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# Families of supersingular abelian surfaces

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*Dedicated to Professor Masayoshi Nagata on his 60th birthday*

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## Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $\mathscr{A}_{2,1}$ be the coarse moduli scheme of principally polarized abelian surfaces over $k$. We study the set

$$V \subset \mathscr{A}_{2,1}$$

of principally polarized supersingular abelian surfaces over $k$. This paper is a continuation of the previous one by T. Ibukiyama, T. Katsura and F. Oort (cf. [5], which will be referred to as [IKO]).

The main point of this paper is to make explicit, and to exploit the methods of Oort [15] and of Moret-Bailly [11] for constructing families of (principally polarized) supersingular abelian surfaces over the projective line $\mathbf{P}^1$. We show that any component of the supersingular locus $V$ of $\mathscr{A}_{2,1}$ is the image of such a family (cf. Corollary 2.2). Furthermore, it follows that for any irreducible component $W$ of $V$ this construction determines a group $G \subset \mathrm{Aut}(\mathbf{P}^1)$ such that

$$\mathbf{P}^1 \to \mathbf{P}^1/G \simeq \tilde{W} \to W \subset V \subset \mathscr{A}_{2,1},$$

where $\tilde{W}$ is the normalization of $W$ (cf. Section 4). For $p \geqslant 3$ the group $G$ is a subgroup of the symmetric group $S_6$ of degree six (cf. Corollary 4.4). This enables us to relate the number of irreducible components of $V$ with a certain class number (cf. Theorem 5.7), and using a computation by K. Hashimoto and T. Ibukiyama (cf. [4], and see also Katsura and Oort [8]), we conclude that

$V$ is irreducible if and only if $p \leqslant 11$.

In Section 6, we compute the number of automorphisms of abelian surfaces when the polarization consists of the join of two supersingular elliptic curves. This information and methods of Igusa [7] and of [IKO] enable us to determine all ramification groups which appear in the morphisms

$$\mathbf{P}^1 \to \mathbf{P}^1/G \simeq \tilde{W}$$

(cf. Section 7). In this way we obtain a second proof for the (ir)reducibility of $V$, but in this way by purely geometric methods. We conclude by determining the groups $G$ which appear for small characteristics.

It seems that the structure of the supersingular locus $V$ of $\mathscr{A}_{2,1}$ has been described rather precisely in this way. We come back to this question for dimension three in our paper [8].

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## §1. Preliminaries and construction of families

In this section, we fix notations and prove some easy lemmas which we need later. Then, we give a survey of the construction of families of supersingular abelian surfaces (cf. Oort [15]) and resume the properties of these families by virtue of Moret-Bailly [11].

Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $X$, $Y$ be non-singular complete algebraic varieties over $k$. We denote by $k(X)$ the rational function field of $X$. For two Cartier divisors $D_1$ and $D_2$ on $X$, $D_1 \sim D_2$ (resp. $D_1 \equiv D_2$) means that $D_1$ is linearly equivalent (resp. algebraically equivalent) to $D_2$. We denote by $id_X$ the identity mapping from $X$ to $X$. For a morphism $f$ from $X$ to $Y$, we denote by $\deg f$ the degree of $f$. We mean by a curve a non-singular complete irreducible curve over $k$, unless otherwise mentioned. By the curve defined by an equation $f(x, y) = 0$ we mean the non-singular complete model of the curve defined by the equation $f(x, y) = 0$. For a curve $\tilde{C}$, we denote by $\tilde{C}^{(p)}$ the image of the Frobenius morphism of $\tilde{C}$. Conversely, for a curve $C = \tilde{C}^{(p)}$, we denote $\tilde{C}$ by $C^{(1/p)}$.

For an abelian variety $A$ over $k$ we denote by End $(A)$ the ring of endomorphisms of $A$. We denote by $\mathrm{Aut}_v(A)$ (resp. Aut $(A)$) the group of automorphisms of $A$ as a variety (resp. as an abelian variety). We have the natural exact sequence

$$0 \longrightarrow A \longrightarrow \mathrm{Aut}_v(A) \xrightarrow{\gamma} \mathrm{Aut}\,(A) \longrightarrow 0. \tag{1.1}$$

For an effective divisor $\Theta$ on $A$ we denote by $\mathrm{Aut}_v(A, \Theta)$ (resp. Aut $(A, \Theta)$) the subgroup of $\mathrm{Aut}_v(A)$ (resp. Aut$(A)$) whose elements induce automorphisms of the subscheme $\Theta$ (resp. whose elements $\theta$ preserve the polarization $\Theta$, i.e., $\theta^*(\Theta) \equiv \Theta$). For a group $G$ and elements $g_i$ ($i = 1, 2, \cdots, n$ with a positive integer $n$) of $G$, we denote by $\langle g_1, \cdots, g_n\rangle$ the subgroup of $G$ generated by $g_i$'s ($i = 1, 2, \cdots, n$). We denote by $|G|$ the order of $G$. We denote by $\iota$ the inversion of $A$. We set for an effective divisor $\Theta$ with $\iota^*(\Theta) = \Theta$

$$RA_v(A, \Theta) = \mathrm{Aut}_v(A, \Theta)/\langle\iota\rangle \quad \text{and} \quad RA(A, \Theta) = \mathrm{Aut}\,(A, \Theta)/\langle\iota\rangle.$$

The group $RA(A, \Theta)$ is called a reduced group of automorphisms of a polarized abelian variety $(A, \Theta)$.

Now, let $A$ be an abelian surface, and let $\Theta$ be a principal polarization on $A$. Then, $\Theta$ is given by a (not necessarily irreducible) curve of genus two, and $(A, \Theta)$ is isomorphic to the (generalized) Jacobian variety $(J(\Theta), \Theta)$. By Weil [21, Satz 2], we have two possibilities for $\Theta$:

(i) $\Theta$ is a non-singular complete curve of genus two,

(ii) $\Theta$ consists of two elliptic curves $E'$ and $E''$ which intersect transversally at a point.

In both cases, we have the isomorphism

$$\mathrm{Aut}_v(A, \Theta) \simeq \mathrm{Aut}\,(A, \Theta) \tag{1.2}$$

induced by $\gamma$ in (1.1). A smooth curve of genus two is a two-sheeted covering of the projective line $\mathbf{P}^1$. For such a curve $C$ we denote again by $\iota$ the generator of the Galois group of the algebraic extension $k(C)/k(\mathbf{P}^1)$. For a reducible curve $C = E' \cup E''$ as in (ii) we denote by $\iota$ the automorphism of $C$ which induces inversions of $E'$ and $E''$. In these cases, we denote by Aut $(C)$ the group of automorphisms of $C$, and we set

$$RA(C) = \mathrm{Aut}\,(C)/\langle \iota \rangle.$$

The group $RA(C)$ is called a reduced group of automorphisms of a (not necessarily irreducible) curve $C$ of genus two. The inversion $\iota$ of $C$ induces the inversion of the (generalized) Jacobian variety $J(C)$. Then, by (1.2) we have

$$\begin{gathered} \mathrm{Aut}\,(C) \simeq \mathrm{Aut}_v(J(C), C) \simeq \mathrm{Aut}\,(J(C), C) \\ \text{and} \\ RA(C) \simeq RA(J(C), C). \end{gathered} \tag{1.3}$$

We denote by $\mathbf{F}_{p^i}$ the finite field with $p^i$ elements. Throughout this paper, we fix a supersingular elliptic curve $E$ over $k$ such that

$$E \text{ is defined over } \mathbf{F}_p \text{ and End } (E) \text{ is defined over } \mathbf{F}_{p^2}. \tag{1.4}$$

For the existence of such a supersingular elliptic curve, see Waterhouse [20, Theorem 4.1.5].

For an abelian variety $A$, we denote by $A^t$ the dual of $A$. For an invertible sheaf (or a divisor) $L$ on $A$, we have the morphism $\varphi_L \colon A \to A^t$ defined by $x \mapsto T_x^*L \otimes L^{-1}$, where $T_x$ is the translation by an element $x$ of $A$. We set

$$K(L) = \mathrm{Ker}\ \varphi_L.$$

For abelian varieties $A$, $B$ and a homomorphism $f\colon B \to A$, we have the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle \varphi_{f^*L}}\downarrow & & \downarrow{\scriptstyle \varphi_L} \\ B^t & \xleftarrow{f^t} & A^t \end{array}, \tag{1.5}$$

where $f^t$ is the dual homomorphism of $f$. For a product of $n$ supersingular elliptic curves, we have the following remarkable results.

THEOREM 1.1 (Deligne). *For any supersingular elliptic curves* $E_j$ $(j = 1, 2, \cdots, 2n$ with $n \geqslant 2)$, *the abelian variety* $E_1 \times \cdots \times E_n$ *is isomorphic to* $E_{n+1} \times \cdots \times E_{2n}$.

For the proof, see Shioda [19, Theorem 3.5].

We denote by $\alpha_p$ the local–local group scheme

$$\alpha_p = \operatorname{Spec} k[\alpha]/(\alpha^p)$$

of rank $p$ with a co-multiplication $\alpha \mapsto \alpha \otimes 1 + 1 \otimes \alpha$.

THEOREM 1.2 (Oort). *Let* $E'$ *be a supersingular elliptic curve. Then, any supersingular abelian surface* $A$ *is isomorphic to* $(E' \times E')/i(\alpha_p)$ *with a suitable immersion* $i\colon \alpha_p \hookrightarrow E' \times E'$.

This follows from Theorem 1.1 and Oort [16, Corollary 7]. The proofs of the following lemmas are obvious (for Lemma 1.3, use Oort [16, Theorem 2]).

LEMMA 1.3. *Let* $A = E_1 \times E_2$ *be an abelian surface with supersingular elliptic curves* $E_1$ *and* $E_2$. *Let* $i\colon \alpha_p \hookrightarrow A$ *be an immersion such that* $B = A/i(\alpha_p)$ *is not isomorphic to a product of two elliptic curves. Then, the subgroup scheme which is isomorphic to* $\alpha_p$ *is unique in* $B$. *Moreover, the natural mapping* $A \to B \to B/\alpha_p$ *is nothing but the Frobenius morphism* $F_A$.

LEMMA 1.4. *Let* $A = E_1 \times E_2$ *be an abelian surface with supersingular elliptic curves* $E_1$ *and* $E_2$. *Let* $i\colon \alpha_\varrho \hookrightarrow A$ *be an arbitrary immersion. Set* $B = A/i(\alpha_p)$ *and let* $\pi_0\colon A \to B$ *be the natural projection. Set* $N = \pi_0(\alpha_p \times \alpha_p)$. *If an automorphism* $\theta$ *of* $B$ *induces an automorphism of* $N$, *then* $\theta$ *lifts to a unique automorphism* $\tilde{\theta}$ *of* $A$ *such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\theta}} & A \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{\theta} & B \end{array} \tag{1.6}$$

*Moreover, the order of* $\theta$ *is equal to the order of* $\tilde{\theta}$.

LEMMA 1.5. *Under the same notations as in Lemma 1.4, assume that* $B$ *is not isomorphic to a product of two elliptic curves. Then, any automorphism* $\theta$ *of* $B$ *lifts to an automorphism* $\tilde{\theta}$ *of* $A$ *such that the diagram* (1.6) *commutes.*

*Proof.* By the uniqueness of a subgroup scheme $\alpha_p$ in $B$, the automorphism $\theta$ induces an automorphism of $N$ in Lemma 1.4. Therefore, this corollary follows from Lemma 1.4. Q.E.D.

Now, we give a survey of the construction of families of supersingular abelian surfaces by virtue of Moret-Bailly [11]. We assume char $k = p \geqslant 3$. Let $E_1$ and $E_2$ be supersingular elliptic curves. We set $A = E_1 \times E_2$. We have the natural immersion

$$\alpha_p \times \alpha_p \hookrightarrow A = E_1 \times E_2. \tag{1.7}$$

We denote by $T_A$ the tangent space of $A$ at the origin, and by $S$ the projective line $\mathbf{P}(T_A)$ obtained from $T_A$. We set

$$\begin{aligned} K_S &= \alpha_p \times \alpha_p \times S \simeq \operatorname{Spec} k[\alpha]/(\alpha^p) \times \operatorname{Spec} k[\beta]/(\beta^p) \times S, \\ A_S &= E_1 \times E_2 \times S. \end{aligned} \tag{1.8}$$

We consider a subgroup scheme $H$ of $K_S = \operatorname{Spec} \mathcal{O}_S[\alpha, \beta]/(\alpha^p, \beta^p)$ defined by the equation $Y\alpha - X\beta = 0$, where $(X, Y)$ is a homogeneous coordinate of $S$. We set $\mathscr{X} = A_S/H$. Then, we have the following diagram:

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "\Delta"] & A_S \arrow[r, "\pi"] \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \mathscr{X} \arrow[r] \arrow[d, "q"] & 1 \text{ (exact)}, \\ & A & & S \simeq \mathbf{P}^1 & \end{tikzcd} \tag{1.9}$$

where $\Delta$ is the natural immersion, $\pi$ is the canonical projection, $pr_1$ and $pr_2$ are projections, and $q$ is the induced morphism. Using Moret-Bailly [11, p. 138–p. 139] and the fact that $p \geqq 3$, we see that there exists an invertible sheaf $L$ (resp. a divisor) on $A$ such that

$$\begin{aligned} &\text{(i) } L \text{ is symmetric, i.e., } \iota^*L \simeq L \text{ (resp. } \iota^*L = L\text{), and} \\ &\text{(ii) } K(L) \simeq \alpha_p \times \alpha_p. \end{aligned} \tag{1.10}$$

Then, there exists an invertible sheaf $M$ on $\mathscr{X}$ such that $\pi^*(M) = pr_1^*(L)$ (cf. Moret-Bailly [11, p. 130]). Using this $M$, we can construct an effective divisor $D$ on $\mathscr{X}$ over $S$ such that

$$\mathcal{O}_{\mathscr{X}}(D) \simeq M \otimes q^*\mathcal{O}_S((p-1)/2). \tag{1.11}$$

We can show that $D$ is a non-singular surface. Setting $D' = \pi^{-1}(D)$, we have

$$\mathcal{O}_{A_S}(D') \simeq pr_1^*(L) \otimes pr_2^*\mathcal{O}_S((p-1)/2). \tag{1.12}$$

For a point $x$ of $S$, we set

$$\mathscr{X}_x = q^{-1}(x),\ D_x = D \cap q^{-1}(x),\ (A_S)_x = pr_2^{-1}(x) \text{ and } H_x = \Delta^{-1}pr_2^{-1}(x).$$

We denote by $\Delta_x$ (resp. $\pi_x$) the homomorphism from $H_x$ to $(A_S)_x$ (resp. from $(A_S)_x$ to $\mathscr{X}_x$) induced by $\Delta$ (resp. $\pi$). We have $D_x^2 = 2$, hence, $D_x$ gives a principal polarization on $\mathscr{X}_x$. Therefore, $D_x$ is either a non-singular curve of genus two or a reducible curve composed of two elliptic curves which intersect transversally at a point. Thus, $q$: $\mathscr{X} \to S$ with $D$ is a family of principally polarized supersingular abelian surfaces. The number of degenerate fibers of $q|_D$: $D \to S$ is given by

$$5p - 5. \tag{1.13}$$

For the details of these facts, see Moret-Bailly [11].

*Remark 1.6*. In case char $k = 2$, we have also a similar family of principally polarized supersingular abelian surfaces to the one in (1.9) by another method (see Moret-Bailly [10]).

## §2. The locus of supersingular abelian surfaces

In this section, we assume char $k = p \geqslant 3$. Let $X$ be a finite union of varieties on which a finite group $G$ acts faithfully. Then, we call $X$ a Galois covering of $X/G$ with Galois group $G$. Let $\mathscr{A}_{2,1}$ (resp. $\mathscr{A}_{2,1,n}$, $(n, p) = 1$) defined over $k$. We have a Galois covering

$$\varphi_n\colon \mathscr{A}_{2,1,n} \to \mathscr{A}_{2,1}. \tag{2.1}$$

The Galois group is isomorphic to $PSp(4, \mathbf{Z}/n) = Sp(4, \mathbf{Z}/n)/\langle \pm 1 \rangle$ (cf. Mumford and Fogarty [13, p. 190]). In particular, in case $n = 2$, the Galois group is isomorphic to the symmetric group $S_6$ of degree six. We set $\varphi = \varphi_2$. As is well-known, the scheme $\mathscr{A}_{2,1,n}$ is a fine moduli scheme for $n \geqslant 3$ (cf. Mumford and Fogarty [13, p. 139]). We denote by $V$ (resp. $V_n$) the locus of supersingular abelian surfaces in $\mathscr{A}_{2,1}$ (resp. $\mathscr{A}_{2,1,n}$). Every component of $V$ and of $V_n$ is a rational curve (cf. Oort [15, p. 177]).

We consider an abelian surface $A = E_1 \times E_2$ with supersingular elliptic curves $E_1$ and $E_2$. Let $L$ be a polarization on $A$ which satisfies Condition (1.10). Then, as in Section 1, we get a family of principally polarized supersingular abelian surfaces $q: \mathscr{X} \to S$. This family is an abelian scheme. We consider a subscheme $\mathscr{X}[n] = \operatorname{Ker}[n]_{\mathscr{X}}$ over $S$, where $[n]_{\mathscr{X}}$ is the multiplication by an integer $n$ in $\mathscr{X}$. If $n$ is not divisible by $p$, then $q|_{\mathscr{X}[n]}$: $\mathscr{X}[n] \to S \simeq \mathbf{P}^1$ is an étale covering. Since $\mathbf{P}^1$ is simply connected, $\mathscr{X}[n]$ becomes a disjoint union of sections. Therefore, for the family $q: \mathscr{X} \to S$, we can put a level $n$-structure with a positive integer $n$ which is not divisible by $p$. Using this structure, we get a morphism

$$\psi_n: S \to \mathscr{A}_{2,1,n} \text{ (resp. } \psi = \psi_1: S \to \mathscr{A}_{2,1}). \tag{2.2}$$

Since the family $q: \mathscr{X} \to S$ is not a trivial family (cf. Oort [15] and Moret-Bailly [11, p. 131]), the image of this morphism gives a component of $V_n$ (resp. $V$). Conversely, we have the following theorem (see also Ekedahl [3, III, Theorem 1.1]).

