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## Ergodic $Z_2$ -extensions over rational pure point spectrum, category and homomorphisms

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**Abstract.** Let  $\mathscr{W}$  be the space of all  $Z_2$ -extensions over a fixed automorphism with a rational pure point spectrum, endowed with the uniform topology. We prove that a set of “typical” points of  $\mathscr{W}$  coincides with the class of those  $Z_2$ -extensions which are measure-theoretically isomorphic to Morse sequences. The factor problem is studied.

### Introduction

Skew products were introduced to ergodic theory by Anzai [2] in connection with the problem of isomorphism. It was shown in [1] that any ergodic automorphism can be represented as a skew product of one of its factor-automorphisms with a family of automorphisms.

However, up to now, a great deal of the attention has been devoted to the study of some simpler forms of skew products, namely to the so called  $G$ -extensions over automorphisms with pure point spectra (for instance [2], [10], [11], [19], [21]).

In the present paper we deal with  $Z_2$ -extensions over automorphisms with rational pure point spectra. Recall some well-known examples of them: generalized Morse sequences [12], continuous substitutions on two symbols [4], Mathew-Nadkarni’s examples [18] with partly continuous Lebesgue spectrum of multiplicity two, Helson-Parry’s constructions [8].

Morse sequences are very useful for constructing counterexamples in ergodic theory (for further details see [16]). We hope that our present paper emphasizes once more the peculiar role of the class of Morse dynamic systems in ergodic theory. Our main result here seems to confirm that the combinatorial approach to ergodic theory (in Jacob’s sense [9]) is of a rather general nature. In 1981 J. Kwiatkowski posed the question whether any “typical”  $Z_2$ -extension over a rational pure point spectrum is a Morse sequence. The first part of the paper positively answers this question. We prove that in the space of  $Z_2$ -extensions of a fixed ergodic automorphism with rational pure point spectrum, the set of extensions isomorphic to

Morse sequences is of the second category (in the weak topology). To prove this we use Katok-Stepin's theory of odd approximation of  $Z_2$ -extensions [11]. But what seems most important is that by analyzing the proof one can observe that in fact we show more: if an ergodic  $Z_2$ -extension over  $T$  is oddly approximated with speed  $o(1/n^2)$  then it is measure-theoretically isomorphic to a Morse sequence. So in this case, Katok-Stepin's theory has quite a combinatorial nature.

In the second part of the paper we treat the factor problem. It is not hard to see that the class of ergodic  $Z_2$ -extensions of automorphisms with pure point spectra is closed under taking factors. It turns out that if  $Sp(T_\theta)$  is the group of all  $p^k$ -roots of unity,  $k \geq 1$ , and  $p$  is a prime number, then the only factors of  $T_\theta$  are those with discrete spectrum. The main result of the sections states that if  $x = b^0 \times b^1 \times \dots$  is a regular Morse sequence and the set of blocks  $\{b^0, b^1, \dots\}$  is finite then the only proper factors of  $x$  are those with discrete spectrum. In particular all continuous substitutions on two symbols enjoy this property.

## I. Definitions and remarks

Let  $(X, \mathcal{B}, \mu)$  be a Lebesgue space and  $T: (X, \mathcal{B}, \mu) \rightarrow (X, \mathcal{B}, \mu)$  be an automorphism. By  $Sp(T)$  we denote the group of all eigenvalues of  $T$ , i.e.  $\lambda \in Sp(T)$  iff  $fT = \lambda f$  for some  $0 \neq f \in L^2(X, \mu)$ . We recall here the result of [3] stating that  $\exp(2\pi i/m) \in Sp(T)$  iff there is a  $T$ -stack of height  $m$  which is a partition of  $X$ , i.e. there is a partition of  $X$  of the form  $\{A, TA, \dots, T^{m-1}A\}$ , for some  $A \in \mathcal{B}$ . Obviously the ergodicity of  $T$  implies that there is only one  $T$ -stack of height  $m$  filling up  $X$  (recording its elements if necessary). We will denote it by  $D^m = (D_0^m, \dots, D_{m-1}^m)$ . Moreover, if  $k|m$  then  $D^k \leq D^m$ , i.e.  $D^m$  is finer than  $D^k$ .

Let  $\{n_t: t \geq 0\}$  be a sequence of natural numbers such that  $n_t | n_{t+1}$ ,  $t \geq 0$  and  $\lambda_{t+1} = n_{t+1}/n_t$ ,  $t = 0, 1, \dots$ ,  $\lambda_0 = n_0 \geq 2$ . Denote by  $G\{n_t: t \geq 0\}$  the group of all roots of unity generated by  $\{\exp(2\pi i/n_t): t \geq 0\}$  (every infinite group of roots of unity can be obtained in this way).

**DEFINITION 1.** An ergodic automorphism  $T$  with discrete spectrum has *rational pure point spectrum* if  $Sp(T) = G\{n_t: t \geq 0\}$ .

These automorphisms were characterized in [17] as follows

**LEMMA 1.**  $T$  has rational pure point spectrum with  $Sp(T) = G\{n_t: t \geq 0\}$  iff there is a sequence of partitions  $\{D^n\}$  of  $X$ , where  $D^n$  are  $T$ -stacks of heights  $n$ , resp. and  $D^n \nearrow \varepsilon$ .

By  $Z_2 = \{0, 1\}$  we mean the group of all integers mod 2 equipped with Haar measure  $m(0) = m(1) = 1/2$ .

Let  $\theta: X \rightarrow Z_2$  be any measurable map.

Let  $\tilde{\mu}$  be the product measure  $\mu \times m$  on  $Y = X \times Z_2$ .

DEFINITION 2. By  $Z_2$ -extension of  $T$  with respect to  $\theta$  we mean the automorphism  $T_\theta: Y^{\mathbb{Z}}$  defined by the formula

$$T_\theta(x, i) = (Tx, \theta(x) + i), \quad x \in X, \quad i \in Z_2. \tag{1}$$

Following [11], [19],  $T_\theta$  is ergodic iff there is no measurable function  $f: X \rightarrow K$  ( $K$  denotes the unit circle) such that

$$f(Tx) = (-1)^{\theta(x)}f(x).$$

Let  $\sigma: Y^{\mathbb{Z}} \rightarrow Y^{\mathbb{Z}}, \sigma(x, i) = (x, i + 1)$  for each  $(x, i) \in Y$ . Then  $\sigma$  is a  $\tilde{\mu}$ -preserving automorphism and  $\sigma T_\theta = T_\theta \sigma$ .

It is not hard to see that

$$L^2(Y, \tilde{\mu}) = \mathcal{C} \oplus \mathcal{D} \tag{2}$$

where  $\mathcal{D} = \{f \in L^2(Y, \tilde{\mu}): f\sigma = f\} = L^2(X, \mu), \mathcal{C} = \mathcal{D}^\perp = \{f \in L^2(Y, \tilde{\mu}): f\sigma = -f\}$ .

