/2), \quad x \in \varpi^i \tilde{\mathfrak{g}}(R). \tag{1.21}$$

If $c(x)$ is written in the form $1 + y$ with y in $\varpi^i \tilde{\mathfrak{g}}(R)$, then $\Omega_\alpha(c(x)) = \psi(\langle x, \alpha\rangle)$. The character group $\{K_i/K_{i+j}\}^{\wedge}$ is identified with the group $\varpi^{-(i+j)}\mathfrak{g}(R)/\varpi^{-i}\mathfrak{g}(R)$.

Assume $j = 1$. Take Ω_α of the form (1.21), with $\alpha \in \mathfrak{g}$. Multiplication by $\varpi^{(i+1)}$ allows us to identify $\varpi^{-(i+1)}\mathfrak{g}(R)/\varpi^{-i}\mathfrak{g}(R)$ with $\mathfrak{g}(\mathbb{F}_q)$, so we can view $\varpi^{(i+1)}\alpha \bmod \mathfrak{p}$ as an element in $\mathfrak{g}(\mathbb{F}_q)$. Decompose this element into its semisimple and nilpotent parts

$$\varpi^{(i+1)}\alpha \bmod \mathfrak{p} = \mathbf{s} + \mathbf{n}. \tag{1.22}$$

When $\mathbf{s}$ is not a scalar matrix we shall soon define a group L and a representation $\Omega_{\mathbf{s}}$ of L. The collection of the $(L, \Omega_{\mathbf{s}})$'s and their G-conjugates shall be the nondegenerate representations of unramified type.

In order to define the groups L, we recall the standard parabolic subgroups of G. Given a subset $I \subset \{a, b\}$, let $\mathbf{P}_I = \mathbf{M}_I\mathbf{U}_I$ be the Levi decomposition (1.7). Let $\mathfrak{m}_I(\mathbb{F}_q)$, $\mathfrak{u}_I(\mathbb{F}_q)$ and $\mathfrak{p}_I(\mathbb{F}_q)$ be the $\mathbb{F}_q$-rational points of the Lie algebra of $\mathbf{M}_I$, $\mathbf{U}_I$ and $\mathbf{P}_I$ respectively.

Consider (1.22). Let α' be a K-conjugate of $\varpi^{(\iota+1)}\alpha$ mod $\not{p}$ so that,

$$\alpha' \in \mathfrak{p}_I(\mathbb{F}_q), \quad \text{but} \quad \alpha' \notin \mathfrak{p}_{I'}(\mathbb{F}_q) \tag{1.23}$$

for all proper subsets I' of I. Let $\alpha' = \mathbf{s}' + \mathbf{n}'$ be the Jordan decomposition. The semisimple part $\mathbf{s}'$ lies in $\mathfrak{m}_I(\mathbb{F}_q)$, while the nilpotent part $\mathbf{n}'$ lies in $\mathfrak{u}_I(\mathbb{F}_q)$. Let $\underline{\mathbf{U}}_I$ be the unipotent subgroup opposite to $\mathbf{U}_I$, i.e., if θ is the Cartan involution of inverse transpose, then $\underline{\mathbf{U}}_I = \theta(\mathbf{U}_I)$. For $m \in \mathbb{N}$, let $M_{I,m}$, $U_{I,m}$ and $\underline{U}_{I,m}$ be the intersections of K_m with $\mathbf{M}_I(F)$, $\mathbf{U}_I(F)$ and $\underline{\mathbf{U}}_I(F)$ respectively.

We now define the nondegenerate representations of unramified type. Let L be the group

$$L = \underline{U}_{I,\iota+1} M_{I,\iota} U_{I,\iota}. \tag{1.24}$$

The group L is in fact a filtration subgroup (1.15) of the parahoric subgroup P_I. The restrictions of $\Omega_{\alpha'}$ and $\Omega_{\mathbf{s}'}$ to L are equal. Denote this character by $\Omega_{\mathbf{s}'}$.

Definition 1.2. A *nondegenerate representation of unramified type* in standard position is a pair $(L, \Omega_{\mathbf{s}'})$ as in (1.23, 24).

We proceed to define the nondegenerate representations of ramified type. The basic scheme of things here parallel the unramified case. We construct a subgroup L related to K_ι and a character Ω of L satisfying certain properties. The kernel of Ω shall contain $K_{\iota+1}$ and the restriction of Ω to K_ι will be represented by a nilpotent element of $\mathfrak{g}(\mathbb{F}_q)$.

Let $i \in \mathbb{N}$. Consider the five classes of sets $\tilde{\alpha}$ described in (1.25). Here, u, v, $\varepsilon \in R^\times$, ε a nonsquare, and $a, b, c, d, e, f \in R$. In (1.25b), $a^2 - bc$, $d^2 - ef \in R^\times$.

$$\varpi^{-\iota-1}\left\{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ u\varpi & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \not{p} & \not{p} & R & R \\ \not{p} & \not{p} & \not{p} & R \\ \not{p} & \not{p} & \not{p} & \not{p} \\ \not{p}^2 & \not{p} & \not{p} & \not{p} \end{bmatrix}\right\} \tag{1.25a}$$

$$\varpi^{-\iota-1}\left\{\begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & a \\ d\varpi & e\varpi & 0 & 0 \\ f\varpi & d\varpi & 0 & 0 \end{bmatrix} + \begin{bmatrix} \not{p} & \not{p} & \not{p} & \not{p} \\ \not{p} & \not{p} & \not{p} & \not{p} \\ \not{p}^2 & \not{p}^2 & \not{p} & \not{p} \\ \not{p}^2 & \not{p}^2 & \not{p} & \not{p} \end{bmatrix}\right\}, \tag{1.25b}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ u\varpi & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}\right\} \tag{1.25c}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ u\varpi & 0 & 0 & 0 \\ 0 & v\varpi & 0 & 0 \\ 0 & 0 & -u\varpi & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}\right\} \tag{1.25d}$$

$$\varpi^{-i-1}\left\{\begin{bmatrix} \varpi c & 0 & 0 & 1 \\ 0 & \varpi c & \varpi u & 0 \\ 0 & \varpi u\varepsilon & \varpi c & 0 \\ \varpi^2 v^2 \varepsilon & 0 & 0 & \varpi c \end{bmatrix} + \begin{bmatrix} \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^3 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 \end{bmatrix}\right\} \quad (i \geqslant 2). \tag{1.25e}$$

We have written each of the five sets in (1.25a–e) in the form

$$\varpi^{-i-1}\alpha + \tilde{\mathfrak{l}}, \tag{1.26}$$

where $\alpha \in \mathfrak{g}$, and $\tilde{\mathfrak{l}}$ is a matrix of ideals. The convention for this notation is that $\tilde{\mathfrak{l}}$ is the set of elements in $\tilde{\mathfrak{g}}(R)$ with the entries in the indicated ideals. Thus, in (1.25a), elements of $\tilde{\mathfrak{l}}$ have their (1, 1) entry in $\varpi^{-i-1}\mathfrak{p}$, their (1, 4) entry in $\varpi^{-i-1}R$, their (4, 1) entry in $\varpi^{-i-1}\mathfrak{p}^2$ and so forth. We shall use similar notation later to describe other sets of matrices.

Note that $\tilde{\mathfrak{l}}$ is an R-submodule of $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{l}}^*$ be the dual lattice (c.f. (1.17)). For each $\tilde{\alpha}$ as in (1.25), let $\tilde{L}$ be the group

$$\tilde{L} = 1 + \tilde{\mathfrak{l}}^* = \{1 + x \mid x \in \tilde{\mathfrak{l}}^*).$$

Let α, $\mathfrak{l}$ and $\mathfrak{l}^*$ denote the intersection of $\tilde{\alpha}$, $\tilde{\mathfrak{l}}$ and $\tilde{\mathfrak{l}}^*$ respectively with $\mathfrak{g}$. Note that $\mathfrak{l}^*$ is also the dual lattice of $\mathfrak{l}$ in $\mathfrak{g}$. Let L be the intersection of $\tilde{L}$ with G. It is easy to check that

$$\tilde{L} = c(\tilde{\mathfrak{l}}^*) \quad \text{and} \quad L = c(\mathfrak{l}^*).$$

The group L is a filtration subgroup (1.15) of some parahoric subgroup P_I. Define a character Ω_α on $\tilde{L}$, hence L, in analogy with (1.21) by the formula

$$\Omega_\alpha(c(x)) = \psi(\langle x, -\alpha\rangle/2), \quad x \in \tilde{\mathfrak{l}}.$$

The group L contains K_{i+1} and the restriction of Ω_α to K_{i+1} is trivial. In cases (1.25b, c, d, e), the group L contains K_i and the restriction of Ω_α to K_i is represented in $\mathfrak{g}(\mathbb{F}_q)$ by reducing elements in $\varpi^{i+1}\alpha$ mod $\not{p}$. In case (1.25b) (resp. (1.25c, d, e)), $\varpi^{i+1}\alpha$ mod $\not{p}$ is equal to $aE_{1,3} + bE_{1,4} + cE_{2,3} + aE_{2,4}$ (resp. $E_{1,4}$) mod $\not{p}$. In case (1.25a), the group L does not contain K_i, but the restriction of Ω_α to the intersection $L \cap K_i$ can be extended to K_i. The extensions of Ω_α to K_i will be represented in $\mathfrak{g}(\mathbb{F}_q)$ by elements of the form

$$E_{1,2} + E_{2,3} - E_{3,1} + aE_{1,3} + aE_{2,4} + bE_{1,4} \bmod \not{p}.$$

These elements are K-conjugate to $\varpi^{i+1}\alpha$ mod $\not{p}$. In every case, the extension of Ω_α to K_i is represented by a nilpotent element $\mathbf{n}$.

DEFINITION 1.3. A *nondegenerate representation of ramified type* in standard position is a pair (L, Ω_α), with $\tilde{\alpha}$ of the form (1.25).

2. Nondegenerate representations as lowest *K*-types

Consider an admissible irreducible representation (π, V) of G. In this section we show some twist $\pi \otimes \chi$, χ a one-dimensional representation of G, must contain a standard minimal nondegenerate representation (L, σ). The approach taken here is the same as the one used in [M1] for $U(2, 1)$.

Given an irreducible admissible representation (π, V), define the *level* of π to be the minimum i such that V^{K_i}, the space of vectors in V fixed by K_i, is nonzero. The representation π is said to have *minimal level under twisting* if the level of π is less than or equal to the level of all twists $\pi \otimes \chi$. It is clear we can assume π has minimal level under twisting. Let $i + 1$ be the minimal level. Choose a nonzero vector v fixed by K_{i+1} and let $K_i v$ be the span of K_i transforms of v. This finite dimensional space can be decomposed into irreducible K_i/K_{i+1} subspaces. If we select a vector from an irreducible component of $K_i v$ and replace v by this vector, we can assume $K_i v$ is an irreducible K_i/K_{i+1} space. If $i = 0$, then π restricted to K contains a representation σ of $K/K_1 = \mathbf{G}(\mathbb{F}_q)$. By Harish-Chandra's philosophy of cusp forms [HC], there is a parahoric subgroup $P \subset K$ as well as a cuspidal representation Ω of P such that the restriction of σ to P contains Ω. In particular, π must contain a nondegenerate representation.

Suppose $i \geqslant 1$. As in Section 1, we identify the characters of K_i/K_{i+1} with $\mathfrak{g}(\mathbb{F}_q)$. The group K_i/K_{i+1} acts on $K_i v$ by a character Ω_α. For $k \in K$, it is clear that $\pi(k)(K_i v)$ is also an irreducible K_i/K_{i+1} subspace. Indeed, if we set $\alpha' = k\alpha k^{-1}$, then the group K_i/K_{i+1} acts on $\pi(k)(K_i v)$ by the character $\Omega_{\alpha'}$. By taking an appropriate k, we can assume α' satisfies (1.23). Let L be the group (1.24), and let $\Omega_{\mathbf{s}'}$ be the restriction of Ω_α to L. If $\mathbf{s}'$ is not a scalar, the pair $(L, \Omega_{\mathbf{s}'})$ is by definition a minimal nondegenerate representation of unramified type. Assume $\mathbf{s}'$ is a scalar. Then, it is easily seen that there is a one-dimensional character χ of G trivial on K_{i+1}, and represented by $-\mathbf{s}'$ on K_i/K_{i+1}. The representation $\pi \otimes \chi$ still have level $i+1$. Furthermore, the $\pi \otimes \chi$ action of K_i/K_{i+1} on $K_i v$ is a character $\Omega_{\mathbf{n}}$, $\mathbf{n}$ an upper triangular nilpotent element of $\mathfrak{g}(\mathbb{F}_q)$. Consider the three cases delineated by the rank of $\mathbf{n}$. In the following analysis, we define two groups L_+ and L. The two groups will change during the course of the analysis, but L_+ will always be normal in L and the quotient will always be abelian.