THEOREM 2.1. *Any component of $V_n$ ($n \geqslant 2$, $(n, p) = 1$) can be obtained by the method in* (2.2) *with a suitable level n-structure and a polarization D as in* (1.11) *obtained from a suitable divisor L which satisfies Condition* (1.10). *Moreover, as L, we can take an effective divisor composed of two elliptic curves.*

*Proof.* Let $W$ be an irreducible component of $V_n$, $x$ a general point of $W$, and $(A_x, C_x, \eta_x)$ the principally polarized supersingular abelian surface with polarization $C_x$ and level $n$-structure $\eta_x$ which corresponds to the point $x$. By the generality of $x$, we see that $W$ is only one irreducible component of $V_n$ on which the point $x$ lies. Since the number of points in $\mathscr{A}_{2,1,n}$ which correspond to a product of two supersingular elliptic curves are finite (cf. Narasimhan and Nori [14], and also see [IKO, Theorem 2.10]), the abelian surface $A_x$ is not isomorphic to a product of two supersingular elliptic curves. Therefore, there exists in $A_x$ the unique subgroup scheme which is isomorphic to $\alpha_p$ (cf. Oort [16, Theorem 2]). We have the following diagram:

$$\begin{array}{ccc} A_x & \xrightarrow{\pi_x} & A_x/\alpha_p \\ {\scriptstyle \varphi_{C_x}} \downarrow & & \\ (A_x)^t & \xleftarrow{\pi_x^t} & (A_x/\alpha_p)^t, \end{array} \tag{2.3}$$

where $\pi_x$ is the canonical projection. We set $\varphi_{C_x}(C_x) = C$. Since $\varphi_{C_x}$ is an isomorphism, the divisor $C$ gives a principal polarization on $(A_x)^t$. It is clear that $A_x/\alpha_p$ and $(A_x/\alpha_p)^t$ are isomorphic to a product of two supersingular eliptic curves, and that $\pi_x \circ \varphi_C \circ \pi_x^t$ is nothing but the Frobenius morphism (see Lemma 1.3). We set $A = (A_x/\alpha_p)^t$ and $L = (\pi_x^t)^{-1}(C)$. We can assume that $C_x$ is symmetric. Then, the divisor $C$ and $L$ are also symmetric. Moreover, denoting by $F_A$ the Frobenius morphism of $A$, we have

$$K(L) \;=\; \operatorname{Ker} \varphi_L \;=\; \operatorname{Ker} \pi_x \circ \varphi_C \circ \pi_x^t \simeq \operatorname{Ker} F_A \simeq \alpha_p \times \alpha_p.$$

Therefore, the divisor $L$ satisfies Condition (1.10). Using this $L$ on $A$, we can construct a family $q\colon \mathscr{X} \to \mathbf{P}^1$ as in (1.9). By our construction, one of the fibres of this family is isomorphic to $(A_x, C_x) = (A, C)$. Now, we can choose the level $n$-structure of the family which coincides with $\eta_x$ at $(A_x, C_x)$. Then, the image of the morphism of $\psi_n$ which is obtained by this family as in (2.2) passes through the point $x$. By the uniqueness of the irreducible component of $V_n$ which passes through $x$, this image coincides with $W$. Hence, the former part of this theorem was proved.

Next, let $q\colon \mathscr{X} \to S$ be a family in (1.9), and let $D$ be a relative polarization on $\mathscr{X}$ obtained by the divisor $L$ on $A$ which satisfies Condition (1.10). By (1.13), the family $q|_D\colon D \to S$ has $5p - 5$ degenerate fibres. Each degenerate fibre consists of two elliptic curves which intersect each other transversally at a point. Let $C' = E' + E''$ be one of these fibres with elliptic curves $E'$ and $E''$. Then, using the notations in (1.9), we see that $\pi^{-1}(C')$ is linearly equivalent to $L$ as divisors on $A$ by (1.12). Therefore, to construct the family $q\colon \mathscr{X} \to S$, we can use $\pi^{-1}(C') = \pi^{-1}(E') + \pi^{-1}(E'')$ instead of $L$. Q.E.D.

COROLLARY 2.2. *Any component of $V$ can be obtained by the method in* (2.2) *with a polarization $D$ in* (1.11) *obtained from a suitable $L$ which satisfies Condition* (1.10). *Moreover, as $L$, we can choose an effective divisor composed of two elliptic curves.*

*Proof.* This follows from Theorem 2.1 and (2.1). Q.E.D.

THEOREM 2.3. (i) *For any point $x$ of $S$, the tangent mapping $(\mathrm{d}\psi_n)_x$ is injective for $n \geqslant 3$. In particular, every branch of the image of $\psi_n$ is non-singular for $n \geqslant 3$.*

(ii) *The morphism $\psi_n$ is generally an immersion for $n \geqslant 3$.*

*Proof.* (i) In this proof, we set $T = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$. Let $t: T \to S$ be a tangent at a point $x$ of $S$. We have the exact sequence

$$1 \longrightarrow H \times_S T \xrightarrow{\Delta'} A \times_k T \xrightarrow{\pi'} \mathscr{X} \times_S T \longrightarrow 1 \tag{2.4}$$

which is induced by (1.9). Suppose $(d\psi_n)_x(t) = 0$. Since $\mathscr{A}_{2,1,n}$ $(n \geqslant 3)$ is a fine moduli scheme, we have the natural homomorphism

$$\varrho: \mathscr{X} \times_S T \to \mathscr{X}_x \times_k T.$$

The morphism $\pi_x \times id_T$ coincides with $\varrho \circ \pi'$ on the closed fiber. Therefore, by the rigidity lemma (cf. Mumford and Fogarty [13, Proposition 6.1]), we have

$$\pi_x \times id_T = \varrho \circ \pi'.$$

This means that we have a factorization of $\Delta'$ such that the following diagram commutes:

$$\begin{tikzcd} \Delta': H \times_S T \arrow[r] \arrow[dr] & \alpha_p \times_k T \arrow[r] \arrow[d] & A \times_k T. \arrow[dl] \\ & T & \end{tikzcd} \tag{2.5}$$

Therefore, we conclude that we have a factorizationof $t$ such that $t$: $T \to \operatorname{Spec} k \to S$. Therefore, we have $t = 0$. Hence, the tangent mapping $(d\psi_n)_x$ is injective for $n \geqslant 3$.

(ii) We set $W = \psi_n(S)$, and denote by $\tilde{W}$ the normalization of $W$. Then, we have the following natural decomposition:

$$\begin{tikzcd} S \arrow[rr, "\psi_n"] \arrow[dr, "\tilde{\psi}_n"'] & & W, \\ & \tilde{W} \arrow[ur, "\Psi"'] & \end{tikzcd}$$

where $\Psi$ is the birational morphism obtained from the normalization of $W$. By (i), the tangent mapping $d\psi_n$ is injective at every point of $S$. Therefore, the mapping $d\tilde{\psi}_n$ is also injective at every point of $S$. Hence, the morphism $\tilde{\psi}_n$ is an étale morphism. Since $S$ is a non-singular rational curve, the curve $\tilde{W}$ is also a non-singular rational curve. Since the rational curve is simply connected, we conclude that $\tilde{\psi}_n$ is an isomorphism. Thus, the morphism $\psi_n$ is generically an immersion. Q.E.D.

The following theorem is due to N. Koblitz. He proved it, using the theory of deformation (see Koblitz [9, p. 193]).

THEOREM 2.4 (Koblitz). *Assume $n \geqslant 3$. Let $\psi_n(x)$ ($x \in S$) be a point which corresponds to a supersingular abelian surface ($A$, $C$, $\eta$) with principal polarization $C$ and level n-structure $\eta$.*

(i) *If $A$ is not isomorphic to a product of two supersingular elliptic curves, then there exists only one component of $V_n$ which passes through $\psi_n(x)$. Moreover, the locus $V_n$ of supersingular abelian surfaces in $\mathscr{A}_{2,1,n}$ is non-singular at $\psi_n(x)$.*

(ii) *If $A$ is isomorphic to a product of two supersingular elliptic curves, then there exist just $p+1$ branches of $V_n$ which passes through $\psi_n(x)$. These branches intersect each other transversally.*

*Remark 2.5.* By a method similar to the proof of Theorem 2.3, we can prove this theorem except the final statement of (ii). We omit details.

*Remark 2.6.* In case char $k = 2$, we can also construct $\psi_n$ with a positive integer $n$ ($n \geqslant 2$, $(n, p) = 1$) and $\psi$ in (2.2), using Remark 1.6, and we can show Theorems 2.1, 2.3, 2.4 and Corollary 2.2 by a similar method.

In Section 4, we treat the case of $\mathscr{A}_{2,1,2}$ (cf. Corollary 5.4).

THEOREM 2.7. *The number of irreducible components of $V$ is equal to the number of isomorphism classes of the families $a$: $\mathscr{X} \rightarrow S$ with relative polarization $D$ given as in* (1.9).

*Proof.* Let $\mathscr{F}$ be the set of representatives of isomorphism classes of the families $q$: $\mathscr{X} \rightarrow S$ with relative principal polarization $D$ given as in (1.9). We denote by $\mathscr{V}$ the set of irreducible components of $V$. By (2.2) we have a mapping

$$\mathscr{G} \rightarrow \mathscr{V}$$

By Theorem 2.1, this mapping is surjective. Let $q_i$: $\mathscr{X}_i \rightarrow S_i$ with relative polarization $D_i$ ($i = 1, 2$) be two families in $\mathscr{F}$ such that the images in $\mathscr{V}$ coincide. Let $x$ be a general point of the corresponding irreducible component of $V$. Then, we can find a point $x_1$ of $S_1$ (resp. $x_2$ of $S_2$) such that $\psi(x_1) = x$ (resp. $\psi(x_2) = x$) by the morphism $\psi$ as in (2.2). We have an isomorphism

$$\theta\colon ((\mathscr{X}_1)_{x_1}, (D_1)_{x_1}) \simeq ((\mathscr{X}_2)_{x_2}, (D_2)_{x_2}).$$

By the generality of $x$, neither $(\mathscr{X}_1)_{x_1}$ nor $(\mathscr{X}_2)_{x_2}$ is isomorphic to a product of two supersingular elliptic curves. Therefore, by Theorem 1.2 and Lemma 1.5, there exists an automorphism $\tilde{\theta}$ of $E \times E$ such that the following diagram commutes:

$$\begin{array}{ccc} E \times E & \xrightarrow{\tilde{\theta}} & E \times E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ (\mathscr{X}_1)_{x_1} & \xrightarrow{\theta} & (\mathscr{X}_2)_{x_2}, \end{array}$$

where $\pi_1$ and $\pi_2$ are purely inseparable homomorphisms of degree $p$. By our construction, $\tilde{\theta}(\pi_1^{-1}((D_1)_{x_1}))$ is algebraically equivalent to $\pi_2^{-1}((D_2)_{x_2})$. Therefore, by a suitable translation $T_x$ with $x \in E \times E$, $T_x(\tilde{\theta}(\pi_1^{-1}((D_1)_{x_1})))$ is linearly equivalent to $\pi_2^{-1}((D_2)_{x_2})$. Since $\pi_1^{-1}((D_1)_{x_1})$ (resp. $\pi_2^{-1}((D_2)_{x_2})$) satisfies Condition (1.10), the family $q_1\colon \mathscr{X}_1 \to S_1$ (resp. $q_2\colon \mathscr{X}_2 \to S_2$) is reconstructed by the divisor $\pi_1^{-1}((D_1)_{x_1})$ (resp. $\pi_2^{-1}((D_2)_{x_2})$) (cf. the proof of Theorem 2.1). Hence, these two families are isomorphic to each other, that is, the mapping $\mathscr{F} \to \mathscr{V}$ is injective. Q.E.D.

## §3. Standard divisors

Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $E_1$ and $E_2$ be elliptic curves defined over $k$.

*Definition 3.1.* We call an abelian surface $E_1 \times E_2$ with polarization $E_1 + E_2$ a principally polarized abelian surface of degenerate type.

PROPOSITION 3.2. *The number of principally polarized supersingular abelian surfaces of degenerate type is up to isomorphism equal to $h(h + 1)/2$, where $h$ is the number of supersingular elliptic curves.*

*Proof.* This follows easily from Theorem 1.1. For details, see [IKO, Section 3]. Q.E.D.

The number $h$ of supersingular elliptic curves is explicitly given by

$$h = (p - 1)/12 + \left\{1 - \left(\frac{-3}{p}\right)\right\}\Big/3 + \left\{1 - \left(\frac{-4}{p}\right)\right\}\Big/4, \tag{3.1}$$

where $(l/p)$ denotes the Legendre symbol (cf. Deuring [1, p. 266] and Igusa [6]). For an element $a$ of $k$, we have the endomorphism of $\alpha_p = \operatorname{Spec} k[\alpha]/(\alpha^p)$:

$$a\colon \alpha_p \to \alpha_p$$

defined by

$$\begin{array}{ccc} a^*\colon k[\alpha]/(\alpha^p) & \to & k[\alpha]/(\alpha^p). \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ \alpha & \longmapsto & a\alpha \end{array}$$

For two supersingular elliptic curves $E_1$ and $E_2$, we have an immersion

$$(1, a)\colon \alpha_p \hookrightarrow \alpha_p \times \alpha_p \subset E_1 \times E_2. \tag{3.2}$$

By $(1, \infty)$, we mean the immersion defined by $(0, 1)$. Since the image $(\lambda, \lambda a)(\alpha_p)$ with a non-zero element $\lambda$ of $k$ coincides with $(1, a)(\alpha_p)$, we can regard $(1, a)$ as a point on the projective line $S = \mathbf{P}^1$, and we call "$a$" a direction. An element $a$ of $k \cup \{\infty\}$ is called a good direction of $E_1 \times E_2$ if the quotient surface $(E_1 \times E_2)/(1, a)(\alpha_p)$ is isomorphic to a product of two supersingular elliptic curves. By Oort [15, Introduction], we have $p^2 + 1$ good directions for $E_1 \times E_2$. Let $C$ be a (not necessarily irreducible) curve of genus two which gives a symmetric principal polarization on $E_1 \times E_2$. By the same method as in Moret-Bailly [11, p. 139] for the case $E_1 + E_2$, there exists $p + 1$ directions $a$ among good directions such that the divisor $pC$ descends to $(E_1 \times E_2)/(1, a)\alpha_p$ as a divisor $L$ which satisfies Condition (1.10). Such $a$'s are called very good directions of $(E_1 \times E_2, C)$. It is easy to see that in case $C = E_1 + E_2$ neither 0 nor $\infty$ is a very good direction. In case $E_1 = E_2$, good directions are defined by

$$a^{p^2} = a \quad \text{or} \quad a = \infty \tag{3.3}$$

(cf. Oort [16, Introduction]). In case $E_1 = E_2$ and $C = E_1 + E_2$, very good directions are defined by

$$a^{p+1} = -1 \tag{3.4}$$

(cf. Moret-Bailly [11, p. 139]).

An element $\theta$ of $\operatorname{Aut}(E_1 \times E_2)$ acts on the set of directions of $E_1 \times E_2$ (resp. good directions of $E_1 \times E_2$), and if $\theta$ is an element of

Aut $(E_1 \times E_2, C)$, it acts on the set of very good directions of $(E_1 \times E_2, C)$ with symmetric principal polarization $C$ as follows:

if $\theta \circ (1, a) = (b_1, b_2)$ for a direction $a$ with elements $b_1, b_2$ of $k$, then the action of $\theta$ on the set is given by

$$a \mapsto b_2/b_1.$$

We write this action by $\theta(a) = b_2/b_1$. The action of an element $\theta'$ of $\mathrm{Aut}_v(E_1 \times E_2)$ on the set of directions of $E_1 \times E_2$ (resp. good directions of $E_1 \times E_2$, resp. very good directions of $(E_1 \times E_2, C)$ if $\theta' \in \mathrm{Aut}_v(E_1 \times E_2, C)$) is given by the action of $\gamma(\theta')$ (cf. (1.1)). For a very good direction $a$ of $(E_1 \times E_2, C)$, we denote by Aut $(E_1 \times E_2, C, a)$ the subgroup of Aut $(E_1 \times E_2, C)$ whose elements preserve the very good direction $a$. We set

$$RA(E_1 \times E_2, C, a) = \mathrm{Aut}\,(E_1 \times E_2, C, a)/\langle\iota\rangle$$

with the inversion $\iota$ of $E_1 \times E_2$.

Let $E$ be the elliptic curve in (1.4). We set

$$\tilde{A} = E \times E.$$

For an element $a$ of $k$, we consider an immersion

$$(1, a)\colon \alpha_p \hookrightarrow \alpha_p \times \alpha_p \subset E \times E = \tilde{A}. \tag{3.5}$$

Then, by (3.3), an element $a$ is a good direction if and only if $a \in \mathbf{F}_{p_2}$ or $a = \infty$.

LEMMA 3.3. *Under the notations as above, let $a$, $b$ be two good directions. Then, there exists an automorphism $\theta$ of $E \times E$ such that*

$$(1, a) = \theta \circ (1, b). \tag{3.6}$$

*Proof.* We consider the natural restriction

$$r\colon \mathrm{End}\,(E) \to \mathrm{End}\,(\alpha_p). \tag{3.7}$$

Then, by Oort [16, Lemma 5], we have $r(\mathrm{End}\,(E)) = \mathbf{F}_{p^2}$. We denote by id the indentity of End $(E)$. In case $a \neq \infty$ and $b \neq \infty$, we take an element

$u$ of End $(E)$ such that $r(u) = a - b$. Then, we have an automorphism

$$\begin{array}{cccc} \theta\colon E \times E & \longrightarrow & E \times E & \\ \cup\!\!\!\!\!\rule{0pt}{0pt} \ \ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} \mathrm{id} & 0 \\ u & \mathrm{id} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. & \end{array} \tag{3.8}$$

This automorphism satisfies (3.6). In case $a = \infty$ and $b \neq \infty$ (resp. $a \neq \infty$ and $b = \infty$), we take an element $u$ of End $(E)$ such that $r(u) = -b$ (resp. $r(u) = a$). Then, the automorphism

$$\theta = \begin{pmatrix} u & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \left( \text{resp. } \theta = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & u \end{pmatrix} \right)$$

satisfies (3.6). In case $a = \infty$ and $b = \infty$, we can take

$$\theta = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Q.E.D.

Let $\mathscr{E} = \{E_\lambda\}_{\lambda = 1, 2, \cdots, h}$ (for the definition of $h$, see (3.1)) be a set of representatives of isomorphism classes of supersingular elliptic curves defined over $k$. Using Theorem 1.1, for each pair $(E_m, E_n)$ $(E_m, E_n \in \mathscr{E}, m \leqslant n)$, we fix an isomorphism

$$\kappa_{m,n}\colon E_m \times E_n \xrightarrow{\sim} E \times E = \tilde{A}. \tag{3.9}$$

We also fix a very good direction $a$ of $(E \times E, E \times \{0\} + \{0\} \times E)$, and for each good direction $b$ of $E \times E$, we fix an automorphism $\theta_{a,b}$ which satisfies (3.6). Let $(E_m \times E_n, E_m + E_n, b)$ be a triple with a very good direction $b$ of $(E_m \times E_n, E_m + E_n)$. Then, by the isomorphism in (3.9), we can consider that this triple exists in $E \times E$. Moreover, $\kappa_{m,n}(b)$ becomes one of good directions of $E \times E$. For the sake of simplicity, we write again $(E_m \times E_n, E_m + E_n, b)$ instead of $(\kappa_{m,n}(E_m) \times \kappa_{m,n}(E_n), \kappa_{m,n}(E_m) + \kappa_{m,n}(E_n), \kappa_{m,n}(b))$. Using the automorphism $\theta_{a,b}$, we can turn $E_m \times E_n$ in $E \times E$ so that the very good direction $b$ of $(E_m \times E_n, E_m + E_n)$ may coincide with the direction $a$. By this method, we have $p + 1$ triples $(\theta_{a,b}(E_m) \times \theta_{a,b}(E_n), \theta_{a,b}(E_m) + \theta_{a,b}(E_n), a)$, using $(E_m \times E_n, E_m + E_n)$.

*Definition 3.4*. For supersingular elliptic curves $E_1'$ and $E_2'$ (resp. $E_1''$ and $E_2''$), let $b'$ (resp. $b''$) be a very good direction of $(E_1' \times E_2', E_1' + E_2')$ (resp.

$(E_1'' \times E_2'', E_1'' \times E_2''))$. A triple $(E_1' \times E_2', E_1' + E_2', b')$ is said to be isomorphic to a triple $(E_1'' \times E_2'', E_1'' + E_2'', b'')$ if there exists an isomorphism $\theta$ from $E_1' \times E_2'$ to $E_1'' \times E_2''$ such that $\theta(E_1' + E_2') = E_1'' + E_2''$ and $\theta(b') = b''$.