From now on  $T$  will be an ergodic automorphism with rational pure point spectrum,  $\text{Sp}(T) = G\{n_i: i \geq 0\}$ , and  $T_\theta$  be a  $Z_2$ -extension of  $T$ .

REMARK 1. If  $T_\theta$  is ergodic and has any point spectrum on  $\mathcal{C}$ , then  $T_\theta$  has discrete spectrum.

Proof. Let  $H \subseteq L^2(Y, \tilde{\mu})$  be the space generated by all eigenfunctions of  $T_\theta$ . Then

$$H \not\supseteq \mathcal{D} = L^2(X, \mu), \tag{3}$$

$$H \text{ is a unitary subring [23].} \tag{4}$$

Consider the partition  $\eta$  of  $Y$ , where

$$A \in \eta \text{ iff } A = \{(x, 0), (x, 1)\}, \quad x \in X. \tag{5}$$

Then  $\eta$  is  $T_0$ -invariant and measurable and the corresponding factor-automorphism  $T_0/\eta$  is isomorphic to  $T$ . Moreover, the unitary subring corresponding to  $\eta$  is contained in  $H$ , so that  $\xi$  is the partition corresponding to  $H$  [23], then

$$\eta \leq \xi, \tag{6}$$

$$T_0/\xi \text{ has discrete spectrum [23].} \tag{7}$$

But the ergodicity of  $T_0$  implies that the number of elements in any atom of  $\xi$  is constant and common for a.e. atom of  $\xi$ . Therefore from (3), (5) and (6) it follows that  $\xi = \varepsilon$ . Thus by (7)  $T_0$  has discrete spectrum.

Note that Remark 1 may be reformulated as follows: if  $T_0$  is ergodic, then

$$T_0 \text{ has partly continuous spectrum iff } \text{Sp}(T_0) = \text{Sp}(T). \tag{8}$$

If this is the case, we call  $T_0$  a continuous  $Z_2$ -extension.

Let us consider a class of well-known examples of ergodic  $Z_2$ -extensions usually called generalized Morse sequences [12]. We briefly recall their definition referring for further properties to [12], [14], [16].

Let  $B = (b_0, \dots, b_{n-1})$ ,  $C = (c_0, \dots, c_{m-1})$  be blocks (finite sequences of 0 and 1) with lengths  $|B| = n$  and  $|C| = m$ . By  $B \times C$  we mean the juxtaposition of blocks  $B \times C = B^{c_1} \dots B^{c_{m-1}}$ , where  $B^0 = B$ ,  $B^1 = \tilde{B} = (b_0 + 1, \dots, b_{n-1} + 1)$ . By  $\text{fr}(B, C)$  we denote the cardinality of the set  $\{i: 0 \leq i \leq |C| - |B|, B = C[i, i + |B| - 1]\}$ , where  $C[r, s] = (c_r, c_{r+1}, \dots, c_s)$ . If  $|B| = |C| = n$ , then  $d(B, C) = \text{card} \{i: 0 \leq i \leq n - 1, B[i] \neq C[i]\}/n$ .

Now, let  $b^0, b^1, b^2, \dots$  be blocks starting with zero,  $|b^i| = \lambda_i \geq 2$  and let  $x = b^0 \times b^1 \times b^2 \times \dots$ .

**DEFINITION 3.**  $x$  is said to be a *Morse sequence* if

- i) infinitely many of the  $b^i$ 's are different from  $0 \dots 0$ ,
- ii) infinitely many of the  $b^i$ 's are different from  $01 \dots 010$ ,
- iii)  $\sum_{i \geq 0} r_i = \infty$ ,  $r_i = \min(1/\lambda_i \text{ fr}(0, b^i), 1/\lambda_i \text{ fr}(1, b^i))$ ,  $i \geq 0$ .

We extend  $x$  to two-sided sequence  $\omega \in \{0, 1\}^Z$  [12] preserving the almost periodicity condition. Let  $\mathcal{O}_x = \{\tau^i \omega: i \in Z\}^{cl}$ , where the closure is taken in  $\{0, 1\}^Z$  and  $\tau$  is the shift transformation. It is known [12] that  $(\mathcal{O}_x, \tau)$  is strictly ergodic. The unique (ergodic)  $\tau$ -invariant measure we denote by  $\mu_x$ . Let  $P = (P_0, P_1)$  be the zero-time partition,  $P_i = \{y \in \mathcal{O}_x: y[0] = i\}$ . Then  $P$  is a generator of  $\tau$  on  $\mathcal{O}_x$ . Let  $\sigma$  be the mirror map on  $\mathcal{O}_x$ , i.e.  $\sigma(y) = \tilde{y}$ ,  $\tilde{y}[i] = y[\tilde{i}]$ . Then  $\sigma$  is an automorphism of  $(\mathcal{O}_x, \mu_x)$   $\sigma\tau = \tau\sigma$ . Let

$c_t = b^0 \times \cdots \times b^t$ ,  $n_t = |c_t| = \lambda_0 \times \cdots \times \lambda_t$ ,  $t \geq 0$  and let  $(T, X, \mu)$  be an ergodic automorphism with discrete spectrum and  $\text{Sp}(T) = G\{n_t; t \geq 0\}$ .

REMARK 2 [14]. For every Morse sequence  $x = b^0 \times \dots$  there is a measurable  $\theta: X \rightarrow Z_2$  such that  $x$  is isomorphic to  $T_\theta$  (more precisely, the  $\tau$  associated with  $x$  is isomorphic to  $T_\theta$ ). If no confusion can arise we will speak about properties of  $x$  instead of properties of  $(\mathcal{O}_x, \tau, \mu_x)$ .

A Morse sequence  $x = b^0 \times b^1 \times \dots$  is called *continuous* if  $\text{Sp}(x) = G\{n_t; t \geq 0\}$  [12].

REMARK 3 [12].  $x$  is continuous iff either

- a) infinitely many of the  $\lambda_i$ 's are even, or
- b)  $\sum_{i \geq 0} \omega(b^i) = \infty$ .

Notice that a) can be strengthened as follows: if infinitely many of the  $\lambda_i$ 's are even, then every ergodic  $Z_2$ -extension is continuous (see [12] p. 348).

Let us observe that any constant Morse sequence  $x = b \times b \times \dots$  is continuous. The class of all continuous substitutions on two symbols [4] coincides with the class of all constant Morse sequences. A larger subclass of continuous Morse sequences is the class of regular Morse sequences [14], where  $x = b^0 \times b^1 \times \dots$  is *regular* provided that there is  $\varrho > 0$  such that

$$\frac{1}{2} - \varrho > \max(\mu_{x_t}(00), \mu_{x_t}(01)) > \varrho; \quad t \geq 0, \quad (9)$$

where  $x_t = b^t \times b^{t+1} \times \dots$ .

