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## KODAIRA SINGULARITIES AND AN EXTENSION OF ARNOLD'S STRANGE DUALITY

W. Ebeling \* and C.T.C. Wall

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### Introduction

In Arnold's survey article [4] he calls attention to a "strange duality" among the 14 "exceptional unimodal" singularities of surfaces in  $\mathbb{C}^3$ . Although a beautiful interpretation of this duality was given by Pinkham

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[47], it remains somewhat mysterious. In independent work in early 1982 the two authors discovered an extension of this duality embracing on one hand series of bimodal singularities and on the other, complete intersection surfaces in  $\mathbb{C}^4$ . Following Nakamura [44,45], we related this to Hirzebruch–Zagier duality of hyperbolic (alias cusp) singularities.

Our extension of the duality is described in section 5.2 (following some remarks in sections 2.3 and 4.2). The duals of complete intersection triangle singularities are not themselves singularities, but are virtual ( $n = -1$ ) cases of sequences (e.g.  $W_{1,n}; n \geq 0$ ) of bimodal singularities. We associate to these well-defined Milnor lattices, and show that all numerical features of Arnold’s strange duality continue to hold.

We also discovered some other symmetries between singularities related to the duality of hyperbolic singularities, embracing other Kodaira singularities, too. So the appearance of pairs of bimodal exceptional singularities with isomorphic Milnor lattices [16] turned out to be part of a more extensive symmetry between singularities of in general different embedding dimensions (see again section 5.2).

Therefore it seemed appropriate to consider the whole class of Kodaira singularities. These are described below, and one of our objectives is to extend to them as far as possible results known for the hypersurface cases, and to seek alternative characterisations analogous to the many known [14] for “rational double points”. A limited success is attained here, but it does become clear that this class of singularities has many beautiful properties.

The paper is divided into five sections. The first introduces Kodaira singularities in general, and the second the strange duality. Next we consider embeddings in euclidean space (and equations). Finally we study the associated quadratic forms: first from the viewpoint of synthesis from local data, and then from the Dynkin diagram approach.

## 1. Singularities of Kodaira type

### 1.1. Enumeration

Kodaira’s enumeration [31] of exceptional fibres in pencils of elliptic curves shows intriguing parallels with other well-known lists (Lie groups etc.). It was discovered ten years ago by Kulikov [32], Laufer [35] and Reid [53] that the same exceptional sets  $E$  but with different neighbourhoods gave minimal resolutions for an important class of singularities. These we shall call *Kodaira singularities* (see also [30]).

We recall Kodaira’s classification. The table given is adapted from [31, p. 604, Table 1] and gives some insight into the structure of the classification. In this table  $A$  is the local monodromy matrix of the elliptic fibration. The cases are distinguished by the conjugacy class of  $A$  in  $SL_2(\mathbb{Z})$ : we obtain all those with  $|\text{tr } A| \leq 2$ , save for the restriction  $b \geq 0$ .

Name	Matrix $A$	$c$	Type of $A^{-1}$	Value of $j$
$I_0$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	$I_0$	Regular
$I_b$	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	$\infty$	–	Pole of order $b$ ( $b > 0$ )
$I_0^*$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	2	$I_0^*$	Regular
$I_b^*$	$\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$	$\infty$	–	Pole of order $b$ ( $b > 0$ )
II	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	II*	0
II*	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	II	0
III	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	III*	1
III*	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	III	1
IV	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	IV*	0
IV*	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	IV	0

The order of  $A$  is denoted by  $c$ : it will reappear in section 3.4.

We now describe the exceptional fibres in each case, and also introduce our notation for the Kodaira singularities.

*Case  $I_0(d)$ :*  $E$  is an elliptic curve. Its normal bundle has Chern number  $-d$  ( $d = 0$  for the Kodaira fibre,  $d > 0$  for the singularity).

*Case  $I_n(b_1, \dots, b_n)$ :*  $E$  is a cycle of rational curves, each meeting both its neighbours. For  $n = 2$  we interpret this to mean that the two curves intersect (transversely) twice; for  $n = 1$ , that the curve has a single point of (transverse) self-intersection. The normal degrees  $-b_1, \dots, -b_n$  of the curves (in order) satisfy  $b_i \geq 2$ . If all the  $b_i$  equal 2, we have the Kodaira fibre. (Note that for  $n = 1$  the curve has normal degree  $-b_1$ , but selfintersection number  $-b_1 + 2$ .)

*Case II( $k$ ):*  $E$  is a rational curve with a cusp; the normal degree is  $-(k + 2)$ , with  $k \geq 1$ .

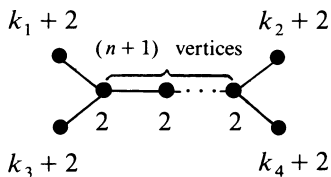
*Case III( $k_1, k_2$ ):*  $E$  consists of two rational curves touching in a point (intersection number 2). The normal degrees are  $-(k_1 + 2)$ ,  $-(k_2 + 2)$  where  $k_1 \geq 0$ ,  $k_2 \geq 0$ ,  $k_1 + k_2 > 0$ .

*Case IV( $k_1, k_2, k_3$ ):*  $E$  consists of three rational curves, meeting (pairwise transversely) in a single point: the normal degrees are  $-(k_i + 2)$ .

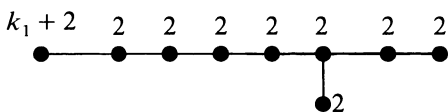
In the remaining cases, all components of  $E$  are rational and nonsingular;  $E$  has normal crossings; no two components meet more than once. We can thus describe  $E$  by a graph  $\Gamma_E$  with one vertex for each component of  $E$ : two vertices are joined iff the corresponding compo-

nents meet. We weight this by writing next to each vertex the integer  $b$  such that  $-b$  is the degree of the corresponding component.

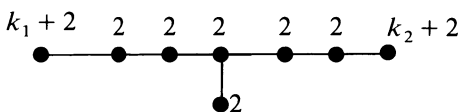
Case  $I_n^*(k_1, k_2, k_3, k_4)$ : Dual graph an extended Dynkin graph of type  $\tilde{D}_{n+4}$ ,



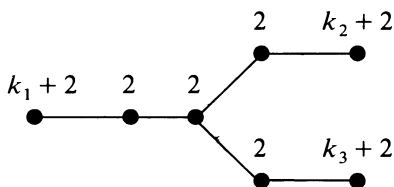
Case  $II^*(k_1)$ : (type  $\tilde{E}_8$ )



Case  $III^*(k_1, k_2)$ : (type  $\tilde{E}_7$ )



Case  $IV^*(k_1, k_2, k_3)$ : (type  $\tilde{E}_6$ )



In each case, the  $k_i$  are nonnegative integers, not all 0: the case when all are 0 gives the fibre in Kodaira's list.

The Kodaira singularities have two basic intrinsic properties: they are elliptic (of genus 1) and Gorenstein (there is a local regular 2-form). For a discussion in terms of these properties, see [53] and [35].

The above symbols do not quite provide a complete classification. For cases  $I_0, I_0^*$  there is a "modulus": in case  $I_0$ , this can be taken as the  $j$ -invariant of the elliptic curve  $E$ ; for  $I_0^*$  as given by the cross-ratio  $\lambda$  of the set of 4 points (in some order) where the outer curves meet the central one. The precise classification in this case depends on consideration of changing the order of the points, and on which of the  $k_i$  are distinct.

For a given case, given  $k_i$  (or  $b_i$ ), and (where relevant) a given  $\lambda$ , the

singularity is determined up to a finite ambiguity. The number of cases is [33] 1 for types  $I_n$  ( $n \geq 0$ ),  $I_n^*$  ( $n > 0$ ); 2 for types II, III, IV and  $I_0^*$ ; and 3 for types  $II^*$ ,  $III^*$  and  $IV^*$ . For types  $I_0$ , II, III, IV,  $I_0^*$ ,  $II^*$ ,  $III^*$ ,  $IV^*$  just one of these cases gives a singularity with  $\mathbb{C}^*$  action.

### 1.2. Uniformisation

The survey article of Neumann [46] describes the different types of uniformisation available for surface singularities, and gives a bijection between such uniformisations and geometric structures (in the style of Thurston [60,61]) on the 3-manifold which is the boundary of a neighbourhood of the singularity.