LEMMA 3.5. *Let $b$, $b'$ be very good directions of $(E_m \times E_n, E_m + E_n)$. The triple $(\tilde{A}, \theta_{a,b}(E_m) + \theta_{a,b}(E_n), a)$ is isomorphic to the triple $(\tilde{A}, \theta_{a,b'}(E_m) + \theta_{a,b'}(E_n), a)$ if and only if $(E_m \times E_n, E_m + E_n, b)$ is isomorphic to $(E_m \times E_n, E_m + E_n, b')$.*

*Proof.* Obvious.

We consider the set of divisors $\theta_{a,b}(E_m) + \theta_{a,b}(E_n)$, where $b$ runs through very good directions of $(E_m \times E_n, E_m + E_n)$. We say that $\theta_{a,b}(E_m) + \theta_{a,b}(E_n)$ is isomorphic to $\theta_{a,b'}(E_m) + \theta_{a,b'}(E_n)$ if there exists an element $\theta$ of $\mathrm{Aut}\,(E_m \times E_n, E_m + E_n)$ such that $\theta_{a,b'}(E_m) + \theta_{a,b'}(E_n) = \theta(\theta_{a,b}(E_m) + \theta_{a,b}(E_n))$. We denote by $\tilde{\mathscr{D}}(E_m, E_n)$ the set of representatives of isomorphism classes of divisors $\theta_{a,b}(E_m) + \theta_{a,b}(E_n)$ with very good directions $b$ of $(E_m \times E_n, E_m + E_n)$. The number of elements of $\tilde{\mathscr{D}}(E_m, E_n)$ is equal to the number of orbits of $RA(E_m \times E_n, E_m + E_n)$ in $p + 1$ very good directions of $(E_m \times E_n, E_m + E_n)$, which will be calculated in Section 6. We set

$$\tilde{\mathscr{D}} = \bigcup_{1 \leqslant m \leqslant n \leqslant h} \tilde{\mathscr{D}}(E_m, E_n). \tag{3.10}$$

By definition, any two triples $(\tilde{A}, C, a)$ and $(\tilde{A}, C', a)$ with $C, C' \in \tilde{\mathscr{D}}$ and $C \neq C'$ are not isomorphic to each other.

Now, we consider the following mapping:

$$\tilde{\pi}\colon \tilde{A} = E \times E \to (E \times E)/(1, a)(\alpha_p) = A. \tag{3.11}$$

Since $a$ is a very good direction of $E \times E$, the abelian surface $A$ is also isomorphic to $E \times E$. We set

$$\begin{aligned} \mathscr{D}(E_m, E_n) &= \{\tilde{\pi}(C)\colon C = \theta_{a,b}(E_m) + \theta_{a,b}(E_n) \in \tilde{\mathscr{D}}(E_m, E_n)\} \\ \mathscr{D} &= \bigcup_{1 \leqslant m \leqslant n \leqslant h} \mathscr{D}(E_m, E_n). \end{aligned} \tag{3.12}$$

Then, any divisor $L$ in $\mathscr{D}$ consists of two supersingular elliptic curves $E'$ and $E''$, that is, $L = E' + E''$, such that $E' \cap E'$ is equal to the subgroup scheme $\tilde{\pi}(\alpha_p \times \alpha_p) \simeq \alpha_p$ of $A$. All divisors in $\mathscr{D}$ satisfy Condition (1.10).

Moreover, by our choice, there does not exist any element of $\mathrm{Aut}_v(A)$ which transforms a divisor in $\mathscr{D}$ to another divisor in $\mathscr{D}$.

*Definition 3.6.* We call a divisor in $\mathscr{D}$ a standard divisor.

LEMMA 3.7. *Let $L' = E' + E''$ be an effective divisor on $A$ with supersingular elliptic curves $E'$ and $E''$ such that $L'$ satisfies Condition* (1.10). *Then, there exist a unique standard divisor $L$ on $A$ and an element $\theta$ of* $\mathrm{Aut}_v(A)$ *such that $\theta(L) = L'$.*

*Proof.* By Condition (1.10), two elliptic curves $E'$ and $E''$ intersect only at one point. Therefore, by a suitable translation $T_x$ of $A$, we can assume that $E_0' = T_x E'$ and $E_0'' = T_x E''$ intersect at the origin. The divisor $E_0' + E_0''$ also satisfies Condition (1.10). We can find elliptic curves $E_m$ and $E_n$ in $\mathscr{E}$ such that $E_0' \simeq E_m$ and $E_0'' \simeq E_n$. We may assume $m \leqslant n$. Using these isomorphisms, we have an isomorphism

$$\varrho\colon E_m \times E_n \simeq E_0' \times E_0''.$$

Let $\pi'\colon E_0' \times E_0'' \to A$ be the natural homomorphism. We set $\pi'' = \pi' \circ \varrho$. Then, we have the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \alpha_p \arrow[r, "i"] & E_m \times E_n \arrow[r, "\pi''"] \arrow[d, "\kappa_{m,n}", "\wr"'] & A \arrow[r] & 0 \text{ (exact)}, \\ & & \tilde{A} = E \times E & & \end{tikzcd} \tag{3.13}$$

where $i$ is an immersion. The immersion $i$ can be written as $i = (1, b)$ with a very good direction $b$ of $(E \times E, \kappa_{m,n}(E_m) + \kappa_{m,n}(E_n))$. We set $C = \theta_{a,b}(\kappa_{m,n}(E_m)) + \theta_{a,b}(\kappa_{m,n}(E_n))$. Then, by the definition of $\tilde{\mathscr{D}}$, there exists an element $\tilde{\theta}$ of $\mathrm{Aut}_v(\tilde{A})$ such that $\tilde{\theta}(C)$ is an element of $\tilde{\mathscr{D}}$. Hence, we have the following diagram:

$$\begin{tikzcd} E_m \times E_n \arrow[r, "\kappa_{m,n}"] \arrow[d, "\pi''"'] & \tilde{A} \arrow[r, "\theta_{a,b}"] & \tilde{A} \arrow[r, "\tilde{\theta}"] & \tilde{A} \arrow[d, "\tilde{\pi}"] \\ A \arrow[rrr, dashed, "\theta'"] & & & A, \end{tikzcd} \tag{3.14}$$

where $\theta'$ is the induces isomorphism. We set $L = \tilde{\pi}(\tilde{\theta}(C))$ and $\theta = T_x^{-1} \circ (\theta')^{-1}$. Then, we see $L \in \mathscr{D}$ and $\theta(L) = L'$. The uniqueness of $L$ follows from the definition of $\mathscr{D}$. Q.E.D.

Using Theorem 2.1, Lemma 3.7 and Remark 2.6, we have the following:

COROLLARY 3.8. *Any component of* $V_n$ $(n \geqslant 2, (n, p) = 1)$ *and of* $V$ *can be obtained by the method in* (2.2) *with a divisor in* $\mathscr{D}$.

## §4. Groups of automorphisms of families

In this section, we assume char $k = p \geqslant 3$, and we use the same notations as in Section 3. As in (1.9) in Section 1, using the abelian surface $A$ in (3.11), we have a family $q\colon \mathscr{X} \to S \simeq \mathbf{P}^1$ of principally polarized supersingular abelian surfaces with relative polarization $D$. We examine the group of automorphisms of this family which preserve the relative polarization $D$. The family $pr_2|_{D'}\colon D' = \pi^{-1}(D) \to S$ has $5p - 5$ reducible fibres. We can consider these reducible fibres as divisors on $A$. By (1.12) they are linearly equivalent to each other and satisfy Condition (1.10). We denote by $\mathscr{B}(\mathscr{X}, D)$ the set of such $5p - 5$ divisors on $A$. For the sake of simplicity, we set $\mathscr{B} = \mathscr{B}(\mathscr{X}, D)$. By Lemma 3.7, for any divisor $L'$ in $\mathscr{B}$ there exist an element $\theta$ of $\mathrm{Aut}_v(A)$ and a standard divisor $L$ of $\mathscr{D}$ such that $\theta(L) = L'$. We set

$$\left\{\begin{array}{rcl} \mathscr{D}(\mathscr{B}) & = & \{L \in \mathscr{D}\colon \text{there exist } L' \in \mathscr{B} \text{ and } \theta \in \mathrm{Aut}_v(A) \\ & & \text{such that } L' = \theta(L)\}, \\ \Gamma(\mathscr{B}) & = & \{\theta \in \mathrm{Aut}_v(A)\colon \text{the automorphism } \theta \text{ induces a} \\ & & \text{permutation of elements of } \mathscr{B}\}, \\ R\Gamma(\mathscr{B}) & = & \Gamma(\mathscr{B})/\langle \iota \rangle, \end{array}\right. \tag{4.1}$$

where $\iota$ is the invertion of $A$. It is clear that $\Gamma(\mathscr{B})$ and $R\Gamma(\mathscr{B})$ are finite groups. Since $\iota$ acts trivially on $\mathscr{B}$, the group $R\Gamma(\mathscr{B})$ acts on $\mathscr{B}$. For an element of $\mathscr{B}$ and an element $x = \theta(L)$ of $\mathscr{D}(\mathscr{B})$ with $\theta \in \mathrm{Aut}_v(A)$, it is easy to see that $R\Gamma(\mathscr{B})_x$ is isomorphic to $R\Gamma(\mathscr{B})_L$. Considering the orbits, we have the following equality:

$$5p - 5 = \sum_{x \in \mathscr{D}(\mathscr{B})} (|R\Gamma(\mathscr{B})|/|R\Gamma(\mathscr{B})_x|), \tag{4.2}$$

where $R\Gamma(\mathscr{B})_x$ denotes the stabilizer of $R\Gamma(\mathscr{B})$ at $x \in \mathscr{D}(\mathscr{B})$. In case $x = \tilde{\pi}(\theta_{a,b}(E_m) + \theta_{a,b}(E_n))$, by Lemma 3.5 we have an isomorphism

$$R\Gamma(\mathscr{B})_x = RA(E_m \times E_n, E_m + E_n, b). \tag{4.3}$$

In Section 6, we will calculate the order of $RA(E_m \times E_n, E_m + E_n, b)$. Using the exact sequence (1.1), we set

$$\Gamma(\mathscr{B})' = \gamma(\Gamma(\mathscr{B})). \tag{4.4}$$

It is easy to see that $\Gamma(\mathscr{B})$ does not contain translations. Therefore, we have

$$\Gamma(\mathscr{B}) \simeq \Gamma(\mathscr{B})'. \tag{4.5}$$

The group $\Gamma(\mathscr{B})'$ acts on the tangent space $T_A$ of $A$ at the origin. By the action, the group $\Gamma(\mathscr{B})'$ acts on $S = \mathbf{P}(T_A)$. Therefore, the group $\Gamma(\mathscr{B})'$ acts on the family $pr_2\colon A_S \to S$. Considering the restriction on the action of $\Gamma(\mathscr{B})'$ on $A_S$ to $\alpha_p \times \alpha_p \times S$, we have an action of $\Gamma(\mathscr{B})'$ on the family $pr_2|_H\colon H \to S$. Hence, we have an action of $\Gamma(\mathscr{B})'$ on the family $q\colon \mathscr{X} \to S$. This action preserves the relative polarization $D$. Conversely, suppose that $\sigma$ is an automorphism of the family $q\colon \mathscr{X} \to S$ which preserves the relative polarization $D$. Let $x$ be a point on $S$ such that $\mathscr{X}_x$ is not isomorphic to a product of two supersingular elliptic curves. Then, we have the isomorphism

$$\sigma_x\colon \mathscr{X}_x \to \mathscr{X}_{\sigma(x)}$$

which is induced by $\sigma$. By Lemma 1.5, we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\tilde{\sigma}_x} & A \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi_{\sigma(x)}} \\ \mathscr{X}_x & \xrightarrow{\sigma_x} & \mathscr{X}_{\sigma(x)}, \end{array} \tag{4.6}$$

which are obtained from (1.9) with $A = E \times E$ and an automorphism $\tilde{\sigma}_x$ of $A$. By assumption, we have $\sigma_x(D_x) \equiv D_{\sigma(x)}$. Since $\sigma_x(D_x)$ and $D_{\sigma(x)}$ are a principal polarization on $\mathscr{X}_{\sigma(x)}$, the divisor $\sigma_x(D_x)$ is transformed into $D_{\sigma(x)}$ by a suitable translation on $\mathscr{X}_{\sigma(x)}$. Therefore, the divisor $\tilde{\sigma}_x(\pi_x^{-1}(D_x))$ is transformed into $(\pi_{\sigma(x)})^{-1}(D_{\sigma(x)})$ by a suitable translation on $A$. Therefore, the composition of $\tilde{\sigma}_x$ and the translation is an element of $\Gamma(\mathscr{B})$, hence, $\tilde{\sigma}_x$ is an element of $\Gamma(\mathscr{B})'$. Let $\sigma'$ be an automorphism of the family $q\colon \mathscr{X} \to S$ which is induced by $\tilde{\sigma}_x$ as above. We consider the automorphism $\tau = \sigma' \circ \sigma^{-1}$. Then, $\tau$ induces the identity on $\mathscr{X}_x$. Therefore, by the rigidity lemma (cf. Mumford and Fogarty [13, Proposition 6.1]), $\tau$ is the identity on $\mathscr{X}$, that is, $\sigma = \sigma'$. Hence, the group $\Gamma(\mathscr{B})'$ is isomorphic to the group of automorphisms of the family $q\colon \mathscr{X} \to S$ which preserve the relative polarization $D$. By (4.5), we have the following theorem.

THEOREM 4.1. *The group of automorphisms of the family $q\colon \mathscr{X} \to S$ which preserve the relative polarization $D$ is isomorphic to the group $\Gamma(\mathscr{B})$.*

Let $n$ ($n \geqslant 2$) be an integer which is not divisible by $p$. As in Section 2, we choose a level $n$-structure on $q\colon \mathscr{X} \to S$. We denote by $G_n$ the Galois group of the covering $\varphi_n\colon \mathscr{A}_{2,1,n} \to \mathscr{A}_{2,1}$. Let $W_n$ (resp. $W$) be the component of $V_n$ (resp. $V$) which is obtained from the family $q\colon \mathscr{X} \to S$ with polarization $D$ and level $n$-structure as above. We set

$$G(W_n) = \{g \in G_n\colon g(W_n) \subset W_n\}.$$

Since $S$ is birationally equivalent to $W_n$ by Theorem 2.3 (ii), the group $G(W_n)$ acts on $S$. Since the stabilizer of $G(W_n)$ at a general point of $W_n$ is trivial (cf. Ibukiyama, Katsura and Oort [5, Propositions 1.3, 1.13 and Theorem 3.3]), $W_n/G(W_n)$ is birationally equivalent to $W$. Hence, we see that

$$S/G(W_n) \text{ is birationally equivalent to } W. \tag{4.7}$$

THEOREM 4.2. *Under the notations as above, the group $G(W_n)$ is isomorphic to $R\Gamma(\mathscr{B})$ for $n \geqslant 3$.*

*Proof.* Since $\mathscr{A}_{2,1,n}$ ($n \geqslant 3$) is a fine moduli scheme, we have the following cartesian diagram:

$$\begin{array}{ccc} \mathscr{X} & \longrightarrow & U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle u} \\ S & \longrightarrow & \mathscr{A}_{2,1,n}, \end{array} \tag{4.8}$$

where $u\colon U \to \mathscr{A}_{2,1,n}$ is the universal family of principally polarized abelian surfaces with level $n$-structure. Let $G_U$ be the group of automorphisms of the family $u\colon U \to \mathscr{A}_{2,1,n}$ which preserve the relative polarization, and let $\iota_U$ be the inversion of this family. We denote by $G_{U,W}$ the subgroup of $G_U$ which consists of automorphisms of the family $u|_{u^{-1}(W_n)}\colon u^{-1}(W_n) \to W_n$.

$$G(W_n) \simeq G_{U,W}/\langle \iota_U \rangle.$$

Since $G_{U,W}$ acts on the family $q\colon \mathscr{X} \to S$ by (4.8), we have an injective homomorphism $G_{U,W} \hookrightarrow \Gamma(\mathscr{B})' \simeq \Gamma(\mathscr{B})'$, hence, $G(W_n) \hookrightarrow R\Gamma(\mathscr{B})$. Therefore, we have morphisms

$$S/G(W_n) \to S/R\Gamma(\mathscr{B}) \to W \subset \mathscr{A}_{2,1}.$$

By (4.7), we see that $S/G(W_n)$ is birationally isomorphic to $S/R\Gamma(\mathscr{B})$. Hence, $G(W_n)$ is isomorphic to $R\Gamma(\mathscr{B})$ for $n \geqslant 3$. Q.E.D.

Now, we investigate the level 2-structure.

THEOREM 4.3. *$G(W_2)$ is isomorphic to $R\Gamma(\mathscr{B})$.*

*Proof.* We have a Galois covering

$$\varphi_{2,4}\colon \mathscr{A}_{2,1,4} \to \mathscr{A}_{2,1,2} \tag{4.9}$$

(cf. Mumford and Fogarty [13, p. 140]). By a suitable choice of the level 4-structure, we have the following commutative diagram:

$$\begin{tikzcd} & W_4 \arrow[r] \arrow[d] & \mathscr{A}_{2,1,4} \arrow[d] \\ S \arrow[ur, "\psi_4"] \arrow[r, "\psi_2"] & W_2 \arrow[r] & \mathscr{A}_{2,1,2}, \end{tikzcd} \tag{4.10}$$

where $\psi_2(S) = W_2$ and $\psi_4(S) = W_4$. Corresponding to this diagram, we have the exact sequence of groups

$$1 \to N \to G(W_4) \to G(W_2) \to 1 \tag{4.11}$$

with a normal subgroup $N$ of $G(W_4)$. Let $\sigma$ be an element of $N$ which is not the identity. Since $W_4$ is birationally equivalent to $\mathbf{P}^1$, the automorphism $\sigma$ has a fixed point $x$ on $W_4$. Let $(A', C, \eta)$ be the principally polarized supersingular abelian surface with level 4-structure $\eta$ corresponding to $x$. We may assume that $C$ is a (not necessarily irreducible) curve of genus two and it is symmetric. Then, $\sigma$ induces an element $\sigma'$ of $RA(A', C)$. The group $RA(A', C)$ acts on the set of points of order two of $A'$. Since $\sigma$ is an element of $N$, $\sigma'$ fixes all points of order two of $A'$. Considering the list of automorphisms of curves of genus two in Igusa [7], this is impossible if $C$ is a non-singular irreducible curve of genus two. Therefore, $(A', C)$ and $\sigma'$ are of the following type:

$$\left\{\begin{array}{l} A' = E_1 \times E_2,\ C = E_1 + E_2 \text{ with supersingular elliptic} \\ \qquad \text{curves } E_1 \text{ and } E_2, \\[2ex] \sigma' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \langle \iota \rangle, \end{array}\right. \tag{4.12}$$

where $-1$ (resp. 1) is the inversion of $E_1$ (resp. the identity of $E_2$). We set

$$\tau' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By Theorem 4.2, the automorphism $\sigma$ is induced by an element of $R\Gamma(\mathscr{B})$ with a suitable $\mathscr{B}$ as in (4.1). Therefore, by the construction of the family, there exists an automorphism $\tau$ of $A = E \times E$ with $E$ as in (1.4) such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\pi'} & A' \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau'} \\ A & \xrightarrow{\pi'} & A', \end{array}$$

where $\pi'$ is the morphism induced by the morphism $\pi$ as in (1.9). Here, we have Ker $\pi' \simeq \alpha_p$, and $(\pi')^{-1}(C)$ satisfies Condition (1.10) and $\tau((\pi')^{-1}(C))$ is algebraically equivalent to $(\pi')^{-1}(C)$. On the other hand, the automorphism $\tau'$ fixes two directions which are tangents to $E_1$ and $E_2$, respectively. Therefore, it is easy to see that $K((\pi')^{-1}(C))$ is isomorphic to either Ker $[p]_{E_1}$ or Ker $[p]_{E_2}$ (cf. Lemma 1.4), which is not isomorphic to $\alpha_p \times \alpha_p$. A contradiction. Hence, we have $N = \{1\}$, and $G(W_4) \simeq G(W_2)$. Now, this theorem follows from Theorem 4.1. Q.E.D.