## II. Theorem on category

In the class  $\mathcal{W}$  of all  $Z_2$ -extensions of  $T$  we introduce some topology. Namely

$$\varrho(T_\theta, T_{\theta'}) = \mu(\theta^{-1}(1) \Delta \theta'^{-1}(1)) \quad [11]. \quad (10)$$

Simultaneously, we have the uniform topology

$$d(T_\theta, T_{\theta'}) = \tilde{\mu}\{(x, i): T_\theta(x, i) \neq T_{\theta'}(x, i)\} \quad [7]. \quad (11)$$

It is clear that these topologies coincide. The class  $\mathcal{W}$  is completely metrizable in the uniform topology because  $\varrho$  is a complete metric. In other words  $\mathcal{W}$  is a closed subspace of the class of all automorphisms of  $(Y, \tilde{\mu})$  endowed with the uniform topology, so it has Baire property.

Our goal is to prove the following:

**THEOREM 1.** *The class of all  $Z_2$ -extensions which are isomorphic to Morse sequences is of the second category in  $\mathcal{W}$ .*

Notice that from Theorem 1 it follows that the class of all ergodic  $Z_2$ -extensions is residual in  $\mathcal{W}$  (cf. [20]).

The proof of Theorem 1 goes by steps.

*Step 1.* The concept of odd approximation [11].

We say  $F \subseteq X$  is oddly approximated with respect to  $D^{n_i} \nearrow \varepsilon$  with a speed  $o(g(n))$  if for some subsequence  $\{n_{i_k}\}$  there exist sets  $F_k$  consisting of an odd number of atoms of  $D^{n_{i_k}}$  such that

$$\mu(F \Delta F_k) = o(g(n_{i_k})) \tag{12}$$

(we recall here that  $n_{i_k}$  is the number of atoms of the  $T$ -stack  $D^{n_{i_k}}$  of height  $n_{i_k}$ ).

It is known [11] that the collection of all measurable functions  $\theta: X \rightarrow Z_2$  such that  $\theta^{-1}(1)$  is oddly approximated with a fixed speed contains a dense  $G_\delta$ -set. Therefore the set of all ergodic  $Z_2$ -extensions is residual since if  $\theta$  is oddly approximated with speed  $o(1/n)$ , then it has simple spectrum [11].

**REMARK.** The proof of density of ergodic  $G$ -extensions can also be found in [10] for a more general situation.

We would like to briefly explain the notion of odd approximation with ‘sufficiently high speed’ in our situation.

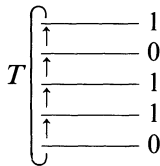


Fig. 1

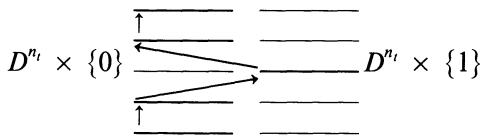


Fig. 2

If the speed of approximation is sufficiently high then on each level  $D_i^{n_i}$  of  $D^{n_i}$  the function  $\theta$  is constant apart from a set of a small measure (Fig. 1). Thus for  $T_\theta$ , two  $T_\theta$ -stacks arise (one of them is denoted by fat dashes on Fig. 2). If we want to assure that  $T_\theta$  admits a cyclic approximation, we must show that the top of the first  $T_\theta$ -stack is carried by  $T_\theta$  on the base of the second one. To show this we must show that there is an odd number of 1’s on Fig. 1.

*Step 2.* Let  $\theta: X \rightarrow Z_2$  be measurable and oddly approximated (i.e.  $\theta^{-1}(1)$  is oddly approximated) with speed  $o(1/n^2)$ . Then there is a sequence of two  $T_\theta$ -stacks  $\bar{C}^{n_i}(0)$  and  $\bar{C}^{n_i}(1)$ ,  $\bar{C}^{n_i}(j) = \{\bar{C}_i^{n_i}: i = 0, 1, \dots, n_i - 1\}$ ,  $j = 0, 1$  such that

$$\sigma \bar{C}_i^{n_i}(0) = \bar{C}_i^{n_i}(1), \quad i = 0, 1, \dots, n_i - 1, \quad (13)$$

$$\bar{C}^{n_i} \rightarrow \varepsilon \quad (\text{not necessarily monotonically}). \quad (14)$$

Indeed, let  $E_i^{n_i}$  be the subset of  $D_i^{n_i}$  consisting of all  $x \in D_i^{n_i}$  such that  $\mu(E_i^{n_i}) > 1/2$  and  $\theta|_{E_i^{n_i}}$  is constant. Consider the set

$$E_i = \bigcup_{i=0}^{n_i-2} T^{-i}(D_i^{n_i} \setminus E_i^{n_i}).$$

Thus  $E_i \subseteq D_0^{n_i}$  and

$$\mu(E_i) \leq \sum_{i=0}^{n_i-2} \mu(D_i^{n_i} \setminus E_i^{n_i}). \quad (15)$$

From our assumption

$$n_i^2 \mu(F_i \Delta \theta^{-1}(0)) \xrightarrow{i} 0, \quad F_i \subseteq D^{n_i}, \quad (16)$$

$$n_i^2 \mu(F'_i \Delta \theta^{-1}(1)) \xrightarrow{i} 0, \quad F'_i \subseteq D^{n_i}.$$

In particular (16) implies

$$n_i^2 \mu(D_i^{n_i} \Delta E_i^{n_i}) \xrightarrow{i} 0, \quad i = 0, \dots, n_i - 2. \quad (17)$$

From (15) and (17) it follows that

$$n_i \mu(E_i) \leq n_i(n_i - 1) \max_{0 \leq i \leq n_i-2} \mu(D_i^{n_i} \setminus E_i^{n_i}) \xrightarrow{i} 0. \quad (18)$$

So, the sets  $(D_0^{n_i} \setminus E_i) \times \{i\}$ ,  $i = 0, 1$  can be taken as the bases of two  $T_\theta$ -stacks of heights  $n_i$ . Putting  $\bar{C}_0^{n_i}(i) = (D_0^{n_i} \setminus E_i) \times \{i\} \cup E_i \times \{i\}$ ,  $i = 0, 1$ , we obtain two  $T_\theta$ -stacks with the property (13). It is clear that (18)



implies the property (14) because the partitions

$$B^{n_t} = \{D_i^{n_t} \times \{j\}: i = 0, 1, \dots, n_t - 1, j = 0, 1\}, \quad t \geq 0,$$

generate all measurable sets in  $Y$ .

REMARK. For simplicity we have assumed in the proof above that the subsequence  $\{n_{t_k}\}$  from (12) is equal to the sequence  $\{n_t\}$ .