For isolated surface singularities there are just 4 models available; we consider them in turn.

A geometric structure of type  $S^3$  comes from a finite group  $\Gamma$  acting freely on  $S^3$ : thus  $\Gamma \subset U_2$ . As is well known [68] the quotient  $\mathbb{C}^2/\Gamma$  is Gorenstein if and only if  $\Gamma \subset SU_2$ . These cases give rise to the well-known list  $A_n$ ,  $D_n$ ,  $E_n$  of singularities [14,15].

For a geometric structure of type Nil we have a discrete cocompact subgroup  $\Gamma$  of the isometry group  $O_2 \cdot \text{Nil}$  acting freely on Nil. The corresponding singularity is Gorenstein if and only if  $\Gamma \subset \text{Nil}$  [12]. Up to deformation,  $\Gamma$  then belongs to one of a sequence

$$\Gamma_n = \begin{pmatrix} 1 & \mathbf{Z} & \mathbf{Z} \\ 0 & 1 & n\mathbf{Z} \\ 0 & 0 & 1 \end{pmatrix}, \quad n \in \mathbb{N}.$$

The corresponding singularity is then of type  $I_0(n)$  above; conversely, we obtain all these. These are the “simple-elliptic” singularities of Saito [54].

Neumann describes in [46] all singularities with geometric structure of type Sol: he obtains two classes. The second class are easily seen not to be Gorenstein. The first give precisely the singularities of type  $I_n$  ( $n \geq 1$ ) above. These are commonly called “cusp singularities” after their appearance in the work of Hirzebruch [23]: however, we prefer to avoid this term.

There remains the much more extensive class of structures of type  $\widetilde{SL}_2$ . Again, it has been shown by Dolgachev [12] and others that the corresponding singularity is Gorenstein if and only if it is the (compactification of the) quotient of the model (universal cover of the bundle of nonzero tangent vectors of the hyperbolic plane  $\mathfrak{H}$ ) by a free discrete cocompact subgroup of  $\widetilde{SL}_2$ . All isolated surface singularities admitting a  $\mathbb{C}^*$  action which is “good” in the sense that, for any  $P$ ,  $t.P$  tends to the singular point as  $t \rightarrow 0$ , except those already discussed, belong to this class: in particular, all those defined by a weighted homogenous system of equations.

Given such a  $\mathbb{C}^*$ -action, one usually describes the situation by “Seifert invariants” as follows. The surface  $T$  with genus  $\bar{g}$  is the orbit space of the action. The non-free  $\mathbb{C}^*$ -orbits, where the stabiliser has order  $p_i$ , give rise to points  $P_i \in T$  marked with  $p_i$ . The action of the stabiliser on the tangent space defines an invariant  $q_i \bmod p_i$ . It is better to consider  $T$  as orbifold (in the sense of Thurston [60]; see also [55]): as such, it has Euler characteristic

$$\chi = 2 - 2\bar{g} + \sum (p_i^{-1} - 1).$$

The link of the singularity is a circle bundle over this orbifold, with characteristic class

$$e = \beta + \sum q_i p_i^{-1},$$

where  $\beta$  is a certain integer. The first homology group of the link is the sum of  $2\bar{g}$  copies of  $\mathbb{Z}$  and a finite group of order  $\Delta = |e| \prod p_i$ . A minimal resolution consists of  $T$  with normal degree  $\beta$ , with a chain of rational curves for each point  $P_i$ . The (negatives of the) normal degrees in the chain are obtained by expanding  $p_i/q_i$  as a continued fraction.

In the Gorenstein case, these data simplify as follows. There is an integer  $R$  such that  $\chi = eR$ . For each  $i$ ,  $q_i$  is inverse to  $R \bmod p_i$ .

We can easily calculate all these from the resolution for cases of Kodaira type. They are listed in the table. In each of these cases,  $\bar{g} = 0$ . We recall that (for given  $\lambda$  in the  $I_0^*$  cases), just one of the 2 (or 3) isomorphism classes admits a good  $\mathbb{C}^*$ -action.

The numbers  $p_i$  are called the Dolgachev numbers of the singularity, after [10,11]. Observe that all cases where  $\bar{g} = 0$ , there are just 3 exceptional orbits,  $R = 1$  and  $\chi < 0$  appear in cases II, III and IV above.

### 1.3. The fundamental cycle

It is shown in [53] and [35] that the minimal resolution of an elliptic Gorenstein singularity carries a 2-cycle  $Z$  with the following properties:

- (1) For divisors  $D$  supported on the exceptional set  $E$ ,

$$h^1(\mathcal{O}_D) = h^1(\mathcal{O}_E) \Leftrightarrow D \geq Z.$$

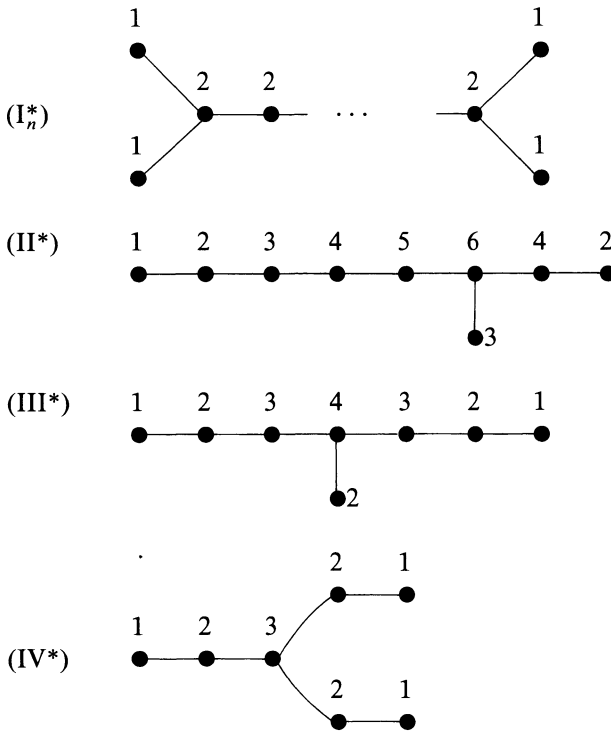
Case	$p_i$	$R$	$\Delta$
II( $k_1$ )	$2, 3, 6 + k_1$	1	$k_1$
III( $k_1, k_2$ )	$2, 4 + k_1, 4 + k_2$	1	$k_1 k_2 + 2(k_1 + k_2)$
IV( $k_1, k_2, k_3$ )	$3 + k_1, 3 + k_2, 3 + k_3$	1	$3\sum k_i + 2\sum k_i k_j + \prod k_i$
$I_0^*(k_1, k_2, k_3, k_4)$	$2 + k_1, 2 + k_2, 2 + k_3, 2 + k_4$	1	$4\sum k_i + 4\sum k_i k_j + 3\sum k_i k_j k_l + 2\prod k_i$
II*( $k_1$ )	$2, 3, 6 + 5k_1$	5	$k_1$
III*( $k_1, k_2$ )	$2, 4 + 3k_1, 4 + 3k_2$	3	$3k_1 k_2 + 2(k_1 + k_2)$
IV*( $k_1, k_2, k_3$ )	$3 + 2k_1, 3 + 2k_2, 3 + 2k_3$	2	$3\sum k_i + 4\sum k_i k_j + 4\prod k_i$

(2) If the singularity occurs on a projective variety  $\bar{X}$  with resolution  $M \supset E$ , then for some line bundle  $L$  on  $\bar{X}$  the canonical bundle  $K_M = f^*L \otimes \mathcal{O}_M(-Z)$ .

(3)  $Z$  is the unique positive divisor supported on  $E$  with positive arithmetic genus.

(4)  $Z$  is the minimal cycle satisfying  $Z \cdot C \leq 0$  for all components  $C$  of  $E$ .

It is easy to write down  $Z$  for singularities of Kodaira type: for the original Kodaira elliptic curve we have  $Z^2 = 0$ . Thus for types  $I_n$ , II, III and IV,  $Z$  is a sum of the components  $C$  of  $E$ , each with multiplicity 1. The multiplicities in the remaining cases are as follows:



The fundamental cycle  $Z$  determines in turn the invariant  $D = -Z^2$  which we will call the *grade*. It is shown in [53] and [35] that (a) the degree (or multiplicity) of the singular point is  $\max(D, 2)$ , (b) its embedding dimension is  $\max(D, 3)$ . Thus for  $D \leq 3$  we have hypersurface singularities; for  $D = 4$  we have complete intersections in  $\mathbb{C}^4$ . The cases  $D > 4$  do not give complete intersections; however [9] if  $D = 5$  we have Pfaffian singularities in  $\mathbb{C}^5$ .

