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Roots of multiplicative functions

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Abstract. An arithmetic function is called *linear* if it is completely multiplicative, and *quadratic* if it is the Dirichlet convolution of two linear functions. If g is an arithmetic function then the n-th root of g, denoted $g^{1/n}$, is a function f such that $g = f \cdots f$ (n factors), where "$\cdot$" denotes Dirichlet convolution. In this paper, explicit formulas are given for the n-th roots of linear and quadratic functions, and for the inverses of these roots.

If m and n are positive integers, then $g^{m/n}$ is defined to be the m-th power (i.e. m factors in the Dirichlet convolution) of $g^{1/n}$. Formulas are given for these "rational roots" of linear functions.

1. Introduction

An arithmetic function is any mapping f from the positive integers into a subset of the complex numbers. An arithmetic function f which is not identically zero is called multiplicative if

$$f(mn) = f(m)f(n) \quad \text{whenever} \quad (m, n) = 1,$$

and it is called linear if

$$f(mn) = f(m)f(n) \quad \text{for all} \quad m \text{ and } n.$$

The Dirichlet convolution, $f*g$, of two arithmetic functions f and g is defined by the relation

$$f*g(n) = \sum_{d|n} f(d)g(n/d).$$

It is well-known that the set of all multiplicative functions, under the operation $*$, is an abelian group: $E(n) = [1/n]$ is the identity element and f^{-1}

defined by

$$f^{-1}(1) = 1/f(1)$$

$$f^{-1}(n) = \frac{-1}{f(1)} \sum_{\substack{d|n \\ d>1}} f(d)f^{-1}(n/d), \quad n > 1$$

is the inverse of f.

While the set of all linear functions is not a group, note that f is linear iff

$$f^{-1}(p^a) = 0$$

for all primes p and integers $a > 1$; and hence a linear function is completely determined by its values at $f(p)$ for each prime p (see [1]). This statement generalizes to: f is the convolution of n linear functions iff

$$f^{-1}(p^a) = 0$$

for all primes p and all integers $a > n$; and hence such a function is completely determined by $f(p), f(p^2), \ldots, f(p^n)$ for each prime p (see [2]).

By the notation f^r, r a positive integer, we mean r copies of f under Dirichlet convolution. rearick [3] showed that if f is any multiplicative function and r is any integer, then there exists a unique multiplicative function g such that $g^r = f$; in fact he proves this result for r a real number and g defined in terms of his exponential and logarithmic operators. For r a positive integer and $g^r = f$, g is called the rth root of f. Kemp [4] considered roots of arithmetic functions of more than one variable.

In this paper we are interested in determining explicitly the roots of linear and quadratic functions (a quadratic function is the Dirichlet convolution of two linear functions) in terms of the functions themselves.

2. Square roots

By the square root of a multiplicative function f we mean the unique multiplicative function that is the solution to the functional equation $g^2 = f$. We will write $g = f^{1/2} = \sqrt{f}$.

THEOREM 2.1. *If f is linear, then for $F = f^{1/2}$*

$$F(p^n) = \frac{1}{2^{2n}} \binom{2n}{n} f(p^n), \tag{1}$$

for p prime and n a non-negative integer.

Proof. By induction on n.

$$F(1) = 1 = \frac{1}{2^0}\binom{0}{0} f(p^0).$$

Assume

$$F(p^k) = \frac{1}{2^{2k}}\binom{2k}{k} f(p^k)$$

for p a prime and $0 \leqslant k < n$.

$$f(p^n) = (f(p))^n = F*F(p^n) = \sum_{i=0}^{n} F(p^i)F(p^{n-i})$$

$$= 2F(p^n) + \sum_{i=1}^{n-1} \frac{1}{2^{2i}}\binom{2i}{i} f(p^i) \frac{1}{2^{2(n-i)}}\binom{2(n-i)}{n-i} f(p^{n-i})$$

$$= 2F(p^n) + \frac{(f(p))^n}{2^{2n}} \sum_{i=1}^{n-1} \binom{2i}{i}\binom{2(n-i)}{n-i}$$

$$= 2F(p^n) + \frac{(f(p))^n}{2^{2n}} \sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} - \frac{2(f(p))^n}{2^{2n}}\binom{2n}{n}.$$

Since

$$\sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} = 2^{2n},$$

simplification yields equation (1).