By  $\bar{C}^{n-1}$  we will mean two stacks of height 1,

$$\bar{C}^{n-1}(i) = \{X \times \{i\}\}, \quad i = 0, 1.$$

Step 3. For  $\theta$  as in Step 2 there is a sequence of two  $T_\theta$ -stacks  $C_i^{n_t}(j)$ ,  $i = 0, \dots, n_t - 1$ ,  $t = -1, 0, 1, \dots$  such that

$$\sigma C_i^{n_t}(0) = C_i^{n_t}(1), \quad (19)$$

$$C^{n_t} \nearrow \varepsilon. \quad (20)$$

The proof of Step 3 is in some sense a modification of Goodson's considerations from [6] (Theorem 3). Indeed, (20) follows directly from that theorem. Now, fix  $t \geq 0$  and consider all 2- $T_\theta$ -stacks  $C^{m_t}$  (i.e.  $C^{m_t}$  is a disjoint union of  $C^{m_t}(0)$  and  $C^{m_t}(1)$  where  $C^{m_t}(i) = \{C_0^{m_t}(i), \dots, C_{n_t-1}^{m_t}(i)\}$ ,  $i = 0, 1$  are  $T_\theta$ -stacks) satisfying  $C^{m_t} \leq \bar{C}^{n_t+1}$ . Then it is not hard to see that if  $\bar{C}^{m_t}$  realizes the minimum of the set

$$d_t = \min \left\{ \sum_{i=0}^{n_t-1} \sum_{k=0}^1 \tilde{\mu}(C_i^{m_t}(k) \Delta \bar{C}_i^{n_t}(k)): C^{m_t} \leq \bar{C}^{n_t+1}, C^{m_t} \text{ is a 2-}T_\theta\text{-stack} \right\}$$

then  $\sigma \bar{C}_i^{m_t}(0) = \bar{C}_i^{m_t}(1)$  and therefore (19) holds.

REMARK 4. Our considerations are restricted by the conditions:

$$\sigma C_i^{m_t}(0) = C_i^{m_t}(1), C_i^{m_t}(0) \cup C_i^{m_t}(1) = D_i^{n_t}, d_t < \delta_t \quad \text{and} \quad \sum_{t \geq 0} \delta_t < \infty.$$

This last condition may be obtained by passing to a subsequence if necessary. Observe also that the condition  $C_i^{n_t}(0) \cup C_i^{n_t}(1) = D_i^{n_t}$  allows us to assume  $C_0^{n_t+1}(i) \subseteq C_0^{n_t}(i)$ ,  $t \geq 0$ ,  $i = 0, 1$ .

Step 4. Let  $\theta$  be as in Step 2. Denote  $Q_i = C^{n^{-1}}(i)$ ,  $i = 0, 1$ . Then there is a sequence  $x = b^0 \times b^1 \times \dots$  such that  $Q$ - $s$ -names of a.e.  $z \in Y$  are sectors of  $x$ .

From the preceding step we have

$$C_i^{n_t}(j) = 1/2n_t, \quad t = -1, 0, 1, \dots, \quad i = 0, \dots, n_t - 1, \quad j = 0, 1, \quad n_{-1} = 1. \tag{21}$$

We define a sequence of blocks  $b^0, b^1, b^2, \dots$  satisfying  $b^i[0] = 0$ ,  $|b^i| = \lambda_i = n_{i+1}/n_i$ , as follows

DEFINITION OF  $b^0$ . We divide  $Q_i$  into  $\lambda_0$  pieces  $C_j^{n_0}(k)$  of measure  $1/2n_0$  using the condition  $Q \leq C^{n_0}$  (Fig. 3).

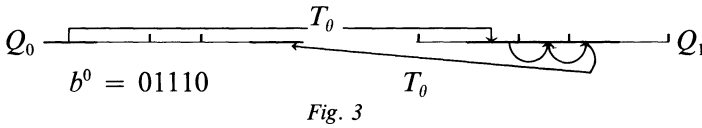
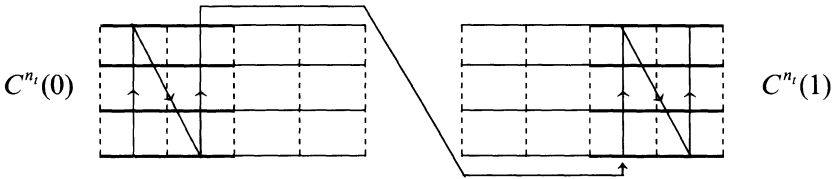


Fig. 3

We look at the trajectory of  $C_0^{n_0}(0)$  and put  $b^0[i] = b^0[i + 1]$  iff  $T_0(C_0^{n_0}(0))$  and  $C_i^{n_0}(0)$  are contained in the same atom  $Q_s$ ,  $s = 0, 1$ .

Let us suppose  $b^0, \dots, b^t$  are already defined.

DEFINITION OF  $b^{t+1}$



$$b^{t+1} = 0011$$

Fig. 4

We divide  $C_0^{n_t}(i)$  into  $\lambda_{t+1}$  pieces  $C_{j_{n_t}}^{n_{t+1}}(k_j)$ ,  $j = 0, 1, \dots, \lambda_{t+1} - 1$  and we look at the trajectory of  $C_0^{n_{t+1}}(0)$  (Fig. 4). We put  $b^{t+1}[i] = b^{t+1}[i + 1]$  iff  $T_0^{n_t}(C_{j_{n_t}}^{n_{t+1}}(0))$  and  $C_i^{n_{t+1}}(0)$  are contained in the same atom  $C_0^{n_t}(s)$ ,  $s = 0, 1$ .

From the definition of  $x = b^0 \times b^1 \times \dots$  it follows that

- $Q$ - $n_0$ -name of  $y$  from  $C_0^{n_0}(0)$  is equal to  $b^0$ ,
- $Q$ - $n_1$ -name of  $y$  from  $C_0^{n_1}(0)$  is equal to  $b^0 \times b^1 = c_1$ ,
- ...
- $Q$ - $n_t$ -name of  $y$  from  $C_0^{n_t}(0)$  is equal to  $b^0 \times b^1 \times \dots \times b^t = c_t$ ,  $t \geq 0$ .

Moreover, since  $\sigma Q_0 = Q_1$ ,  $Q$ - $n$ -name of  $\sigma y = (Q$ - $n$ -name of  $y$ ) $^{\sim}$ ,  $n \geq 0$ .

This implies that  $Q$ - $n$ -names of a.e.  $y \in Y$  are sectors of  $x = b^0 \times b^1 \times \dots$ , unless  $b^t = 0 \dots 0$ ,  $t \geq t_0$ . But this situation is excluded since  $T_\theta$  is ergodic (if  $b^t = 0 \dots 0$ ,  $t \geq t_0$ , then  $C_0^{n_{t_0}}(0) \cup \dots \cup C_{n_{t_0}-1}^{n_{t_0}}(0)$  is  $T_\theta$ -invariant).

*Step 5.* Either  $b^t = 01 \dots 010$ ,  $t \geq t_0$  or  $x = b^0 \times b^1 \times \dots$  is a Morse sequence (continuous or not).