Observe that the normal degrees for a singularity of Kodaira type are obtained from those for a Kodaira elliptic curve by decreasing the



Case	$I_0(d)$	$I_n(b_1, \dots, b_n)$	$II(k_1)$	$III(k_1, k_2)$	$IV(k_1, k_2, k_3)$	$I_n^*(k_1, k_2, k_3, k_4)$
$D$	$d$	$\Sigma(b_i - 2)$	$k_1$	$k_1 + k_2$	$k_1 + k_2 + k_3$	$k_1 + k_2 + k_3 + k_4$
Case	$II^*(k_1)$	$III^*(k_1, k_2)$	$IV^*(k_1, k_2, k_3)$			
$D$	$k_1$	$k_1 + k_2$	$k_1 + k_2 + k_3$			

numbers for certain components of multiplicity 1 in  $Z$ . Thus  $Z^2$  is decreased by the same amount. We can now see that, in the notation above, the grade is as indicated in the table.

Thus the singularities of a given type with grade  $D$  are classified by partitions of  $D$  into the appropriate number of parts. We shall see later that these partitions correspond to Arnold’s series of singularities.

The embedding result comes from considering the line bundle  $L = \mathcal{O}_Z(-Z)$ : then [53]  $h^0(L) = D$ , and  $H^0(L)$  generates  $\oplus H^0(nL)$  if  $D \geq 3$ . If  $D = 2$ , the cokernel of  $H^0(L) \otimes H^0(L) \rightarrow H^0(2L)$  has dimension 1; and if  $D = 1$  our three generators are in  $H^0(L)$ ,  $H^0(2L)$  and  $H^0(3L)$ , respectively.

## 2. Duality

### 2.1. Hyperbolic singularities: Inoue–Hirzebruch surfaces and duality

By “hyperbolic singularity” we mean those of type  $I_n$  with  $n > 0$ . The link of  $I_n(b_1, \dots, b_n)$  is a torus bundle over a circle, with monodromy

$$A(\mathbf{b}) = \begin{pmatrix} b_n & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix},$$

where each  $b_i \geq 2$ , and at least one is  $> 2$ . Indeed, this is a normal form for conjugacy classes of matrices in  $SL_2(\mathbb{Z})$  with trace  $\geq 3$ . The conjugate by a matrix of determinant  $-1$  can be put in the form  $A(\mathbf{b}^*)$ ; then  $\mathbf{b}^*$  is defined to be the sequence *dual* to  $\mathbf{b}$ . The singularity link corresponding to  $I_n(\mathbf{b}^*)$  is thus homeomorphic to that for  $I_n(\mathbf{b})$ , but with reversed orientation. It is not difficult to fit the two singularities together to obtain a complete surface with two isolated singular points [25b,37,44,45,52].

These are called Inoue–Hirzebruch surfaces, or hyperbolic Inoue surfaces. We observe parenthetically that there also exist [25a] parabolic Inoue surfaces, obtained by compactifying a  $I_0(n)$  singularity by a “Kodaira fibre” of type  $I_n(2, 2, \dots, 2)$ , and forming in some sense a limiting case of the above.

To obtain the duality explicitly we distinguish the  $b_r$  equal to 2 from those exceeding 2, and write the sequence (up to cycling reordering) as

$$\mathbf{b} = \underbrace{2, \dots, 2}_{k_1^* - 1}, k_1 + 2, \underbrace{2, \dots, 2}_{k_2^* - 1}, k_2 + 2, \dots, k_g + 2.$$

We then have [24]

$$\mathbf{b}^* = k_1^* + 2, \underbrace{2, \dots, 2}_{k_1 - 1}, k_2^* + 2, \underbrace{2, \dots, 2}_{k_2 - 1}, \dots, k_g^* + 2, \underbrace{2, \dots, 2}_{k_g - 1}.$$

Observe that since not all the  $b_r$  are 2, this is well defined. Each of the  $k_i, k_i^*$  is  $\geq 1$ . This duality, and the alternative notation  $k_i, k_i^*$  is the key to understanding the further relations below.

Instead of matrices, we could have worked with continued fractions. It follows from [23] that if

$$\xi = b_n - \frac{1}{b_{n-1}} - \dots - \frac{1}{b_2} - \frac{1}{b_1} - \frac{1}{b_n} - \dots,$$

then

$$\xi - 1 = k_g + \frac{1}{k_g^*} + \frac{1}{k_{g-1}} + \dots + \frac{1}{k_1} + \frac{1}{k_1^*} + \frac{1}{k_g} + \dots.$$

In terms of the new notation, we find

$$D = \sum (b_r - 2) = \sum k_i,$$

so that the formula  $D = \sum k_i$  now holds in all cases. We also have

$$n = \text{length of sequence } \mathbf{b} = \sum k_i^* = D^* \text{ (say).}$$

Thus  $D^*$  is the grade of the dual singularity, which is of type  $I_D$ . We also need

$$\sum_{i=1}^g b_i = \sum_{i=1}^g (k_i^* - 1)2 + (k_i + 2) = \sum_{i=1}^g (2k_i^* + k_i) = 2D^* + D;$$

of course also  $\sum b_i^* = 2D + D^*$ .

We now consider duality, restricted to complete intersections, following Nakamura [44]. If both  $I_n(\mathbf{b})$  and its dual are complete intersections, then  $\sum_{i=1}^g k_i \leq 4$  and  $\sum_{i=1}^g k_i^* \leq 4$ . Since  $k_i, k_i^* \geq 1$ , cases are as follows:

- $g = 1$ ; each of  $\mathbf{k}, \mathbf{k}^*$  is 1, 2, 3, or 4.
- $g = 2$ ; each of  $\mathbf{k}, \mathbf{k}^*$  is (1, 1), (1, 2), (1, 3) or (2, 2).
- $g = 3$ ; each of  $\mathbf{k}, \mathbf{k}^*$  is (1, 1, 1) or (1, 1, 2).
- $g = 4$ ; each of  $\mathbf{k}, \mathbf{k}^*$  is (1, 1, 1, 1).

We also need a cyclic order, in which terms of  $\mathbf{k}$  alternate with those of  $\mathbf{k}^*$ . For  $g \leq 2$ , such an order is clearly unique: the only time when it is not is when  $\mathbf{k} = \mathbf{k}^* = (1, 1, 2)$ : the cyclic orders (1, 1, 1, 1, 2, 2) resp. (1, 1, 2, 1, 1, 2) correspond to  $\mathbf{b} = \mathbf{b}^* = (2, 3, 3, 4)$  resp. (2, 3, 4, 3).

A list of these 38 cases is given in Table 1. For some purposes we shall extend the list to  $D \leq 5, D^* \leq 5$ . Thus the possibilities for  $\mathbf{k}, \mathbf{k}^*$  are:

- $g = 1$ ; 1, 2, 3; 4; 5.
- $g = 2$ ; (1, 1), (1, 2); (1, 3), (2, 2); (1, 4), (2, 3).
- $g = 3$ ; (1, 1, 1); (1, 1, 2); (1, 1, 3), (1, 2, 2).
- $g = 4$ ; (1, 1, 1, 1); (1, 1, 1, 2).
- $g = 5$ ; (1, 1, 1, 1, 1).

Of these 82 cases, the cyclic order of  $\mathbf{k}, \mathbf{k}^*$  is non-unique in just 10, in each of which there are 2 possibilities. These arise when  $\mathbf{k}, \mathbf{k}^*$  are among

- $g = 3$ ; (1, 1, 2); (1, 1, 3), (1, 2, 2).
- $g = 4$ ; (1, 1, 1, 2).

There are thus 92 cases in all.

We observe finally that the first homology group of the link is the sum of an infinite cyclic group (corresponding to the base circle) and the cokernel of the map represented by  $A - I$ : the order of the latter is

$$|\det(A - I)| = |2 - \text{trace } A|;$$

- $n = 1$ :  $b_1 - 2$ .
- $n = 2$ :  $b_1 b_2 - 4$ .
- $n = 3$ :  $b_1 b_2 b_3 - b_1 - b_2 - b_3 - 2$ , etc.