Case 1: $\operatorname{rank}(\mathbf{n}) = 3$. We can take $\mathbf{n}$ of the form

$$\mathbf{n} = E_a + E_b \bmod \mathfrak{p}.$$

In the notation of (1.11) and (1.13), it is easily verified that

$$L_+ = B_{4i} \tag{2.1a}$$

is contained in the kernel of $\Omega_{\mathbf{n}}$. Let

$$L = B_{4i-1}. \tag{2.1b}$$

The characters of L/L_+ can be realized as in (1.21) by the cosets

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -a \\ c\varpi & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & R & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & R \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix}\right\}, \tag{2.2}$$

$a, b, c \in R$. Let $v \neq 0$ be a vector in V fixed by L_+. Write Lv for the span of the L transforms of v. The decomposition of Lv into L subspaces yields characters with $a = b = 1$. If $c \neq 0 \bmod \mathfrak{p}$, the representation Ω of L on Lv is nondegenerate of type (1.25a). On the other hand, if $c = 0 \bmod \mathfrak{p}$, the vector $w = \pi(t)v$ transforms under L as a character with $b = 0$. The decomposition of $K_i w$ into K_i/K_{i+1} subspaces give vectors which transform under K_i/K_{i+1} by characters $\Omega_{\mathbf{n}}$, $\mathbf{n} \in \mathfrak{g}(\mathbb{F}_q)$ nilpotent of rank 2.

Case 2: rank$(\mathbf{n}) = 2$. We can assume $\mathbf{n}$ has the form

$$\mathbf{n} \;=\; AE_{a+b} + BE_{2a+b} + CE_b \bmod \wp, \quad \text{with} \quad A^2 - BC \neq 0 \bmod \wp.$$

The kernel of $\Omega_{\mathbf{n}}$ contains the group

$$L_+ \;=\; P_{\{s_1\},2i}. \tag{2.3a}$$

Take $v \neq 0$ fixed by L_+. Set

$$L \;=\; P_{\{s_1\},2i-1}. \tag{2.3b}$$

Realize the character group $\{L/L_+\}^\wedge$ as the cosets

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & z & x \\ \varpi d & \varpi e & 0 & 0 \\ \varpi f & \varpi d & 0 & 0 \end{bmatrix} + \begin{bmatrix} \wp & \wp & \wp & \wp \\ \wp & \wp & \wp & \wp \\ \wp^2 & \wp^2 & \wp & \wp \\ \wp^2 & \wp^2 & \wp & \wp \end{bmatrix}\right\}, \tag{2.4}$$

$x, y, z, d, e, f \in R$. Decompose Lv, as in the rank three case, into irreducible L/L_+ subspaces and choose a nonzero vector, again relabeled v, which transforms by a character with $x = A$, $y = B$ and $z = C$. If $d^2 - ef \neq 0 \bmod \wp$, the representation Ω of L on Lv is nondegenerate of type (1.25b). The alternative is $d^2 - ef = 0 \bmod \wp$. Both L and L_+ are normalized by the element t in (1.0). Consequently, L acts on the vector $w = \pi(t)v$ according to the character

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & d & e \\ 0 & 0 & f & d \\ \varpi A & \varpi B & 0 & 0 \\ \varpi C & \varpi A & 0 & 0 \end{bmatrix} + \begin{bmatrix} \wp & \wp & \wp & \wp \\ \wp & \wp & \wp & \wp \\ \wp^2 & \wp^2 & \wp & \wp \\ \wp^2 & \wp^2 & \wp & \wp \end{bmatrix}\right\}.$$

In turn, K_i/K_{i+1} acts on w as a character $\Omega_{\mathbf{n}'}$, with $\mathbf{n}' \in \mathfrak{g}(\mathbb{F}_q)$ a nilpotent element of rank 1 or 0. If $\mathbf{n}'$ is of rank 0 then (π, V) possesses a nonzero K_i-fixed vector contradicting the hypothesis of minimal level $i + 1$. We therefore can assume $\mathbf{n}'$ has rank 1.

Case 3: rank($\mathbf{n}$) = 1. Here, we write $\mathbf{n}$ as

$$\mathbf{n} = E_{2a+b} \bmod \mathfrak{p}.$$

The kernel of $\Omega_{\mathbf{n}}$ contains

$$L_+ = K \cap \left\{ 1 + \varpi^{l} \begin{bmatrix} R & R & R & R \\ R & R & R & R \\ R & R & R & R \\ \mathfrak{p} & R & R & R \end{bmatrix} \right\}. \tag{2.5a}$$

Take $v \neq 0$ again to be fixed by L_+. Set

$$L = K \cap \left\{ 1 + \varpi^{l} \begin{bmatrix} R & R & R & \mathfrak{p}^{-1} \\ R & R & R & R \\ R & R & R & R \\ R & R & R & R \end{bmatrix} \right\}, \tag{2.5b}$$

and represent the characters of L/L_+ by the cosets

$$\varpi^{-l-1} \left\{ \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varpi b & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \end{bmatrix} \right\}, \tag{2.6}$$

$a, b \in R$. As in the previous two cases, Lv decomposes into irreducible L/L_+ subspaces. Choose a nonzero vector, again called v, which transforms by a character with $a = 1$. If $b \neq 0 \bmod \mathfrak{p}$, the representation Ω of L on Lv is nondegenerate of type (1.25c). If $b = 0 \bmod \mathfrak{p}$, the group

$$L_+ = P_{\{s_0\},3l-1} \tag{2.7a}$$

fixes v. Redefine L to be

$$L = P_{\{s_0\},3l-2}. \tag{2.7b}$$

The cosets

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & a \\ \varpi b & 0 & 0 & 0 \\ \varpi c & 0 & 0 & 0 \\ 0 & \varpi c & -\varpi b & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}\right\}, \tag{2.8}$$

a, b, $c \in R$, realize the characters of L/L_+. The decomposition of Lv into irreducible L subspaces, produces characters with $a = 1$. We can in fact assume $c = 0 \bmod \mathfrak{p}$. To see this, let $\tilde{\alpha}$ denote the set (2.8). The element $u(x) = u_b(x)$, $x \in R$, (c.f. (1.3)) normalizes L and L_+ and $u(x)\tilde{\alpha}u(x)^{-1}$ is equal to

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & a \\ \varpi(b+cx) & 0 & 0 & 0 \\ \varpi c & 0 & 0 & 0 \\ 0 & \varpi c & -\varpi(b+cx) & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}\right\}.$$

Hence, if $v \neq 0$ transforms under L/L_+ by a character with $c \neq 0 \bmod \mathfrak{p}$, then $w = \pi(u(-b/c))v$ transforms under L/L_+ by a character with $b = 0$. Since s_0 normalizes both L and L_+, the vector $\pi(s_0)w$ transforms under L/L_+ by a character with $a = 1$ and $c = 0$. This proves our assertion that we can assume $c = 0$. As an important consequence, we conclude the group

$$L_+ = B_{4i-2} \tag{2.9a}$$

fixes a nonzero vector v. Set

$$L = B_{4i-3}. \tag{2.9b}$$

The cosets

$$\varpi^{-i-1}\left\{\begin{bmatrix} 0 & 0 & 0 & a \\ \varpi b & 0 & 0 & 0 \\ 0 & \varpi c & 0 & 0 \\ 0 & 0 & -\varpi b & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \end{bmatrix}\right\}, \tag{2.10}$$

$a, b, c \in R$, realize $\{L/L_+\}^\wedge$. The characters of L which appear in Lv have $a = 1$. Assume v transforms under L by such a character. If b, $c \neq 0 \bmod \not{p}$, the representation of L on Lv is nondegenerate of type (1.25d). If $c = 0 \bmod \not{p}$, then $\pi(t)v$ is fixed by K_i in contradiction to the hypothesis on minimality of level. If $b = 0 \bmod \not{p}$, the group

$$L_+ = J_{2i-1} \tag{2.11a}$$

fixes v. If $i = 1$, this group is the radical J_1 of the "nonstandard" maximal compact subgroup J in (1.11). Let σ be the representation of J/J_1 on Jv. By Harish-Chandra's philosophy of cusp forms [HC], there is a parahoric subgroup $P \subset J$ and a cuspidal representation Ω of P/P_1 such that the restriction of σ to P contains Ω. Hence again, the representation (π, V) contains a nondegenerate representation. Consider the alternative $i \geqslant 2$. Set

$$L = J_{2i-2} \tag{2.11b}$$

and realize $\{L/L_+\}^\wedge$ by the cosets

$$\varpi^{-i-1}\left\{\begin{bmatrix} \varpi(c+b) & 0 & 0 & a \\ 0 & \varpi(c+g) & \varpi d & 0 \\ 0 & \varpi e & \varpi(c-g) & 0 \\ \varpi^2 f & 0 & 0 & \varpi(c-b) \end{bmatrix} + \begin{bmatrix} p^2 & \not{p} & \not{p} & \not{p} \\ \not{p}^2 & \not{p}^2 & \not{p}^2 & \not{p} \\ \not{p}^2 & \not{p}^2 & \not{p}^2 & \not{p} \\ \not{p}^3 & \not{p}^2 & \not{p}^2 & \not{p}^2 \end{bmatrix}\right\}, \tag{2.12}$$

$a, b, c, d, e, f, g \in R$. We can assume L acts on v by a character with $a = 1$. Let $\tilde{\alpha}$ be the set (2.12). Both L and L_+ are normalized by $\underline{u}(x) = u_{-2a-b}(x)$, $x \in R$, and $\underline{u}(x)\tilde{\alpha}\underline{u}(x)^{-1}$ is

$$\varpi^{-i-1}\left\{\begin{bmatrix} \varpi(c+b') & 0 & 0 & a \\ 0 & \varpi(c+g) & \varpi d & 0 \\ 0 & \varpi e & \varpi(c-g) & 0 \\ \varpi^2 f' & 0 & 0 & \varpi(c-b') \end{bmatrix} + \begin{bmatrix} p^2 & \not{p} & \not{p} & \not{p} \\ \not{p}^2 & \not{p}^2 & \not{p}^2 & \not{p} \\ \not{p}^2 & \not{p}^2 & \not{p}^2 & \not{p} \\ \not{p}^3 & \not{p}^2 & \not{p}^2 & \not{p}^2 \end{bmatrix}\right\},$$

where $b' = b - ax$ and $f' = f + 2bx - ax^2$. Observe that the discriminant $af + b^2$, is invariant, i.e., $af + b^2 = af' + b^2$. In our situation $a = 1 \bmod \not{p}$. If the discriminant is a square, say y^2, then $w = \pi(\underline{u}((b \pm y)/a))v$

transforms under L by a character of type (2.12) with $f = 0$. This means $\pi(d(\varpi, 1, 1))w$ is a K_l fixed vector contradicting the minimal level hypothesis. Hence, we can assume v transforms according to a character with $a = 1$, $b = 0$ and f a nonsquare unit. We can take f to be ε. Consider the eigenvalues of

$$\begin{bmatrix} (c+g) & d \\ e & (c-g) \end{bmatrix}.$$

We claim they do not belong to $\mathbb{F}_q$. To see this note that the parahoric subgroup $P_{\{s_1\}}$ normalizes L and L_+. If the eigenvalues lie in $\mathbb{F}_q$, then there exists an element $k \in P_{\{s_1\}}$ so that $w = \pi(k)v$ transforms under L by a character with $e = 0 \bmod \mathfrak{p}$. This means $\pi(t)v$ is a K_l fixed vector, in contradiction to the minimality of level hypothesis. So, up to a twist by a one dimensional character of $GSp(4)$, there exists a vector $v \neq 0$ which transforms under L by the character

$$\varpi^{-l-1}\left\{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \varpi u & 0 \\ 0 & \varpi u \varepsilon & 0 & 0 \\ \varpi^2 \varepsilon & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathfrak{p}^2 & \mathfrak{p} & \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^3 & \mathfrak{p}^2 & \mathfrak{p}^2 & \mathfrak{p}^2 \end{bmatrix}\right\},$$

$u \neq 0 \bmod \mathfrak{p}$. This is of course a nondegenerate representation of type (1.25e).