Indeed, suppose infinitely many of the  $b^t$ 's are different from  $01 \dots 010$  (and  $0 \dots 0$  by Step 4). All we have to show is that

$$\sum_{t \geq 0} r_t = \infty \quad (\text{see Definition 3}).$$

Let us observe that for the sequence  $\{C^{n_t}\}$  we have defined in Step 3,

$$\varrho(\bar{C}^{n_t}, C^{n_t}) \xrightarrow{t} 0 \quad (22)$$

holds (see the proof of Theorem 3 in [6]). Hence

$$2n_t \tilde{\mu}(\bar{C}_0^{n_t}(i) \Delta C_0^{n_t}(i)) \xrightarrow{t} 0, \quad i = 0, 1, \quad (23)$$

since  $\bar{C}^{n_t}$  and  $C^{n_t}$  are  $2$ - $T_\theta$ -stacks. Moreover,  $\{\bar{C}^{n_t}\}$  has the property

$$2n_t \tilde{\mu}(T_\theta^{n_t}(\bar{C}_0^{n_t}(0)) \Delta \bar{C}_0^{n_t}(1)) \xrightarrow{t} 0. \quad (24)$$

Since

$$\tilde{\mu}(T_\theta^{n_t}(\bar{C}_0^{n_t}(0)) \Delta C_0^{n_t}(1)) < \tilde{\mu}(T_\theta^{n_t}(\bar{C}_0^{n_t}(0)) \Delta \bar{C}_0^{n_t}(1)) + \tilde{\mu}(\bar{C}_0^{n_t}(1) \Delta C_0^{n_t}(1))$$

and

$$\tilde{\mu}(T_\theta^{n_t}(C_0^{n_t}(0)) \Delta C_0^{n_t}(1)) < \tilde{\mu}(T_\theta^{n_t}(\bar{C}_0^{n_t}(0)) \Delta C_0^{n_t}(1))$$

$$+ \tilde{\mu}(T_\theta^{n_t}(\bar{C}_0^{n_t}(0)) \Delta T_\theta^{n_t}(C_0^{n_t}(0))),$$

we have

$$2n_t \tilde{\mu}(T_\theta^{n_t}(C_0^{n_t}(0)) \Delta C_0^{n_t}(1)) \xrightarrow{t} 0. \quad (25)$$

Now, we show that

$$\frac{1}{\lambda_{t+1}} (\text{fr } (00, b^{t+1}) + \text{fr } (11, b^{t+1})) \leq 2n_t \tilde{\mu}(T_\theta^{n_t}(C_0^{n_t}(0)) \Delta C_0^{n_t}(1)). \quad (26)$$

Indeed, if  $b^{t+1}[i] = b^{t+1}[i + 1]$ , then  $T_\theta^{n_t}(C_{i_t}^{n_t+1}(j)) \subseteq C_0^{n_t}(j)$ ,  $j = 0, 1$ . With any such a pair  $(i, i + 1)$  we can assign the level  $C_{i_t}^{n_t+1}(0)$  with the measure equal to  $1/2n_t$  and  $C_{i_t}^{n_t+1}(0) \subseteq T_\theta^{n_t}(C_0^{n_t}(0)) \Delta C_0^{n_t}(1)$ . So,  $1/2n_{t+1}(\text{fr } (00, b^{t+1}) + \text{fr } (11, b^{t+1})) \leq \tilde{\mu}(T_\theta^{n_t}(C_0^{n_t}(0)) \Delta C_0^{n_t}(1))$ .

From (25) and (26) it follows that

$$\frac{1}{\lambda_{t+1}} (\text{fr } (00, b^{t+1}) + \text{fr } (11, b^{t+1})) \xrightarrow{t} 0$$

and therefore  $x$  is a Morse sequence.

*Step 6.*  $Q$  is a generator for  $T_\theta$  as soon as  $x$  is a Morse sequence.

We will show that for a.e.  $y, y' \in Y$  there is an  $n \in N$  such that  $Q$ - $n$ -names of  $y$  and  $y'$  are different. Indeed, suppose that  $y, y'$  have the property from Step 4, i.e. their  $Q$ - $n$ -names are sectors of  $x$ . Moreover

$$y = \bigcap_{i \geq 0} C_{i_t}^{n_t}(j_t), \quad y' = \bigcap_{i \geq 0} C_{i'_t}^{n_t}(j'_t), \quad \text{since } C^{n_t} \not\rightarrow \varepsilon.$$

Now, let us take  $\mathcal{O}_x$ . Then we have also a sequence of two  $\tau$ -stacks  $E^{n_t}(j)$ ,

$$E_{i_t}^{n_t}(j) = \{y \in \mathcal{O}_x: y[-i + kn_t, -i + (k + 1)n_t - 1] = c_i \text{ or}$$

$$\tilde{c}_i, y[-i, -i + n_t - 1] = \sigma^j c_i\}$$

and the corresponding partitions  $E^{n_t} \not\rightarrow \varepsilon$  [14]. Let  $z = \bigcap_{i \geq 0} E_{i_t}^{n_t}(j_t)$  and  $z' = \bigcap_{i \geq 0} E_{i'_t}^{n_t}(j'_t)$ . Since  $y \neq y'$ , there is  $t$  such that  $(i_t, j_t) \neq (i'_t, j'_t)$  and therefore  $z \neq z'$ . Moreover, any  $Q$ - $n$ -name of  $y$  ( $y'$ ) is equal to  $P$ - $n$ -name of  $z$  ( $z'$ ). But  $P$  is a generator of  $\tau$ , so  $P$ - $n$ -names of  $z$  and  $z'$  are different for some  $n$  and therefore for that  $n$ , the  $Q$ - $n$ -names of  $y$  and  $y'$  are different.

*Step 7.* If  $x = b^0 \times b^1 \times \dots$  determined by the  $\theta$  is a Morse sequence then  $T_\theta$  and  $x$  are isomorphic as dynamical systems.

By Step 4 and 6 we can find a point  $y \in Y$  which is generic for  $T_\theta$  and  $Q$ - $s$ -name of  $y$  is always a sector of  $x$ ,  $s \geq 0$ . Next, we select  $z \in \mathcal{O}_x$  such that  $\tau$ - $P$ - $s$ -name of  $z$  is equal to  $Q$ - $s$ -name of  $y$ ,  $s \geq 0$ . Hence, there are two

generic points  $y$  for  $T_0$  and  $z$  for  $x$  such that  $Q$ - $\infty$ -name of  $y = P$ - $\infty$ -name of  $z$ . Since  $Q$  and  $P$  are generators,  $T_0$  and  $x$  must be isomorphic.