## 2.2. The parameters $p_i$

We next consider the equations of the hypersurface hyperbolic singularities. According to [28,29] these are of the form

$$x^p + y^q + z^r + xyz = 0,$$

where  $p^{-1} + q^{-1} + r^{-1} < 1$  and  $p, q, r$  are related to  $\mathbf{b}$  as in the following table.

Similarly, in the complete intersection case we can take as equations

$$w^p + y^r + xz = 0, \quad x^q + z^s + wy = 0.$$

$p, q, r (p \leq q \leq r)$	$\mathbf{b}$	$\mathbf{k}$	$\mathbf{k}^*$	$\mathbf{b}^*$
2, 3, $r (r \geq 7)$	$2^{r-7}, 3$	1	$r-6$	$r-4$
2, 4, $r (r \geq 5)$	$2^{r-5}, 4$	2	$r-4$	$2, r-2$
2, $q, r (q \geq 5)$	$2^{q-5}, 3, 2^{r-5}, 3$	1,1	$q-4, r-4$	$q-2, r-2$
3, 3, $r (r \geq 4)$	$2^{r-4}, 5$	3	$r-3$	$2, 2, r-1$
3, $q, r (q \geq 4)$	$2^{q-4}, 3, 2^{r-4}, 4$	2,1	$q-3, r-3$	$2, q-1, r-1$
$p, q, r (p \geq 4)$	$2^{p-4}, 3, 2^{q-4}, 3, 2^{r-4}, 3$	1,1,1	$p-3, q-3, r-3$	$p-1, q-1, r-1$

Depending how many of  $p, q, r$  and  $s$  equal 2 there are five cases for  $\mathbf{b}$  (corresponding to the five possible partitions  $\mathbf{k}$  of 4); in each case,

$$\mathbf{b}^* = (p, q, r, s).$$

We are thus led to introduce a new notation, modifying  $\mathbf{b}^*$  (we regard  $\mathbf{k}, \mathbf{k}^*$  as the basic notation). Recall that  $D$  is the length of the series  $\mathbf{b}^*$ .

If  $D = 1$ , define  $p_1^* = 2, p_2^* = 3, p_3^* = 6 + k_1^* = 4 + b_1^*$ .

If  $D = 2$ , set  $p_1^* = 2, p_2^* = 4 + k_2^* = 2 + b_1^*, p_3^* = 4 + k_1^* = 2 + b_2^*$ .

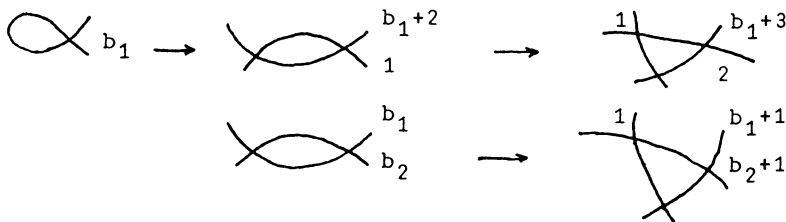
If  $D = 3$ , set  $p_i^* = 1 + b_i^* = 3 + k_{4-i}^*$ , for  $i = 1, 2, 3$ .

If  $D = 4$ , set  $p_i^* = b_i^*$  for  $1 \leq i \leq 4$ .

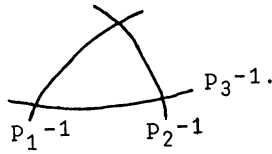
Note here that  $k_i^*$  is set equal to 0 for  $i > g$ . Also note that, in the four cases,

$$\sum p_i^* = \left. \begin{array}{l} 11 + k_1^* \\ 10 + k_1^* + k_2^* \\ 9 + \sum_{i=1}^3 k_i^* \\ \sum_{i=1}^4 b_i^* \end{array} \right\} = 12 + \sum k_i^* - \sum k_i, \left. \begin{array}{l} \\ \\ \\ = 2D + D^* \end{array} \right\} = 12 + D^* - D.$$

Interchanging the  $k_i$  and the  $k_i^*$  we correspondingly define  $p_i$  if  $D^* \leq 4$ , and have  $\sum p_i = 12 + D - D^*$ .



Schema 1.



Schema 2.

We introduced the  $p_i^*$  so that if  $D \leq 3$  the equation took the form

$$x_1^{p_1^*} + x_2^{p_2^*} + x_3^{p_3^*} + x_1 x_2 x_3 = 0.$$

It turns out that the parameters  $p_i$  are related to the minimal resolution which is *good* in the sense that each component is embedded, no two meet twice, and all crossings are normal. For types  $I_1, I_2$  these are described in Schema 1. So for each of types  $I_1, I_2, I_3$  we obtain a triangle as in Schema 2.

### 2.3. Strange duality

We establish bijections between the indicated classes of singularities of Kodaira type by

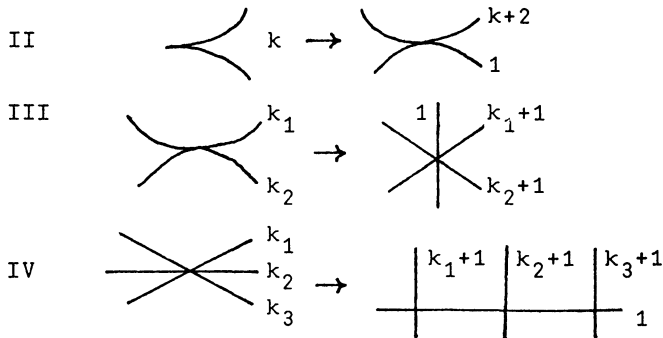
$$II(k_1) \rightarrow^\beta I_1(k_1 + 2),$$

$$III(k_1, k_2) \rightarrow^\beta I_2(k_1 + 2, k_2 + 2),$$

$$IV(k_1, k_2, k_3) \rightarrow^\beta I_3(k_1 + 2, k_2 + 2, k_3 + 2).$$

Since these preserve  $D$ , hypersurfaces correspond to hypersurfaces, complete intersections to the same.

Observe that minimal good resolutions are obtained as in the following figure.



Schema 3.

Thus for each singularity of type II, III or IV, the minimal good resolution has the form

$$- \left| \begin{array}{c|c|c} p_1 & p_2 & p_3 \\ \hline & & 1 \end{array} \right|,$$

where  $\mathbf{p}$  is the sequence defined in section 2.2.

Indeed, in the quasihomogeneous cases, we can now identify the component marked 1 with the central sphere  $S^2$ , and see that there are three exceptional orbits and that the numbers  $p_i$  above coincide with those described in section 1.2 and often known as “Dolgachev numbers” after [10].

If we now denote by  $\delta$  the duality of section 2.1, then  $\beta^{-1}\delta\beta$  defines a duality among those singularities of classes II, III and IV for which  $\beta^{-1}$  is defined. Since  $\mathbf{b}^*$  has length at most 3 if and only if  $D \leq 3$ , i.e., we have a hypersurface singularity, these are precisely the hypersurface singularities of these classes. Now  $\delta_2 = \beta^{-1}\delta\beta$  is the same as the strange duality of Arnol’d [4]. We recall that a satisfying geometrical interpretation using compactification by K3 surfaces was given by Pinkham [49].

We also have correspondences

$$\gamma_n : \mathbf{I}_4(k_1 + 2, k_2 + 2, k_3 + 2, k_4 + 2) \rightarrow \mathbf{I}_n^*(k_1, k_2, k_3, k_4);$$

bijection for  $n > 0$  (for  $n = 0$  we lose the cyclic order). These will be seen below to play a similar role to the above. There is also the trivial bijection

$$\mathbf{II}(k_1) \rightarrow {}^\epsilon \mathbf{II}^*(k_1),$$

$$\mathbf{III}(k_1, k_2) \rightarrow {}^\epsilon \mathbf{III}^*(k_1, k_2),$$

$$\mathbf{IV}(k_1, k_2, k_3) \rightarrow {}^\epsilon \mathbf{IV}^*(k_1, k_2, k_3),$$

which can be composed with  $\beta$  or with  $\delta_2$ . We shall study these further in the section on quadratic forms.

Observe, however, that each of  $\beta$ ,  $\gamma_n$ ,  ${}^\epsilon$  preserves the sequence  $\mathbf{k}$ , and hence  $D$ .