In conclusion, our analysis has shown

THEOREM 2.1. *For any admissible representation π of $GSp(4)$, there is a one dimensional character χ of $GSp(4)$ so that $\pi \otimes \chi$ contains a nondegenerate representation in standard position.*

We turn now to the question of when two nondegenerate representations can occur in the same irreducible representation of G. Define two nondegenerate representations (L, Ω), (L', Ω') to be *associate* if

(i) $L = P$, $L' = P'$ are parahoric subgroups, $P/P_1 \cong P'/P_1'$ and $\Omega \cong \Omega'$, or (2.13)

(ii) $\Omega = \Omega_{\mathbf{s}}, \Omega = \Omega_{\mathbf{s}'}$ with $\mathbf{s}, \mathbf{s}'$ of the form (1.23, 24) or (1.25) and some element of $\mathbf{s}$ is conjugate to some element of $\mathbf{s}'$.

THEOREM 2.2. *Suppose (π, V) is an irreducible admissible representation of G. If (L, Ω), (L', Ω') are two nondegenerate representations contained in π, then they are associate.*

Proof. We use the intertwining principle. Let W_Ω and $W_{\Omega'}$ respectively be the Ω and Ω' subspaces of V. Let $E_{\Omega'}$ denote the projection from V onto $W_{\Omega'}$. Since π is irreducible, there is a $g \in G$ such that

$$I = E_{\Omega'}\pi(g): W_\Omega \to W_{\Omega'}$$

is nonzero, and for $h \in L \cap gL'g^{-1}$, $I\Omega(h) = \Omega'(g^{-1}hg)I$. We consider three cases according to whether none, one or both L, L' are parahoric subgroups. Case 1, $\Omega = \Omega_{\mathbf{s}}$, $\Omega' = \Omega_{\mathbf{s}'}$. Write $\mathbf{s} = s + \mathfrak{l}$, $\mathbf{s}' = s' + \mathfrak{l}$. Observe that

$$L \cap gL'g^{-1} = c(\mathfrak{l}^* \cap g\mathfrak{l}'^*g^{-1}).$$

Therefore, if $y \in \mathfrak{l}^* \cap g\mathfrak{l}'^*g^{-1}$, then

$$\psi(\langle y, -s\rangle/2) = \Omega_{\mathbf{s}}(c(y)) = \Omega_{\mathbf{s}'}(g^{-1}c(y)g) = \psi(\langle y, -gs'g^{-1}\rangle/2).$$

This means $\operatorname{tr}(y(s - gs'g^{-1})) \in R$ for all $y \in \mathfrak{l}^* \cap g\mathfrak{l}'^*g^{-1}$. We conclude $s - gs'g^{-1} \in \{\mathfrak{l}^* \cap g\mathfrak{l}'^*g^{-1}\}^* = \mathfrak{l} + g\mathfrak{l}'g^{-1}$, i.e., $\mathbf{s}$ and $g\mathbf{s}'g^{-1}$ intersect.
Case 2, $\Omega = \Omega_{\mathbf{s}}$ and $L' = P'$ a parahoric subgroup. We show Ω and Ω' cannot both occur in π by showing that Ω and the trivial representation of P_1' cannot occur simultaneously in π. The trivial character of P_1'/P_2' can realized as $\Omega_{\mathbf{s}'}$ with $\mathbf{s}' = \mathfrak{l}$ and $P_1' = c(\mathfrak{l}^*)$. By the same reasoning as in case 1, some element of $\mathbf{s}$ must be conjugate to some element of $\mathbf{s}'$. This is impossible because the minimum valuation of the eigenvalues of each element in $\mathbf{s}$ differ from those of $\mathbf{s}'$ (see Theorem 6.1 [HM2]).
Case 3, L and L' both parahoric subgroups. The reasoning again is based on the intertwining principle. It has already been done by Harish Chandra (see [HC]). □

3. Level one representations

In this and the next two sections, we shall give a fairly detailed description of the irreducible representations of G. In light of Theorem 2.1, it suffices to describe the irreducible representations of G which contain a given nondegenerate representation (L, Ω). To accomplish this goal, we investigate the Hecke algebra $\mathscr{H}(G//L, \Omega)$. In preparation for this analysis, we briefly recall some relevant facts about Hecke algebras and their place in our classification of the representations of G.

Let Q be an arbitrary open compact subgroup of G and let (σ, V_σ) be an irreducible representation of Q. The Hecke algebra $\mathscr{H} = \mathscr{H}(G//Q, \sigma)$ consists of the functions $f: G \to \operatorname{End} V_\sigma$ such that

(i) f is compactly supported

(ii) $f(kgk') = \sigma(k)f(g)\sigma(k')$, $g \in G$ and $q, q' \in Q$.

An element g in G is said to lie in the support of $\mathscr{H}$ if there is a $f \in \mathscr{H}$ which is nonzero at g. The support set of $\mathscr{H}$ is denoted by supp $\mathscr{H}$. The algebra $\mathscr{H}$ is of course an algebra under convolution. If Haar measure is normalized so that vol $(Q) = 1$, then the function

$$e_\sigma(g) = \begin{cases} \sigma(g) & g \in Q \\ 0 & g \notin Q \end{cases}$$

is the identity element of $\mathscr{H}$. If (π, V) is any admissible representation of G, consider the tensor product space $W = V \otimes V_\sigma$. The Hecke algebra $\mathscr{H}$ acts on W by the formula

$$\pi(f)(v \otimes w) = \int_G \pi(g)v \otimes f(g)w \, dg,$$

$f \in \mathscr{H}$, $v \in V$ and $w \in V_\sigma$. Observe that if $k \in Q$, then

$$\begin{aligned} \pi(f)(\pi(k)v \otimes \sigma(k)w) &= \int_G \pi(g)\pi(k)v \otimes f(g)\sigma(k)w \, dg \\ &= \int_G \pi(gk)v \otimes f(gk)w \, dg \\ &= \pi(f)(v \otimes w) \\ &= (\pi(k) \otimes \sigma(k))\pi(f)(v \otimes w). \end{aligned}$$

In particular, the actions of $\mathscr{H}$ and Q on W commute; hence, $\mathscr{H}$ acts on the finite dimensional space $E = W^Q$. It is not difficult to see that the dimension of E is equal to the multiplicity of the contragredient representation σ' in π and additionally, if π is irreducible, then the representation of $\mathscr{H}$ on E is also irreducible. The process of passing from V to E is a bijection between the irreducible admissible representations of G which contain σ' with positive multiplicity and the irreducible finite dimensional representations of E. As an upshot of these considerations and Theorem 2.1, the problem of determining the representations of G is equivalent to determining the representations of the Hecke algebras $\mathscr{H}(G//L, \Omega)$, where (L, Ω) is a nondegenerate representation. We show these Hecke algebras (actually very closely related Hecke algebras $\mathscr{H}(G//J, \sigma)$) are isomorphic to Hecke algebras of smaller groups. This allows a transfer of representations between the groups.

We also remark that $\mathscr{H}$ possess a natural involutive *-operation. It is defined for $f \in \mathscr{H}$ by

$$f^*(g) = f(g^{-1})^*. \tag{3.1}$$

The * on the right side is the adjoint operation on End V_σ. The inner product

$$\langle f_1, f_2 \rangle = \operatorname{tr} \{f_1 * f_2^*(1)\}. \tag{3.2}$$

on $\mathscr{H}$ provides a context in which the Plancherel Formula can be formulated (see Appendix 1 [HM1]). The Hecke algebra isomorphisms established shall preserve the L_2 structure of the algebras and so the transfer of representations will preserve the property of square integrability. The isomorphisms shall also preserve the support of the Hecke algebras. This means the transfer of representations will preserve supercuspidality (see [HM1]).

We now give a description of the Hecke algebras $\mathscr{H} = \mathscr{H}(G//P, \Omega)$ when P is a parahoric subgroup of the form (1.11) and Ω is a cuspidal representation of P/P_1.

We need to review various properties of the Bruhat decomposition [BT]. Let $W^{\text{aff}} = N/\mathbf{A}(R)$ be the affine Weyl group defined in (1.8). For $w \in N$, let $\mathbf{w}$ denote the image of w in W^{aff}. Each double coset of B in G can be written in the form BwB, $w \in N$. Two double cosets BwB and $Bw'B$ are equal precisely when $\mathbf{w} = \mathbf{w}'$ in W^{aff}. This decomposition is written symbolically as

$$G = BW^{\text{aff}}B. \tag{3.3}$$

The length of an element w in N is the integer $\ell(w)$ such that

$$[BwB : B] = q^{\ell(w)}. \tag{3.4}$$

The length can also be interpreted as the number of reflections, i.e., elements of S, in a reduced expression for $\mathbf{w}$. The elements in S all have length 1. Also

$$\begin{aligned} &\text{(i) } BwBw'B = Bww'B, \quad \ell(w) + \ell(w') = \ell(ww') \\ &\text{(ii) } BwBsB = BwB \cup BwsB, \quad s \in S \text{ and } \ell(ws) = \ell(w) - 1. \end{aligned} \tag{3.5}$$

We remark that it is possible to choose a set of representatives $\{w\}$ for $\mathbf{W}^{\text{aff}}$ in a fashion so that if w and w' are the representatives for $\mathbf{w}$ and $\mathbf{w}'$, then

$$ww' \text{ is the representative for } \mathbf{ww}' \text{ when } \ell(w) + \ell(w') = \ell(ww'). \quad (3.6)$$

Similar results hold for parahoric subgroups. In the parahoric case, if P is generated by B and the set of reflections $I \subset S$, let $W_I \subset W^{\text{aff}}$ be the subgroup generated by the reflections in I. Then, the double cosets of P in G are in one–one correspondence with the double cosets of W_I in W^{aff}.

For a parahoric subgroup $P = P_I$, let

$$\begin{aligned} M = {} & \text{subgroup of } P \text{ generated by } \mathbf{A}(R) \text{ and the subgroups} \\ & U_{c,i} \subset P \text{ such that } U_{-c,-i} \subset P. \end{aligned} \quad (3.7)$$

If $I \subset \{s_0, s_1\}$, then $M = \mathbf{M}_I(R)$ and $P/P_1 = \mathbf{M}_I(\mathbb{F}_q)$. In all cases $P_I = MP_{I,1}$. The next result gives a condition w must satisfy in order to lie in the support of $\mathscr{H}$.

THEOREM 3.1. (Harish-Chandra). *A necessary condition for PwP to lie in the support of $\mathscr{H}(G//P, \Omega)$, Ω a cuspidal representation of P/P_1 is that w normalizes M, i.e.*

$$wMw^{-1} = M.$$

Proof. For $w \in W^{\text{aff}}$, let $H = \{wPw^{-1} \cap P\} P_1/P_1$. The group H is a parabolic subgroup of P/P_1. Suppose $w \in \operatorname{supp} \mathscr{H}$, and $H \neq P/P_1$. Let U be the unipotent radical of H. The representation Ω^w of $\{wPw^{-1} \cap P\}$ is trivial on $wPw^{-1} \cap P_1$ and so can be viewed as a representation of H. This representation is trivial on U and when induced to P/P_1 will intertwine with Ω. This of course contradicts cuspidality of Ω. Therefore, $H = P/P_1$. This is equivalent to $wMw^{-1} = M$. □

We turn to the individual cases in (1.11).