COROLLARY 4.4. *$R\Gamma(\mathscr{B})$ is isomorphic to a subgroup of $S_6$.*

*Proof.* Since $G_2 \simeq PSp(4, \mathbb{Z}/2)$ is isomorphic to $S_6$ and $G(W_2)$ is a subgroup of $G_2$, this corollary follows from Theorem 4.3. Q.E.D.

COROLLARY 4.5. *Under the notations in Section 2, the morphism $\psi_2$ is generically an immersion.*

*Proof.* Since $G(W_2)$ is isomorphic to $G(W_4)$ as above, this corollary follows from Theorem 2.3 (ii). Q.E.D.

*Remark 4.6.* Considering the family commented in Remark 1.6, we see that Theorem 4.1 also holds in case $p = 2$, and that $S/\Gamma(\mathscr{B})$ is the normalization of $V$. We omit the details.

## §5. The number of irreducible components of $V_n$ and of $V$

Let $k$ be an algebraically closed field of characteristic $p \geqslant 3$, unless otherwise mentioned. Let

$$\mathscr{F} = \{q_\lambda\colon \mathscr{X}_\lambda \to S \text{ with relative polarization } D_\lambda\}_{\lambda=1,2,\cdots,H'}$$

be the set of representatives of isomorphism classes of the families with relative polarization $D_\lambda$ given as in (1.9). By Theorem 2.7, the number $H'$ is equal to the number of irreducible components of $V$. We set

$$\mathscr{B}_\lambda = \mathscr{B}(\mathscr{X}_\lambda, D_\lambda),\ \mathscr{D}_\lambda = \mathscr{D}(\mathscr{B}_\lambda),\ \Gamma_\lambda = \Gamma(\mathscr{B}_\lambda) \text{ and } R\Gamma_\lambda = R\Gamma(\mathscr{B}_\lambda). \tag{5.1}$$

As in (4.2),

$$5p - 5 = \sum_{x \in \mathscr{D}_\lambda} |R\Gamma_\lambda| / |(R\Gamma_\lambda)_x|. \tag{5.2}$$

We note that $\mathscr{D} = \bigcup_\lambda \mathscr{D}_\lambda$, and that for every $x \in \mathscr{D}$, there exists only one $\lambda$ such that $x \in \mathscr{D}_\lambda$. Therefore, using this correspondence, we have a surjective mapping:

$$\Lambda\colon \mathscr{D} \to \{1, 2, \cdots, H'\}. \tag{5.3}$$

Considering the action of $RA(E_m \times E_n, E_m + E_n)(E_m, E_n \in \mathscr{E}, m \leqslant n)$ on the set of $p + 1$ very good directions of $(E_m \times E_n, E_m + E_n)$, we have

$$p + 1 = \sum_{x \in \mathscr{D}(E_m, E_n)} |RA(E_m \times E_n, E_m + E_n)| / |(R\Gamma_{\Lambda(x)})_x|. \tag{5.4}$$

By the mass formula for supersingular elliptic curves (cf. Deuring [1, Sections 5 and 10]):

$$\sum_{E_m \in \mathscr{E}} 1/|\mathrm{Aut}\,(E_m)| = (p - 1)/24, \tag{5.5}$$

we have

$$\sum_{1 \leqslant m \leqslant n \leqslant h} 1/|RA(E_m \times E_n, E_m + E_n)| = \{(p - 1)/24\}^2 \tag{5.6}$$

(for details, see [IKO, Section 3.1]). We denote by $|V_n|$ the number of irreducible components of $V_n$.

THEOREM 5.1. *Assume* $(n, p) = 1$. *Then,*

$$|V_n| = |PSp(4, \mathbb{Z}/n)|(p^2 - 1)/2880$$

*Proof.* The group $PSp(4, \mathbb{Z}/n)$ acts on the set of irreducible components of $V_n$ (cf. Section 2). Therefore, considering the stabilizer of each component, by Theorem 2.3 (ii) and Corollary 4.5 we have

$$\begin{aligned}
|V_n| &= \sum_{\lambda=1}^{H'} |PSp(4, \mathbb{Z}/n)|/|R\Gamma_\lambda| \quad \text{(by Theorems 4.2 and 4.3)} \\
&= \{|PSp(4, \mathbb{Z}/n)|/(5p - 5)\} \left\{ \sum_{\lambda=1}^{H'} \sum_{x \in \mathscr{D}_\lambda} 1/|(R\Gamma_\lambda)_x| \right\} \quad \text{(by 5.2))} \\
&= \{|PSp(4, \mathbb{Z}/n)|/(5p - 5)\} \left\{ \sum_{x \in \mathscr{D}} 1/|(R\Gamma_{\Lambda(x)})_x| \right\} \\
&= \{|PSp(4, \mathbb{Z}/n)|/(5p - 5)\} \left\{ \sum_{1 \leqslant m \leqslant n \leqslant h} \sum_{x \in \mathscr{D}(E_m, E_n)} 1/|(R\Gamma_{\Lambda(x)})_x| \right\} \\
&= \{|PSp(4, \mathbb{Z}/n)|/(5p - 5)\} \\
&\quad \times \left\{ \sum_{1 \leqslant m \leqslant n \leqslant h} (p + 1)/|RA(E_m \times E_n, E_m + E_n)| \right\} \quad \text{(by 5.4))} \\
&= \{|PSp(4, \mathbb{Z}/n)|(p^2 - 1)\}/2880 \quad \text{(by 5.6)).} \qquad \text{Q.E.D.}
\end{aligned}$$

COROLLARY 5.2. *Let $q$ be an odd prime number different from $p$. Then,*

$$|V_2| = (p^2 - 1)/4 \text{ and } |V_q| = q^4(q^4 - 1)(q^2 - 1)(p^2 - 1)/5760.$$

*Proof.* Since we have $|PSp(4, \mathbb{Z}/2)| = |S_6| = 720$ and $|PSp(4, \mathbb{Z}/q)| = q^4(q^4 - 1)(q^2 - 1)/2$, this corollary follows from Theorem 5.1. Q.E.D.

Let $\mathscr{S}_n$ be the set of points in $V_n$ which correspond to principally polarized supersingular abelian surfaces $(A, C, \eta)$ with principal polarization $C$ and level $n$-structure $\eta$ such that $A$ is isomorphic to a product of two supersingular elliptic curves.

THEOREM 5.3. *Assume* $(n, p) = 1$ *and* $n \geqslant 3$. *Then,*

$$|\mathscr{S}_n| = |PSp(4, \mathbb{Z}/n)|(p-1)(p^2+1)/2880.$$

*Proof.* For an abelian surface $E \times E$ with a supersingular elliptic curve $E$, we have $p^2 + 1$ good directions (see (3.3)). Therefore, by construction, on each family $q_\lambda \colon \mathscr{X}_\lambda \to S$ there exist just $p^2 + 1$ fibres $(\mathscr{X}_\lambda)_x$ which are isomorphic to $E \times E$ (cf. Theorem 1.1). If we consider the set $\mathscr{F}$ of all such families, by Theorems 2.3 (ii) and (2.4) (ii) each $p + 1$ such fibres correspond to the same point in $\mathscr{A}_{2,1,n}$ $(n \geqslant 3)$. Hence, by Theorem 5.1, the number $|\mathscr{S}_n|$ is equal to $\{(p^2+1)|PSp(4, \mathbb{Z}/n)|(p^2-1)/2880\}/(p+1) = |PSp(4, \mathbb{Z}/n)|(p-1)(p^2+1)/2880$. Q.E.D.

By the same way as in the proof of Corollary 5.2, we have the following:

COROLLARY 5.4. *Let* $q$ *be an odd prime number different from* $p$. *Then,*

$$|\mathscr{S}_q| = q^4(q^4-1)(q^2-1)(p-1)(p^2+1)/5760.$$

THEOREM 5.5. *Assume* $p \geqslant 3$. *Then,*

$$|\mathscr{S}_2| = (p-1)(p^2+5p-4)/4.$$

*Proof.* We consider the Galois covering in (4.9). Corresponding to this covering, we have an exact sequence of groups

$$1 \to \tilde{N} \to PSp(4, \mathbb{Z}/4) \to PSp(4, \mathbb{Z}/2) \to 1.$$

The group $\tilde{N}$ acts on $\mathscr{S}_4$, and we have $\mathscr{S}_2 \simeq \mathscr{S}_4/\tilde{N}$. Let $\sigma$ be an element of $\tilde{N}$ such that $\sigma$ has at least one fixed point in $\mathscr{S}_4$. Then, by the same method as in the proof of Theorem 4.2, we see that the fixed point corresponds to a principally polarized abelian surface $(A, C, \eta)$ with level 4-structure $\eta$ such that

$$\begin{cases} A = E_1 \times E_2 \text{ with supersingular elliptic curves } E_1 \text{ and } E_2, \\ C = E_1 + E_2, \text{ and} \\ \sigma \text{ induces either } \sigma' \text{ in (4.12) or the identity of } RA(A). \end{cases} \tag{5.7}$$

Conversely, if $(A, C)$ and $\sigma'$ are given as in (5.7), then $\sigma'$ gives an element $\sigma$ of $\tilde{N}$ which fixes a point in $\mathscr{S}_4$ corresponding to $(A, C, \eta)$ with principal polarization $C$ and any level 4-structure $\eta$. Therefore, by (1.13), (3.3) and Theorem 2.4 (ii), we have

$$\begin{aligned}|\mathscr{S}_2| &= \{(5p-5)|V_4|/(p+1)\}(2/|\tilde{N}|) \\ &\quad + \{\{(p^2+1)-(5p-5)\}|V_4|/(p+1)\}(1/|\tilde{N}|) \\ &= \{(p^2+5p-4)/(p+1)\}\{|V_4|/|\tilde{N}|\}.\end{aligned}$$

Since $|\tilde{N}| = |PSp(4, \mathbb{Z}/4)|/|PSp(4, \mathbb{Z}/2)| = |PSp(4, \mathbb{Z}/4)|/720$, by Theorem 5.1 we have the desired result. Q.E.D.

We now give an algebraic proof of the mass formula in our special case (cf. Eichler [2, Satz 1], see also Hashimoto and Ibukiyama [4, (I), Section 3]).

THEOREM 5.6. (*Eichler*). *The following equality holds*:

$$\sum_{(A,C)} 1/|RA(C)| = (p-1)(p^2+1)/2880, \tag{5.8}$$

*where $(A, C)$ runs through isomorphism classes of principally polarized supersingular abelian surfaces such that $A$ is isomorphic to a product of two supersingular elliptic curves.*

*Proof.* The group $G_3 = PSp(4, \mathbb{Z}/3)$ acts on the set $\mathscr{S}_3$. We denote by $\mathscr{S}_3'$ the set of representatives of orbits. Let $(A, C, \eta)$ be the principally polarized abelian surface with principal polarization $C$ and level 3-structure $\eta$ which corresponds to $x \in \mathscr{S}_3$. Then, the group $RA(C) \simeq RA(A, C)$ is isomorphic to the stabilizer $(G_3)_x$. Considering the orbits of $G_3$ in $\mathscr{S}_3$, by Corollary 5.4 we have

$$9(p-1)(p^2+1) = \sum_{s \in \mathscr{S}_3'} |G_3|/|(G_3)_x|.$$

Since $\mathscr{S}_3'$ corresponds to the set of representatives of isomorphism classes of $(A, C)$, we have the formula (5.8). Q.E.D.

Let $B$ be a definite quaternion algebra over the field $\mathbf{Q}$ of rational numbers with discriminant $p$. We regard $B^2$ as a left vector space over $B$. We denote by $H_2(1, p)$ the class number of the non-principal genus in $B^2$ (for details, see Shimura [18]). This number $H_2(1, p)$ is explicitly calculated in Hashimoto and Ibukiyama [4, (II)] (see also [IKO, Remark 2.17]). To prove the following theorem, Theorem 2.15 in [IKO] plays an important role.

THEOREM 5.7. *Assume* char $k = p > 0$. *The number* $H'$ *of irreducible components of* $V$ *is equal to* $H_2(1, p)$.

*Proof.* For $p = 2$, this follows from Igusa [7] (see also Section 8, (1)). Now, we assume $p \geqslant 3$. Let $E$ be a supersingular elliptic curve as in (1.4). We set $A = E \times E$. Let $a$ be a direction of $A$ such that $A' = (E \times E)/(1, a)(\alpha_p)$ is not isomorphic to a product of two supersingular elliptic curves. We consider the natural projection:

$$\pi: A \to A'.$$

We denote by NS($A$) (resp. NS($A'$)) the Néron–Severi group of $A$ (resp. $A'$). For an element $C'$ of NS($A'$), $C' > 0$ means that the divisor class $C'$ contains an effective divisor. We set

$$\mathscr{P}' = \{C' \in \mathrm{NS}(A'): (C')^2 = 2, C' > 0\}$$

and

$$\mathscr{P} = \pi^{-1}(\mathscr{P}') = \{\pi^{-1}(C'): C' \in \mathscr{P}'\}.$$

We define an equivalent relation $\approx$ on $\mathscr{P}$ as follows:

$$C_1 \approx C_2 (C_1, C_2 \in \mathscr{P}) \text{ if and only if } g^*(C_1) \equiv C_2 \text{ for some } g \text{ of Aut } (A).$$

By Oort [16, Theorem 2], we see that a representative divisor of any element of $\mathscr{P}$ satisfies Condition (1.10). Therefore, by (2.2) we have the natural mapping

$$\Phi: \mathrm{P}/\approx \;\to \{\text{irreducible components of } V\}.$$

By Theorems 1.1 and 1.2, on each irreducible component of $V$ there exists a point which corresponds to the abelian surface $A'$ with a suitable polarization $C'$. Considering the divisor $\pi^{-1}(C')$, we can reconstruct the original irreducible component as in the proof of Theorem 2.1. Therefore, the mapping $\Phi$ is surjective. Suppose that $C_1 = \pi^{-1}(C_1')$ and $C_2 = \pi^{-1}(C_2')$ are two effective divisors of $\mathscr{P}$ such that $\Phi(C_1) = \Phi(C_2)$. Then, by the

construction of the family in (1.9), we can find an effective divisor $L_1'$ of $\mathscr{P}'$ which satisfies the following two properties:
(i) $\pi^{-1}(L_1') = L_1$ is linearly equivalent to $C_1$,
(ii) there exists an element $g'$ of Aut $(A')$ such that $(g')^*(L_1')$ is algebraically equivalent to $C_2'$.
By Lemma 1.5, we can find an element of $g$ of Aut $(A)$ such that $\pi \circ g = g' \circ \pi$. Since $g^*(C_1)$ is linearly equivalent to $g^*(L_1)$ and $g^*(L_1)$ is algebraically equivalent to $C_2$, we conclude that $g^*(C_1)$ is algebraically equivalent to $C_2$. Hence, the mapping $\Phi$ is injective. Since we know that $|\mathscr{P}/\approx|$ is equal to $H_2(1, p)$ (cf. [IKO, Theorem 2.15]), the number of irreducible components of $V$ is equal to $H_2(1, p)$. Q.E.D.

THEOREM 5.8. *Assume* char $k = p > 0$. *The supersingular locus $V$ is irreducible if and only if $p \leqslant 11$.*

*Proof.* This follows from the explicit formula of $H_2(1, p)$ (cf. Hashimoto and Ibukiyama [4, (II)], and see also [IKO, Remark 2.17]). Q.E.D.

*Remark 5.9.* We will write explicitly the number $H_2(1, p)$ for small $p$'s in Table 4 in Section 7 (see also Hashimoto and Ibukiyama [4, (II), p. 698]).

## §6. Groups of automorphisms of principally polarized supersingular abelian surfaces of degenerate type

In this section, we assume char $k = p \geqslant 2$. Let $E_1$ and $E_2$ be two elliptic curves. Then, we have the following two cases:

$$\mathrm{Aut}\,(E_1 \times E_2, E_1 + E_2) \simeq \mathrm{Aut}\,(E_1) \times \mathrm{Aut}\,(E_2) \quad \text{if } E_1 \not\simeq E_2, \tag{6.1}$$

$$1 \to \mathrm{Aut}\,(E_1) \times \mathrm{Aut}\,(E_2) \to \mathrm{Aut}\,(E_1 \times E_2, E_1 + E_2) \to \mathbb{Z}/2 \to 1 \quad \text{(exact)} \quad \text{if } E_1 \simeq E_2. \tag{6.2}$$

Moreover, if $E_1 \simeq E_2$, then we have

$$\mathrm{Aut}\,(E_1 \times E_2, E_1 + E_2) \simeq \langle \mathrm{Aut}\,(E_1) \times \mathrm{Aut}\,(E_2), \sigma\rangle, \tag{6.3}$$

where $\sigma$ is the automorphism of $E_1 \times E_2$ defined by

$$\begin{array}{cccc} \sigma\colon & E_1 \times E_2 & \longrightarrow & E_1 \times E_2. \\ & \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ & (x, y) & \longrightarrow & (y, x) \end{array} \tag{6.4}$$

Here, we identify $E_1$ with $E_2$ through the isomorphism $E_1 \simeq E_2$. For an elliptic curve $E'$, we denote by $j(E')$ the $j$-invariant of $E'$. In case $p \neq 2, 3$, we denote by $E_\omega$ (resp. $E_i$) the elliptic curve defined by

$$E_\omega\colon Y^2 = X^3 - 1 \text{ (resp. } E_i\colon Y^2 = X^3 - X). \tag{6.5}$$

We have $j(E_\omega) = 0$ (resp. $j(E_i) = 1728$). As is well-known, we have

$$\mathrm{Aut}\,(E_\omega) \simeq \mathbb{Z}/6 \text{ (resp. } \mathrm{Aut}\,(E_i) \simeq \mathbb{Z}/4).$$

The elliptic curve $E_\omega$ (resp. $E_i$) is supersingular if and only if $p \equiv 2 \pmod 3$ (resp. $p \equiv 3 \pmod 4$). In case $p = 2$ (resp. $p = 3$), there exists a unique supersingular elliptic curve (cf. Deuring [1]). We denote by $\xi$ (resp. $\zeta$) a primitive twelfth root of unity (resp. a primitive eighth root of unity). By (3.4), we have the following lemmas.

LEMMA 6.1. *Assume $E_1 \simeq E_2$. Then, $\pm\sqrt{-1}$ are very good directions of $(E_1 \times E_2, E_1 + E_2)$ if and only if $p \equiv 1 \pmod 4$.*

LEMMA 6.2. *Assume $p \equiv 2 \pmod 3$ (resp. $p \equiv 3 \pmod 4$). Then, the elements $\xi, \xi^3, \xi^5, \xi^7, \xi^9, \xi^{11}$ (resp. $\zeta, \zeta^3, \zeta^5, \zeta^7$) are very good directions of $(E_\omega \times E_\omega, E_\omega \times \{0\} + \{0\} \times E_\omega)$ (resp. $(E_i \times E_i, E_i \times \{0\} + \{0\} \times E_i)$) if and only if $p \equiv 5 \pmod{12}$ (resp. $p \equiv 3 \pmod 8$).*

The group $RA(E_1 \times E_2, E_1 + E_2)$ acts on the set of $p + 1$ very good directions of $(E_1 \times E_2, E_1 + E_2)$. For a very good direction $a$ of $(E_1 \times E_2, E_1 + E_2)$, the stabilizer at $a$ is given by $RA(E_1 \times E_2, E_1 + E_2, a)$. Since we know the structure of the group of automorphisms of an elliptic curve (cf. Deuring [1, Section 5]), by (6.1), (6.2) and above lemmas, we have the following list for supersingular elliptic curves $E_1$ and $E_2$.