### 3. Embeddings

#### 3.1. General remarks

The Kodaira singularities with  $D \leq 3$  are given by equations in  $\mathbb{C}^3$ . These equations had been obtained by Arnol’d [4] in his classification of singularities. Arnol’d’s 1-modal singularities are precisely those of types

$I_n$ , II, III and IV; his 2-modal singularities are those of types  $I_n^*$ ,  $II^*$ ,  $III^*$  and  $IV^*$ .

The cases  $D = 4$  appear as complete intersections in  $\mathbb{C}^4$ , hence as  $f^{-1}(\mathbf{0})$  for some  $f: (\mathbb{C}^4, \mathbf{0}) \rightarrow (\mathbb{C}^2, \mathbf{0})$ . We contemplate classifying such maps up to  $\mathcal{X}$ -equivalence [40] (or contact equivalence). Say that  $f$  has *modality*  $r$  if for a versal unfolding

$$F: (\mathbb{C}^4 \times \mathbb{C}^a, \mathbf{0}) \rightarrow (\mathbb{C}^2 \times \mathbb{C}^a, \mathbf{0})$$

of  $f$ , all the germs in some neighbourhood of  $\mathbf{0}$  fall into a finite number of families, each depending on at most  $r$  parameters, for  $\mathcal{X}$ -equivalence. Say that  $f$  has *strict modality*  $r$  if the same holds for multigerms (all singular, with the same target). One of the main conclusions of [66] is that a germ has strict modality 1 if and only if the singularity  $f^{-1}(\mathbf{0})$  is of Kodaira type.

These cases also are tabulated below (and this tabulation was one of the origins of the present paper), in the notation of [66], which is a slight modification of Arnold's [4]. All the cases, other than those of type  $I_n$ , belong to well-defined series. We find by inspection that the series is determined by the numbers  $k_i$  defined in our notation. If the  $k_i$  are arranged in decreasing order, and any which equal zero are omitted, we find the following table:

$k$	1	2	1,1	3	2,1	1,1,1	4	3,1	2,2	2,1,1	1,1,1,1
Series	$E$	$Z$	$W$	$Q$	$S$	$U$	$J'$	$L$	$K'$	$M$	$I$

We recall that in these cases  $D = \sum k_i$ .

We also consider below the cases  $D = 5$ . According to [9], each of these cases is defined by 5 equations which are given by the Pfaffians of a skew-symmetric  $5 \times 5$  matrix. We shall obtain such a matrix (it is not, of course, unique) in each case.

### 3.2. The hyperbolic case

We have already given normal forms for equations here, namely

$$x_1^{p_1^\dagger} + x_2^{p_2^\ddagger} + x_3^{p_3^\ddagger} + x_1 x_2 x_3 = 0$$

if  $D \leq 3$ , while if  $D = 4$  we take

$$x_1^{b_1^\dagger} + x_3^{b_3^\ddagger} + x_2 x_4 = x_1 x_3 + x_2^{b_2^\ddagger} + x_4^{b_4^\ddagger} = 0.$$

In the case  $D = 5$  it is easy to verify (this has also been noted by

Nakamura) that a suitable matrix is

$$\begin{pmatrix} 0 & x_4 & -x_2^{b_2^*-1} & x_5^{b_5^*-1} & -x_3 \\ -x_4 & 0 & x_5 & -x_3^{b_3^*-1} & x_1^{b_1^*-1} \\ x_2^{b_2^*-1} & -x_5 & 0 & x_1 & -x_4^{b_4^*-1} \\ -x_5^{b_5^*-1} & x_3^{b_3^*-1} & -x_1 & 0 & x_2 \\ x_3 & -x_1^{b_1^*-1} & x_4^{b_4^*-1} & -x_2 & 0 \end{pmatrix}.$$

We set  $p_i^* = b_i^* - 1$  for  $1 \leq i \leq 5$  and define correspondingly  $p_i$ .

As we shall shortly see in the other cases, the nature of the terms of lowest order here is determined by the sequence  $k$ . As usual, the cases  $D \leq 2$  are somewhat anomalous.

If  $D = 1$ , then  $p_1^* = 2$ ,  $p_2^* = 3$  and we have the alternative normal form

$$x_1^{k_1^*+6} + x_1^2 x_2^2 + x_2^3 + x_3^2.$$

If  $D = 2$ , again  $p_1^* = 2$  and we have the alternative form

$$x_1^{k_1^*+4} + x_1^2 x_2^2 + x_2^{k_2^*+4} + x_3^2.$$

If  $k = (1, 1)$ , the only term of order 4 in  $x_1, x_2$  is  $x_1^2 x_2^2$ ; if  $k = 2$ ,  $k_2^* = 0$ , so we have  $x_1^2 x_2^2 + x_2^4$ .

If  $D = 3$ , all terms have order  $\geq 3$ , and those of degree 3 define a plane cubic curve which is:

- if  $k = (1, 1, 1)$ , a triangle,
- if  $k = (2, 1)$ , a conic and chord,
- if  $k = (3)$ , a nodal cubic.

Similarly for  $D = 4$  or  $5$  we find that the terms of lowest degree (namely 2) in the equations define a set of  $g$  smooth rational curves, of degree  $k_1, \dots, k_g$  respectively, with each meeting the next one simply, all intersection points being distinct.

If we substitute  $b_i^* = 2$  for each  $i$  in the above formulae, we obtain a singularity of type  $I_0(D)$ : in some formal sense, we can consider  $I_0(D)$  as dual to  $I_D(2, 2, \dots, 2)$ : compare the remark in section 2.1 about parabolic Inoue surfaces. However, the general singularity of type  $I_0(n)$  involves a modulus. The equations for  $n \leq 3$  are well known; for  $n = 4$  we can take a generic pencil of quadrics and for  $n = 5$  a generic skew-symmetric matrix which is linear in  $x_1, x_2, x_3, x_4$  and  $x_5$ .

It is interesting to observe the relation between the varieties associated in this way to hyperbolic or other Kodaira singularities with the same  $k$ . For  $D = 3$ , we have:



$k$	3	2,1	1,1,1
Hyperbolic	nodal cubic	conic + chord	triangle
Other	cuspidal cubic	conic + tangent	concurrent lines
Series	$Q$	$S$	$U$

For  $D = 4$ , the pencils of quadrics can be described by Segre symbols, and we find:

$k$	4	3,1	2,2	2,1,1	1,1,1,1
Hyperbolic	2,1,1	2,2	(1,1),1,1	2,(1,1)	(1,1),(1,1)
Other	3,1	4	(2,1),1	(3,1)	1,1,1;1
Series	$J'$	$L$	$K'$	$M$	$I$

### 3.3. Equations for Kodaira singularities: reduction methods

There are (at least) three ways of obtaining explicit equations for embedded singularities of the various types. First, for the quasihomogeneous singularities of types II, III, IV, IV\*, III\* and II\* we can write down generators and relations for the ring of automorphic forms of the given hyperbolic triangle, using a modification of the method of Milnor [42] (which will be described elsewhere). Second, one can resolve a surface given by equations explicitly (and hence infer step by step the terms needed in the equations to produce the desired result). Third, we have reduction procedures. We recall from [67,7.9] that if

$$f(x, y, z) = yx^2 + 2xb(y, z) + c(y, z),$$

set

$$\Delta_x f(w, y, z) = w^2 - b(y, z)^2 + yc(y, z);$$

if

$$f(x, y, z, w) = (xy + a(y, z, w), xz + b(y, z, w)),$$

set

$$L_x f(y, z, w) = yb(y, z, w) - za(y, z, w).$$

Then the exceptional sets  $S, T$  in the minimal resolutions of  $f$  and  $\Delta_x f$  (resp.  $L_x f$ ) are isomorphic, but one of the normal degrees is changed by 1. The same argument gives a reduction procedure for Pfaffian singular-

ities: if  $f$  is defined by the Pfaffians of

$$\begin{pmatrix} 0 & x_5 & p & q & r \\ -x_5 & 0 & u & v & w \\ -p & -u & 0 & x_3 & x_2 \\ -q & -v & -x_3 & 0 & x_1 \\ -r & -w & -x_2 & -x_1 & 0 \end{pmatrix}$$

(where  $p, q, r, u, v, w$  are functions of  $x_1, x_2, x_3$  and  $x_4$ ), we define  $P_{x_5}f = (ux_1 - vx_2 + wx_3, px_1 - qx_2 + rx_3)$  and the same conclusion obtains.