*Step 8, proof of Theorem 1.* As we have seen in Step 5 it is possible  $x = b^0 \times b^1 \times \dots$ ,  $b^t = 01 \dots 010$   $t \geq t_1$ . If this is the case we see that  $T_0^{n_t}(C_0^{n_t}(0)) = C_0^{n_t}(1)$  and therefore we can define a  $T_0$ -stack of height  $2n_t$ , so  $\exp(2\pi i/2n_t) \in \text{Sp}(T_0)$ ,  $t \geq t_1$ . Thus, from Remark 1,  $T_0$  has pure point spectrum and  $\text{Sp}(T_0) = G\{n'_t: t \geq 0\}$  where  $n'_t = n_t$  for  $t < t_0$ ,  $n'_t = 2n_t$  for  $t \geq t_0$ , where  $t_0$  is the smallest natural number such that  $\lambda_{t_0}$  is odd. It is a consequence of the fact that if  $\exp(2\pi i/2n_t) \in \text{Sp}(T_0)$  then  $\exp(2\pi i/2n_{t-1}) = \exp(2\pi i\lambda_t/2n_t) \in \text{Sp}(T_0)$ .

Let  $y = \beta^0 \times \beta^1 \times \dots$  be a Morse sequence  $|\beta^i| = |b^i|$ ,  $i \geq 0$  and  $\sum_{i \geq 0} \omega(\beta^i) < \infty$  (see Remark 3). Then  $y$  has a discrete spectrum and  $\text{Sp}(y) = \text{Sp}(T_0)$ . Therefore we can “replace”  $x$  by some Morse sequence. Now, our proof is complete by Step 1, Step 5 and Step 7.

### III. On the factor problem for $Z_2$ -extensions

A motivation to study the factors problem lies in the following:

**PROPOSITION 1.** *Let  $T_0$  be a continuous, ergodic  $Z_2$ -extension and let  $U: (Y', \tilde{\mu}')^{\mathbb{Z}}$  be a factor of it with partly continuous spectrum. Then there are  $T': (X', \mu')^{\mathbb{Z}}$  with discrete spectrum and  $\theta': X' \rightarrow Z_2$  measurable, such that  $U$  is isomorphic to  $T'_\theta$ .*

*Proof.* We will use Pickel's and Kushnirenko's theorems [13, 22] concerning sequence entropy. If  $T_0$  is a continuous, ergodic  $Z_2$ -extension, then  $\sup_{A \subseteq N} h_A(T_0) = \log 2$  [22]. Furthermore,  $\sup_{A \subseteq N} h_A(U) \leq \sup_{A \subseteq N} h_A(T_0)$ . But if  $U$  does not have discrete spectrum then  $\sup_{A \subseteq N} h_A(U) > 0$  [13]. So using Pitskel's result once more we obtain  $\sup_{A \subseteq N} h_A(U) = \log 2$ . We recall that 2 is then the number of elements in  $\varphi^{-1}(x')$  (a.e.  $x'$ ) where  $\varphi: (Y, T_0, \tilde{\mu}) \rightarrow (X', T', \mu')$  establishes a homomorphism between  $T_0$  and its maximal factor with discrete spectrum.

Now, let  $T_0$  be a continuous  $Z_2$ -extension and  $\eta$  be the partition described by (5). Then  $\eta$  is  $T_0$ -invariant and measurable and  $T_0/\eta$  is isomorphic to  $T$ . This factor (and all the factors which are determined by subgroups of  $\text{Sp}(T)$ ) has discrete spectrum.

In the sequel we consider proper factors of a given  $Z_2$ -extension.

It was observed in [16] that some Morse sequences have the only factors with discrete spectra. This can be generalized as follows.

PROPOSITION 2. Let  $p$  be a fixed prime number and let  $\text{Sp}(T) = G\{n_t; t \geq 0\}$  have the property  $\lambda_t = p^{k_t}$ ,  $k_t \geq 1$ ,  $t \geq 0$ . Then every ergodic continuous  $Z_2$ -extension has no factors with partly continuous spectrum.

*Proof.* First, let us observe that if  $U$  is a factor with partly continuous spectrum (via  $\varphi$ ) of  $T_0$ , then  $\text{Sp}(U)$  is an infinite subgroup of  $\text{Sp}(T)$ . Indeed, otherwise  $U$  would be a  $Z_2$ -extension of some ergodic transformation defined on a finite space. But there is no ergodic transformation with partly continuous spectrum on a finite space. So, we may assume  $\text{Sp}(U) = \text{Sp}(T)$  since no other infinite subgroup of  $\text{Sp}(T)$  exists.

Now let  $\{\bar{D}^{n_t}\}$  be a sequence of  $U$ -stacks of heights  $n_t$  corresponding to  $\text{Sp}(U)$ . Then  $\varphi^{-1}(\bar{D}^{n_t})$  is again a  $T_0$ -stack of height  $n_t$ , so it is equal to  $D^{n_t}$ ,  $t \geq 0$ . Therefore  $\sigma$ -algebra generated by  $\{\varphi^{-1}(\bar{D}^{n_t})\}_{t \geq 0}$  contains a  $\sigma$ -algebra of  $\eta$ -measurable sets, because the latter  $\sigma$ -algebra is generated by  $\{D^{n_t}\}_{t \geq 0}$ . As a conclusion we have  $\eta \leq \xi$ , where  $\xi$  is  $T_0$ -invariant measurable partition corresponding to the factor  $U$ . Hence, either  $\xi = \eta$ , a contradiction to continuity of  $U$ , or  $\xi = \varepsilon$ , and  $U$  is then isomorphic to  $T_0$ .

The above investigations might suggest that the case  $\lambda_t = p^{k_t}$ ,  $t \geq 0$  is the only case where all factors of  $T_0$  have discrete spectra. However, this is not true as the following theorem shows.

THEOREM 2. If  $x = b^0 \times b^1 \times \dots$  is a regular Morse sequence and  $\lambda_i \leq r$ ,  $i \geq 0$ , then all factors of  $x$  have discrete spectrum.

Before the proof we establish some auxiliary facts.

Let  $x = b^0 \times b^1 \times \dots$  be a continuous Morse sequence. By  $\eta_x$  we mean the measurable partition ( $\tau$ -invariant) corresponding to the maximal factor with discrete spectrum. Hence

$$A \in \eta_x \text{ iff } A = \{z, \tilde{z}\}, \quad z \in \mathcal{O}_x. \tag{27}$$

Then by Proposition 1, any proper factor of  $x$  with partly continuous spectrum is also a  $Z_2$ -extension. Thus it is of the form  $T_0: (X \times Z_2, \tilde{\mu})^2$ , where  $T: (X, \mu)^2$  has discrete spectrum with an infinite group of eigenvalues and  $\text{Sp}(T) \not\subseteq G\{n_t; t \geq 0\}$ . Consider now the factor of  $T_0$  generated by the partition  $Q = (X \times \{0\}, X \times \{1\})$  (i.e. the  $\sigma$ -algebra corresponding to  $\bigvee_{-\infty}^{+\infty} (T_0^{-i}Q)$ ). We assert that this factor has partly continuous spectrum. Indeed, otherwise  $\bigvee_{-\infty}^{+\infty} T_0^{-i}Q \leq \eta$  (see (5)) since any factor with discrete spectrum is canonical [19] and  $\eta$  determines the maximal factor of  $T_0$  with discrete spectrum. But, then  $X \times \{0\}, X \times \{1\}$  would be  $\eta$ -measurable, i.e.