For $N = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$, p_i primes and a_i positive integers, let $\varrho(N) = a_1 + a_2 + \cdots + a_r$ and

$$B(N) = \prod_{i=1}^{N} \binom{2a_i}{a_i}.$$

COROLLARY 2.2. *For f a linear function, F the square root of f, and N a positive integer,*

$$F(N) = \frac{B(N)f(N)}{2^{2\varrho(N)}}.$$

COROLLARY 2.3. *For n a positive integer,*

$$2^{2\varrho(n)} = \sum_{d|n} B(d)B\left(\frac{n}{d}\right) = B*B(n). \tag{2}$$

Proof. Let f be a linear function.

$$f(n) = f^{1/2}*f^{1/2}(n) = \sum_{d|n} (f^{1/2}(d)f^{1/2}(n/d).$$

So by Corollary 2.2,

$$f(n) = \sum_{d|n} \frac{B(d)f(d)B(n/d)f(n/d)}{2^{2\varrho(d)}2^{2\varrho(n/d)}}.$$

Now, f is linear and $\varrho(d) + \varrho(n/d) = \varrho(n)$, so,

$$f(n) = \frac{f(n)}{2^{2\varrho(n)}} \sum_{d|N} B(d)B(n/d),$$

and (2) follows immediately.

Theorem 2.1 generalizes to

THEOREM 2.4. *If f is a quadratic function (say $f = g*h$, g and h linear) and F is the square root of f, then for p a prime and n a positive integer:*

$$F(p^n) = \frac{1}{2^{2n}} \sum_{j=0}^{n} \binom{2j}{j} \binom{2(n-j)}{n-j} g(p^j)h(p^{n-j}). \tag{3}$$

Proof.

$$F*F = f = g*h = (g^{1/2}*g^{1/2})*(h^{1/2}*h^{1/2}) = (g^{1/2}*h^{1/2})*(g^{1/2}*h^{1/2}).$$

Rearick's result about the uniqueness of roots implies $F = g^{1/2} * h^{1/2}$. Hence, by Theorem 2.1,

$$F(p^n) = \sum_{j=0}^{n} \frac{1}{2^{2j}} \binom{2j}{j} g(p^j) \frac{1}{2^{2(n-j)}} \binom{2(n-j)}{n-j} h(p^{n-j})$$

$$= \frac{1}{2^{2n}} \sum_{j=0}^{n} \binom{2j}{j} \binom{2(n-j)}{n-j} g(p^j) h(p^{n-j}).$$

Using the notation $\iota_s(n) = n^s$, $d(n) = \iota_0 * \iota_0(n)$ (the number of divisors of n) and $\sigma(n) = \iota_1 * \iota_0(n)$ (the sum of divisors of n) are quadratic functions. It is immediate that $d^{1/2}(n) = 1$ for all n and

$$\sigma^{1/2}(p^k) = \frac{1}{2^{2k}} \sum_{j=0}^{k} \binom{2j}{j} \binom{2(k-j)}{k-j} p^j.$$

THEOREM 2.5. *For f a linear function, p a prime and n a non-negative integer,*

$$f^{-1/2}(p^n) = \frac{-1}{2^{2n}(2n-1)} \binom{2n}{n} f(p^n). \tag{4}$$

Proof. For $n = 0, f^{-1/2}(1) = 1$.