These reduction procedures allow us to deal with whole classes of singularities simultaneously. We shall suppose that they give bijections between isomorphism classes of germs, though the arguments above only yield bijections of topological types.

Case	II	III	IV	I <sub>0</sub> *
A	$x_1^2 + ax_2x_1$	$x_2x_1 + ax_1^3$	$x_1^3 + ax_2x_1^2$	$\lambda x_2^2 + \mu x_2x_1^2 + \nu x_1^4 + ax_2x_1^3$
Case	IV*	III*	II*	I <sub>n</sub> * ( $n \geq 1$ )
A	$x_1^5 + ax_2x_1^3 + a'x_2x_1^4$	$x_2x_1^3 + ax_1^6 + a'x_1^7$	$x_1^6 + ax_2x_1^4 + a'x_2x_1^5$	$x_2^2 + x_1^{4+n}$

Case  $g = 1$ . Define a function  $A$  depending on cases as in the table. Then the equations for the various series are:

$$k = (1), \quad x_3^2 + x_2^3 + x_1^5 A\left(\frac{x_2}{x_1}, x_1\right) = 0,$$

$$k = (2), \quad x_3^2 + x_1x_2^3 + x_1^3 A(x_2, x_1).$$

$$k = (3), \quad x_1x_3^2 + x_2^3 + x_1^2 A(x_2, x_1),$$

$$k = (4), \quad (x_1x_4 - x_2^2, x_2x_4 + x_3^2 + x_1A(x_2, x_1)),$$

$$k = (5), \quad \begin{pmatrix} 0 & x_5 & A(x_2, x_1) & -x_4 & x_3 \\ -x_5 & 0 & x_4 & x_2 & 0 \\ -A(x_2, x_1) & -x_4 & 0 & x_3 & x_2 \\ x_4 & -x_2 & -x_3 & 0 & x_1 \\ -x_3 & 0 & -x_2 & -x_1 & 0 \end{pmatrix}.$$

The weighted homogeneous cases are those with  $a = a' = 0$ ; the other cases are those with  $a \neq 0$  or (II\*, III\*, IV\*)  $a = 0$  and  $a' \neq 0$ . For type I<sub>0</sub><sup>\*</sup>, the cubic  $t^3 + \lambda t^2 + \mu t + \nu = 0$  must have distinct roots. We can even include the hyperbolic cases here by setting  $A = -3x_2 + 2x_1 + x_1^{\lambda \dagger + 1}$ .

Case  $g = 2$ . Here we define the function  $B$  as in the table.

The equations are given by:

$$\mathbf{k} = (1, 1), \quad x_3^2 + x_2^4 - x_1^3 B(x_2, x_1),$$

Case	III	IV	I <sub>0</sub> <sup>*</sup>	IV*	III*
$B$	$x_1^2 + ax_2^2$	$x_2x_1 + ax_1^3$	$\lambda x_2^2 + \mu x_1^3 + ax_2^2x_1$	$x_2x_1^2 + ax_1^4 + a'x_1^5$	$x_1^4 + ax_2^2x_1 + a'x_2^2x_1^2$

Case	I <sub>n</sub> <sup>*</sup> ( $k_1, 2, k_2, 2$ )	I <sub>n</sub> <sup>*</sup> ( $k_1, k_2, 2, 2$ ) ( $n$ odd)	I <sub>n</sub> <sup>*</sup> ( $k_1, k_2, 2, 2$ ) ( $n$ even)
$B$	$x_2^2 + x_1^{n+3}$	$2x_2^2 - x_1^3 + x_2x_1^{(n+3)/2}$	$2x_2^2 - x_1^3 + x_2^2x_1^{n/2}$

$$\mathbf{k} = (2, 1), \quad x_1x_3^2 + 2x_3x_2^2 + x_1^2B(x_2, x_1),$$

$$\mathbf{k} = (3, 1), \quad (x_1x_4 - 2x_2x_3, x_3^2 + x_2x_4 + x_1B(x_2, x_1)),$$

$$\mathbf{k} = (2, 2), \quad (x_1(x_3 + x_4) + 2x_2^2, x_3x_4 - x_1B(x_2, x_1)),$$

$$\mathbf{k} = (4, 1), \quad \begin{pmatrix} 0 & x_5 & B(x_2, x_1) & -x_4 & x_3 \\ -x_5 & 0 & x_4 & 2x_3 & 0 \\ -B(x_2, x_1) & -x_4 & 0 & x_3 & x_2 \\ x_4 & -2x_3 & -x_3 & 0 & x_1 \\ -x_3 & 0 & -x_2 & -x_1 & 0 \end{pmatrix},$$

$$\mathbf{k} = (3, 2), \quad \begin{pmatrix} 0 & -B(x_2, x_1) & x_5 & 0 & x_4 \\ B(x_2, x_1) & 0 & -(x_3 + x_4) & x_3 & x_2 \\ -x_5 & x_3 + x_4 & 0 & 2x_2 & 0 \\ 0 & -x_3 & -2x_2 & 0 & x_1 \\ -x_4 & -x_2 & 0 & -x_1 & 0 \end{pmatrix}.$$

The rôles of  $a, a'$  are as before. For type I<sub>0</sub><sup>\*</sup>,  $\mu(\lambda^2 + 4\mu) \neq 0$ .

Case  $g = 3$ . Define  $C(x_1, x_2, x_3)$  as in the table.

Here the term  $Q$  – as indeed the  $x_2^2$  appearing in cases IV and IV\* – can be any quadratic in  $x_2$  and  $x_3$  which is *not* a linear combination of  $3x_2^2 - x_3^2$  and  $x_2x_3$ . The term  $L$  is a linear expression in  $x_2$  and  $x_3$  which is *not* a multiple of the chosen factor  $A$  ( $= x_2, x_2 \pm x_3$ ) of  $x_2^3 - x_2x_3^2$  (a corresponding remark applies for type I<sub>0</sub><sup>\*</sup>).

The normal form for  $k = (1, 1, 1)$  is

$$x_2^3 - x_2x_3^2 + x_1^2C;$$

for  $k = (2, 1, 1)$  we have

$$(2x_1x_4 + x_2^2 - x_3^2, 2x_2x_4 - x_1C).$$

Case	IV	$I_0^*$	IV*	$I_n^*$
C	$x_1^2 + ax_2^2$	$(\lambda x_2 + \mu x_3)x_1$ $+ ax_3x_1^2$	$x_1^3 + ax_2^2$ $+ a'x_1x_2^2$	$x_1A + \begin{cases} x_1^{(n-1)/2}Q, & n \text{ odd} \\ x_2^{(n+2)/2}L, & n \text{ even} \end{cases}$

Here there is less symmetry.

$$k = (3, 1, 1), \quad \begin{pmatrix} 0 & x_5 & -C & -2x_4 & 0 \\ -x_5 & 0 & 2x_4 & -x_2 & -x_3 \\ C & -2x_4 & 0 & x_3 & x_2 \\ 2x_4 & x_2 & -x_3 & 0 & x_1 \\ 0 & x_3 & -x_2 & -x_1 & 0 \end{pmatrix};$$

$$k = (2, 2, 1), \quad \begin{pmatrix} 0 & -C & x_5 & 0 & x_2 + x_3 \\ C & 0 & x_4 & -x_4 & x_2 - x_3 \\ -x_5 & -x_4 & 0 & -x_2 & 0 \\ 0 & x_4 & x_2 & 0 & x_1 \\ -(x_2 + x_3) & -(x_2 - x_3) & 0 & -x_1 & 0 \end{pmatrix}.$$

Note here that  $P$ -reduction gives an equation in the  $(2, 1, 1)$  series but with  $x_2 + x_3$  (rather than  $x_2$ ) as the preferred factor of  $x_2^3 - x_2x_3^2$ .

Case  $g = 4$ . We define  $E(x_2, x_1)$  to be, resp:

$(I_0^*) ax_1^3 + x_2^2$ ;  $(I_n^*, n \geq 2 \text{ even}) x_1^{(n+2)/2}$ ;  $(I_n^*, n \text{ odd}) x_2x_1^{(n+1)/2}$ .