$P = P_{\{s_0,s_1\}}$. Here, the group M in (3.7) is $\mathbf{G}(R)$ and $P/P_1 = \mathbf{G}(\mathbb{F}_q)$. Let $Z \subset G$ be the group of scalar matrices. It is an immediate consequence of Theorem 3.1 that $\operatorname{supp} \mathscr{H}(G//P, \Omega) = ZK$. Set $G' = Z$ and $P' = G' \cap P$. For $z \in Z$, let e_z (resp. f_z) denote the function in $\mathscr{H}' = \mathscr{H}(G'//P')$ (resp. $\mathscr{H} = \mathscr{H}(G//P, \Omega)$) whose support is $P'zP'$ (resp. PzP) and whose value at z is 1 (resp. the identity operator). Then, PROPOSITION 3.2. *The map* $\eta\colon \mathscr{H}' \to \mathscr{H}$ *given by*

$$\eta(e_z) = f_z$$

*is a *-isomorphism of algebras.*

REMARKS. (a) The center Z of G is of course not compact. In order to speak of square-integrable and supercuspidal representations of G mod Z, let I be the identity matrix in G, and let T denote the subgroup of G generated by the scalar ϖI, i.e.,

$$T = \{\varpi^n I | n \in \mathbb{Z}\}. \tag{3.8}$$

The group G/T has a compact center and given an irreducible representation (π, V) of G, there is a unique unramified one dimensional character of G such that $\pi \otimes \chi$ factors to a representation of G/T. The representation π is square integrable (resp. supercuspidal) mod Z precisely when $\pi \otimes \chi$ is as a representation of G/T. The formal degree of π is then the formal degree of $\pi \otimes \chi$.

(b) A result such as Proposition 3.2 yields a similar result for the group G/T. This is done by extending Ω trivially to T and integrating functions in $\mathscr{H}(G'//P')$ (resp. $\mathscr{H}(G//P, \Omega)$) over T to obtain functions in $\mathscr{H}(G'//TP')$ (resp. $\mathscr{H}(G//TP, \Omega)$). The map η will integrate over T to yield a *-isomorphism from $\mathscr{H}(G'//TP')$ to $\mathscr{H}(G//TP, \Omega)$. In our specific situation, since $G' = TP'$, the algebra $\mathscr{H}(G'//TP')$ is the complex numbers $\mathbb{C}$. This means that any irreducible admissible representation (π, V) of G which contains Ω on restriction to TP is in fact equivalent to $\mathrm{ind}_{TP\uparrow G}\Omega$. In particular, as a representation of G/T, π is supercuspidal and has formal degree

$$d_\pi = (\deg \Omega)/\mathrm{vol}\,(TP/T). \tag{3.9}$$

The formal degree is of course related to the normalization of Haar measure via vol (TP/T).

$P = P_{\{s_0, s_2\}}$. Recall that

$$P/P_1 = \{(g_1, g_2) \in \mathbf{GL}_2(\mathbb{F}_q) \times \mathbf{GL}_2(\mathbb{F}_q) | \det g_1 = \det g_2\}. \tag{3.10}$$

The element t of (1.9) determines an automorphism of P/P_1; it interchanges g_1 and g_2. Let Z' be the group generated by Z and t. The support of $\mathscr{H} = \mathscr{H}(G//P, \Omega)$ is easily seen to be either ZP or $Z'P$ depending on whether Ω is fixed under the automorphism induced by t. Set $G' = Z$ or Z' accordingly. Let $P' = G' \cap P$. For $z \in Z$, let e_z (resp. f_z) be the function in $\mathscr{H}' = \mathscr{H}(G'//P')$ (resp. $\mathscr{H}$) with value 1 (resp. identity operator) at z and support zJ' (resp. zJ). In case $G' = Z'$, let $\mathfrak{J}$ be an element in End V_Ω such that i) $\mathfrak{J}\Omega(g) = \Omega(tgt^{-1})\mathfrak{J}$ and ii) $\mathfrak{J}^2 = I$, the identity operator. Let e_{zt} (resp. f_{zt}) be the function in $\mathscr{H}'$ (resp. $\mathscr{H}$) with value 1 (resp. operator $\mathfrak{J}$) at zt and support $P'ztP'$ (resp. $PztP$).

Proposition 3.3. *The map* η: $\mathscr{H}' \to \mathscr{H}$ *defined by*

$$\eta(e_z) = f_z \quad \textit{and} \quad \eta(e_{zt}) = f_{zt}$$

*is a *-isomorphism of algebras.*

The remarks after Proposition 3.2 apply here also. Extend Ω to TP by letting Ω act trivially on T. If $G' = Z$, then any irreducible admissible representation (π, V) of G which contains Ω on restriction to TP is equivalent to $\mathrm{ind}_{TP\uparrow G}\Omega$. As a representation of G/T, π is supercuspidal and its formal degree is given by (3.9). If $G' = Z'$, then $\mathrm{ind}_{TP\uparrow G}\Omega$ decomposes into two supercuspidal representations, corresponding to the two extensions of Ω to $Z'P$. The formal degree of these two representations is

$$(\deg \Omega)/\mathrm{vol}\ (Z'P/T). \tag{3.11}$$

Case $P = B = P_\varnothing$. The group M in (3.7) is $\mathbf{A}(R)$. For convenience, we identify P/P_1 and $\mathbf{A}(\mathbb{F}_q)$. Under this identification, a cuspidal representation of P/P_1 is a linear character Ω of the group $\mathbf{A}(\mathbb{F}_q)$. The character Ω can be factored into the product of three characters χ, θ and φ. In the notation of (1.1), χ, θ and φ are characters which depend only on the components a, b and c respectively. The normalizer of $\mathbf{A}(R)$ is the group N of monomial matrices. Up to conjugacy, the stabilizer Stab (Ω) of Ω in N is one of five groups:

$$\begin{array}{rl} \text{(i)} & N \\ \text{(ii)} & A = \mathbf{A}(F) \\ \text{(iii)} & A \cup s_0 A \\ \text{(iv)} & A \cup s_1 A \\ \text{(v)} & A \cup s_1 A \cup s_0 s_1 s_0 A \cup s_0 s_1 s_0 s_1 A. \end{array} \tag{3.12}$$

Let sgn denote the order 2 character of $\mathbb{F}_q^\times$. The relation among the characters χ, θ and φ in the various cases are:

$$\begin{aligned}
&\text{(i) } \chi = \theta/\varphi = 1\\
&\text{(ii) } \Omega \text{ regular}\\
&\text{(iii) } \chi \neq \theta/\varphi = 1\\
&\text{(iv) } \chi = \theta/\varphi \neq \text{sgn or } 1\\
&\text{(v) } \chi = \theta/\varphi = \text{sgn}.
\end{aligned} \tag{3.13}$$

We give a description of the Hecke algebra $\mathcal{H} = \mathcal{H}(G//B, \Omega)$ based on these five cases.

Observe that since Ω is one dimensional, for $g \in \operatorname{supp} \mathcal{H}$, there is a unique function $e_g \in \mathcal{H}$ with support BgB and $e_g(g) = 1$. By the Bruhat decomposition any double coset BgB is of the form BwB for some $w \in W^{\text{aff}}$. Therefore, if $W' = \operatorname{Stab}(\Omega)$, then

$$\operatorname{supp} \mathcal{H} = BW'B.$$

REMARK. We shall see that only cases i) and v) contain discrete series representations of G/T. In each case, G/T has four such representations. These are described in (3.14), (3.16), and (3.58).

Case (i). Up to twisting by a one dimensional character of G, we can assume Ω is the trivial character. The following result of Iwahori and Matsumoto [1] gives a description of $\mathcal{H}$ in terms of generators and relations.

THEOREM 3.4. *The algebra $\mathcal{H}(G//B)$ is generated by*

$$e_s,\ s \in \{s_0, s_1, s_2, t\}.$$

The elements e_w satisfy the relations

$$\begin{aligned}
&\text{(a) } e_w * e_{w'} = e_{ww'} \quad \ell(ww') = \ell(w) + \ell(w')\\
&\text{(b) } e_s * e_s = q + (q-1)e_s \quad s \in \{s_0, s_1, s_2\}\\
&\text{(c) (i) } e_{s_0} * e_{s_1} * e_{s_0} * e_{s_1} = e_{s_1} * e_{s_0} * e_{s_1} * e_{s_0}\\
&\quad\ \text{(ii) } e_{s_1} * e_{s_2} * e_{s_1} * e_{s_2} = e_{s_2} * e_{s_1} * e_{s_2} * e_{s_1}\\
&\text{(d) (i) } e_t * e_t = e_{\varpi I}\\
&\quad\ \text{(ii) } e_t * e_{s_0} = e_{s_2} * e_t\\
&\quad\ \text{(iii) } e_t * e_{s_1} = e_{s_1} * e_t.
\end{aligned}$$

There are four square integrable representations of G/T which contain the trivial representation of B. We describe the representation of $\mathcal{H}$ corresponding to these representations. In the first two cases, the so-called *special* *representations*, the representation π of $\mathscr{H}$ is one dimensional. In terms of the generators given in (1.6) and (1.9)

$$\pi(e_s) = -1 \quad s \in S \quad \text{and} \quad \pi(e_t) = \pm 1. \tag{3.14}$$

The formal degree of the two special representations are both

$$\begin{aligned} d_\pi &= \left\{ \sum_{w \in W^{\text{aff}}/T} q^{-\ell(w)} \right\}^{-1} \text{vol}\,(B)^{-1} \\ &= \{(q^3 - 1)(q - 1)/\{2(q^2 + 1)(q + 1)^2\}\}/\text{vol}\,(B). \end{aligned} \tag{3.15}$$

The remaining two square integrable representations were first constructed by Borel [B]. The representation of $\mathscr{H}$ is two dimensional in each case. Here,

$$\pi(e_{s_0}) = \begin{bmatrix} q & 0 \\ 0 & -1 \end{bmatrix} \quad \pi(e_{s_1}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.16}$$

$$\pi(e_{s_2}) = \begin{bmatrix} -1 & 0 \\ 0 & q \end{bmatrix} \quad \pi(e_t) = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

These two representations have the same formal degree. It is

$$\{(q - 1)^2/\{2q(q + 1)^2\}\}/\text{vol}\,(B). \tag{3.17}$$

Upon restriction to $Sp(4) \subset G$, these representations decompose into two irreducible components.

Case (*ii*) Ω *regular*. To describe the structure of $\mathscr{H}$, which will turn out to be a group algebra, we need a preliminary result. Let

$$\begin{aligned} U_+ &= \prod_{c>0} U_{c,0} \quad \text{and} \quad U_- = \prod_{c<0} U_{c,1}. \\ A^+ &= \{a \in A \mid a^{-1} U_+ a \subset U_+\} \\ A^- &= \{a \in A \mid a^{-1} U_- a \subset U_-\} = \{a \mid a^{-1} \in A^+\}. \end{aligned} \tag{3.18}$$

Note that $B = \mathbf{A}(R)\, U_+ U_-$.

PROPOSITION 3.5. *For $h \in A$, and $a \in A^+$ or A^-,*

$$BhBaB \cap BAB = BhaB.$$

Proof. We recall a sharp form of the Bruhat decomposition (3.3). For $w \in N$ and $c \in \Phi$, let $w(c)$ be the root so that $wU_c w^{-1} = U_{w(c)}$. Let

$$\begin{aligned} C_+ &= \{c \in \Phi \mid w(U_c \cap B)w^{-1} \nsubseteq B\} \\ C_- &= \{c \in \Phi \mid w^{-1}(U_c \cap B)w \nsubseteq B\} \\ C_0 &= \{c \in \Phi \mid w(U_c \cap B)w^{-1} = U_{w(c)} \cap B\} \\ C_0 &= w^{-1}C_0 w. \end{aligned} \tag{3.19}$$

Each element g in BwB can be written uniquely in the form

$$g = u_- dwu_0 u_+ = u_- u_0' wd' u_+, \tag{3.20}$$

where d, $d' \in \mathbf{A}(R)$ and u_-, u_+, u_0 and u_0' respectively are a product of elements in $\{U_c \mid c \in C_-\}$, $\{U_c \mid c \in C_+\}$, $\{U_c \mid c \in C_0\}$ and $\{U_c \mid c \in C_{0'}\}$. Similar decompositions hold for parahoric subgroups $P_{S'}$ and w's in the normalizer of the group $M_{S'}$. We now prove the proposition when $a \in A^+$. Each $g \in BaB$ can be written uniquely as

$$g = u_L a d u_U,$$

with $u_L \in U_-$, $u_U \in U_+$ and $d \in \mathbf{A}(R)$. This means $BhBaB$ is a union

$$BhBaB = \cup\, Bhu_L aB \quad (u_L \in U_-).$$

Suppose $x \in hBa \cap BAB$, say $x = bh'b'$. Write this as

$$hu_L a = u_- u_+ dh'd'u_+' u_-',$$

where u_-, $u_-' \in U_-$; u_+, $u_+' \in U_+$; and d, $d' \in \mathbf{A}(R)$. Thus

$$u_-^{-1} hu_L a u_-'^{-1} = u_+ dh'd'u_+'. \tag{3.21}$$

It is obvious from (3.21) that $BhBaB \cap BAB = BhaB$. □

Every element of A is the product of an element in A^+ and an element in A^-; therefore,

$$f_g * f_{g'} \;=\; \text{multiple of } f_{gg'} \tag{3.22}$$

is an immediate consequence of Proposition 3.5. Our next goal is to establish a quantitative version of (3.22) to obtain the precise multiplicative structure of $\mathscr{H}$. To do this, let

$$\begin{aligned} d_+ &= \varpi^{-1}(E_{1,1} + E_{2,2}) + (E_{3,3} + E_{4,4}) \\ &= t^{-1}s_0s_1s_0 d(-1, -1, 1) \\ d_- &= d_+^{-1} \\ h_+ &= \varpi^{-1}E_{1,1} + (E_{2,2} + E_{3,3}) + \varpi E_{4,4} = s_2s_1s_0s_1 \\ h_- &= h_+^{-1}. \end{aligned} \tag{3.23}$$