In Table 1 we set $A = E_1 \times E_2$ and $C = E_1 + E_2$. We denote by $a$ a very good direction of $(A, C)$.

## §7. Automorphisms of families and ramification groups

In this section, we assume char $k = p \geqslant 3$, unless otherwise mentioned. Let $E$ be a supersingular elliptic curve as in (1.4). Throughout this section, $E'$ and $E''$ mean suitable supersingular elliptic curves.

Let $W$ be an irreducible component of $V$ in $\mathscr{A}_{2,1}$. By Corollary 2.2, there exists a family

$$q\colon \mathscr{X} \to S = \mathbf{P}^1 \text{ with a relative polarization } D \tag{7.1}$$

*Table 1.*

Case (I) $E_1 \not\simeq E_2$

| | | $\lvert RA(A, C)\rvert$ | $\lvert RA(A, C, a)\rvert$ | Number of orbits of very good directions | Number of elements in each orbit |
|---|---|---|---|---|---|
| $p \geqslant 5$ | $E_1 \not\simeq E_\omega, E_\iota$<br>$E_2 \not\simeq E_\omega, E_\iota$ | 2 | 1 | $(p+1)/2$ | 2 |
| | $E_1 \not\simeq E_\omega, E_\iota$<br>$E_2 \simeq E_w$<br>$(p \equiv 2 \pmod 3)$ | 6 | 1 | $(p+1)/6$ | 6 |
| | $E_1 \not\simeq E_\omega, E_\iota$<br>$E_2 \simeq E_\iota$<br>$(p \equiv 3 \pmod 4)$ | 4 | 1 | $(p+1)/4$ | 4 |
| | $E_1 \simeq E_\omega$<br>$E_2 \simeq E_\iota$<br>$(p \equiv 11 \pmod{12})$ | 12 | 1 | $(p+1)/12$ | 12 |
| $p = 2$ or 3 | No such cases | | | | |

Case (II) $E_1 \simeq E_2$

| | | | $\lvert RA(A, C)\rvert$ | Very good direction $a$ | $\lvert RA(A, C, a)\rvert$ | Number of orbits of very good directions | Number of elements in each orbit |
|---|---|---|---|---|---|---|---|
| $p \geqslant 5$ | $E_1 \not\simeq E_\omega, E_\iota$ | $p \equiv 1 \pmod 4$ | 4 | $a = \pm\sqrt{-1}$ | 2 | 1 | 2 |
| | | | | $a \neq \pm\sqrt{-1}$ | 1 | $(p-1)/4$ | 4 |
| | | $p \equiv 3 \pmod 4$ | 4 | any | 1 | $(p+1)/4$ | 4 |
| | $E_1 \simeq E_\omega$ | $p \equiv 5 \pmod{12}$ | 36 | $a = \xi, \xi^3, \xi^5, \xi^7, \xi^9, \xi^{11}$ | 6 | 1 | 6 |
| | | | | $a \neq \xi, \xi^3, \xi^5, \xi^7, \xi^9, \xi^{11}$ | 3 | $(p-5)/12$ | 12 |
| | | $p \equiv 11 \pmod{12}$ | 36 | any | 3 | $(p+1)/12$ | 12 |
| | $E_1 \simeq E_\iota$ | $p \equiv 3 \pmod 8$ | 16 | $a = \zeta, \zeta^3, \zeta^5, \zeta^7$ | 4 | 1 | 4 |
| | | | | $a \neq \zeta, \zeta^3, \zeta^5, \zeta^7$ | 2 | $(p-3)/8$ | 8 |
| | | $p \equiv 7 \pmod 8$ | 16 | any | 2 | $(p+1)/8$ | 8 |
| $p = 3$ | $j(E_1) = 0$ | | 144 | any | 36 | 1 | 4 |
| $p = 2$ | $j(E_1) = 0$ | | 576 | any | 192 | 1 | 3 |

as in (1.9) obtained from a standard divisor $L$ on $A = E \times E$ which gives $W$ by the method indicated in (2.2). We use the notations in (4.1). The group $\Gamma(\mathscr{B})$ induces the group of automorphisms of the family (7.1) (cf. Section 4). We set

$$G: = R\Gamma(\mathscr{B}). \tag{7.2}$$

Then, by Proposition 4.1, we have $G \subset \mathrm{Aut}\,(\mathbf{P}^1)$, and the morphism

$$S = \mathbf{P}^1 \xrightarrow{\tilde{\psi}} \mathbf{P}^1/G = \tilde{W} \xrightarrow{\Psi} W \subset \mathscr{A}_{2,1}, \tag{7.3}$$

where $\tilde{\psi}$ is the natural projection, $\Psi$ is the normalization and $\Psi \circ \tilde{\psi}$ is the morphism $\psi = \psi_1$ described in Section 2. Sometimes, we write $\mathrm{Gal}\,(S \to W)$ instead of $G$.

In this section, we study the following questions:
(1) Which groups $G$ can occur?
(2) Which ramification groups I of $G$ occur?
Note that if $p \geqslant 3$, then by Corollary 4.4, for every component $W$ of $V$ the related group $G$ is a subgroup of $PSp(4, \mathbb{Z}/2) \simeq S_6$:

$$G \subset S_6 \tag{7.4}$$

Lemma 7.1. *Let $k$ be a field of characteristic $p \geqslant 0$, and let $S$ be an algebraic curve over $k$. For a non-singular point $P$ of $S$, let $I$ be a finite subgroup* Aut $(S)$ *of automorphisms of $S$ which fixes the point $P$. If $p = 0$, then $I$ is a cyclic group. If $p > 0$, then there exists a normal subgroup $I_0$ of $I$ such that $|I_0|$ is a power of $p$, and such that $I/I_0$ is a cyclic group of order prime to $p$.*

*Proof.* Let $T^*$ be the co-tangent space of $S$ at $P$. Since $\sigma(P) = P$ for every $\sigma$ of $I$, we obtain a representation of $I$ on the one-dimensional vector space $T^*$. Therefore, we have a homomorphism

$$\mu: I \to k^*.$$

We set $I_0 = \mathrm{Ker}\,\mu$. Since $\mu(I) \simeq I/I_0$ is a subgroup of $k^*$, we see that $I/I_0$ is a cyclic group. Moreover, if $p > 0$, then the order of $I/I_0$ is prime to $p$. When $p = 0$ (resp. $p > 0$), let $\sigma$ be any element of order $n$ of $I_0$ with $(n, p) = 1$. Let $t$ be a regular system of parameters at $P$. We set

$$s = t + \sigma^* t + \cdots + (\sigma^*)^{n-1} t.$$

Since $s = nt$ in $T^*$, we see that $s$ is a regular system of parameters at $P$. Since $s$ is invariant under $\sigma$, the action of $\sigma$ on the local ring of $P$ is trivial. Hence, $\sigma$ is the identity of $I_0$. Hence, $I_0 = \{1\}$ if $p = 0$ (resp. the order of $I_0$ is a power of $p$ if $p > 0$). Q.E.D.

We denote by $V_4$ (resp. $S_n$, resp. $A_n$, resp. $D_{2e}$) Klein's four group (resp. the symmetric group of degree $n$, resp. the alternating group of degree $n$, resp. the dihedral group of order $2e$). We have $V_4 \simeq D_4$ and $S_3 \simeq D_6$.

LEMMA 7.2. *Assume $p \geqslant 7$. Then, the order of the group $G$ as in* (7.2) *is prime to $p$, and the group $G$ is isomorphic to one of the following groups*:

$$\mathbb{Z}/\mathrm{d}\,(1 \leqslant \mathrm{d} \leqslant 6),\ D_{2e}(2 \leqslant e \leqslant 6),\ A_4,\ S_4,\ A_5.$$

*Moreover, the ramification group $I$ of $G$ at a point of $S$ is a cyclic group.*

*Proof.* Since $p \geqslant 7$ and $G \subset S_6$ by (7.4), $p$ does not divide the order of $G$. Hence, there is no wild ramification in $\mathbf{P}^1 \to \mathbf{P}^1/G$, and we can prove the former part in usual way (cf. Pinkham [17, p. 3–p. 5]). The latter part of this lemma follows from Lemma 7.1. Q.E.D.

For our family $q: \mathscr{X} \to S \simeq \mathbf{P}^1$, we have the diagram (1.9) in Section 1. Let $I$ be the stabilizer of $G$ at a point $x$ of $S$. The group $I$ is called a ramification group of $G$ at a branch point $\hat{\psi}(x)$ of $S/G = \tilde{W}$. In this case, we say that the ramification group $I$ occurs at the curve $D_x$. If $I \simeq \mathbb{Z}/n$ $(2 \leqslant n \leqslant 6)$, then the ramification is called a $\mathbb{Z}/n$-ramification. The ramification group $I$ induces a subgroup of the reduced group of automorphisms of $(\mathscr{X}_x, D_x)$. By the definition of $R\Gamma(\mathscr{B})$ with $\mathscr{B} = \mathscr{B}(\mathscr{X}, D)$, the group $I$ also induces a subgroup of $RA(D_x)$:

$$I \hookrightarrow RA(D_x). \tag{7.5}$$

Assume that $\mathscr{X}_x$ is isomorphic to $E \times E$. Then, by (1.9) and (3.11), we have the following diagram:

$$\begin{array}{ccccc} \tilde{A} & \xrightarrow{\tilde{\pi}} & A & \xrightarrow{\pi} & A' \simeq \mathscr{X}_x \\ \| & & \| & & \| \\ E \times E & & E \times E & & E \times E \\ & & \cup & & \cup \\ & & (\pi^{-1})(D_x) & & D_x \end{array} \tag{7.6}$$

with $A = (pr_2)^{-1}(x)$, where $\pi$ and $\tilde{\pi}$ are purely inseparable morphisms of degree $p$, and $\pi \circ \tilde{\pi}$ is the Frobenius morphism. We set $(\pi \circ \tilde{\pi})^{-1}(D_x) = p\tilde{D}_x$. Then, $\tilde{D}_x$ is a (not necessarily irreducible) curve of genus two such that

$$\tilde{D}_x \simeq (D_x)^{(1/p)}. \tag{7.7}$$

The divisor $\pi^{-1}(D_x)$ satisfies Condition (1.10). Since $\pi \circ \tilde{\pi}$ is the Frobenius morphism, the group $I$ also induces a subgroup of $RA(\tilde{D}_x)$:

$$I \hookrightarrow RA(\tilde{D}_x). \tag{7.8}$$

The group $RA(\tilde{D}_x) \simeq RA_v(\tilde{A}, \tilde{D}_x)$ acts on $p + 1$ very good directions of $(\tilde{A}, \tilde{D}_x)$ (cf. Section 3). Therefore, the group $I$ also acts on these very good directions through $RA(\tilde{D}_x)$. Let $a$ be the direction of the natural immersion of Ker $\tilde{\pi} \simeq \alpha_p$ into $\tilde{A} = E \times E$. Then, by our construction, $a$ is a very good direction of $(\tilde{A}, \tilde{D}_x)$ and the action of $I$ on $p + 1$ very good directions of $(\tilde{A}, \tilde{D}_x)$ preserves the direction $a$.

LEMMA 7.3. *Under the notations as above,*

$$I \simeq RA_v(\tilde{A}, \tilde{D}_x, a).$$

*Proof.* The group $RA_v(\tilde{A}, \tilde{D}_x, a)$ is isomorphic to $RA_v(A, \pi^{-1}(D_x))$, which is a subgroup of $R\Gamma(\mathscr{B})$. The group $\Gamma(\mathscr{B})$ induces the group of automorphisms of the family $q$: $\mathscr{X} \to S$ with relative polarization $D$ (cf. Section 4). Therefore, by our construction, we see $I \simeq RA_v(A, \pi^{-1}(D_x))$, hence, $I \simeq RA_v(\tilde{A}, \tilde{D}_x, a)$. Q.E.D.

From here on, we assume $p \geqslant 7$. In this case, the group $I$ is a cyclic group by Lemma 7.2. Let $\sigma$ be a generator of $I$. Then, by (7.8) $\sigma$ acts on the set $\mathbf{P}^1$ of directions of $\tilde{A} = E \times E$ (cf. Section 3). Since the order of $I$ is prime to $p$ in this case, $\sigma$ has two fixed points on $\mathbf{P}^1$. The automorphism $\sigma$ also acts on the set of $p + 1$ very good directions of $(\tilde{A}, \tilde{D}_x)$. The direction $a$ as above is fixed by $\sigma$. From these considerations, we can easily prove the following lemma.

LEMMA 7.4. *Assume $p \geqslant 7$. Under the same notations as above, assume $A' = \mathscr{X}_x \simeq E \times E$. Then, the ramification group $I = \langle\sigma\rangle$ acts on $p + 1$ very good directions of $(\tilde{A}, \tilde{D}_x)$. The automorphism $\sigma$ has at least one fixed* *point and has at most two fixed points on these $p + 1$ very good directions. Moreover, the group $I$ acts freely on the very good directions except the fixed points of $\sigma$.*

We list in Table 2 the ramifications in each group in Lemma 7.2 (for instance, see Pinkham [17, p. 4]).

*Notation 7.5.* Let $G$ be a group as in (7.2). We write

*Table 2.* $p \geqslant 7$

| Group G | Order | Ramification index |
|---|---|---|
| $\mathbb{Z}/n$ $(2 \leqslant n \leqslant 6)$ | $n$ | $(n, n)$ |
| $D_{2e}$ $(2 \leqslant n \leqslant 6)$ | $2e$ | $(2, 2, e)$ |
| $A_4$ | 12 | (2, 3, 3) |
| $S_4$ | 24 | (2, 3, 4) |
| $A_5$ | 60 | (2, 3, 5) |

$$(G;\, e_1', e_2', \text{- - -}\,;\, e_1'', e_2'', \text{- - -}\,)$$

if $e_1'$, $e_2'$, - - - are the orders of the ramification groups at the points of $\tilde{W}$ over which the principal polarizations of type $E' \cup E''$ lie, and $e_1''$, $e_2''$, - - - are the orders of ramification groups at the points of $\tilde{W}$ over which the principal polarizations given by irreducible curves of genus two lie.

Let $C$ be a non-singular irreducible curve of genus two. By Igusa [7], $RA(C)$ is isomorphic to one of the following groups:

$$(0)\{0\},\ (1)\ \mathbb{Z}/2,\ (2)\ S_3,\ (3)\ V_4,\ (4)\ D_{12},\ (5)\ S_4,\ (6)\ \mathbb{Z}/5.$$

In [IKO], Katsura and Oort [8], we said that an irreducible curve $C$ of genus two is in Class (i) $(0 \leqslant i \leqslant 6)$ if $RA(C)$ *contains* the group in (i). In this paper, we use the following definition.

*Definition 7.6.* We say that an irreducible curve $C$ of genus two is of type (i) $(0 \leqslant i \leqslant 6)$ if $RA(C)$ *is isomorphic to* the group in (i).

We note that curves $C$ and $C^{(1/p)}$ are in the same class and of the same type, that is, $RA(C)$ is isomorphic to $RA(C^{(1/p)})$.

*Definition 7.7.* Let $\sigma$ be an element of $S_6$. An element $\tau$ of $S_6$ is said to be of type $\sigma$ if it is conjugate with $\sigma$ in $S_6$.

LEMMA 7.8. *A* $\mathbb{Z}/6$-ramification *appears if and only if* $p \equiv 5 \pmod{12}$, *and in this case it appears once, and we have*

$$(D_{12};\ 2,\ 6;\ 2).$$

*For a generator* $\sigma$ *of the ramification group* $\mathbb{Z}/6$, $\sigma$ *is of type* $(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$.

*Proof.* If $C$ is an irreducible curve of type (i) $(0 \leqslant i \leqslant 6)$ with $i \neq 4$, then $RA(C)$ does not contain a cyclic group of order six. The curve of type (4) is supersingular if and only if $p \equiv 5 \pmod 6$ (cf. [IKO, Proposition 1.11]):

$$p + 1 \equiv 0 \pmod 6. \tag{7.9}$$

The group $RA(C) \simeq D_{12}$ operates on $p + 1$ very good directions of $(J(C), C)$. By (7.9), the group of order six cannot operate on $p + 1$ points with fixed points as in Lemma 7.4. Thus, $\mathbb{Z}/6$-ramifications never appear at any irreducible curve of genus two.

If $C$ is a reducible curve, by Table 1 in Section 6 we see that a $\mathbb{Z}/6$-ramification occurs if and only if $p \equiv 5 \pmod{12}$. In case $p \equiv 5 \pmod{12}$ it appears exactly one. If $G \simeq \mathbb{Z}/6$, then $G$ has two fixed points on $S = \mathbf{P}^1$. Therefore, a $\mathbb{Z}/6$-ramification appears twice on $\tilde{W}$. A contradiction. Therefore, by Lemma 7.2 we have $G \simeq D_{12}$. The group $G$ acts on $5p - 5$ points on $S$ over which the polarizations are reducible. Since $5p - 5 \equiv 8 \pmod{12}$, we have among these $5p - 5$ points two points whose stabilizers are isomorphic to $\mathbb{Z}/6$. The two points transform into each other by $G$. Hence, a $\mathbb{Z}/2$-ramification and a $\mathbb{Z}/6$-ramification appears at some reducible curves and another $\mathbb{Z}/2$-ramification appears once at one curve of type (i) $(1 \leqslant i \leqslant 5)$. The final statement follows from (7.4). Q.E.D.

LEMMA 7.9. *A* $\mathbb{Z}/5$-ramification *appears if and only if* $p \equiv 2$ or $3 \pmod 5$. *In these cases, it appears once, and it appears at the curve of type* (6). *For a generator* $\sigma$ *of the ramification group* $\mathbb{Z}/5$, $\sigma$ *is of type* $(1\ \ 2\ \ 3\ \ 4\ \ 5)$. *The group* $G$ *is*

$$\text{either}\quad G \simeq A_5 \ \text{or}\ G \simeq D_{10}.$$

*If* $G \simeq A_5$ *and*

$$p \equiv \begin{cases} 1 \pmod{12}, \text{ then } (A_5;\ -;\ 2,\ 3,\ 5), \\ 5 \pmod{12}, \text{ then } (A_5;\ 3;\ 2,\ 5), \\ 7 \pmod{12}, \text{ then } (A_5;\ 2;\ 3,\ 5), \\ 11 \pmod{12}, \text{ then } (A_5;\ 2,\ 3;\ 5); \end{cases}$$

*if* $G \simeq D_{10}$, *then*

either $(D_{10}; 2, 2; 5)$ or $(D_{10}; -; 2, 2, 5)$.