We have equations:  $k = (1, 1, 1, 1)$ ,

$$(x_2(x_3 - x_4) + x_1^3, x_3(x_2 - x_4) + x_1E(x_2, x_1));$$

$k = (2, 1, 1, 1)$ ,

$$\begin{pmatrix} 0 & x_5 & E(x_2, x_1) & 0 & x_2 - x_4 \\ -x_5 & 0 & x_1^2 & x_4 - x_3 & 0 \\ -E(x_2, x_1) & -x_1^2 & 0 & x_3 & x_2 \\ 0 & -(x_4 - x_3) & -x_3 & 0 & x_1 \\ -(x_2 - x_4) & 0 & -x_2 & -x_1 & 0 \end{pmatrix}.$$

### 3.4. Weights

The weights in the weighted homogeneous cases above display similar regularities. We first define parameters  $c$  and  $R$  as in the table.

Here  $c$  is the order of the matrix  $A$  of section 1.1; it is also (except for cases II, III, IV) the highest multiplicity of a component in the funda-

Case	II	III	IV	$I_0^*$	IV*	III*	II*
$c$	6	4	3	2	3	4	6
$R$	1	1	1	1	2	3	5

mental cycle;  $R$  equals either 1 or  $c - 1$  (or both) and has the same meaning as in section 1.2.

Next suppose that  $\mathbf{k} = (k_1, k_2, \dots, k_g)$  defines a partition of  $D$ . We define a sequence of  $D$  natural numbers by

$$1, 2, \dots, k_1, 1, 2, \dots, k_2, \dots, 1, 2, \dots, k_g.$$

Rearrange this sequence in increasing order, and change its first term from 1 to 0; we then denote it by

$$\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_D.$$

Then the weights of the coordinates are given by

$$\text{weight}(x_i) = c + \tilde{w}_i R \quad \text{for } 1 \leq i \leq D.$$

In the cases when  $D < 3$  further formulae are needed, and are as follows:

If  $\mathbf{k} = (1)$ , weight  $x_2 = 2(c + R)$ , weight  $x_3 = 3(c + R)$ .

If  $\mathbf{k} = (2)$ , weight  $x_3 = 2c + 3R$ .

If  $\mathbf{k} = (1, 1)$ , weight  $x_3 = 2(c + R)$ .

We also observe that:

if  $g = 1$ , weight  $A = c + 6R$  (all cases),

if  $g = 2$ , weight  $B = c + 4R$  (all cases),

if  $g = 3$ , weight  $C = c + 3R$  (all cases),

if  $g = 4$ , weight  $E = c + 2R$  (only one case).

One can infer the weights of the equations. If  $D \leq 3$ , then of course

$$\text{weight } f = \text{wt } x_1 + \text{wt } x_2 + \text{wt } x_3 + R.$$

If  $D = 4$ ,

$$\text{wt } f_1 + \text{wt } f_2 = \text{wt } x_1 + \text{wt } x_2 + \text{wt } x_3 + \text{wt } x_4 + R$$

whence, by closer examination:

<b>k</b>	1	2	(1,1)	3	(2,1)	(1,1,1)
wt $f$	$6(c + R)$	$4c + 6R$	$4(c + R)$	$3(c + 2R)$	$3c + 4R$	$3(c + R)$
<b>k</b>	4	(3,1)	(2,2)	(2,1,1)	(1,1,1,1)	
wt $f_1$	$2c + 4R$	$2c + 3R$	$2c + 2R$	$2c + 2R$	$2c + 2R$	
wt $f_2$	$2c + 6R$	$2c + 4R$	$2c + 4R$	$2c + 3R$	$2c + 2R$	

Conversely, the only monomials having the desired weights for *all*  $c, R$  are ( $D \leq 3$ ):

<b>k</b>	1	2	(1,1)	3	(2,1)	(1,1,1)
Monomials	$x_1^3, x_3^2$	$x_1x_2^3, x_3^2$	$x_3^2, x_3x_2^2, x_2^4$	$x_2^3, x_1x_3^2$	$x_1x_3^2, x_2^2x_3$	$x_2^3, x_2^2x_3, x_2x_3^2, x_3^3$

defining the initial forms used above. For  $D = 4$ :

<b>k</b>	4	(3,1)	(2,2)	(2,1,1)	(1,1,1,1)
$f_1$	$x_1x_4, x_2^2$	$x_1x_4, x_2x_3$	$x_1x_4, x_1x_3, x_2^2$	$x_1x_4, x_2^2, x_2x_3, x_3^2$	$x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2$
$f_2$	$x_2x_4, x_3^2$	$x_2x_4, x_3^2$	$x_3^2, x_3x_4, x_4^2$	$x_2x_4, x_3x_4$	$x_2^2, x_2x_3, x_2x_4, x_3^2, x_3x_4, x_4^2$

and generic linear combinations of these monomials define a pencil of quadrics with respective Segre symbol

	3,1	4	(2,1),1	(3,1)	1,1,1;1
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In each case, the intersection of the pencil of conics has components with degrees  $k_1, k_2, \dots, k_g$  ( $\sum k_i = 4$ ): all have a common point, and the base locus has a unique singular point.

Analogous considerations apply for  $n = 5$ . In this case, the Pfaffian formulation yields the surface as the intersection of a smooth 5-dimensional hypersurface with the Grassmannian cone in  $\Lambda^2(V^5)$ . The auxiliary space  $V^5$  also admits a  $\mathbb{C}^*$ -action, though its weights may be half-integral. These weights (semiweights) are of the form  $\frac{1}{2}(c + l, R)$ , where the numbers  $l_i$  are given by the table.

<b>k</b>	$l_1$	$l_2$	$l_3$	$l_4$	$l_5$
(5)	-1	1	3	5	7
(4,1)	-1	1	3	3	5
(3,2)	-1	1	1	3	5
(2,2,1)	-1	1	1	3	3
(3,1,1)	0	0	2	2	4
(2,1,1,1)	0	0	2	2	2

These determine the linear part of the skew-symmetric matrix (take it generic subject to these weights) and hence the quadratic part of the defining equations. In turn, these determine a curve of degree 5 in  $\mathbb{P}^4$ : it is again a union of irreducible components with degrees  $k_i$ .

It must be observed that the arbitrariness in the choice of the matrix is appreciably greater than that in the defining equations: this does not however affect the essential uniqueness of the equations given.

## 4. Quadratic forms

### 4.1. Milnor fibrations and smoothings

If  $X$  is a hypersurface singularity, i.e. defined as  $f^{-1}(0)$  for  $f: (\mathbb{C}^{n+1}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ , then Milnor [41] showed that the restriction of  $f$  to  $B_\epsilon \cap f^{-1}(D_\eta - \{0\})$  is a fibration for  $\eta \ll \epsilon \ll 1$ . This yields a deformation of  $X$  to  $X_t = B_\epsilon \cap f^{-1}(t)$  for  $0 < |t| < \eta$ , which is smooth, and is known as the *Milnor fibre*. Moreover, Milnor showed that  $X_t$  has the homotopy type of a bouquet of  $n$ -spheres. Thus the only nontrivial reduced homology group is  $H = H_n(X_t)$  which is free abelian; its rank is known as the Milnor number and denoted by  $\mu$ . Intersection numbers define a bilinear pairing, symmetric if  $n$  is even,  $(,): H \times H \rightarrow \mathbb{Z}$ . The group  $H$  with this pairing is known as the *Milnor lattice*. Since  $X_t$  has trivial normal bundle in  $\mathbb{C}^{n+1}$  it is parallelisable, so the quadratic form on  $H$  is even.

We next seek to extend these results to other classes of singularities. In the case of complete intersections, corresponding results were obtained by Hamm [22]: there is a fibration (not quite as above) whose fibre has all the properties listed above.

Now suppose  $X$  a normal surface singularity. A deformation of  $X$  is a flat mapping  $\pi: Y \rightarrow \mathbb{C}$  with  $\pi^{-1}(0) = X$ ; as above, we make this more precise by embedding  $Y$  in  $\mathbb{C}^{n+1}$  with the singular point of  $X$  at the origin, and then restricting to  $B_\epsilon \cap \pi^{-1} D_\eta$  for sufficiently small  $\epsilon, \eta$ . We say  $X$  deforms to  $X_t = \pi^{-1}t$ ; if  $X_t$  is smooth, it is a smoothing of  $X$ . In particular [30; Cor. 4.6] a singularity  $X$  of Kodaira type deforms to a space (say  $V_t$ ) which has a single singularity of type  $I_0$  with the same degree  $D$ .

