COMPOSITIO MATHEMATICA

JÜRGEN JOST

On the existence of harmonic maps from a surface into the real projective plane

Compositio Mathematica, tome 59, nº 1 (1986), p. 15-19
<http://www.numdam.org/item?id=CM_1986__59_1_15_0>

Compositio Mathematica 59 (1986) 15–19

ON THE EXISTENCE OF HARMONIC MAPS FROM A SURFACE INTO THE REAL PROJECTIVE PLANE

Jürgen Jost

The problem of representing the homotopy classes of maps of a closed Riemann surface Σ to the real projective plane P (endowed with a Riemannian metric) by harmonic maps has been studied in detail in [3], we use that reference as background (with the topological assertions completed by [1]). In the present note, we prove the following

THEOREM: *Let* θ: $\pi_1(\Sigma) \to \pi_1(P)$ *be a non-zero homomorphism.*

(*a*) *If* θ *is nonorientable, any homotopy class representing* θ *contains a minimum of the energy.*

(*b*) *If* θ *is orientable, then at least two* (*of the infinitely many*) *homotopy classes representing* θ *contain a minimum of the energy.*

(*c*) *If P is endowed with its standard* $\mathbb{RP}$ (2) *metric of constant positive curvature, then, given* $x_0 \in \Sigma$, $p_0 \in P$, *there are at least two harmonic maps inducing* θ *and mapping* x_0 *into* p_0.

REMARKS: (i) (a) is new only for the case

$$\deg \theta = \mathrm{O} \bmod 2,$$

since otherwise θ induces only one homotopy class. In any case, θ as in a) can always be represented by a map from Σ onto a closed geodesic in P, cf. [1], and hence also by a harmonic map. It is not clear, however, whether – for an arbitrary metric on P – such a map is energy minimizing.

(ii) Despite the above Theorem and several nonexistence results, cf. [3], there is still a number of homotopy classes where the existence question is open, most notably for oriented maps from nonorientable surfaces into P (except those covered by (b)).

The proof of the results is obtained by modifying the argument of [4] (cf. also [2]; a key idea is due to Wente [6]). The idea is to construct a map v with

$$E(v) < E_\theta + 2 \text{ Area } P, \tag{1}$$

where

$$E(V) := \tfrac{1}{2}\int_{\Sigma} |\mathrm{d}v|^2 \,\mathrm{d}\Sigma$$

is the energy of v, and E_θ denotes the infimum of the energy over all maps inducing θ. As in [4] or [5], we find a map $\tilde{u}\colon \Sigma \to P$ with

$$E(\tilde{u}) = E_\theta. \tag{2}$$

Since θ is nontrivial, $\tilde{u}$ is not a constant map, and hence we can find some $x_1 \in \Sigma$ with

$$\mathrm{d}\tilde{u}(x_1) \neq 0.$$

Let $k\colon U \to \mathbb{C}$ be a conformal chart on some neighbourhood U of $\tilde{u}(x_1)$ with $k(\tilde{u}(x_1)) = 0$.

By Taylor's theorem, $k \circ \tilde{u} \,|\, \partial B(x_1, \epsilon)$ is a linear map up to an error of order $\mathrm{O}(\epsilon^2)$, i.e.

$$|k \circ \tilde{u}(x) - \mathrm{d}(k \circ \tilde{u})(x_1)(x - x_1)| = \mathrm{O}(\epsilon^2) \tag{3}$$

for $x \in \partial B(x_1, \epsilon)$.

We now look at conformal maps of the form

$$w = az + b/z, \quad a, b \in \mathbb{C}, \quad a = a_1 + \mathrm{i}a_2, \quad b = b_1 + \mathrm{i}b_2.$$

The restriction of such a map to a circle $\rho(\cos\theta + \mathrm{i}\sin\theta)$ in $\mathbb{C}$ is given by

$$u = \left(a_1\rho + \frac{b_1}{\rho}\right)\cos\theta + \left(\frac{b_2}{\rho} - a_2\rho\right)\sin\theta$$

$$v = \left(a_2\rho + \frac{b_2}{\rho}\right)\cos\theta + \left(a_1\rho - \frac{b_1}{\rho}\right)\sin\theta,$$

where $w = u + \mathrm{i}v$.

Therefore, we can choose a and b in such a way that w restricted to this circle coincides with any prescribed nontrivial linear map. This map is nonsingular if

$$\rho^4 \neq \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}.$$

W.l.o.g.

$$\rho^4 \leqslant \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2} \tag{4}$$

(otherwise we perform an inversion at the unit circle).

Hence w can be extended as a conformal map from the interior of the circle $\rho(\cos\theta + \mathrm{i}\sin\theta)$ onto the exterior of its image. (If equality holds in (4), then this image is a straight line covered twice, and the exterior is the complement of this line in the complex plane).

We are now in a position to define v.

On $\Sigma \setminus B(x_1, \epsilon)$ we put $v = \tilde{u}$.