*Proof.* Since the order of Aut $(E' \times E'', E' + E'')$ is not divisible by five, a $\mathbb{Z}/5$-ramification cannot appear at any reducible curve. A $\mathbb{Z}/5$-ramification may appear at the curve of type (6). By [IKO, Proposition 1.13], in case $p \geqslant 7$ the curve $C$ of type (6) is supersingular if and only if $p \not\equiv 1 \pmod 5$. If $p \equiv 4 \pmod 5$, the Jacobian variety $J(C)$ is isomorphic to $E \times E$. $RA(C) \simeq \mathbb{Z}/5$ operates at $p + 1$ very good directions of $(J(C), C)$. Since $p + 1 \equiv 0 \pmod 5$, such an action does not exist by Lemma 7.4. Hence, a $\mathbb{Z}/5$-ramification does not appear if $p \equiv 4 \pmod 5$. If $p \equiv 2$ or $3 \pmod 5$, $J(C)$ is supersingular and is not isomorphic to a product of two supersingular elliptic curves. By Theorem 1.2, there exists a unique $\alpha_p$-covering

$$\pi' : E \times E \to J(C).$$

By Lemma 1.5, the action of $\mathbb{Z}/5$ on $(J(C), C)$ lifts to the action on $(E \times E, (\pi')^{-1}(C))$. We may assume that $C$ is symmetric in $J(C)$. Then, $(\pi')^{-1}(C)$ satisfies Condition (2.10). Using this divisor, we can construct a family as in Section 1 in which a $\mathbb{Z}/5$-ramification appears. Conversely, if there exists a family as in Section 1 in which a $\mathbb{Z}/5$-ramification appears, then on $E \times \mathrm{E}$ we can find an effective divisor $L$ which satisfies Condition (1.10) and which is fixed by a group of order five. Since the order of Aut $(E' \times E'', E' + E'')$ is not divisible by five, we see that $L$ is given by an irreducible curve. This means that $L$ is up to isomorphism of $E \times E$ given by $(\pi')^{-1}(C)$ as above with the unique curve of type (6). Hence, the family in which a $\mathbb{Z}/5$-ramification appears uniquely exists if $p \equiv 2$ or $3 \pmod 5$, and a $\mathbb{Z}/5$-ramification appears once. For a generator $\sigma$ of the ramification group $\mathbb{Z}/5$, $\sigma$ is of the type $(1\ 2\ 3\ 4\ 5)$ by (7.4). Suppose $G \simeq \mathbb{Z}/5$. Then, $G$ has two fixed points on $S \simeq \mathbf{P}^1$, and a $\mathbb{Z}/5$-ramification appears twice on $\tilde{W}$. A contradiction. Therefore, we have $G \supsetneqq \mathbb{Z}/5$. By Lemma 7.2, we conclude that $G \simeq A_5$ or $G \simeq D_{10}$. If $p \equiv 1, 5, 7, 11 \pmod{12}$ respectively, then $5p - 5 \equiv 0, 20, 30, 50 \pmod{60}$ respectively. If $G \simeq A_5$, a group of order sixty acts on $5p - 5$ points on $\mathbf{P}^1$ over which reducible polarizations lie. This gives as in the proof of Lemma 7.8 the ramification behavior indicated. If $G \simeq D_{10}$, we have $p^2 + 1 \equiv 0 \pmod{10}$ and $5p - 5 \equiv 0 \pmod{10}$, and the possibilities for the ramification behavior follow. Q.E.D.

*Definition 7.10.* We denote by $E' \cup E''$ a curve composed of two elliptic curves $E'$ and $E''$ whose zero points are identified with transversal crossing. We call six points of exact order two on $E'$ and $E''$ Weierstrass points of $E' \cup E''$.

LEMMA 7.11. *A* $\mathbb{Z}/4$-ramification *does not appear if* $p \equiv \pm 1 \pmod 8$. *If* $p \equiv 3 \pmod 8$, *it appears exactly once. In this case, it appears at a reducible curve* $E' \cup E'$ *and a generator of the ramification group* $\mathbb{Z}/4$ *is of type* $(1\ 2\ 3\ 4)(5\ 6)$. *If* $p \equiv 5 \pmod 8$, *it appears exactly once. In this case, it appears at the curve of type* (5), *and a generator of the ramification group* $\mathbb{Z}/4$ *is of the type* $(1\ 2\ 3\ 4)$. *In these cases, we have either* $G \simeq S_4$ *or* $G \simeq D_8$.

*If* $G \simeq S_4$ *and*

$$p \equiv \begin{cases} 5 \pmod{24}, \text{ then } (S_4;\ 2,\ 3;\ 4), \\ 11 \pmod{24}, \text{ then } (S_4;\ 2,\ 3,\ 4;\ -), \\ 13 \pmod{24}, \text{ then } (S_4;\ 2;\ 3,\ 4), \\ 19 \pmod{24}, \text{ then } (S_4;\ 2,\ 4;\ 3); \end{cases}$$

*if* $G \simeq D_8$ *and*

$$p \equiv \begin{cases} 3 \pmod 8, \text{ then } (D_8;\ 2,\ 2,\ 4;\ -) \text{ or } (D_8;\ 4;\ 2,\ 2), \\ 5 \pmod 8, \text{ then } (D_8;\ 2;\ 2,\ 4). \end{cases}$$

*Proof.* In the case of an irreducible curve, a $\mathbb{Z}/4$-ramification can only appear at the curve $C$ of type (5). By [IKO, Proposition 1.12], this curve is supersingular if and only if $p \equiv 5$ or $7 \pmod 8$. In the case of a reducible curve, a $\mathbb{Z}/4$-ramification can appear if and only if $p \equiv 3 \pmod 8$ by Table 1 in Section 6, and it appears exactly once. If $p \equiv 7 \pmod 8$, $RA(C) \simeq S_4$ acts on $p + 1$ very good directions of $(J(C), C)$. Since $p + 1 \equiv 8$ or $0 \pmod{24}$, a $\mathbb{Z}/4$-ramification does not appear by Lemma 7.4 in this case. If $p \equiv 5 \pmod 8$, $RA(C) \simeq S_4$ acts on $p + 1$ very good directions of $(J(C), C)$. Since $p + 1 \equiv 6$ or $14 \pmod{24}$, each cyclic subgroup of order four has two fixed directions on these $p + 1$ very good directions by Lemma 7.4. In $RA(C)$ we have three cyclic subgroups of order four. Therefore, these fixed directions transform into each other by $RA(C)$. Therefore, a $\mathbb{Z}/4$-ramification appears exactly once and $\mathbb{Z}/4 \subsetneqq G$ as in the proof of Lemma 7.9. Hence, we conclude $G \simeq S_4$ or $D_4$ by Lemma 7.2. Considering the action of the ramification group $\mathbb{Z}/4$ on six Weierstrass points of $C$, we see that a generator of the ramification group $\mathbb{Z}/4$ is of type $(1\ 2\ 3\ 4)$ (cf. Igusa [7]). If $p \equiv 3 \pmod 8$, a $\mathbb{Z}/4$-ramification appears exactly once at the reducible curve $E_i \cup E_i$ (cf. (6.5)). Thus, we have $\mathbb{Z}/4 \subsetneqq G$. The group

$$\tilde{I} \subset \operatorname{Aut}(E_i \times E_i, E_i \times \{0\} + \{0\} \times E_i, a)$$

with a suitable very good direction $a$ which gives the ramification group $\mathbb{Z}/4$ is generated by

$$\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix} : E_i \times E_i \to E_i \times E_i,$$

where $\sigma$ is a complex multiplication by $\sqrt{-1}$ on $E_i$. Considering the action $\left(\begin{smallmatrix} 0 & \sigma \\ 1 & 0 \end{smallmatrix}\right)$ on Weierstrass points of $E_i \times \{0\} \cup \{0\} \times E_i$, we see that a generator of the ramification group $\mathbb{Z}/4$ is of type $(1\ 2\ 3\ 4)(5\ 6)$. Ramification behavior of the groups, if they appear, is computed as in Lemmas 7.8 and 7.9. Q.E.D.

*Notation 7.12.*

$$[a_1, a_5, a_7, a_{11}, a_{13}, a_{17}, a_{19}, a_{23}; 24] := a_i \quad \text{if } p \equiv i \pmod{24}.$$

*Notation 7.13.* We denote by $n_2$ (resp. $n_3$) the number of isomorphism classes of irreducible supersingular curves $C$ of genus two with $RA(C) \simeq S_3$ (resp. $RA(C) \simeq V_4$).

The numbers $n_2$ and $n_3$ are explicitly calculated in [IKO, Theorem 3.3]:

$$n_2 = [([p/3] + 1)/2] - [0, 2, 1, 1, 1, 1, 0, 2; 24],$$

$$n_3 = [([p/4] + 1)/2] - [0, 2, 1, 1, 1, 1, 0, 2; 24],$$

where $[r]$ means the integral part of a rational number $r$.

LEMMA 7.14. *The following is the complete list of all* $\mathbb{Z}/3$-ramifications *which can appear*:

if $p \equiv 5 \pmod{12}$, then $(p - 5)/12$ times at $E_\omega \cup E_\omega$,
if $p \equiv 11 \pmod{12}$, then $(p + 1)/12$ times at $E_\omega \cup E_\omega$,
if $p \equiv 1 \pmod 3$, then $n_2$ times at the curves of type (2),
if $p \equiv 7$ or $13 \pmod{24}$, then once also at the curve of type (5).

*Proof.* For reducible curves, we can read off all ramification groups from Table 1 in Section 6. Since the orders of $RA(C)$ of curves $C$ of type (0), (1), (3) or (6) are not divisible by three, these curves are excluded. As we saw in Lemma 7.8, a $\mathbb{Z}/6$-ramification appears at the curve of type (4) if and only if $p \equiv 5 \pmod{12}$ and a $\mathbb{Z}/3$-ramification does not appear at this curve. The curve $C$ of type (5) is supersingular if and only if $p \equiv 5$ or $7 \pmod 8$. The group $RA(C) \simeq S_4$ acts on $p+1$ very good directions of $(J(C), C)$. If $p \equiv 5$ or $23 \pmod{24}$, then elements of order three act on these very good directions without fixed points by Lemma 7.4. If $p \equiv 7$ or $13 \pmod{24}$, then elements of order three act on these very good directions with fixed points. By this fact, we see that a $\mathbb{Z}/3$-ramification appears exactly once at the curve of type (5) if $p \equiv 7$ or $13 \pmod{24}$. In a similar way, we see that a $\mathbb{Z}/3$-ramification appears exactly once at each curve of type (2) if $p \equiv 1 \pmod 3$, and a $\mathbb{Z}/3$-ramification does not appear at any curve of type (2) if $p \equiv 2 \pmod 3$. Q.E.D.

*Definition* 7.15. Let $C$ be a non-singular complete curve of genus two or a curve as in Definition 7.10. Let $\sigma$ be an element of Aut $(C)$. Suppose that the order of $\sigma$ is two. We say that $\sigma$ is long if the action of $\sigma$ on Weierstrass points of $C$ is a permutation of type $(1\ 2)(3\ 4)(5\ 6)$; we say that $\sigma$ is short if the action on Weierstrass points of $C$ is of type $(1\ 2)(3\,4)$.

LEMMA 7.16. *A* $\mathbb{Z}/2$-ramification *does not appear at* $C = E_\omega \cup E_\omega$. *All* $\mathbb{Z}/2$-ramifications *at* $C = E_i \cup E_i$ *with* $p \equiv 3 \pmod 4$ *are short. All* $\mathbb{Z}/2$-ramifications *at* $C = E' \cup E'$ *with* $j(E') \neq 0,\ 1728$, *and with* $p \equiv 1 \pmod 4$ *are long.*

*Proof.* The first statement follows from Table 1 in Section 6. If $p \equiv 1 \pmod 4$, for every $E'$ with $j(E') \neq 0,\ 1728$ there exists exactly one point on $\tilde{W}$ with ramification group $\mathbb{Z}/2$ (cf. Table 1 in Section 6). It is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : E' \cup E' \to E' \cup E',$$

where 1 (resp. $-1$) is the identity (resp. the inversion) of $E'$. On the Weierstrass points of $C = E' \cup E'$, this gives a permutation which is long. If $p \equiv 3 \pmod 4$, every ramification group $\mathbb{Z}/2$ for a reducible curve is given by $C = E_i \cup E_i$ with

$$\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} : E_i \cup E_i \to E_i \cup E_i$$

with $\sqrt{-1}$ of Aut $(E_i)$, the complex multiplication of $\sqrt{-1}$ on $E_i$ as before. The action of this automorphism on the Weierstrass points of $C = E_i \cup E_i$ is short. Q.E.D.

*Remark 7.17.* Both matrices in the proof of the previous lemma are of order four.

PROPOSITION 7.18. *Let $C$ be an irreducible curve of genus two. Let $\sigma$ be an element of order two of $RA(C)$, and let $\tilde{\sigma}$ be an element of* Aut $(C)$ *which gives $\sigma$ in $RA(C)$.*

(*i*) *If $\sigma$ is short, then the order of $\tilde{\sigma}$ is equal to four.*
(*ii*) *If $\sigma$ is long, then the order of $\tilde{\sigma}$ is equal to two.*
(*iii*) *If $C$ is supersingular and a ramification group $\mathbb{Z}/2$ is given by $\sigma$, then $\sigma$ is short.*

*Proof.* (i) In this case, we can assume that $\sigma$ fixes the points 0 and $\infty$ of $\mathbf{P}^1$ with respect to a suitable coordinate $X$ of $\mathbf{A}^1$ in $\mathbf{P}^1$. Then, the curve $C$ is given in the form

$$Y^2 = X(X^2 - a)(X^2 - b)$$

with $a, b \in k^*$; $a \neq 0, 1$; $b \neq 0, 1$; $a \neq b$, and the automorphism $\tilde{\sigma}$ is given by

$$\tilde{\sigma}\colon X \mapsto -X,\ Y \mapsto \sqrt{-1}\,Y.$$

Hence, we have ord $\tilde{\sigma} = 4$.

(ii) In this case, $C$ is given in the form

$$Y^2 = (X^2 - a)(X^2 - b)(x^2 - c)$$

and $\tilde{\sigma}$ is given in the form

$$\tilde{\sigma}\colon X \mapsto -X,\ Y \mapsto Y.$$

Hence, we have ord $\tilde{\sigma} = 2$.

(iii) Let $\sigma \in RA(C)$ be an element of order two. Since $\sigma$ has exactly two fixed points on $\mathbf{P}^1 \simeq C/\langle\iota\rangle$, $\sigma$ cannot be of type (1 2). Suppose $\sigma$ is long. Then, by (ii) there exists an element $\tilde{\sigma}$ of order two of Aut $(C)$ such that $\tilde{\sigma}$ gives $\sigma$ in $RA(C)$. As in Igusa [7, p. 648] and [IKO, the proof of Proposition 1.3], we have a separable morphism of degree four

$$\pi_1 \colon J(C) \to E_\sigma \times E_\tau, \tag{7.10}$$

where $E_\sigma = C/\langle\tilde{\sigma}\rangle$, $E_\tau = C/\langle\iota \circ \tilde{\sigma}\rangle$ and they are elliptic curves. Ker $\pi_1$ is contained in the group of elements of order two of $J(C)$. Therefore, there exists an isogeny $\pi_2$: $E_\sigma \times E_\tau \to J(C)$ such that $\pi_1 \circ \pi_2 = [2]_{E_\sigma \times E_\tau}$. The automorphism $\tilde{\sigma}$ acts on $J(C)$ and induces the action on $E_\sigma \times E_\tau$ in (7.10). The action of $\tilde{\sigma}$ on $E_\sigma \times E_\tau$ is given by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : E_\sigma \times E_\tau \to E_\sigma \times E_\tau,$$

where 1 (resp. $-1$) is the identity of $E_\sigma$ (the inversion of $E_\tau$). We have the commutative diagram:

$$\begin{tikzcd}
E_\sigma \times E_\tau \arrow[r, "\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] \arrow[d, "\pi_2"'] & E_\sigma \times E_\tau \arrow[d, "\pi_2"] \\
J(C) \arrow[r, "\sigma'"] \arrow[d, "\pi_1"'] & J(C) \arrow[d, "\pi_1"] \\
E_\sigma \times E_\tau \arrow[r, "\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"'] & E_\sigma \times E_\tau,
\end{tikzcd} \tag{7.11}$$

where $\sigma'$ is an isomorphism induced by $\sigma_2$. The isomorphism $\sigma'$ is different from $\tilde{\sigma}$ by a translation by an element of order two. Since $(\sigma')^*(C)$ is algebraically equivalent to $C$, $\sigma_2^* \circ \pi_2^*(C)$ is algebraically equivalent to $\pi_2^*(C)$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
E_\sigma \times E_\tau \arrow[r, "\sigma_2"] \arrow[d, "\varphi_{E_\sigma + E_\tau}"'] & E_\sigma \times E_\tau \arrow[d, "\varphi_{E_\sigma + E_\tau}"] \\
E_\sigma^t \times E_\tau^t & E_\sigma^t \times E_\tau^t \arrow[l, "\sigma_2^t"'] \\
E_\sigma \times E_\tau \arrow[r, "\sigma_2"] \arrow[u, "\varphi_{\pi_2^*(C)}"] & E_\sigma \times E_\tau . \arrow[u, "\varphi_{\pi_2^*(C)}"']
\end{tikzcd}$$

Since $\varphi_{E_\sigma + E_\tau}$ is an isomorphism, we have

$$\varphi^{-1}_{E_\sigma + E_\tau} \circ \varphi_{\pi^*(C)} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha \in \operatorname{End}(E_\sigma)$, $\delta \in \operatorname{End}(E_\tau)$, $\beta \in \operatorname{Hom}(E_\tau, E_\sigma)$ and $\gamma \in \operatorname{Hom}(E_\sigma, E_\tau)$. Therefore, we have

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Hence, we have $\beta = 0$ and $\gamma = 0$. Since $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right)$ is given by a polarization, we see that $\alpha$ and $\delta$ are integers (cf. Mumford [12, p. 190, (3)]). It is easy to see that

$$\varphi^{-1}_{E_\sigma + E_\tau} \circ \varphi_{E_\sigma} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\varphi^{-1}_{E_\sigma + E_\tau} \circ \varphi_{E_\tau} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we have

$$\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \varphi^{-1}_{E_\sigma + E_\tau} \circ \varphi_{\delta E_\sigma + \alpha E_\tau}.$$

Hence, we have

$$\pi_2^*(C) \equiv \delta E_\sigma + \alpha E_\tau.$$

We denote by $pr_\tau$ the projection from $E_\sigma \times E_\tau$ to the second factor $E_\tau$, and by $o_\tau$ the origin of $E_\tau$. We have the equality of intersection numbers

$$(C \cdot \pi_1^*(E_\sigma)) = (C \cdot \pi_1^* \circ pr_\tau^*(o_\tau)) = \deg pr_\tau \circ \pi_1|_C = 2.$$

We have also

$$(\pi_1 \circ \pi_2)^*(E_\sigma) = [2]^*_{E_\sigma + E_\tau}(E_\sigma) \equiv 4E_\sigma.$$

Therefore, we have

$$\begin{aligned} 4(E_\sigma \cdot \pi_2^*(C)) &= ((\pi_1 \circ \pi_2)^*(E_\sigma) \cdot \pi_2^*(C)) \\ &= \deg \pi_2 \cdot (\pi_1^*(E_\sigma) \cdot C) = 2^2 \cdot 2. \end{aligned}$$

Hence, we have $\alpha = (E_\sigma \cdot \pi_2^*(C)) = 2$. Similarly, we have $\delta = (E_\tau \cdot \pi_2^*(C)) = 2$, that is, we have

$$\pi_2^*(C) \equiv 2(E_\sigma + E_\tau). \tag{7.12}$$

The tangent space at the origin of $E_\sigma \times E_\tau$ is isomorphic to the tangent space at the origin of $J(C)$ by the homomorphism $(\pi_2)_*$ induced by $\pi_2$. By (7.12), we see that $a \in k^*$ is a very good direction of $(E_\sigma \times E_\tau, E_\sigma + E_\tau)$ if and only if $(\pi_2)_*(a)$ is a very good direction of $(J(C), C)$. Let $(\pi_2)_*(a)$ be a very good direction of $(J(C), C)$ which is fixed by the action of $\tilde{\sigma}$. Then, by the definition of the action of $\sigma$ and the diagram (7.11), $a$ is a very good direction of $(E_\sigma \times E_\tau, E_\sigma + E_\tau)$ fixed by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Hence, we have $(1, a) = (1, 0)$ or $(0, 1)$. This contradicts the result in Moret-Bailly [11, p. 139]. Hence, $\sigma$ is short. Q.E.D.