Assume equation (4) holds for $0 \leqslant k < n$.

$$0 = f^{-1/2} * f^{1/2}(p^n) = \sum_{k=0}^{n} f^{-1/2}(p^k) f^{1/2}(p^{n-k}).$$

So,

$$f^{-1/2}(p^n) = -\sum_{k=0}^{n-1} f^{-1/2}(p^k) f^{1/2}(p^{n-k})$$

$$= -\sum_{k=0}^{n-1} \frac{1}{2^{2k}} \left(\frac{-1}{2k-1}\right) \binom{2k}{k} \frac{1}{2^{2(n-k)}} \binom{2(n-k)}{n-k} f(p^n)$$

$$= \frac{1}{2^{2n}} \sum_{k=0}^{n-1} \frac{1}{(2k-1)} \binom{2k}{k} \binom{2(n-k)}{n-k} f(p^n)$$

$$= \frac{f(p^n)}{2^{2n}} \sum_{k=0}^{n} \frac{1}{(2k-1)} \binom{2k}{k} \binom{2(n-k)}{n-k}$$

$$+ \frac{(-1)}{2^{2n}} \frac{1}{(2n-1)} \binom{2n}{n} f(p^n)$$

$$= \frac{-1}{2^{2n}(2n-1)} \binom{2n}{n} f(p^n),$$

since

$$\sum_{k=0}^{n} \frac{1}{(2k-1)} \binom{2k}{k} \binom{2(n-k)}{n-k} = 0 \quad \text{for } n \geqslant 1.$$

The analogous result for quadratic functions is:

THEOREM 2.6. *If $f = g * h$ (g and h linear) and $F = f^{1/2}$, then for p a prime and n a positive integer,*

$$F^{-1}(p^n) = \frac{1}{2^{2n}} \sum_{j=0}^{n} \frac{1}{(2j-1)(2(n-j)-1)} \times \binom{2j}{j} \binom{2(n-j)}{n-j} g(p^j)h(p^{n-j}). \tag{5}$$

Functions that are the Dirichlet convolution of a linear function and the inverse of a linear function are called totient functions. Euler's ϕ function is the most famous totient. Combining Theorems 2.1 and 2.6 we can easily prove:

THEOREM 2.7. *If f is a totient, say $f = g * h^{-1}$ (g and h linear), then for p a prime and n a positive integer, the square root of f is given by:*

$$F(p^n) = \frac{-1}{2^{2n}} \sum_{j=0}^{n} \frac{1}{2(n-j)-1} \binom{2j}{j} \binom{2(n-j)}{n-j} g(p^j)h(p^{n-j}). \tag{6}$$

Since $\phi(p^n) = \iota_1 * \iota_0^{-1}(p^n) = (p-1)p^{n-1}$,

$$\phi^{1/2}(p^n) = \frac{-1}{2^{2n}} \sum_{j=0}^{n} \frac{1}{(2(n-j)-1)} \binom{2j}{j} \binom{2(n-j)}{n-j} p^j. \tag{7}$$

Examine (7) to note that

$$\phi^{1/2}(p^n) = \frac{(p-1)}{2^{2n}} P_n(p)$$

where $P_n(x)$ is a polynomial.

3. Rational roots of linear functions

In this section the results of Section 2 are generalized to rational roots of linear functions.

THEOREM 3.1. *If f is linear and $F = f^{1/t}$ for a positive integer t, then for p a prime and n a positive integer,*

$$F(p^n) = \frac{(t+1)(2t+1)\ldots((n-1)t+1)}{t^n n!} f(p^n). \tag{8}$$

Proof. Note that $F(1) = 1 = f(1)$, and $F(p) = (1/t)f(p)$. Now by induction, it is easy to see that $F(p^n) = K(t;n)f(p^n)$, where $K(t;n)$ is a rational number independent of p. Defining $K(t;0) = 1$, we need to show $K(t;n)$ is the coefficient of $f(p^n)$ in equation (8).

By the definition of Dirichlet convolution,

$$\begin{aligned} f(p^n) &= \sum F(p^{i_1})F(p^{i_2}) \cdots F(p^{i_t}) \\ &= \sum K(t;i_1)f(p^{i_1}) \cdots K(t;i_t)f(p^{i_t}), \end{aligned}$$

where the summation is over all choices of non-negative $i_1, i_2, \ldots, i_t$ such that $i_1 + i_2 + \cdots + i_t = n$. Moreover, since f is linear, we must have

$$\sum_{\substack{i_x \geq 0 \\ i_1 + \cdots + i_t = n}} K(t;i_1) \cdots K(t;i_t) = 1. \tag{9}$$

As noted at the beginning of this proof, $K(t;\ 1) = 1/t$. This also follows directly from (8) with $n = 1$.