PROPOSITION 3.6. *For* $h \in A$

$$f_{d_+} * f_h = q^{\{\ell(d_+)+\ell(h)-\ell(d_+h)\}/2} f_{d_+h} = f_h * f_{d_+}$$

$$f_{h_+} * f_h = q^{\{\ell(h_+)+\ell(h)-\ell(h_+h)\}/2} f_{h_+h} = f_h * f_{h_+}.$$

Proof. We know that $f_{d_+} * f_h$ is equal to a multiple of f_{d_+h}. To determine the multiple, we evaluate $f_{d_+} * f_h$ at d_+h. We find

$$f_{d_+} * f_h(d_+h) = \int_G f_{d_+}(g) f_h(g')\,dg, \quad \text{where} \quad gg' = d_+h.$$

The integrand is nonzero if and only if $g \in Bd_+B$ and $g' \in BhB$. Thus, we need to determine those g (resp. g') in Bd_+B (resp. BhB) such that $gg' = d_+h$. We use (3.20). In our particular case of Bd_+B, each element g can be written uniquely as

$$g = udd_+v,$$

where $d \in \mathbf{A}(R)$, $v \in U_+U_{-a,1}$ and $u \in U_{-b,1}U_{-a-b,1}U_{-2a-b,1}$. Similarly, for h, let C_-, $C_{0'}$ and C_+ be the sets in (3.19). Each element g' of BhB is uniquely of the form

$$g' = u'hd'v',$$

where u' (resp. v') is a product of elements in U_c, $c \in C_- \cup C_{0'}$ (resp. U_c, $c \in C_+$) and $d' \in \mathbf{A}(R)$. The product $gg' = d_+ h$ is thus equivalent to $udd_+ vu'hd'v' = d_+ h$, i.e.,

$$dvu'd' = d_+^{-1}u^{-1}d_+ hv'^{-1}h^{-1}. \tag{3.24}$$

With the obvious notation, write $u = u_{-b}u_{-a-b}u_{-2a-b}$. Then, (3.24) implies $u_c \in U_{c,2}$ when $c \notin C_+$. This is the only condition placed on u. Therefore, $f_{d_+} * f_h(d_+ h) = q^{3-\beta}$, where $\beta = \#\{\{-b, -a-b, -2a-b\} \cap C_+\}$, i.e.,

$$f_{d_+} * f_h = q^{\{\ell(d_+)+\ell(h)-\ell(d_+ h)\}/2} f_{d_+ h}.$$

The other relations are proved by the same type of reasoning. □

Since d_+ and h_+ generate A mod $\mathbf{A}(R)$, we conclude

PROPOSITION 3.7. *For* $w, w' \in A$,

$$f_w * f_{w'} = q^{\{\ell(w)+\ell(w')-\ell(ww')\}/2} f_{ww'}.$$

Let $G' = A$ and $B' = A \cap B$. In this setup, the support of $\mathscr{H}$ is $BG'B$ and the Hecke algebra $\mathscr{H}' = \mathscr{H}(G'//B')$ is the group algebra. Let e_w, $w \in A$, denote the characteristic function of wB'. The next result exhibits an isomorphism between $\mathscr{H}$ and $\mathscr{H}'$ similar to the isomorphisms found in [HM1] and [M1].

COROLLARY 3.8. *The map* η: $\mathscr{H}' \to \mathscr{H}$ *defined by*

$$\eta(e_w) = q^{-\ell(w)/2} f_w$$

*is a *-isomorphism of algebras.*

REMARK. The map η of course effects a transfer between the irreducible representations of G containing Ω' and the irreducible representations, i.e., characters of G'/B'. Furthermore, since η is a *-isomorphism, the transfer of representations will preserve Plancherel Measure (see [M1]). An important consequence of the transfer is the following: since G'/T has no discrete series, there are none of G/T containing Ω'.

Case (*iii*) $\chi \neq \theta/\varphi = 1$. Let

$$
\begin{aligned}
r_0 &= s_0 \\
r_1 &= E_{1,1} - \varpi^{-1}E_{2,3} + \varpi E_{3,2} + E_{4,4} = s_1 s_2 s_1 \\
&h_+, h_- \text{ as in (3.23)} \\
t' &= \varpi E_{1,1} + E_{2,3} + \varpi E_{3,2} - E_{4,4} = s_1 s_0 t.
\end{aligned}
\tag{3.25}
$$

The elements r_0 and r_1 generate an infinite dihedral subgroup $W_r' \subset W'$. Let W_s' be the subgroup of W' generated by h_+ and ϖI. The group W_s' is in the center of W' and

$$W_r' \cap W_s' = \{1\}, \quad W' = W_r'W_s' \cup t'W_r'W_s'. \tag{3.26}$$

In particular, any element w' of W' can be written uniquely in the form

$$w' = dwt'^m, \quad d \in W_r', \quad w \in W_s' \quad \text{and} \quad m \in \{0, 1\}. \tag{3.27}$$

Let ℓ' be the length function on W_r', i.e., for $w \in W_r'$, $\ell'(w)$ is equal to the number of r_0's and r_1's in a reduced word expression for w. It is easy to verify that $\ell'(ww') = \ell'(w) + \ell'(w')$, $w, w' \in W_h'$, iff $\ell(ww') = \ell(w) + \ell(w')$. Consequently

$$f_w * f_{w'} = f_{ww'} \quad \text{when} \quad w, w' \in W_r' \quad \text{and} \quad \ell'(ww') = \ell'(w) + \ell'(w'). \tag{3.28}$$

We prove an analogue of Proposition 3.7.

PROPOSITION 3.9. *For $d \in W_s'$, $w \in W_r'$, f_d and f_w commute. For $g \in \{1, t'\}$*

$$\{q^{-\ell(d)/2}f_d\} * \{q^{-\ell(w)/2}f_w\} * \{q^{-\ell(g)/2}f_g\} = q^{-\ell(dwg)/2}f_{dwg}.$$

Proof. We first establish some special cases of the Proposition. We can assume the element d is a power of h_+ or h_-. If h is either h_+ or h_-, we have $Bh^mB = \{BhB\}^m$; hence

$$f_{h^m} = f_h^m. \tag{3.29}$$

Consider $f_{h_+} * f_{r_0 r_1}$. Since

$$\begin{aligned} Bh_+ Br_0 r_1 B &= Bs_2 s_1 s_0 s_1 Bs_0 s_1 s_2 s_1 B = Bs_2 s_1 s_0 s_1 s_0 Bs_1 s_2 s_1 B \\ &= (Bs_2 s_1 s_0 s_1 s_0 B \cup Bs_2 s_0 s_1 s_0 B) s_2 s_1 B \\ &= Bs_2 s_1 s_0 s_1 s_0 s_2 s_1 B \cup Bs_2 s_0 s_1 s_0 s_2 s_1 B, \end{aligned} \tag{3.30}$$

we have $Bh_+ Br_0 r_1 B \cap \operatorname{supp} \mathscr{H} = Bh_+ r_0 r_1 B$, so $f_{h_+} * f_{r_0 r_1}$ is a multiple of $f_{h_+ r_0 r_1}$. Evaluating $f_{h_+} * f_{r_0 r_1}$ at $h_+ r_0 r_1$ gives

$$f_{h_+} * f_{r_0 r_1} = q f_{h_+ r_0 r_1}. \tag{3.31}$$

Similarly, for $h \in \{h_+, h_-\}$,

$$\begin{aligned} f_h * f_{r_0 r_1} &= q f_{h r_0 r_1} = f_{r_0 r_1} * f_h \\ f_h * f_{r_1 r_0} &= q f_{h r_1 r_0} = f_{r_1 r_0} * f_h, \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} f_h * f_{r_0} &= f_{h r_0} = f_{r_0} * f_h \\ f_h * f_{r_1} &= q f_{h r_1} = f_{r_1} * f_h. \end{aligned} \tag{3.33}$$

We are now ready to prove the proposition in general. Consider the case when $g = 1$. Write d as h^m, with $m \geqslant 0$ and $h \in \{h_+, h_-\}$. Write w in the form

$$\begin{aligned} &\text{(i) } (r_0 r_1)^s k, \quad s \geqslant 0 \quad \text{and} \quad k \in \{1, r_0\} \quad \text{or} \\ &\text{(ii) } (r_1 r_0)^s k, \quad s \geqslant 0 \quad \text{and} \quad k \in \{1, r_1\}. \end{aligned} \tag{3.34}$$

If $m \geqslant 1$ and $s \geqslant 1$, then

$$\begin{aligned} &\text{(i) } Bh^m wB = Bh(r_0 r_1) BBh^{m-1} (r_0 r_1)^{s-1} kB, \quad w \text{ type (3.34i)} \\ &\text{(ii) } Bh^m wB = Bh(r_1 r_0) BBh^{m-1} (r_1 r_0)^{s-1} kB, \quad w \text{ type (3.34ii)}. \end{aligned} \tag{3.35}$$

Thus, under these conditions,

$$\begin{aligned} \text{(i) } f_{h^m w} &= f_{h(r_0 r_1)} * f_{h^{m-1}(r_0 r_1)^{s-1} k} \\ &= q^{-1} f_h * f_{r_0 r_1} * f_{h^{m-1}(r_0 r_1)^{s-1} k} \\ \text{(ii) } f_{h^m w} &= f_{h(r_1 r_0)} * f_{h^{m-1}(r_1 r_0)^{s-1} k} \\ &= q^{-1} f_h * f_{r_1 r_0} * f_{h^{m-1}(r_1 r_0)^{s-1} k}. \end{aligned} \tag{3.36}$$

That f_d and f_w commute and the formula for the convolution now follows by repeated applications of (3.28, 32, 33, 36). The proof in the case $g = t'$ is more tedious but follows the same method. We omit the details. □

Since $Bt'Bt'B = Bh_{-}\varpi IB = Bh_{-}B\varpi IB$, we have

$$f_{t'} * f_{t'} = f_{h_{-}\varpi I} = f_{h_{-}} * f_{\varpi I}. \tag{3.37}$$

The elements $f_{t'}, f_{\varpi I}, f_{r_0}$ and f_{r_1} therefore generate $\mathscr{H}$. In order to determine the exact structure of $\mathscr{H}$, we need to compute the relations among $f_{t'}, f_{\varpi I}$, f_{r_0} and f_{r_1}. We merely state the relations and omit the cumbersome but straightforward computations.

$$\begin{aligned} f_{t'} * f_{t'^{-1}} &= q^2 f_1 \\ f_{r_0} * f_{r_0} &= q f_1 + (q-1) f_{r_0} \\ f_{t'} * f_{r_0} * f_{t'^{-1}} &= q f_{r_1}. \end{aligned} \tag{3.38}$$

Using the notation of (1.7), let G' be the group

$$G' = \mathbf{M}_{\{s_0\}}(F). \tag{3.39}$$

The group W' is the Weyl group of G'. The group $B' = G' \cap B$ is an Iwahori subgroup of G'. Let e_w, $w \in W'$, be the characteristic function of $B'WB'$. Combining (3.28, 29, 37, 38), and Proposition 3.9 together, we get

Proposition 3.10. *The map* η: $\mathscr{H}'(G'//B') \to \mathscr{H}(G//B, \Omega)$ *defined by*

$$\eta(e_w) = \{\text{vol}\,(BwB)/\text{vol}\,(B'wB')\}^{-1/2} f_w, \quad w \in W''$$

*is a *-isomorphism of algebras.*

The remark after Corollary 3.8 applies here: G'/T has no discrete series; hence, there are none of G/T containing Ω'.