On $B(x_1, \epsilon - \epsilon^2)$,we choose a conformal map w into the extended complex plane $\overline{\mathbb{C}}$ as above that coincides on $\partial B(x_1, \epsilon - \epsilon^2)$ with the linear map $(1/(1-\epsilon))\, \mathrm{d}(k \circ \tilde{u})(x_1)$. We identify $\overline{\mathbb{C}}$ with S^2 via stereographic projection and let

$$\pi\colon S^2 \to \mathbb{RP}(2)$$

be the standard projection, normalized by $\pi(0) = \tilde{u}(x_1)$. (We have of course identified the conformal structure on P with the standard $\mathbb{RP}(2)$ structure, so that $\pi\colon S^2 \to P$ is conformal). We then put $v = \pi \circ w$ on $B(x_1, \epsilon - \epsilon^2)$.

Finally, on $B(x_1, \epsilon) \setminus B(x_1, \epsilon - \epsilon^2)$ we interpolate between $\tilde{u}$ and $\pi \circ w$. Introducing polar coordinates (r, ϕ), we define

$$f(\phi) := k \circ \tilde{u}(\epsilon, \phi)$$

$$g(\phi) := \mathrm{d}(k \circ \tilde{u})(x_1)(\epsilon, \phi) = \frac{1}{1-\epsilon}\mathrm{d}(k \circ \tilde{u})(x_1)(\epsilon - \epsilon^2, \phi)$$

and

$$t(r, \phi) := (f(\phi) - g(\phi)) \cdot \frac{r}{\epsilon^2} + \frac{1}{\epsilon}(g(\phi) - (1-\epsilon) f(\phi))$$

and finally

$$v = k^{-1} \circ t \quad \text{on } B(x_1, \epsilon) \setminus B\big(x_1, \epsilon - \epsilon^2\big).$$

It is straightforward to calculate (cf. [4] and [5; 4.5]).

$$E(k^{-1} \circ t) = \mathrm{O}(\epsilon^3). \tag{5}$$

Therefore

$$E(v)=E(\tilde{u}\,|\,\Sigma\setminus B(x_1,\epsilon))+E\big(\pi\circ w\,|\,B(x_1,\epsilon-\epsilon^2)\big)$$
$$+E\big(k^{-1}\circ t\,|\,B(x_1,\epsilon)\setminus B(x_1,\epsilon-\epsilon^2)\big).$$

The first term contributes (cf. (3))

$$E(\tilde{u})-\mathrm{O}(\epsilon^2)=E_\theta-\mathrm{O}(\epsilon^2)$$

since $\mathrm{d}\tilde{u}(x_1)\neq 0$, the second one at most

$$2\ \mathrm{Area}(P),$$

since $\pi\circ w$ is conformal of degree 2, and the third one is controlled by (5).

Altogether

$$E(v)\leqslant E_\theta-\mathrm{O}(\epsilon^2)+2\ \mathrm{Area}(P)+\mathrm{O}(\epsilon^3)$$

and if we choose ϵ small enough, v satisfies (1), and the proof of a) and b) is complete, arguing as in [4] and observing that $\tilde{u}$ and v exhaust enough homotopy classes, cf. [3].

For the proof of c) we fix $x_0\in\Sigma$ and $p_0\in P$ and minimize the energy only among maps u with

$$u(x_0)=p_0. \tag{6}$$

(6) does not constitute a restriction, since by assumption the isometry group of the image is transitive, and hence any $w\colon\Sigma\to P$ can be transformed into a map satisfying (6) and having the same energy, simply by composition with an isometry of P that maps $w(x_0)$ onto p_0.

Therefore, we can work in the category of homotopy classes with base point, and the argument of the proof of (a), (b) then shows that the minimum of energy is attained in at least two classes, since now attaching of a sphere leads to a new class in any case.

This proves (c).

Acknowledgement

I gratefully acknowledge the support of my work by the Centre for Mathematical Analysis in Canberra and the SFB 72 in Bonn.

References

[1] F. ADAMS: Maps from a surface to the projective plane. *Bull. London Math. Soc.* 14 (1982) 533–534.

[2] H. BREZIS and J.M. CORON: Large solutions for harmonic maps in two dimensions. *Comm. Math. Phys.* 92 (1983) 203–215.

[3] J. EELLS and L. LEMAIRE: On the construction of harmonic and holomorphic maps between surfaces. *Math. Ann.* 252 (1980) 27–52.

[4] J. JOST: The Dirichlet problem for harmonic maps from a surface with boundary onto a 2-sphere with non-constant boundary values. *J. Diff. Geom.* 19 (1984) 393–401.

[5] J. JOST: Harmonic maps between surfaces. *Springer Lecture Notes in Math.*, 1062 (1984).

[6] H. WENTE: A general existence theorem for surfaces of constant mean curvature. *Math. Z.* 120 (1971) 277–288.

(Oblatum 27-VIII-1984)

Mathematisches Institut
Ruhruniversität
Postfach 102148
D-4630 Bochum
BRD