COROLLARY 7.19. (*i*) *Curves of type* (1) *do not give any* $\mathbb{Z}/2$-ramification.
(*ii*) *Curves of type* (2) *do not give any* $\mathbb{Z}/2$-ramification.
(*iii*) *Curves of type* (3) *do not give any* $\mathbb{Z}/2$-ramification *if* $p \equiv 3 \pmod 4$. *Each curve of type* (3) *gives exactly once a* $\mathbb{Z}/2$-ramification *if* $p \equiv 1 \pmod 4$.

*Proof.* For curves of type (1) or (2) every 2-torsion element in $RA(C)$ is long. Therefore, (i) and (ii) follow from Proposition 7.18. For a curve $C$ of type (3), $RA(C) = V_4$ has two long and one short 2-torsion elements. This group acts on $p + 1$ very good directions of $(J(C), C)$. If $p \equiv 3 \pmod 4$, by Lemmas 7.2 and 7.4 the subgroups of $RA(C)$ of order two give in all no or four fixed directions on these very good directions. By Proposition 7.18, we conclude that $RA(C)$ gives no fixed directions. If $p \equiv 1 \pmod 4$, the subgroups of $RA(C)$ of order two gives in all two or six fixed directions. Therefore, by Proposition 7.18, it gives exactly two fixed directions.Since the order of $RA(C)$ is four, the two fixed directions transform into each other by $RA(C)$. Hence, a $\mathbb{Z}/2$-ramification appears exactly once for each curve $C$ in this case. Q.E.D.

COROLLARY 7.20. *If* $p \equiv 5 \pmod{12}$, *the normalization of the component of* $V$ *corresponding to the group* $D_{12}$ *has a* $\mathbb{Z}/2$-ramification *at a curve of type* $E' \cup E'$ *with* $j(E') \neq 0, 1728$.

*Proof.* Since we have an injection $D_{12} \to S_6$, $D_{12}$ acts on six points as permutation. Therefore, the order of the stabilizer of each point is equal to two. Therefore, the number of elements of order two of $D_{12}$ which is not long is at most three. We denote by $N$ the normal subgroup of order six of $D_{12}$. Then, we have an element of order two of $D_{12}\backslash N$ which is long. Hence, this proposition follows from Lemma 7.16 and Proposition 7.18 (iii). Q.E.D.

LEMMA 7.21. (*i*) *If* $D_{10} \to S_6$, *then all* 2-torsion *elements in* $D_{10}$ *are short.*
(*ii*) *If* $D_8 \hookrightarrow S_6$ *with* $\delta \mapsto (1\ 2\ 3\ 4)$, *then* $D_8\backslash\langle\delta\rangle$ *has short and long* 2-torsion *elements.*
(*iii*) *If* $D_8 \hookrightarrow S_6$ *with* $\delta \mapsto (1\ 2\ 3\ 4)(5\ 6)$, *then either all elements* $D_8\backslash\langle\delta\rangle$ *are short, or* $D_8$ *contains an element of type* $(2\ 4)$.

*Proof.* Considering the action of groups on six points, we get this lemma by straightforward calculation. Q.E.D.

We summarize in Table 3 the results on the numbers of branch points which can appear. We denote by $E_\omega$ (resp. $E_i$) the elliptic curve with $j(E_\omega) = 0$ (resp. $j(E_i) = 1728$) as in (6.5). In the following table, we denote by $E'$ supersingular elliptic curves with $j(E') \neq 0, 1728$.

For small prime numbers we list in Table 4 the total number of branch points for the groups determined by the irreducible components of $V$.

## §8. Examples

Let $k$ be an algebraically closed field of characteristic $p \geqslant 2$. Every irreducible component $W$ of the supersingular locus $V$ of $\mathscr{A}_{2,1}$ is given in the form

$$\psi\colon S = \mathbf{P}^1 \to \mathbf{P}^1/G = \tilde{W} \to W \subset \mathscr{A}_{2,1} \tag{8.1}$$

as in (2.2), where $G$ is the group as in (7.2), and where $\tilde{W}$ is the normalization of $W$ (for the case $p = 2$, see Remarks 1.5 and 4.6). For a point $x$ of $S$, we denote by $G_x$ the stabilizer of $G$ at $x$ as before. For a point $y$ of $\tilde{W}$ we also

*Table 3.* $p \geqslant 7$

<table>
<tr><td colspan="3">Ramification group \ $p$ mod 24</td><td>1</td><td>5</td><td>7</td><td>11</td><td>13</td><td>17</td><td>19</td><td>23</td></tr>
<tr><td>$\mathbb{Z}/6$</td><td colspan="2">at $E_\omega \cup E_\omega$</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td rowspan="2">$\mathbb{Z}/4$</td><td colspan="2">type (1 2 3 4)(5 6) at $E_\iota \cup E_\iota$</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td colspan="2">type (1 2 3 4) at $C$ of type (5)</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="3">$\mathbb{Z}/3$</td><td colspan="2">at $E_\omega \cup E_\omega$</td><td>0</td><td>$(p-5)/12$</td><td>0</td><td>$(p+1)/12$</td><td>0</td><td>$(p-5)/12$</td><td>0</td><td>$(p+1)/12$</td></tr>
<tr><td colspan="2">at $C$ of type (5)</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td colspan="2">at $C$ of type (2)</td><td>$n_2$</td><td>0</td><td>$n_2$</td><td>0</td><td>$n_2$</td><td>0</td><td>$n_2$</td><td>0</td></tr>
<tr><td rowspan="4">$\mathbb{Z}/2$</td><td>long</td><td>at $E' \cup E'$</td><td>$(p-1)/12$</td><td>$(p-5)/12$</td><td>0</td><td>0</td><td>$(p-1)/12$</td><td>$(p-5)/12$</td><td>0</td><td>0</td></tr>
<tr><td rowspan="3">short</td><td>at $E_\iota \cup E_\iota$</td><td>0</td><td>0</td><td>$(p+1)/8$</td><td>$(p-3)/8$</td><td>0</td><td>0</td><td>$(p-3)/8$</td><td>$(p+1)/8$</td></tr>
<tr><td>at $C$ of type (4)</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td>at $C$ of type (3)</td><td>$n_3$</td><td>$n_3$</td><td>0</td><td>0</td><td>$n_3$</td><td>$n_3$</td><td>0</td><td>0</td></tr>
<tr><td>$\mathbb{Z}/5$</td><td colspan="2">at $C$ of type (6)</td><td colspan="8">0 if $p \equiv 1$ or 4 (mod 5)<br>1 if $p \equiv 2$ or 3 (mod 5)</td></tr>
</table>

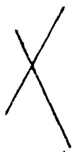
*Table 4.* $p \geqslant 7$

| Ramification group | | $p$ | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{Z}/6$ | $E_\omega \cup E_\omega$ | | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| $\mathbb{Z}/5$ | type (6) | | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 |
| $\mathbb{Z}/4$ | $E_\iota \cup E_\iota$ | | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| | type (5) | | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 |
| $\mathbb{Z}/3$ | $E_\omega \cup E_\omega$ | | 0 | 1 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 3 | 0 | 4 | 4 | 5 | 1 |
| | type (5) | | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| | type (2) | | 0 | 0 | 1 | 0 | 3 | 0 | 0 | 4 | 5 | 0 | 7 | 0 | 0 | 0 | 9 |
| $\mathbb{Z}/2$ | long | $E' \cup E'$ | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 3 | 3 | 0 | 0 | 4 | 0 | 5 |
| | short | $E_\iota \cup E_\iota$ | 1 | 1 | 0 | 0 | 2 | 3 | 0 | 4 | 0 | 0 | 5 | 6 | 0 | 7 | 0 |
| | | type (4) | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| | | type (3) | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 4 | 4 | 0 | 0 | 5 | 0 | 7 |
| total | | | 3 | 3 | 6 | 6 | 6 | 6 | 9 | 9 | 15 | 12 | 14 | 11 | 17 | 13 | 23 |
| $H_2(1, p)$ | | | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 5 | 4 | 5 | 4 | 6 | 5 | 8 |

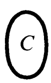
denote by $G_y$ the corresponding inertia group. We denote by $\delta_y$ the degree of local difference at $y$ of $\tilde{W}$. For a point $x$ of $S$, we denote $\delta_{\psi(x)}$ by $\delta_x$. We have

$$\begin{aligned} &\delta_x \geqslant |G_x| - 1 \text{ for } x \text{ of } \tilde{W} \text{ (resp. of } S), \\ &\text{and the equality holds if } |G_x| \text{ is prime to } p. \end{aligned} \tag{8.2}$$

In this section, for small characteristics $p$ we determine which groups $G$ appear, and in which way the ramifications, the groups and the curves are related. In the picture below, we use the following symbols:

(*a*):  $E_1$ $E_2$ principally polarized abelian surface $E_1 \times E_2$ with polarization $E_1 + E_2$, where $E_1$ and $E_2$ are supersingular elliptic curves,

(b):  $C$ principally polarized abelian surface $(J(C), C)$ with a non-singular curve $C$ of genus two such that $J(C)$ is isomorphic to a product of two supersingular elliptic curves,

(c):  principally polarized abelian surface $(J(C), C)$ with a non-singular curve $C$ of genus two such that $J(C)$ is supersingular and is not isomorphic to a product of two elliptic curves.

In a family $q$: $\mathscr{X} \to \mathbf{P}^1$ with relative polarization $D$ as in (1.9) we have $p^2 + 1$ fibres of type (*a*) or (*b*), and we have $5p - 5$ fibres of type (*a*) as we have seen in (1.13).

(1) $p = 2$

In this case, the supersingular locus was already studied by Igusa [7, pp. 615–616], and it turned out to be irreducible. In Moret-Bailly [10], we find a more refined description. There exists exactly one family $q$: $\mathscr{X} \to S$ of principally polarized abelian surfaces as in Moret-Bailly [10]. Exactly five fibres of $q$: $\mathscr{X} \to S$ are principally polarized abelian surfaces of degenerate type. Let $x_1, \text{---}, x_5$ be the corresponding points. In case $p = 2$, there is only one isomorphism class of supersingular elliptic curves defined by

$$E \colon Y^2 + Y = X^3.$$

There is only one standard divisor as in Section 3, which corresponds to the family $q$: $\mathscr{X} \to S$. Let $\mathscr{G}$ be the group of automorphisms of the family $q$: $\mathscr{X} \to S$ preserving the relative polarization. By the uniqueness of the standard divisor, $\mathscr{G}$ acts transitively on the set $\{x_1, \text{- - -}, x_5\}$, and by Table 1 in Section 6 we see that the order of the stabilizer at $x_1$ is equal to $192 \times 2 = 384$. Therefore, we have

$$|\mathscr{G}|/384 = 5, \quad \text{hence } |\mathscr{G}| = 1920.$$

If a curve of genus two in characteristic two is supersingular, its normal form is $Y^2 + Y = X^5 + \alpha X^3$ (cf. Igusa [7, p. 615]), hence by Igusa [7, p. 645] we see that a general fibre of $q: \mathscr{X} \to S$ has a group of automorphisms of order $2^4 \times 2$ as principally polarized abelian surface. Thus, $G = \text{Gal}\,(\mathbf{P}^1 \to \tilde{V})$ has order

$$|G| = 1920/(2^4 \times 2) = 60.$$

This group $G$ acts transitively on the set $\{x_1, \text{- - -}, x_5\}$ and the inertia group $G$ has order $384/(2^4 \times 2) = 12$. There is exactly one (isomorphism class of an) irreducible curve $C$ which has more automorphisms than a general supersingular curve:

$$C\colon Y^2 + Y = X^5.$$

In Igusa [7, p. 645], we have $|RC(C)| = 16 \times 5$. Let $\{y_1, \text{- - -}, y_n\}$ be the points of $S$ over which there is such a fibre. Then the inertia group at such a point has order $160/(2^4 \times 2) = 5$, hence at such points the covering is tamely ramified, and the local difference at such points equals $\delta_y = 4$. The covering $\tilde{\psi}: S = \mathbf{P}^1 \to \tilde{V}$ ramifies exactly at the points corresponding to the following principally abelian surfaces:

(a) $(E \times E, E \times \{0\} + \{0\} \times E)$ with
 $|G_x| = 12$, $x = \tilde{\psi}(x_i)$ $(1 \leqq i \leqq 5)$ and local different $\delta_x$;
 note that $\delta_x \geqq 12$,

(b) $(J(C), C)$ with
 $|G_y| = 4$, $y = \tilde{\psi}(y_j)$ $(i \leqq j \leqq n)$ and $\delta_y = 4$.

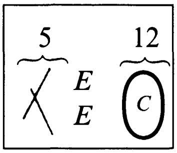
The Zeuthen–Hurwitz–Hasse formula reads in this case:

$$-2 = 60 \times (-2) + (60/12)\delta_x + n \times \delta_y.$$

Note that $n$ is a multiple of 12. We see that this is only possible with $n = 12$ and $\delta_x = 14$. Thus there is one orbit of points corresponding to $C$.

Summarizing, we have:

$\mathscr{X}$




$$\downarrow q$$

$$S \quad \underset{x_i \qquad y_j}{\rule{6em}{0.4pt}} \xrightarrow{\tilde{\psi}} \tilde{V} \longrightarrow V \subset \mathscr{A}_{2,1}$$

$$|G_x| = 12,\ \delta_x = 14; \quad |G_y| = 5,\ \delta_y = 4; \quad \deg \tilde{\psi} = 60.$$

THEOREM 8.1. *Assume $p = 2$. Under the notations as above,*

$$G \simeq A_5.$$

*Proof.* The group $G$ acts on the set $\{x_1, \text{- - -}, x_5\}$ as permutation. Since any element of $G$ which is not the identity has at most two fixed points on $\mathbf{P}^1$, the action of $G$ on $\{x_1, \text{- - -}, x_5\}$ is faithful. Since $|G| = 60$, we conclude $G \simeq A_5$. Q.E.D.

*Remark 8.2.* We give here another proof of Theorem 8.1 and a remark on the defining field of some special points on $S$. Let $\sigma$ be an element of $\mathscr{G}_{x_1}$ with $\operatorname{ord} \sigma = 3$. Then, $\sigma$ permutes $\{x_2, \text{- - -}, x_5\}$. It has a fixed point in $\{x_2, \text{- - -}, x_5\}$, say $\sigma(x_2) = x_2$. We choose an isomorphism

$$\theta \colon S \xrightarrow{\sim} \mathbf{P}^1$$

such that $\theta(\{x_3, x_4, x_5\}) = \mathbf{P}^1(\mathbf{F}_2)$. Then, we have $\sigma \in PGL(2, \mathbf{F}_2)$. Therefore, we have $x_1, x_2 \in \mathbf{P}^1(\mathbf{F}_4)$, and

$$\theta(\{x_1, \text{- - -}, x_5)\} = \mathbf{P}^1(\mathbf{F}_4).$$

Thus, we have an injective homomorphism

$$G \hookrightarrow PGL(2, \mathbf{F}_4).$$

Since $|G| = 60 = |PGL(2, \mathbf{F}_4)|$, we have

$$G \simeq PGL(2, \mathbf{F}_4) \simeq A_5.$$

Note that under $\theta$ the points $y_j$ $(1 \leqslant j \leqslant 12)$ are mapped onto $\mathbf{P}^1(\mathbf{F}_{16})\backslash\mathbf{P}^1(\mathbf{F}_4)$.

Now, assume $p \geqslant 3$. The number of irreducible components of $V_2 \subset \mathscr{A}_{2,1,2}$ equals $(p^2 - 1)/4$ (cf. Corollary 5.2). For every irreducible component $W$ of $V$ in $\mathscr{A}_{2,1}$, and an irreducible component $W_2$ of $V_2$ in $\mathscr{A}_{2,1,2}$ over $W$, we have a family $q\colon \mathscr{X} \to S \simeq \mathbf{P}^1$ and a group $G$ as in (7.2) such that $S$ is isomorphic to the normalization $\tilde{W}_2$ of $W_2$ (cf. Corollary 4.5). We have the morphism

$$\psi\colon S \to S/G \simeq \tilde{W} \to W \subset \mathscr{A}_{2,1}.$$

We fix a prime number $p$. We use the notations in Section 5. We set

$$G_\lambda\colon = R\Gamma(\mathscr{B}_\lambda) \quad (\lambda = 1, \text{- - -}, H' = H_2(1, p))$$

(cf. Theorem 5.7). Since $S \simeq \tilde{W}_2$, we have

$$\sum_{\lambda=1}^{H'} 720/|G_\lambda| = (p^2 - 1)/4 \tag{8.3}$$

(cf. Theorem 5.1 and Corollary 5.2).

(2) $p = 3$

There exists up to isomorphism exactly one supersingular elliptic curve $E$ defined by

$$E\colon Y^2 = X^3 - X.$$

We have only one standard divisor (cf. Section 3 and Table 1 in Section 6). Therefore, we have $H' = 1$ (see also Remark 5.9). Thus, by (8.3) we have

$$|G| = 4 \cdot 720/(p^2 - 1) = 360.$$

The family $q\colon \mathscr{X} \to S$ with relative polarization $D$ has $5p - 5 = 10$ fibres of principally polarized abelian surfaces of degenerate type, say at $x_1$, - - - , $x_{10}$ of $S$. By Table 1 in Section 6, we see

$$|G_{x_i}| = 72/2 = 36,$$

hence, $G$ operates transitively on $\{x_1, \cdots, x_{10}\}$. The branch locus of $\tilde{\psi}$: $S = \mathbf{P}^1 \to \tilde{W}$ is at the points corresponding to
(a) $(E \times E, E \times \{0\} + \{0\} \times E)$
(b) $(J(C), C)$, $C$: $Y^2 = X^5 - 1$ (the curve of type (6)).
The Jacobian variety $J(C)$ is not isomorphic to a product of two supersingular elliptic curves (cf. [IKO, Proposition 1.13]). Let $y_j \in S$ $(1 \leqslant j \leqslant n(360/5)$ with an integer $n)$ be the points of $S$ corresponding to the curve of type (6). Then, since $RA(C) \simeq \mathbb{Z}/5$, we see that

$$G_{y_j} \simeq \mathbb{Z}/5 \quad (1 \leqslant j \leqslant n(360/5))$$

by a similar method as in the proof of Lemma 7.9. By the Zeuthen–Hurwitz–Hasse formula, we have

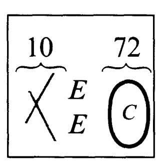
$$-2 = 360 \times (-2) + \sum_{i=1}^{10} \delta_{x_i} + \sum_{j=1}^{72n} \delta_{y_j}$$

with $\delta_{y_j} = 4$ and $\delta_{x_i} = m$, where $m$ and $n$ are some integers. Thus, we have

$$-2 = 360 \times (-2) + 10m + 4 \times 72n.$$

This is possible if and only if $n = 1$ and $m = 43$. We set $x = \tilde{\psi}(x_i)$ $(1 \leqslant i \leqslant 10)$ and $y = \tilde{\psi}(y_j)$ $(1 \leqslant j \leqslant 72)$. Then, we have the following picture of $q$: $\mathscr{X} \to S$ with relative polarization $D$.

$\mathscr{X}$




$\downarrow q$

$S$ $\underset{x_i \qquad y_j}{\overline{\qquad\qquad}} \xrightarrow{\tilde{\psi}} \tilde{V} \longrightarrow V \subset \mathscr{A}_{2,1}$

$|G_x| = 36,\ \delta_x = 43;\quad |G_y| = 5,\ \delta_y = 4;\quad \deg \tilde{\psi} = 360.$

THEOREM 8.3. *Assume $p = 3$. Under the notations as above,*

$$G \simeq A_6.$$

*Proof.* By Corollary 4.4, we have an injective homomorphism $G \hookrightarrow S_6$. Since $|G| = 360$, we conclude $G \simeq A_6$. Q.E.D.