Case (*iv*) $\chi = \theta/\varphi \neq$ sgn or 1. The results in this case are parallel to those of case (iii). The proofs of the results in case (iv) follow the same pattern as in case (iii). As such, we merely state the results and omit the proofs. Let

$$r_0 = \varpi^{-1}E_{1,2} + \varpi E_{2,1} + \varpi^{-1}E_{3,4} + \varpi E_{4,3} = s_2 s_0 s_1 s_0 s_2$$

$$r_1 = s_1$$

$$d_+, d_- \text{ as in (3.23)} \tag{3.40}$$

$$t' = E_{1,2} + \varpi E_{2,1} + E_{3,4} + \varpi E_{4,3} = s_2 s_0 t d(1, -1, -1).$$

The group W' is the Weyl group of $G' = \mathbf{M}_{\{s_1\}}(F)$, and $B' = G' \cap B$ is an Iwahori subgroup of G'. Let e_w again denote the characteristic function of $B'wB'$.

Proposition 3.11. *The map η: $\mathscr{H}'(G'//B') \to \mathscr{H}(G//B, \Omega)$ defined by*

$$\eta(e_w) = \{\mathrm{vol}\,(BwB)/\mathrm{vol}\,(B'wB')\}^{-1/2} f_w, \quad w \in W'$$

*is a *-isomorphism of algebras.*

Again, G'/T has no discrete series and therefore, none of G/T containing Ω'.

Case (*v*) $\chi = \theta/\varphi = \mathrm{sgn}$. Let

$$r_0 = \varpi^{-1}E_{1,3} + \varpi^{-1}E_{2,4} + \varpi E_{3,1} + \varpi E_{4,2} = s_2 s_1 s_2 d(1, 1, -1)$$

$$r_1 = t^{-1} r_0 t = s_0 s_1 s_0 d(1, 1, -1)$$

$$r_0' = \varpi^{-1}E_{1,2} + \varpi E_{2,1} + \varpi^{-1}E_{3,4} + \varpi E_{4,3} = s_2 s_0 s_1 s_0 s_2 \tag{3.41}$$

$$r_1' = s_1$$

$$t' = E_{1,2} + \varpi E_{2,1} + E_{3,4} + \varpi E_{4,3} = s_2 s_0 t d(1, -1, -1).$$

Set

$$\begin{aligned} W_r &= \text{subgroup of } W' \text{ generated by } r_0 \text{ and } r_1, \\ W_r^e &= \text{subgroup of } W' \text{ generated by } W_r \text{ and } t \\ W_{r'} &= \text{subgroup of } W' \text{ generated by } r_0' \text{ and } r_1', \\ W_{r'}^e &= \text{subgroup of } W' \text{ generated by } W_{r'} \text{ and } t'. \end{aligned} \tag{3.42}$$

Both W_r and $W_{r'}$ are infinite dihedral groups generated by two generators. The two groups W_r^e and $W_{r'}^e$ centralize one another and

$$W' = W_r^e W_{r'}^e, \quad W_r^e \cap W_{r'}^e = \text{image of } T \text{ in } W^{\text{aff}}. \tag{3.43}$$

In particular, any element of W' can be written in the form ww', with $w \in W$ (resp. $w' \in W'$) and w, w' are unique modulo T. Let ℓ_r (resp. ℓ_r') be the length function in W_r (resp. $W_{r'}$). An easy verification shows

$$\ell_r(ww') = \ell_r(w) + \ell_r(w') \text{ iff } \ell(ww') = \ell(w) + \ell(w'), \quad w, w' \in W_r$$

$$\tag{3.44}$$

$$\ell_r'(ww') = \ell_r'(w) + \ell_r'(w') \text{ iff } \ell(ww'). = \ell(w) + \ell(w'), \quad w, w' \in W_{r'}$$

Hence, if both $w, w' \in W_r$ (resp. $W_{r'}$), then

$$f_w * f_{w'} = f_{ww'} \tag{3.45}$$

provided their ℓ_r (resp. $\ell_{r'}$) lengths add.

Both ℓ_r and ℓ_r' can be extended to W_r^e and $W_{r'}^e$. Indeed, set $\ell_r(wt) = \ell_r(w)$, $w \in W_r$, and set $\ell_r'(w't') = \ell_r'(w')$, $w' \in W_r'$. It is clear that

$$f_w * f_t = f_{wt}, \quad w \in W_r$$

$$f_t * f_t = f_{\varpi I}, \quad f_t * f_{t^{-1}} = f_1,$$

and

$$f_t * f_{r_1} * f_{t^{-1}} = f_{r_0}. \tag{3.46}$$

Since

$$Br_1Br_1B = B \cup Bs_0B \cup Bs_0s_1s_0B \cup Bs_1s_0s_1B \cup Bs_0s_1s_0s_1B,$$

the convolution $f_{r_1} * f_{r_1}$ is linear combination of f_1, f_{r_1} and $f_{r_1r_1'}$. We have

$$f_{r_1} * f_{r_1}(1) = \int_G f_{r_1}(g) f_{r_1}(g^{-1})\, dg = \text{vol}(Br_1B) = q^3. \tag{3.47}$$

To evaluate $f_{r_1} * f_{r_1}(r_1r_1')$, we evaluate

$$f_{r_1} * f_{r_1}(r_1r_1') = \int_G f_{r_1}(g) f_{r_1}(g')\, dg, \quad \text{where } gg' = r_1r_1'.$$

The integrand is nonzero precisely when g, g' both belong to Br_1B. Write g as

$$g = u_b(D)\,u_{a+b}(E)\,u_{2a+b}(C)\,r_1 m, \quad C,\ D,\ E \in R,\ m \in B,$$

and g' as

$$g' = m' r_1 u_b(D')\,u_{a+b}(E')\,u_{2a+b}(C'), \quad C',\ D',\ E' \in R,\ m' \in B.$$

Let $\mathfrak{p}$ denote the prime ideal in R. Then, $gg' = r_1 r_1'$ is equivalent to E, $E' \in \mathfrak{p}$, $CC' = DD' = -1 \bmod \mathfrak{p}$. Thus,

$$f_{r_1} * f_{r_1}(r_1 r_1') = \sum_{C,D \text{ units}} \chi(-C^{-1})\,\theta(D^{-1})\,\varphi(C) = 0. \tag{3.48}$$

In a similar fashion

$$\begin{aligned} f_{r_1} * f_{r_1}(r_1) &= \sum_{E \text{ unit},\, C} \chi(-E^{-1})\,\theta(-E^{-1})\,\varphi(E) \\ &= (q-1)\,q\chi(-1)\,\theta(-1). \end{aligned} \tag{3.49}$$

Set $\kappa = \chi(-1)\,\theta(-1)$, so κ is ± 1. Combining (3.47, 48, 49) we obtain

$$f_{r_1} * f_{r_1} = q^3 f_1 + (q-1)\,q\kappa f_{r_1}. \tag{3.50}$$

In particular, it follows that the vector space of functions

$$\mathscr{A} = \{f \in \mathscr{H} \,|\, \text{support of } f \subset BW_r^e B\} \tag{3.51}$$

is a subalgebra of $\mathscr{H}(G//B, \Omega)$. In order to identify $\mathscr{A}$, consider $G'' = \mathbf{GL}_2(F)$. Let B'' be the Iwahori group of matrices in $\mathbf{GL}_2(R)$ which are upper triangular mod $\mathfrak{p}$. Let A'' be the diagonal matrices in G'' and let

$$r_0'' = \begin{bmatrix} 0 & \varpi^{-1} \\ \varpi & 0 \end{bmatrix}, \quad r_1'' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad t'' = \begin{bmatrix} 0 & 1 \\ \varpi & 0 \end{bmatrix}. \tag{3.52}$$

Let $N'' = A'' \cup t''A''$ be the monomial matrices, and $W^{\text{aff}''} = N/\{N \cap B\}$ the affine Weyl group of G''. Let e_w be the characteristic function of $B''wB''$. Combining (3.45, 46, 50) yields

PROPOSITION 3.12. *The map* η: $\mathcal{H}(G''//B'') \to \mathcal{A}$ *defined by*

$$\eta(e_{r_0''}) = \{\kappa q\}^{-1} f_{r_0}, \quad \eta(e_{r_1''}) = \{\kappa q\}^{-1} f_{r_1} \quad \text{and} \quad \eta(e_{t''}) = f_t$$

*is a *-isomorphism of algebras.*

In a likewise manner, the vector space of functions

$$\mathcal{B} = \{f \in \mathcal{H} \mid \text{support of } f \subset BW_{r'}^e B\} \tag{3.53}$$

is a subalgebra of $\mathcal{H}(G//B, \Omega)$, and

PROPOSITION 3.13. *Let* $\kappa = \theta(-1)$. *The map* η: $\mathcal{H}(G''//B'') \to \mathcal{B}$ *defined by*

$$\eta(e_{r_0''}) = \{\kappa q^2\}^{-1} f_{r_0'}, \quad \eta(e_{r_1''}) = \kappa^{-1} f_{r_1'} \quad \text{and} \quad \eta(e_{t''}) = q^{-1} f_{t'}$$

*is a *-isomorphism of algebras.*

The vector space

$$\mathcal{C} = \{f \in \mathcal{H} \mid \text{support of } f \subset BTA(R)B\}. \tag{3.54}$$

is contained in the center of $\mathcal{H}$ and also in $\mathcal{A}$ and $\mathcal{B}$. The next result identifies $\mathcal{H}$ as the tensor product $\mathcal{H} = \mathcal{A} \otimes_{\mathcal{C}} \mathcal{B}$.

PROPOSITION 3.14. *For* $w \in W_r^e$, $w' \in W_{r'}^e$; f_w *and* $f_{w'}$ *commute and*

$$q^{-\ell(w)/2} f_w * q^{-\ell(w')/2} f_{w'} = q^{-\ell(ww')/2} f_{ww'}.$$

Proof. The method of proof is basically the same as the one used in Proposition 3.9. Let

$$\begin{aligned} &d_+, d_- \text{ as in (3.23)} \\ &g_+ = \varpi^{-1}E_{1,1} + E_{2,2} + \varpi^{-1}E_{3,3} + E_{4,4} \quad \text{and} \quad g_- = g_+^{-1}. \end{aligned} \tag{3.55}$$

Modulo T, an element $w \in W_r^e$ is one of two forms:

$$\text{(i) } \varpi^a I d^m \quad \text{or} \quad \text{(ii) } \varpi^a I d^m t, \quad m \geqslant 0, d \in \{d_+, d_-\} \tag{3.56}$$

Similarly, an element $w' \in W_{r'}^{e}$ is of the form

$$\text{(i) } \varpi^b I g^n \quad \text{or} \quad \text{(ii) } \varpi^b I g^n t', \quad n \geqslant 0, \quad g = \{g_+, g_-\}. \tag{3.57}$$

Consider w of type (3.56i) and w' of type (3.57(i)). If either m or n is zero the result is trivial. If $m \geqslant 1$ and $n \geqslant 1$, then

$$B\varpi^a I d^m \varpi^b I g^n B = \{B\varpi IB\}^{a+b}\{BdgB\}\{Bd^{m-1}g^{n-1}B\},$$

so

$$f_{\varpi^a I d^m} * f_{\varpi^b I g^n} = (f_{\varpi I})^{a+b} * f_{dh} * f_{d^{m-1}g^{n-1}}.$$

Hence, the proposition will follow for w of type (3.56i) and w' of type (3.57i) provided

$$f_d * f_g = q f_{dg}.$$

This last relation is proved by the same method used to show (3.50). The proof of the proposition for the other three cases of w and w' is similar. We omit the details. □

There are four square integrable representations of G/T which contain Ω'. They correspond to the one dimensional representations π of $\mathscr{H}$ given by

$$\begin{aligned} \pi(f_r) &= -1, \quad r \in \{r_0, r_1, r_0', r_1'\} \\ \pi(f_t) &= \pm 1 \quad \text{and} \quad \pi(f_{t'}) = \pm 1. \end{aligned} \tag{3.58}$$