$\sigma(X \times \{i\}) = X \times \{i\}$ , a contradiction. The factor generated by  $Q$  is isomorphic to a shift dynamical system  $(W, \tau, \nu)$ , where

$$W \subseteq \{0, 1\}^{\mathbb{Z}}, \quad \sigma W = W, \quad \sigma \nu = \nu \quad (\sigma y = \tilde{y}). \quad (28)$$

Moreover if  $\xi$  is the partition of  $W$  given by

$$B \in \xi \text{ iff } B = \{y, \tilde{y}\}, \quad y \in W \quad (29)$$

then  $\xi$  is measurable,  $\tau$ -invariant and the corresponding factor-automorphism  $\tau/\xi$  has discrete spectrum. Furthermore, since  $T_0$  is a proper factor of  $x$ , so is  $(W, \tau, \nu)$ .

Now, suppose that  $\psi: (\mathcal{O}_x, \tau, \mu_x) \rightarrow (W, \tau, \nu)$  establishes a homomorphism. Then

$$\psi^{-1}\xi \leq \eta \quad (30)$$

since  $\psi^{-1}\xi$  induces a factor of  $x$  with discrete spectrum and this factor is canonical.

The condition that  $(W, \tau, \nu)$  has partly continuous spectrum implies  $\psi$  cannot identify any pair  $(z, \tilde{z}) \in \mathcal{O}_x$ , because the set  $\{z \in \mathcal{O}_x: \psi z = \psi \tilde{z}\}$  is  $\tau$ -invariant. So

$$\psi z \neq \psi \tilde{z} \quad \text{a.e. } z \in \mathcal{O}_x. \quad (31)$$

This implies

$$\psi \tilde{z} = \tilde{\psi} z \quad \text{a.e. } z \in \mathcal{O}_x. \quad (32)$$

Indeed, from (29) and (30) it follows that for a.e. pair  $\{z, \tilde{z}\} \subseteq \mathcal{O}_x$  there is a pair  $\{y, \tilde{y}\} \subseteq W$  such that  $\{z, \tilde{z}\} \subseteq \psi^{-1}\{y, \tilde{y}\}$ . Now, (32) follows directly from (31).

Let us fix  $\delta > 0$ . Then from the Birkhoff theorem there exists a code  $\varphi_\delta: \mathcal{O}_x \rightarrow \{0, 1\}^{\mathbb{Z}}$  (i.e.  $\varphi_\delta$  is measurable  $\varphi_\delta \tau = \tau \varphi_\delta$ ,  $z[-k, k] = z'[-k, k]$  implies  $(\varphi_\delta z)[0] = (\varphi_\delta z')[0]$  a.e.  $z, z' \in \mathcal{O}_x$ ,  $k$  is the length of the code) such that

$$d(\varphi z, \varphi_\delta z) < \delta \quad \text{a.e. } z \in \mathcal{O}_x, \quad (33)$$

where  $d(z, z') = \lim_m d(z[-m, m], z'[-m, m])$  if the limit exists.

In view of (32) and (33) we have

$$\liminf_m d((\varphi_\delta z)[-m, m], (\varphi_\delta \tilde{z})[-m, m]) > 1 - 2\delta. \quad (34)$$

The second kind of argument we use in the proof of Theorem 2 is connected with a property of  $\mathcal{O}_x$  which holds for any regular Morse sequence of the form:  $x = b^0 \times b^1 \times \dots, \lambda_i \leq r, i \geq 0$ . Namely

$$\begin{aligned} & \text{There is } \delta > 0 \text{ such that for every } y, y' \in \mathcal{O}_x \\ & y \neq y' \text{ implies } \liminf_m d(y[-m, m], y'[-m, m]) \geq \delta. \end{aligned} \quad (35)$$

This fact is an obvious consequence of Proposition 1 in [15].

*Proof of Theorem 2.* Let  $\delta > 0$  be fixed. Denote the code of  $c_i$  and  $\tilde{c}_i$  (via  $\varphi_\delta$ ) by  $d_i$  and  $\hat{d}_i, n_i > 2k - 1$ . Of course we cannot assume  $\hat{d}_i = \tilde{d}_i$ .

From (34) it follows that

$$d(d_i, \hat{d}_i) > 1 - 3\delta \quad \text{for } t \text{ large enough.} \quad (36)$$

Since  $(W, \tau, \nu)$  is a proper factor,  $\psi$  is not one-to-one. Let

$$\psi z = \psi z'. \quad (37)$$

Hence from (33)

$$\limsup_m d((\varphi_\delta z)[-m, m], (\varphi_\delta z')[-m, m]) \leq d(\varphi_\delta z, \psi z) + d(\varphi_\delta z', \psi z') < 2\delta. \quad (38)$$

Notice that we may assume  $\psi z \neq \psi \tau^s z$  a.e.  $z \in \mathcal{O}_x, s \in \mathbb{Z}$ , because otherwise a.e. point of  $W$  would be periodic, and this is a contradiction since  $(W, \tau, \nu)$  has partly continuous spectrum.

Consequently we have

$$z' \neq \tilde{z}, \tau^s z, \quad s \in \mathbb{Z}. \quad (39)$$

Now, fix  $t$  satisfying (36) and

$$2k/n_i < \delta. \quad (40)$$

We divide  $\varphi_\delta z$  and  $\varphi_\delta z'$  into a juxtaposition of  $d_i, \hat{d}_i$  and some ‘‘holes’’ of length  $2k$  (see Fig. 5).



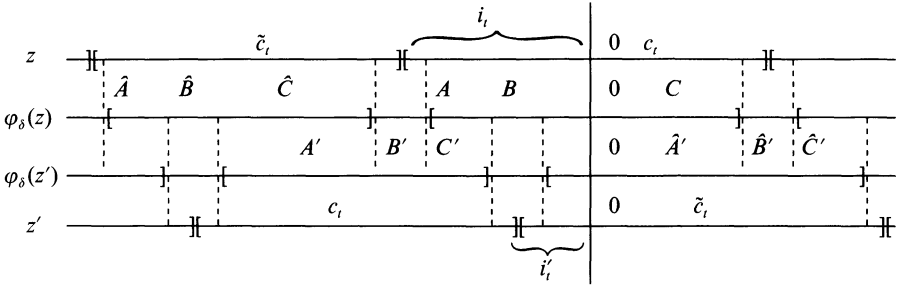


Fig. 5

We obtain a partition of  $d_t$  ( $\hat{d}_t$ ) on  $\varphi_\delta z$  into three blocks  $A, B, C$  ( $\hat{A}, \hat{B}, \hat{C}$ ) and also  $d_t$  ( $\hat{d}_t$ ) on  $\varphi_\delta z'$  into three blocks  $A', B', C'$  ( $\hat{A}', \hat{B}', \hat{C}'$ ) where  $|B| = |B'| = 2k, |A| = |\hat{A}| = |C'| = |\hat{C}'|, |C| = |\hat{C}| = |A'| = |\hat{A}'|$ . Then, from (40) either  $|A| \geq n_t/4$  or  $|C| \geq n_t/4$ . We will consider the case  $|A| \geq n_t/4$ . Thus, from (36) it follows that

$$d(A, \hat{A}) \geq [(1 - 3\delta)|d_t| - \frac{3}{4}|d_t|]/\frac{1}{4}|d_t| = 1 - 12\delta. \tag{41}$$