Now consider the McLaurin series:

$$(1 - X)^{-1/t} = \sum_{n=0}^{\infty} A_n X^n, \tag{10}$$

where

$$A_0 = 1,\ A_1 = \frac{1}{t}, \ldots, A_n = \frac{(1/t + n - 1)}{n} A_{n-1}.$$

Writing A_n explicitly, we obtain

$$A_n = \frac{(t + 1) \cdots ((n - 1)t + 1)}{t^n \cdot n!}.$$

Raising both sides of (10) to the t-th power,

$$(1 - X)^{-1} = \sum_{n=0}^{\infty} C_n X^n$$

where

$$C_n = \sum A_{i_1} A_{i_2} \cdots A_{i_t},$$

where the summation is over all choices of $i_1, \ldots, i_t$ such that $i_1 + \cdots + i_t = n$.

Now the unique McLaurin series for $(1 - X)^{-1}$ is $\Sigma_{n=0}^{\infty} X^n$, i.e., $C_n = 1$ for all $n \geqslant 0$. Thus, $1 = \Sigma A_{i_1} A_{i_2} \cdots A_{i_t}$ where $A_0 = 1$ and $A_1 = 1/t$. This is the same recursive relation as (9) with the same initial conditions and hence (8) is proven.

We now wish to determine $f^{s/t}(p^n)$ in terms of $f(p^n)$ where f is linear. As a special case of the following ($f^s(n) = n^s$) see Beumer [5].

THEOREM 3.2. *If f is a linear function and s is a positive integer, then*

$$f^s(p^n) = \binom{n + s - 1}{s - 1} f(p^n). \tag{11}$$

Proof. By induction on s. Immediate for $s = 1$. Assume (11) holds for $1 \leqslant k < s$.

$$\begin{aligned} f^s(p^n) &= \sum_{i=0}^{n} f^{s-1}(p^i) f(p^{n-i}) \\ &= \sum_{i=0}^{n} \binom{i + s - 2}{s - 2} f(p^i) f(p^{n-i}) \\ &= f(p^n) \sum_{i=0}^{n} \binom{i + s - 2}{s - 2} \\ &= \binom{n + s - 1}{s - 1} f(p^n). \end{aligned}$$

Noting that

$$\binom{n + s - 1}{s - 1} = \frac{\prod_{j=1}^{s-1} (n + j)}{(s - 1)!}$$

leads us to

THEOREM 3.3. *If f is linear, s, t, n are positive integers and p is a prime, then*

$$f^{s/t}(p^n) = \frac{\prod_{j=0}^{n-1} (jt + s)}{n!t^n} f(p^n). \tag{12}$$

Proof. Since $f^s(p^n) = \binom{n+s-1}{s-1} f(p^n)$, we have

$$F(p^n) = K(t, s; n) f(p^n).$$

Again as in Theorem 3.1, examine

$$(1 - X)^{-s/t} = \sum_{n=0}^{\infty} A_n X^n$$

with

$$A_0 = 1, A_1 = \frac{s}{t}, \ldots, A_n = \frac{s(s+t)\cdots(s+(n-1)t)}{n!t^n}.$$

Raising each side to the t-th power, we obtain

$$(1 - X)^{-s} = \sum_{n=0}^{\infty} C_n X^n;$$

$$C_n = \sum_{\substack{\Sigma i_j = n \\ i_j \geqslant 0}} A_{i_1} \cdots A_{i_t}.$$

Now the McLaurin series expansion of

$$(1 - X)^{-s} \text{ is } \sum_{n=0}^{\infty} \binom{s - 1 + n}{n} X^m$$

and hence

$$C_n = \binom{s - 1 + n}{n} = \sum A_{i_1} \ldots A_{i_t}$$

where $A_j = 1 = K(t, s; 0)$ and $A_1 = s/t = K(t, s; 1)$ and hence $A_n = K(t, s; n)$ for all positive n.

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