These representations have formal degree

$$\{(q-1)/\{2(q+1)\}\}^2/\mathrm{vol}\,(B). \tag{3.59}$$

Case $P = P_{\{s_0\}}$. Here, in the notation of (1.7), we have

$$P/P_1 = \mathbf{M}_{\{s_0\}}(\mathbb{F}_q) \tag{3.60}$$

$$= \left\{ \begin{bmatrix} a & & \\ & A & \\ & & d \end{bmatrix} \middle| A \in \mathbf{GL}_2(\mathbb{F}_q) \text{ and } \det A = ad \right\}.$$

A cuspidal representation of P/P_1 is a tensor product $\Omega = \xi \otimes \sigma$, where ξ is a character of $\mathbb{F}_q^\times$, i.e., the a component, and σ is a cuspidal representation of $\mathbf{GL}_2(\mathbb{F}_q)$, the A component. Representatives w for the double cosets PwP which can be in the support of $\mathscr{H} = \mathscr{H}(G//P, \Omega)$ have the form

$$\varpi^m\{\varpi^{-r}E_{1,1} + E_{2,2} + E_{3,3} + \varpi^r E_{4,4}\} \quad \text{or} \tag{3.61a}$$

$$\varpi^m\{(-\varpi)^{-r}E_{1,4} + E_{2,2} + E_{3,3} + -(-\varpi)^r E_{4,1}\} \tag{3.61b}$$

with $m, r \in \mathbb{Z}$. We have selected a slightly complicated form for (3.61b) in order that the representatives (3.61a, b) satisfy (3.6). The elements (3.61a) always belong to supp $\mathscr{H}$ for any pair (ξ, σ). On the other hand, elements (3.61b) lie in supp $\mathscr{H}$ precisely when

(i) ξ is the trivial character and σ is any cuspidal representation
(ii) ξ is the character of order 2 and σ is a cuspidal representation such that σ is equivalent to $\sigma \otimes (\xi \circ \det)$. (3.62)

Up to scalar multiple, each $w \in$ supp $\mathscr{H}$ determines a function f_w whose support is precisely PwP. The set of f_w's is a basis for $\mathscr{H}$. If $w, w' \in$ supp $\mathscr{H}$, and $PwPw'P = Pww'P$, then by Proposition 2.2 in [HM1],

$$f_w * f_{w'} = \text{multiple of } f_{ww'}. \tag{3.63}$$

It shall be convenient to pick the f_w's so that (3.63) holds with the scalar equal to 1. For w an element of the form (3.61a), let f_w be the function in $\mathscr{H}$, with support PwP and

$$f_w(w) = \text{identity operator}. \tag{3.64a}$$

To describe the function f_w, when w is of the form (3.61b), we treat separately the cases (3.62i) and (3.62ii).

Case (3.62(i)). Here, let f_w be the function with support PwP and

$$f_w(w) = \text{identity operator}. \tag{3.64b}$$

Case (3.62(ii)). In order to describe f_w, we need to introduce some preliminary notation. Let V_σ be the representation space of σ. The restriction of σ to $\mathbf{SL}_2(\mathbb{F}_q)$ decomposes into two irreducible components of degree $(q-1)/2$ each. The two representations are distinguished by their character values on the two non-identity unipotent conjugacy classes in $\mathbf{SL}_2(\mathbb{F}_q)$. Let

$$u_+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad u_- = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix},$$

and let

$$\gamma_+ = \{-1 + \sqrt{\{(-1)^{(q-1)/2}q\}}\}$$

$$\gamma_- = \{-1 - \sqrt{\{(-1)^{(q-1)/2}q\}}\}.$$

Then

$$V_\sigma = V_+ \oplus V_-$$

$$\sigma = \sigma_+ \oplus \sigma_-$$

where

$$\begin{array}{llll} \text{trace } \sigma_+(u_+) = \gamma_+ & \text{trace } \sigma_+(u_-) = \gamma_- \\ \text{trace } \sigma_-(u_+) = \gamma_- & \text{trace } \sigma_-(u_-) = \gamma_+ . \end{array} \qquad (3.65)$$

Define $\mathfrak{J}$ to be the element in End V_σ which is the identity on V_+ and minus the identity on V_-. Then

$$\mathfrak{J}\sigma(k)\,\xi(\det(k)) = \sigma(k)\,\mathfrak{J} \quad \text{for} \quad k \in \mathbf{GL}_2(\mathbb{F}_q). \qquad (3.66)$$

We have now developed enough notation to describe f_w. For w of the form (3.61b), let f_w be the function in $\mathscr{H}$ with support PwP and

$$f_w(w) = \mathfrak{J}. \qquad (3.64c)$$

The f_w's in (3.64a, b, c) have been defined so that

$$f_w * f_{w'} = f_{ww'} \quad \text{whenever} \quad PwPw'P = Pww'P.$$

We now describe the structure of $\mathscr{H}$ according to whether the elements of W^{aff} in supp $\mathscr{H}$ have the form (3.61a) or (3.61a, b).

Case 1. supp $\mathscr{H} = \{PwP \mid w$ of the form (3.61a)$\}$. Let h_+ and h_- be as in (3.23). Every element w of the form (3.61a) is uniquely a nonnegative power of h_+ or h_- times a scalar matrix $\varpi^m I$. Let W' be the subgroup of N generated by h_+, and ϖI. Then,

$$\text{supp}\,\mathscr{H} \;=\; PW'P.$$

Write w as $\varpi^m h^s$ with m an integer, $h \in \{h_+, h_-\}$ and s a nonnegative integer. Then, since

$$PwP \;=\; P\varpi^m IP\{PhP\}^s,$$

we have, $f_w = (f_{\varpi I})^m * (f_h)^s$. If $w' = \varpi^{m'} h'^{s'}$ and $h' = h$, then $PsPw'P = Pww'P = Pw'PwP$ and so $f_w * f_{w'} = f_{ww'} = f_{w'} * f_w$. Suppose now $h \neq h'$. The next result determines the product between h_+ and h_- and hence between f_w and $f_{w'}$.

PROPOSITION 3.5. $f_{h_+} * f_{h_-} = q^4 f_I = f_{h_-} * f_{h_+}$.

Proof. The support of $f_{h_+} * f_{h_-}$ is shown to be P by the same reasoning as in Proposition 3.5. We conclude that $f_{h_+} * f_{h_-}$ is a multiple of f_I. The multiple is easily seen to be $q^4 = [Pd_+P: P]$. □

Let G' denote the group

$$G' \;=\; \{g \in \mathbf{A}(F) \mid (2,2) \text{ and } (3,3) \text{ entries are equal}\}, \tag{3.67}$$

and let $P' = G' \cap P$. In this setup, supp $\mathscr{H} = PG'P$, and the Hecke algebra $\mathscr{H}(G'//P')$ is the group algebra of G'/P'. Each double coset of P' in G' is represented by an element w of the form (3.61a). Let e_w be the characteristic function of wP', and let r be as in (3.61a). Then,

COROLLARY 3.16. *The map* η: $\mathscr{H}(G'//P') \to \mathscr{H}(G//P, \Omega)$ *defined by*

$$\eta(e_w) \;=\; q^{-2|r|} f_w$$

*is a *-isomorphism of algebras.*

As in the previous cases, η determines a transfer between the characters of G'/P' (resp. $\{G'/T\}/\{TP'/T\}$) and the irreducible representations of G (resp. G/T) which contain Ω'. There are no square integrabl representations of G/T containing Ω'.

Case 2. supp $\mathscr{H} = \{PwP \mid w \text{ of the form (3.61a, b)}\}$. Let

$$r_0 = E_{1,4} + E_{2,2} + E_{3,3} - E_{4,1} = s_1 s_0 s_1 \quad \text{and} \quad r_1 = s_2. \tag{3.68}$$

The subgroup of W^{aff} generated by r_0 and r_1 is an infinite dihedral group. Let

$$W' = \langle r_0, r_1, \varpi I \rangle \tag{3.69}$$

be the subgroup of W^{aff} generated by r_0, r_1 and ϖI. The support of $\mathscr{H}$ is equal to $PW'P$. We need to determine the multiplication between the f_w's. Let ℓ' be the length function in W'.

LEMMA 3.17. For w, w' as in (3.61a, b), the following are equivalent

(i) $\ell(ww') = \ell(w) + \ell(w')$
(ii) $\ell'(ww') = \ell'(w) + \ell'(w')$
(iii) $Pww'P = PwPw'P$.

Proof. The element (3.61a) is equal to $\varpi^m h_+^r$ if $r \geqslant 0$ and $\varpi^m h_-^{-r}$ if $r < 0$, while the element (3.61b) is equal to $\varpi^m h_+^{r-1} r_1$ if $r \geqslant 1$ and $\varpi^m h_-^{-r} r_0$ if $r \leqslant 0$. Lable these four cases type I, II, III and IV respectively. If w is type I or type IV, then each of the three relations (i), (ii) and (iii) holds precisely when w' is type I or type III respectively. When w is type II or type III, each of the three relations holds exactly when w' is respectively type II or type IV. □

Let κ be the scalar $\sigma(-1)$. The next result determines the structure of $\mathscr{H}$.

THEOREM 3.18. *The elements f_w satisfy*

(a) $f_w * f_{w'} = f_{ww'}$ for $\ell'(ww') = \ell'(w) + \ell'(w')$
(b) (i) for Ω as (3.62i)
$$f_{r_1} * f_{r_1} = q f_1 + (q-1) f_{r_1}$$
$$f_{r_0} * f_{r_0} = q^3 f_1 - (q-1) q \kappa f_{r_0}$$
(b) (ii) for Ω as (3.62ii)
$$f_{r_1} * f_{r_1} = q\xi(-1) f_1$$
$$f_{r_0} * f_{r_0} = q^3 \xi(-1) f_1 + (q^2-1)(\gamma - \gamma')\kappa f_{r_0}$$

Proof. Relation (a) is a consequence of Lemma 3.17 and Proposition 2.2 in [HM1]. To show relation (b) we begin by determining the support of $f_{r_1} * f_{r_1}$ and $f_{r_0} * f_{r_0}$. The support of $f_{r_1} * f_{r_1}$ is contained in the set Pr_1Pr_1P. Observe, as in the proof of Proposition 3.4, that

$$Pr_1Pr_1P = Ps_2(B \cup Bs_0B)s_2P = Ps_2P \cup P \cup Ps_2s_0P \cup Ps_0P = P \cup Ps_2P.$$

This means $f_{r_1} * f_{r_1}$ is a linear combination of f_1 and f_{r_1}. Similarly, the support of $f_{r_0} * f_{r_0}$ is contained in the set Pr_0Pr_0P and

$$Pr_0Pr_0P = P \cup Ps_1P \cup Ps_1s_0s_1P.$$

But Ps_1P is not in the support of Ω. The convolution $f_{r_0} * f_{r_0}$ is therefore, a linear combination of f_1 and f_{r_0}. So, $f_r * f_r$, $r \in \{r_0, r_1\}$, is a linear combination of f_1 and f_r. To determine the precise linear combination, we evaluate $f_r * f_r$ at 1 and r. The value of $f_r * f_r$ at 1 is

$$\begin{aligned} f_r * f_r(1) &= \int_G f_r(g) f_r(g^{-1})\,\mathrm{d}g \\ &= \mathrm{vol}\,(PrP) \int_P \int_P f_r(xry) f_r(y^{-1}r^{-1}x^{-1})\,\mathrm{d}y\,\mathrm{d}x \\ &= \mathrm{vol}\,(PrP)\,\xi(-1). \end{aligned}$$

For $f_r * f_r(r)$, we have

$$f_r * f_r(r) = \int_G f_r(g) f_r(g')\,\mathrm{d}g, \quad gg' = r. \tag{3.70}$$

In order to evaluate $f_r * f_r(r)$ we use the sharp form of the Bruhat decomposition (3.20). We treat separately the two cases $r = r_1$ and $r = r_0$. The double coset Pr_1P has the product decomposition

$$\begin{aligned} Pr_1P &= U_{-2a-b,1} H_1 r_1 U_{-2a-b,1} \\ &= U_{-2a-b,1} r_1 H_1 U_{-2a-b,1}, \end{aligned} \tag{3.71}$$

where H_1 is the set

$$H_1 = M_{\{s_0\}} U_{a,0} U_{a+b,0} U_{-a,1} U_{-a-b,1}.$$

For r_0, we set

$$H_0 = M_{\{s_0\}}.$$

Then we have

$$\begin{aligned} Pr_0P &= U_{2a+b,0} U_{a+b,0} U_{a,0} H_0 r_0 U_{2a+b,0} U_{a+b,0} U_{a,0} \\ &= U_{2a+b,0} U_{a+b,0} U_{a,0} r_0 H_0 U_{2a+b,0} U_{a+b,0} U_{a,0}. \end{aligned} \tag{3.72}$$

We use (3.71) and (3.72) to evaluate (3.70).