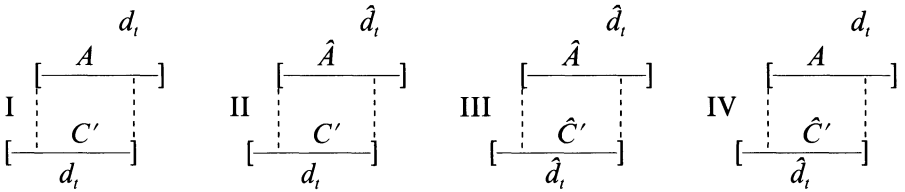


Fig. 6

We have either

$$d(A, C') \geq (1 - 12\delta)/2 \quad \text{or} \quad d(\hat{A}, C') \geq (1 - 12\delta)/2, \tag{42}$$

since  $d(A, C') + d(\hat{A}, C') \geq d(A, \hat{A})$  and (41) hold.

Combining (42) with (38) we see that if for instance situation I (Fig. 6) appears on  $\varphi_\delta z$  and  $\varphi_\delta z'$  then situation II is nearly “excluded”. Since the frequency  $c_t$  and  $\tilde{c}_t$  on  $z$  ( $z'$ ) are within  $\delta$  provided that we consider the places of the form  $i_t + sn_t$  ( $i'_t + sn_t$ ),  $s \in \mathbb{Z}$ , we get the following:

*if the situation I appears (and situation II is nearly “excluded”) then situation III appears (and situation IV is nearly “excluded”).*

Let us turn back to  $z$  and  $z'$  and take into consideration  $\tau^{-(i_t-i'_t)} z$  and  $z'$ . If situations I and III appear, then it follows that below the  $c_t$ 's ( $\tilde{c}_t$ 's) of  $\tau^{-(i_t-i'_t)} z$  there are  $c_t$ 's ( $\tilde{c}_t$ 's) of  $z'$  nearly always

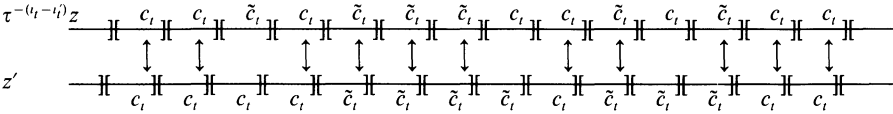


Fig. 7

If situations II and IV appear then it means that below the  $c_i$ 's ( $\tilde{c}_i$ 's) of  $\tau^{-(i_1-i_2)}z$  there are  $\tilde{c}_i$ 's ( $c_i$ 's) of  $z'$  nearly "always", i.e. in the first case

$$\liminf_m d(\tau^{-(i_1-i_2)}z[-m, m], z'[-m, m]) < 100\delta$$

and in the second

$$\liminf_m d(\tau^{-(i_1-i_2)}z[-m, m], \tilde{z}'[-m, m]) < 100\delta.$$

Therefore (31), (35) and (39) give a contradiction for a suitable choice of  $\delta > 0$ .

**COROLLARY 1.** *For any continuous substitution on two symbols the only factors are those with discrete spectrum.*

We finish our considerations by giving a class of ergodic  $Z_2$ -extensions having some partly continuous factors.

Assume  $G\{n_t: t \geq 0\}$  has the property that the  $\lambda_t$ 's are odd,  $t \geq 0$  and in addition  $\theta^{-1}(1)$  is oddly approximated with speed  $o(1/n)$  in such a way that  $T_0$  has partly continuous spectrum.

**REMARK.** Such a  $T_0$  exists. For instance, if we put

$$b^i = \underbrace{01 \dots 01}_{v_i} \dots \underbrace{101 \dots 101}_{v_i} \quad \lambda_i = 2v_i + 1, \quad i \geq 0$$

then  $x = b^0 \times b^1 \times \dots$  is a continuous Morse sequence and admits an odd approximation with a speed depending upon how fast the sequence  $\{\lambda_i\}$  tends to infinity.

Let  $T': (X', \mu')$  be an ergodic dynamical system with discrete spectrum and  $\text{Sp}(T') = G\{n'_t: t \geq 0\}$ ,  $n'_t = 3n_t$ ,  $t \geq 0$ . There is  $\varphi: X' \rightarrow X$ ,  $T\varphi = \varphi T'$ ,  $\mu = \mu' \varphi^{-1}$ . Now define  $\theta': X' \rightarrow Z_2$  putting

$$\theta' = \theta\varphi. \tag{43}$$

We assert  $\theta'^{-1}(1)$  is oddly approximated with the speed  $o(1/n)$ . Indeed

$$\begin{aligned} \mu \left( \theta^{-1}(1) \Delta \bigcup_{r=0}^{2k_1} D_{j_r}^{n'_i} \right) &= \mu' \left( \varphi^{-1} \left( \theta^{-1}(1) \Delta \bigcup_{r=0}^{2k_1} D_{j_r}^{n'_i} \right) \right) \\ &= \mu' \left( \theta'^{-1}(1) \Delta \bigcup_{r=0}^{2k_1} \varphi^{-1} D_{j_r}^{n'_i} \right) \end{aligned}$$

But  $\varphi^{-1} D^{n'_i}$  is a  $T'$ -stack of height  $n_i$  and  $\varphi^{-1} D^{n'_i} \leq D^{n'_i}$  because  $n_i | n'_i$  and moreover any level  $\varphi^{-1} D_{j_r}^{n'_i}$  is a union of three levels of  $D^{n'_i}$ , so

$$\mu \left( \theta^{-1}(1) \Delta \bigcup_{r=0}^{2k_1} D_{j_r}^{n'_i} \right) = \mu' \left( \theta'^{-1}(1) \Delta \bigcup_{s=0}^{2l_1} D_{i_s}^{n'_i} \right).$$

We get  $\theta'^{-1}(1)$  is oddly approximated with the required speed. In particular,  $T'_\theta$  is ergodic and has partly continuous spectrum. Using (43) it is not difficult to verify that  $\varphi \times id: X' \times Z_2 \rightarrow X \times Z_2$  establishes a homomorphism from  $T'_\theta$  to  $T_\theta$ .

Finally, note that if we assume  $\theta^{-1}(1)$  is oddly approximated with speed  $o(1/n^2)$ , then we can obtain a homomorphism between some continuous Morse sequences.

**REMARK.** It would be interesting to characterize all  $Z_2$ -extensions having a factor with partly continuous spectrum.

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