(3) $p = 5$

There exists up to isomorphism exactly one supersingular elliptic curve $E$ defined by

$$E:\ Y^2 = X^3 - 1.$$

There exists up to isomorphism exactly one irreducible supersingular curve $C$ of genus two defined by

$$C:\ Y^2 = X^5 - X$$

such that the Jacobian variety $J(C)$ is isomorphic to $E \times E$. We have $RA(C) \simeq PGL(2, \mathbf{F}_5)$ (cf. Igusa [7, p. 645]), thus $|RA(C)| = 120$. We have only one standard divisor (cf. Section 3 and Table 1 in Section 6). Therefore, we have $H' = 1$ (see also Remark 5.9), and by (8.3) we have

$$|G| = 4 \cdot 720/(p^2 - 1) = 120.$$

We have $5p - 5 = 20$ points $x_1, \cdots, x_{20}$ of $S$ corresponding to

(a) $(E \times E, E \times \{10\} + \{0\} \times E)$.

By Table 1 in Section 6 we have

$$|G_{x_i}| = 12/2 = 6.$$

Therefore, $G$ operates transitively on the set $\{x_1, \cdots, x_{20}\}$. Since $J(C) \simeq E \times E$, the number of points of $S$ corresponding to

(b) $(J(C), C)$

is equal to $(p^2 + 1) - (5p - 5) = 6$. We denote by $y_1, \cdots, y_6$ the points of $S$ which correspond to $(J(C), C)$. Let $t$ be the number of orbits of the action of $G$ on $\{y_1, \cdots, y_6\}$, and let $n_i$ $(1 \leqslant i \leqslant t)$ be the cardinality of these orbits. By the Zeuthen–Hurwitz–Hasse formula we have

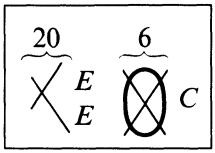
$$-2 = 120 \times (-2) + \sum_{i=1}^{20} \delta_{x_i} + \sum_{j=1}^{6} \delta_{y_j}$$

with $\delta_{x_i} = 5$ $(1 \leqslant i \leqslant 20)$. Thus, we have $\Sigma_{j=1}^{6} \delta_{y_j} = 138$. If $y_j$ is a point in an orbit whose cardinality is $n_i$, then we have

$$\delta_{y_j} \geqslant (120/n_i) - 1$$

by (8.2). Since $n_1 + n_2 + \cdots + n_t = 6$, we see that if $t \geqslant 2$, then $\Sigma_{j=1}^{6} \delta_{y_j} > 6 \cdot 23 = 138$. Thus, we have $t = 1$. Hence, $G$ acts transitively on $\{y_1, \cdots, y_6\}$ and we have $|G_{y_j}| = 20$ $(1 \leqslant j \leqslant 6)$. Moreover, we have $\delta_{y_j} = 138/6 = 23$. We set $x = \tilde{\psi}(x_i)$ $(1 \leqslant i \leqslant 20)$ and $y = \tilde{\psi}(y_j)$ $(1 \leqslant j \leqslant 6)$. Then, we have the following picture of $q\colon \mathscr{X} \to S$ with relative polarization $D$.

$\mathscr{X}$




$\downarrow q$

$S \qquad \underset{x_i \qquad y_j}{\rule{6cm}{0.4pt}} \xrightarrow{\tilde{\psi}} \tilde{V} \longrightarrow V \subset \mathscr{A}_{2,1}$

$$G_x \simeq \mathbb{Z}/6,\ \delta_x = 5; \quad |G_y| = 20,\ \delta_y = 23; \quad \deg \tilde{\psi} = 120.$$

Using Lemma 7.2 and $G_y \subset G \subset S_6$, we see $G_y \simeq \langle \sigma, \tau \rangle$ with suitable elements $\sigma$ and $\tau$ such that ord $\sigma = 4$, ord $\tau = 5$ and $\sigma\tau = \tau^2\sigma$.

THEOREM 8.4. *Assume $p = 5$. Under the notations as above,*

$$G \simeq PGL(2, \mathbf{F}_5) \simeq RA(C).$$

*Proof.* Let $\tau$ be an element of order five of $G_{y_6}$. Then, we have $\tau(y_6) = y_6$ and $\tau$ permutes $\{y_1, \cdots, y_5\}$. We may assume $\tau(y_1) = y_2$. We can choose an isomorphism

$$\theta\colon S \xrightarrow{\sim} \mathbf{P}^1$$

such that $\theta(y_6) = (1:0)$, $\theta(y_1) = (0:1)$ and $\theta(y_2) = (1:1)$. Then, by our choice of coordinates, we have

$$\theta \circ \tau \circ \theta^{-1}(x) = x + 1,$$

where $x$ is an inhomogeneous coordinate of $\mathbf{A}^1 = \mathbf{P}^1 \backslash \{(1:0)\}$. The set $\{y_1, \cdots, y_6\}$ is mapped by $\theta$ onto $\mathbf{P}^1(\mathbf{F}_5)$. Any element $g$ of $G$ induces a permutation of $\{y_1, \cdots, y_6\}$. Therefore, we have an injective homomorphism $g \hookrightarrow PGL(2, \mathbf{F}_5)$. Since $|G| = 120 = |PGL(2, \mathbf{F}_5)|$, we have $G \simeq PGL(2, \mathbf{F}_5)$. Q.E.D.

*Remark 8.5.* As in the proof of Theorem 8.4, any element of $G_{x_i}$ is defined over $\mathbf{F}_5$. Therefore, we see $\theta(x_i) \in \mathbf{P}^1(\mathbf{F}_{25})$. Hence, $\{x_1, \cdots, x_{20}\}$ is mapped onto $\mathbf{P}^1(\mathbf{F}_{25})\backslash\mathbf{P}^1(\mathbf{F}_5)$ by $\theta$.

*Remark 8.6.* In Section 5, we proved that $V$ is irreducible if and only if $p \leqslant 11$ (cf. Theorem 5.8). Here we present another proof in which we do not use any calculation of class numbers.

In case $2 \leqslant p \leqslant 5$, we have seen earlier in this section that $V$ is irreducible. Now we use the results in Section 7 for other prime numbers. In case $7 \leqslant p \leqslant 11$, we have

$$\sum_{\lambda=1}^{H'} 720/|G_\lambda| \leqslant 60$$

by (8.3). Therefore, we have $|G_\lambda| \geqslant 12$. By Lemma 7.2 and Pinkham [17, p. 4], $G_\lambda$ operates with at least three branch points. In Table 4 in Section 7, we have seen that for $p = 7$ or 11 the total number of branch points is equal to three. Thus, $V$ is irreducible in these cases. In case $p \geqslant 13$, we see by Table 3 in Section 7 that the total number of branch points is greater than or equal to six. Since for each family as in (7.1) we have at most three branch points by Table 2 in Section 7, we conclude that $V$ is reducible.

From now on, we assume $p \geqslant 7$. Under the notations in (8.1), we have seen in Lemma 7.2 which groups can appear. We have also seen

$$G \not\simeq \mathbf{Z}/6 \text{ (cf. Lemma 7.8) and } G \not\simeq \mathbf{Z}/5 \text{ (cf. Lemma 7.9).}$$

The supersingular locus $V_2$ in $\mathscr{A}_{2,1,2}$ has $(p^2 - 1)/4$ irreducible components (cf. Corollary 5.2). Let $\{W_\lambda\}_{\lambda=1,\cdots,H'}$ be the set of irreducible components of the supersingular locus $V$ in $\mathscr{A}_{2,1}$.

*Notation 8.7.* Suppose that $W_\lambda$ corresponds to $(G_\lambda;\, e'_{\lambda,1}, e'_{\lambda,2}, \cdots;\, e''_{\lambda,1}, e''_{\lambda,2}, \cdots)$ (cf. Notation 7.5) and that there exists $n_\lambda$ irreducible components of $V_2$ which are mapped by $\varphi$ to $W_\lambda$. Then, we write

$$\begin{aligned}(p^2 - 1)/4 &= n_1(G_1;\, e'_{1,1}, e'_{1,2}, \cdots;\, e''_{1,1}, e''_{1,2}, \cdots)\\ &\quad + \cdots + n_{H'}(G_{H'};\, e'_{H',1}, e'_{H',2} \cdots;\, e''_{H',1}, e''_{H',2}, \cdots).\end{aligned}$$

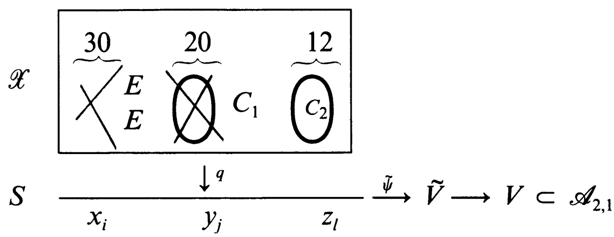
We have

$$(p^2 - 1)/4 = \sum_{\lambda=1}^{H'} n_\lambda \text{ and } n_\lambda|G_\lambda| = |S_6| = 720.$$

By Remark 5.9, Lemmas 7.2, 7.8, 7.9, 7.11, and Formulas (8.3) and (8.4), we have the following examples (4), (5), (6), (7), (8) and (9).

(4) $p = 7$

$$(p^2 - 1)/4 = 12 = 12(A_5; 2; 3, 5).$$




$$\deg \tilde{\psi} = 60.$$

$$E\colon Y^2 = X^3 - X,$$

$$C_1\colon Y^2 = X(X^2 - 1)(X^2 + 1),\ RA(C_1) \simeq S_4,$$

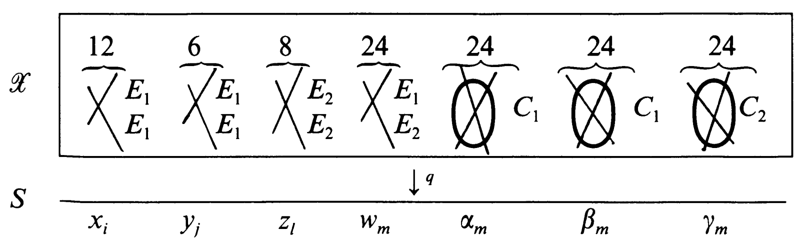
$$C_2\colon Y^2 = X^5 - 1,\ RA(C_2) \simeq \mathbb{Z}/5.$$

We set $\tilde{\psi}(x_i) = x\ (1 \leqslant i \leqslant 30)$, $\tilde{\psi}(y_j) = y\ (1 \leqslant j \leqslant 20)$, $\tilde{\psi}(z_l) = z\ (1 \leqslant l \leqslant 12)$.

$$G \simeq A_5;\ |G_x| = 2,\ |G_y| = 3,\ |G_z| = 5.$$

(5) $p = 11$

$$(p^2 - 1)/4 = 30 = 30(S_4; 2, 3, 4; -).$$




$$S \xrightarrow{\tilde{\psi}} \tilde{V} \longrightarrow V \subset \mathscr{A}_{2,1}$$

$\deg \tilde{\psi} = 24.$

$E_1\colon Y^2 = X^3 - X,$

$E_2\colon Y^2 = X^3 - 1,$

$C_1\colon Y^2 = (X^3 - 1)(X^3 - 3),\ RA(C_1) \simeq S_3,$

$C_2\colon Y^2 = X^6 - 1,\ RA(C_2) \simeq D_{12}.$

We set $\tilde{\psi}(x_i) = x$ $(1 \leqslant i \leqslant 12)$, $\tilde{\psi}(y_j) = y$ $(1 \leqslant j \leqslant 6)$, $\tilde{\psi}(z_l) = z$ $(1 \leqslant l \leqslant 8)$, $\tilde{\psi}(\alpha_m) = \alpha$, $\tilde{\psi}(\beta_m) = \beta$, $\tilde{\psi}(\gamma_m) = \gamma$ $(1 \leqslant m \leqslant 24)$.

$G \simeq S_4$; $|G_x| = 2$, $|G_y| = 4$, $|G_z| = 3$, $|G_w| = |G_\alpha| = |G_\beta| = |G_\gamma| = 1.$

We have $\hat{\psi}(\alpha) \neq \tilde{\psi}(\beta)$ and $\Psi \circ \tilde{\psi}(\alpha) = \Psi \circ \tilde{\psi}(\beta)$.

(6) $p = 13$

$(p^2 - 1)/4 = 42 = 12(A_5; -; 2, 3, 5) + 30(S_4; 2; 3, 4).$

(7) $p = 17$

$(p^2 - 1)/4 = 72 = 12(A_5; 3; 2, 5) + 60(D_{12}; 2, 6; 2).$

(8) $p = 19$

$(p^2 - 1)/4 = 90 = 30 + 60.$

By Lemma 7.8, $D_{12}$ does not appear. Hence, we have

$90 = 30(S_4; 2, 4; 3) + 60(A_4; 2; 3, 3).$

(9) $p = 23$

$(p^2 - 1)/4 = 132.$

We could not decide which of the two cases

$$\begin{aligned} 132 &= 12(A_4;\ 2,\ 3;\ 5) + 120(S_3;\ 2,\ 2,\ 3;\ -) \\ &= 72(D_{10};\ 2,\ 2;\ 5) + 60(A_4;\ 2,\ 3,\ 3;\ -) \end{aligned}$$

holds for this prime number.

(10) $p = 29$
Considering Lemma 7.8 and Table 4 in Section 7, we have two possibilities:

$$\begin{aligned} (p^2 - 1)/4 &= 210 \\ &= 60(D_{12};\ 2,\ 6;\ 2) + 30(S_4;\ 2,\ 3;\ 4) + 120(S_3;\ 3;\ 2,\ 2) \\ &= 60(D_{12};\ 2,\ 6;\ 2) + 90(D_8;\ 2;\ 2,\ 4) + 60(A_4;\ 3,\ 3;\ 2). \end{aligned}$$

LEMMA 8.8. *Assume $p = 29$. There exists up to isomorphism two supersingular curves of genus two with $RA(C) \simeq V_4$, and these two curves are conjugate with each other over the prime field $\mathbf{F}_{29}$.*

*Proof.* The zeros of the polynomial $h(X)$ which was introduced in [IKO, Definition 7.1] give all supersingular curves of type (3), (4) or (5) (cf. [IKO, Proposition 1.9]). In case $p = 29$, this polynomial is of degree seven. It is divisible by

$$(X + 1)(X - 9)(X - 13).$$

The curve with $\beta = -1$ is of type (5) and the curve with $\beta = 9$ or 13 is of type (4). A direct computation shows that for $p = 29$ we have

$$3h(X)/(X + 1)(X - 9)(X - 13) = X^4 - 3X^3 - X^2 - 3X + 1.$$

We can easily show that this polynomial is irreducible in $\mathbf{F}_{29}[X]$. Using zeros of this polynomial, we get two supersingular curves $C_1$, $C_2$ with $RA(C_1) = RA(C_2) \simeq V_2$ such that $C_1$ is not isomorphic to $C_2$ (cf. [IKO, Lemma 1.5]). Let $C_1$ (resp. $C_2$) be given by $\beta = \beta_1$ (resp. $\beta = \beta_2$) as in [IKO, Section 1.3]. Then $\beta_1$ is conjugate with $\beta_2$ over $\mathbf{F}_{29}$ as above. Thus, $C_1$ is conjugate with $C_2$ over $\mathbf{F}_{29}$. Q.E.D.

**Theorem 8.9.** *Assume $p = 29$. Then,*

$$210 = 60(D_{12}; 2, 6; 2) + 30(S_4; 2, 3; 4) + 120(S_3; 3; 2, 2).$$

*Proof.* Suppose

$$210 = 60(D_{12}; 2, 6; 2) + 90(D_8; 2; 2, 4) + 60(A_4; 3, 3; 2).$$

Under the notations in the proof of Lemma 8.8, curves $C_1$ and $C_2$ with $RA(C_i) \simeq V_4$ $(i = 1, 2)$ give $\mathbb{Z}/2$-ramifications. We denote by $\overline{\mathbb{F}}_{29}$ the algebraic closure of $\mathbb{F}_{29}$. The moduli space $\mathscr{A}_{2,1}$ is defined over $\mathbb{F}_{29}$ and the Galois group Gal $(\overline{\mathbb{F}}_{29}/\mathbb{F}_{29})$ operates. By Lemma 8.8, the point of $\mathscr{A}_{2,1}$ which corresponds to $(J(C_1), C_1)$ is transformed into the point which corresponds to $(J(C_2), C_2)$ by a suitable element of Gal $(\overline{\mathbb{F}}_{29}/\mathbb{F}_{29})$, which contradicts the fact that $C_1$ and $C_2$ belong to the different components with different groups. Q.E.D.

(11) $p = 31$
Considering Table 4 in Section 7, we conclude

$$\begin{aligned}(p^2 - 1)/4 &= 240 \\ &= 60(A_4; 2; 3, 3) + 60(A_4; 2; 3, 3) + 120(S_3; 2, 2; 3).\end{aligned}$$

*Remark 8.10.* We have no prime numbers for which we decided that one of the groups $D_{10}$, $D_8$, $V_4$ appears. In case $p \geqslant 7$ and the total number of branch points is not divisible by three (for example all prime numbers $p$ with $43 \leqslant p \leqslant 61$) it follows that from the groups $\mathbb{Z}/3$ and $\mathbb{Z}/2$ at least one of them appears.

*Remark 8.11.* The group $G = \{1\}$ appears for large $p$. If not, from (8.3) it would follow that

$$(p^2 - 1)/1440 \leqslant H' \leqslant (p^2 - 1)/48$$

by $2 \leqslant |G_\lambda| \leqslant 60$ for $\lambda = 1, 2, \cdots, H'$. The first inequality contradicts the asymptotic behavior of $H' = H_2(1, p)$ which has the leading term $(p^2 - 1)/2880$ (cf. Hashimoto and Ibukiyama [4, (II)]). Another way to show this fact, we calculate the total number of branch points. For example, in case $p \equiv 23 \pmod{24}$ and $|G_\lambda| \geqslant 2$ ($\lambda = 1, \cdots, H'$), then by Table 3 in Section 7 we have

$$(p + 1)/12 + (p + 1)/8 \geqslant 2H' \geqslant (p^2 - 1)/720,$$

a contradiction if $p \geqslant 167$.

*Remark 8.12.* We list in Table 5 some prime numbers where the group $G$ appears.

*Table 5.*

$p = 2 \quad G \simeq A_5$

$p = 3 \quad G \simeq A_6$

$p = 5 \quad G \simeq PGL(2, \mathbf{F}_5)$

$p \geqslant 7$

| Group $G$ | $720/\lvert G\rvert$ | appears at $p$ |
|---|---|---|
| $A_5$ | 12 | 7, 13, 17, --- |
| $S_4$ | 30 | 11, 13, 19, 29, --- |
| $D_{12}$ | 60 | iff $p \equiv 5 \pmod{12}$ |
| $A_4$ | 60 | 19, 31, --- |
| $D_{10}$ | 72 | ? > Ibukiyama: NO 20-IV-'85 |
| $D_8$ | 90 | ? |
| $S_3$ | 120 | 29, 31, --- |
| $V_4$ | 180 | ? |
| $\mathbb{Z}/3$ | 240 | at 43, 47, 53, 59, 61, --- |
| $\mathbb{Z}/2$ | 360 | |
| $\{1\}$ | 720 | 167 etc. |

*Remark 8.13.* Professor T. Ibukiyama communicated that he could decide the possibility of $G$ and could also compute, up to isomorphism, the number of families $\mathscr{X} \to S$ as above with group $G$ for each $p$.

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