Case r_1. In (3.70), we write g and g' as

$$g = u_{-2a-b}(\varpi x) r_1 h_1 u_{-2a-b}(\varpi y)$$

$$g' = u_{-2a-b}(\varpi y') h_1' r_1 u_{-2a-b}(\varpi z).$$

The product gg' is equal to r_1 precisely when z is a unit, say D, $x = D^{-1}$ and

$$h_1 u_{-2a-b}(\varpi(y + y')) h_1' = d(D^{-1}, 1, 1) u_{-2a-b}(\varpi D^{-1})$$

Hence,

$$f_{r_1} * f_{r_1}(r_1) = \text{vol}\,(Pr_1P) q^{-1} \sum_D f_{r_1}(r_1) \xi(D) f_{r_1}(r_1)$$

$$= \begin{cases} (q - 1) I & \text{if } \Omega \text{ is of type (3.62i)} \\ 0 & \text{if } \Omega \text{ is of type (3.62ii)} \end{cases}$$

Case r_0. Here, we write g and g' as

$$g = \begin{bmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{bmatrix} r_0 m_0 h_0$$

$$g' = h_0' m_0' r_0 \begin{bmatrix} 1 & X & Y & Z \\ 0 & 1 & 0 & Y \\ 0 & 0 & 1 & -X \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The product gg' is equal to r_0 precisely when z is a unit D, $Z = D^{-1}$, $x = XD$, $y = YD$, and

$$m_0 h_0 h_0' m_0' = \begin{bmatrix} D^{-1} & 0 & 0 & 0 \\ 0 & 1 - XYD & -Y^2D & 0 \\ 0 & X^2D & 1 + XYD & 0 \\ 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} 1 & XD & YD & -D \\ 0 & 1 & 0 & YD \\ 0 & 0 & 1 & -XD \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$u = \begin{bmatrix} 1 - XYD & -Y^2D \\ X^2D & 1 + XYD \end{bmatrix} \text{(a unipotent element).}$$

Then,

$$\begin{aligned} f_{r_0} * f_{r_0}(r_0) &= \operatorname{vol}(Pr_0P)q^{-3}\sum{}' f_{r_1}(r_1)\xi(D^{-1})\sigma(u)f_{r_1}(r_1) \\ &= \operatorname{vol}(Pr_0P)q^{-3}\sum{}' \xi(D^{-1})\sigma(u) \end{aligned} \tag{3.73}$$

The sum is over X, $Y \in R \bmod \wp$ and D a unit mod $\wp$. Let χ_σ denote the character of σ. If Ω is of the form (3.62(i)), then taking the trace of both sides of (3.73) gives

$$\operatorname{trace} f_{r_0} * f_{r_0}(r_0) = \sum{}' \chi_\sigma(u)$$

The character value $\chi_\sigma(u)$ is $(q - 1)\kappa$ when X and Y are zero and $-\kappa$ otherwise. Hence, trace $(f_{r_0} * f_{r_0}(r_0)) = -\kappa(q - 1)^2 q$, and so

$$f_{r_0} * f_{r_0} = q^3 f_1 - (q - 1)q\kappa f_{r_0}.$$

If Ω is of the form (3.62(ii)), we can trace the two sides of (3.73) and use the explicit character values (3.65) to obtain

$$f_{r_0} * f_{r_0} = q^3\xi(-1)f_1 + (q^2 - 1)(\gamma - \gamma')\kappa f_{r_0}.$$

A consequence of Theorem 3.18 is that for Ω of type (3.62i), there is a Hecke algebra isomorphism of $\mathscr{H} = \mathscr{H}(G//P, \Omega)$ in analogy with Propositions 3.2, 3.3, 3.10, 3.11 and Corollary 3.8. Let G' denote the group

$$G' = \left\{ \begin{bmatrix} a & 0 & 0 & b \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ c & 0 & 0 & d \end{bmatrix} \in G \right\}, \tag{3.74}$$

$$B' = G' \cap P = \text{an Iwahori subgroup of } G'.$$

The group W' of (3.69) is the affine Weyl group of G'. Let e_w, $w \in W'$, be the characteristic function of $B'wB'$.

COROLLARY 3.19. *For Ω of type (3.62i), the map η: $\mathscr{H}(G'//B') \to \mathscr{H}(G//P, \Omega)$ defined on generators by*

$$\eta(e_{\varpi I}) = f_{\varpi I}, \quad \eta(e_{r_0}) = (-q\kappa)^{-1} f_{r_0}, \quad \eta(e_{r_1}) = f_{r_1}$$

*is a *-isomorphism of algebras.*

It is now easy to describe the square integrable representations of G/T which contain Ω'. There is one such representation τ. It corresponds to the special representation τ_{St} of G'. The Hecke algebra representation is one dimensional and given by

$$\tau(e_{\varpi I}) = 1, \quad \tau(e_{r_0}) = -1 \quad \text{and} \quad \tau(e_{r_1}) = -1. \tag{3.75}$$

The formal degree d_τ of τ is

$$d_\tau = \{\deg(\Omega)/\mathrm{vol}(P)\}\{d_{St}\,\mathrm{vol}(B')\} = (q-1)^2/\{(q+1)\,\mathrm{vol}(P)\}. \tag{3.76}$$

If Ω is type (3.62(ii)), there are two discrete series representations of G/T containing Ω'. The corresponding representations of $\mathscr{H}$ are

$$\tau(f_{\varpi I}) = 1, \quad \tau(f_{r_0}) = -(\gamma - \gamma')\kappa \quad \text{and} \quad \tau(f_{r_1}) = \pm(\gamma - \gamma'). \tag{3.77}$$

The two representations have formal degree

$$d_\tau = \{(q-1)(q^2-1)\}/\{2(q^2+1)\,\mathrm{vol}(P)\}. \tag{3.78}$$

Case $P = P_{\{s_1\}}$. This case is very similar to the previous case $P = P_{\{s_0\}}$. We merely state the results and outline the proofs. Let

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.79}$$

For $A \in \mathbf{GL}_2(\mathbb{F}_q)$, let A' be the transpose of A. In the notation of (1.7), we have

$$\begin{aligned} P/P_1 &= \mathbf{M}_{\{s_1\}}(\mathbb{F}_q) \\ &= \left\{ \begin{bmatrix} A & \\ & B \end{bmatrix} \,\middle|\, A \in \mathbf{GL}_2(\mathbb{F}_q), \quad \lambda \in \mathbb{F}_q^\times \quad \text{and} \quad B = \lambda J^{-1}(A')^{-1}J \right\} \end{aligned} \tag{3.80}$$

A cuspidal representation of P/P_1 is a tensor product $\Omega = \xi \otimes \sigma$ where ξ is a character $\mathbb{F}_q^\times$ and σ is a cuspidal representation of $\mathbf{GL}_2(\mathbb{F}_q)$. The elements of W^{aff} which can belong to the support of $\mathscr{H} = \mathscr{H}(G//P, \Omega)$ have the form

$$\varpi^m\{\varpi^r E_{1,1} + \varpi^r E_{2,2} + E_{3,3} + E_{4,4}\} \quad \text{or} \tag{3.81a}$$

$$\varpi^m\{E_{1,3} + E_{2,4} + \varpi^r E_{3,1} + \varpi^r E_{4,2}\} \tag{3.61b}$$

$m, r \in \mathbb{Z}$. Let θ denote the automorphism of $\mathbf{GL}_2(\mathbb{F}_q)$ of inverse transpose. Elements of type (3.81a) are always in supp $\mathscr{H}$, while those of type (3.81b) belong to supp $\mathscr{H}$ precisely when

$$\sigma \text{ is a cuspidal representation of } \mathbf{GL}_2(\mathbb{F}_q) \text{ equivalent to } \sigma \circ \theta. \tag{3.82}$$

Let V_σ be the representation space of σ and let

$$\mathfrak{J} = \sum \sigma(X)$$

be the sum over those 2×2 invertible matrices X whose (1, 1) and (2, 2) entries are equal. We have

$$\mathfrak{J}^2 = q^2(q-1)^2 I$$

and

$$\mathfrak{J}\sigma(J\theta(k)J) = \sigma(k)\mathfrak{J} \quad \text{for} \quad k \in \mathbf{GL}_2(\mathbb{F}_q).$$

We now describe the structure of $\mathscr{H}$ according to whether the elements of W^{aff} in supp $\mathscr{H}$ have the form (3.81a) or (3.81a, b).

Case 1. supp $\mathscr{H} = \{PwP \mid w$ of the form (3.81a)$\}$. Set

$$d_+, d_- \text{ as in (3.23)} \tag{3.83}$$

Let W' be the subgroup of W^{aff} generated by d_+ and ϖI, and let G' denote the group

$$G' = \{g \in \mathbf{A}(F) \mid (1, 1) \text{ and } (2, 2) \text{ entries are equal}\}. \tag{3.84}$$

Then,

$$\text{supp } \mathscr{H} = PW'P = PG'P.$$

Let $P' = G' \cap P$. The Hecke algebra $\mathscr{H}' = \mathscr{H}(G'//P')$ is the group algebra of G'/P'. Each double coset of P' in G' is represented by an element w of the form (3.81a). Let f_w (resp. e_w) be the element in $\mathscr{H}$ (resp. $\mathscr{H}'$) with support PwP (resp. wP') and value the identity operator (resp. one) at w. Let r be as in (3.81a). Then, in complete analogy with Proposition 3.15 and Corollary 3.16, we have

PROPOSITION 3.20. $f_{d_+} * f_{d_-} = q^3 f_I = f_{d_-} * f_{d_+}$ and

COROLLARY 3.21. *The map* η: $\mathscr{H}(G'//P') \to \mathscr{H}(G//P, \Omega)$ *defined by*

$$\eta(e_w) = q^{-3|r|/2} f_w$$

*is a *-isomorphism of algebras.*

There are no discrete series representations of G containing Ω'.

Case 2. supp $\mathscr{H} = \{PwP \mid w$ of the form (3.81a, b)$\}$. Take

$$r_0,\ r_1 \text{ and } t \text{ as in (3.41).} \tag{3.85}$$

Let W' be the subgroup of W^{aff} generated by r_0, r_1 and t. The element t conjugates r_0 to r_1. The subgroup W_d of W' generated by r_0 and r_1 is again an infinite dihedral group. The support of $\mathscr{H}$ is equal to $PW'P$. As in case 1, we exhibit a basis for $\mathscr{H}$. Let f_w, $w \in W'$, be the function in $\mathscr{H}$ with support PwP and

$$f_w(w) = \begin{cases} \text{identity operator, } w \text{ type (3.81a)} \\ \{q(q-1)\}^{-1}\mathfrak{J},\ w \text{ type (3.81b).} \end{cases}$$

The f_w's are a basis for $\mathscr{H}$. To determine the multiplication between the f_w's, let ℓ' be the length function in W'. Lemma 3.17 is valid in our present context. In analogy with Theorem 3.18, we have

THEOREM 3.22. *The elements* f_w *satisfy*

(a) $f_w * f_{w'} = f_{ww'}$ *when* $\ell'(ww') = \ell'(w) + \ell'(w')$

(b) $f_{r_1} * f_{r_1} = q^3 f_1 + (q-1)qf_{r_1}$

$f_t * f_{r_1} * f_{t^{-1}} = f_{r_0}$

There are two discrete series representations of G/T containing σ'. The corresponding representations π of $\mathscr{H}$ are both one dimensional. They are given by

$$\pi(f_{r_1}) = -q, \quad \text{and} \quad \pi(f_t) = \pm 1. \tag{3.86}$$

The formal degree of these two representations are

$$d_\pi = \{(q-1)/(2(q+1))\} \operatorname{vol}(P)^{-1}. \tag{3.87}$$

This completes our analysis of the representations of G which contain a level one nondegenerate representation. In Part II we give a classification of those representations of G which contain a nondegenerate representation of higher level.

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