According to [48] if  $D \leq 9$ ,  $V_t$  is smoothable; thus  $X$  is also. We can thus define Milnor fibres for  $X$ . In general they are not unique. However, if  $D = 5$  (so that  $X$  has codimension 3 in  $\mathbb{C}^5$ ) the base space of the semiuniversal deformation of  $X$  is smooth [64], so the  $X_t$  form a connected family and the Milnor lattice is unique. See also [63] for a general reference on smoothings.

**THEOREM 4.1.1:** *Let  $X$  be a normal surface singularity,  $X_t$  a smoothing obtained as above. Then:*

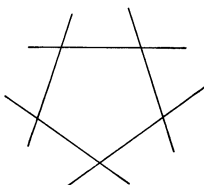
- (i)  $X_t$  has the homotopy type of a finite CW-complex of dimension  $\leq 2$ .
- (ii)  $\text{rk } H_1(X_t, \mathbb{Z}) = 0$ .
- (iii) If  $X$  is Gorenstein, then  $X_t$  is parallelisable.
- (iv) If  $X$  is Kodaira singularity with  $D \leq 5$ , then  $X_t$  is 1-connected and has the homotopy type of a bouquet of 2-spheres.

PROOF: (i) follows from [1].

(ii) is a result due to Greuel and Steenbrink [21].

For a proof of (iii) see e.g. [56] combined with [13].

We are indebted to E. Looijenga for the following proof of (iv). We shall first show that  $X_t$  is 1-connected for a simple elliptic or a hyperbolic singularity  $X$  of grade 5. According to [37; III, (1.7) plus (2.8)] the Milnor fibre of such a singularity is homeomorphic to a rational surface, where an anti-canonical cycle of length 5 is removed:



Let us consider the simple elliptic case first. By the partial classification of [37] a rational surface with an anti canonical cycle of rational curves, each with self-intersection number  $-2$ , of length 5 can be obtained as follows: Consider  $\mathbb{P}^2$  and in  $\mathbb{P}^2$  a cubic curve  $C$  with one node. Blow up four points on the regular part of  $C$ , no three of which are collinear, to get a Del Pezzo surface of degree five with a curve  $\bar{C}$ . Then blow up successively the singular point and four suitable infinitely near points, such that the strict transform of  $C$  becomes the desired anti-canonical cycle  $\bar{C}$ . Call the resulting surface  $Y$ .

Now  $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/3\mathbb{Z}$  and the preimage of a generator is homotopic in  $Y - \bar{C}$  to a loop lying on an exceptional curve of the first kind, which meets exactly one component of  $\bar{C}$  transversally in a single point, and going around that point. Since this loop is contractible in  $Y - \bar{C}$ ,  $Y - \bar{C}$  is simply connected. For the hyperbolic singularities one has to blow up further points on  $\bar{C}$ . But the fundamental group of the complement of the strict transform can have at most more relations.

Now we consider the remaining Kodaira singularities. It was shown in [30] that there is a deformation of  $X$  into a simple-elliptic singularity of grade  $D$  via simultaneous resolution. This means that there is a 3-dimensional manifold  $\mathcal{M}$  and a flat mapping  $\tilde{\omega}: \mathcal{M} \rightarrow \Delta$  to a complex disk, such that the Stein factorisation

$$\mathcal{M} \rightarrow \tau \mathcal{V} \rightarrow \omega \Delta$$

gives a deformation of  $X$ , where the general fibre  $V_t$  has a simple elliptic



singularity  $x_i$  of grade  $D$  and  $Z_i = \tau^{-1}(x_i)$  is an elliptic curve with selfintersection number  $-D$  in  $M_i = \tilde{\omega}^{-1}(t)$ . Now this simple elliptic singularity can be smoothed. Since  $V_i$  is Stein, also  $V_i$  deforms into a smooth space  $X_i$ . Choose a small ball around  $x_i$  which intersects  $X_i$  in a Milnor fibre  $Y_i$  of  $x_i$ . Then we have a homeomorphism  $X_i/Y_i \rightarrow M_i/Z_i$  ( $Y_i$  and  $Z_i$  collapsed to a point). Consider the following diagram:

$$\begin{array}{ccccccc}
 \rightarrow & \pi_1(Y_i) & \rightarrow & \pi_1(X_i) & \rightarrow & \pi_1(X_i, Y_i) & \rightarrow 0 \\
 & & & & & \downarrow & \\
 & & & & & \pi_1(X_i/Y_i) & \\
 & & & & & \uparrow & \\
 \rightarrow & \pi_1(M_i) & \rightarrow & \pi_1(M_i, Z_i) & \rightarrow & 0.
 \end{array}$$

As we saw above,  $\pi_1(Y_i) = 0$ . Therefore also  $\pi_1(X_i, Y_i) \cong \pi_1(X_i/Y_i)$ . If  $X$  is not a hyperbolic singularity, then  $\pi_1(M_i) \cong \pi_1(M_0) \cong \pi_1(E) = 0$ , since  $E$  is then homotopy equivalent with a bouquet of 2-spheres. This implies  $\pi_1(X_i) = 0$ . That  $X_i$  has the homotopy type of a bouquet of  $\mu$  spheres, follows as in [41, Theorem 6.5].  $\square$

#### 4.2. The Milnor number

The Milnor number  $\mu$  was defined in section 4.1 as the second Betti number of any smoothing. A formula was given by Laufer [34] for hypersurface singularities and extended in [59] to all smoothable Gorenstein singularities,

$$\mu = Eu + Z^2 + (12h - 1).$$

Here,  $h$  is the genus. Singularities of Kodaira type are elliptic, so  $h = 1$  and the last term equals 11. We have already introduced  $D$  to denote  $-Z^2$ . Finally,  $Eu$  is the topological Euler characteristic of the minimal resolution. We see by inspection that its values are as follows:

Type	$I_n$	II	III	IV	$I_n^*$	IV*	III*	II*
Eu	$n(=D^*)$	2	3	4	$n+6$	8	9	10

Using these formulae we can thus attach a well-defined integer  $\mu$  to any singularity of Kodaira type,  $\mu = 11 + Eu - D$ . In the hyperbolic case, we have

$$\mu = 11 + D^* - D.$$

Thus for dual singularities  $X$  and  $\delta X$ , we have

$$\mu(X) + \mu(\delta X) = 22.$$

For the parameters  $p_i$  of section 2.2 we had

$$\sum p_i^* = 12 + D^* - D = \mu + 1.$$

We now consider the other transforms of section 2.3. We find that  $\beta$ ,  $\gamma_n$  and  $\epsilon$  all preserve  $D$  while

$$\text{Eu}(\beta X) = \text{Eu}(X) - 1,$$

$$\text{Eu}(\gamma_n X) = \text{Eu}(X) + n + 2,$$

$$\text{Eu}(\epsilon X) = 12 - \text{Eu}(X).$$

Now the strange duality was defined on singularities of type II, III and IV with  $D \leq 3$  by  $\delta_2 = \beta^{-1}\delta\beta$ ; thus (as expected)

$$\mu(X) + \mu(\delta_2 X) = 24.$$

We can extend this to include singularities of type  $I_n$  and  $D \leq 4$  by augmenting  $\beta$  by the bijection  $\gamma_n^{-1}$ . But for this to give the same formula for  $\mu$  we need to substitute  $n = -1$ . Thus our extension of the strange duality involves a “virtual” singularity  $n = -1$  associated to each sequence  $I_n^*(k_1, k_2, k_3, k_4)$  with  $\sum k_i \leq 4$ .

For singularities of types  $II^*$ ,  $III^*$  and  $IV^*$  with  $D \leq 3$  we can define  $\delta_3(X) = \epsilon\beta^{-1}\delta\beta\epsilon^{-1}(X)$ . Then  $\text{Eu}(\delta_3 X) = 11 - D(X)$ , so  $\mu(\delta_3 X) = \mu X$ . If we extend this by replacing  $\beta\epsilon^{-1}$  by  $\gamma_n^{-1}$  for type  $I_n^*$  we find the formula for  $\mu$  is correct if we take  $n = 1$ .

### 4.3. Signatures and mixed Hodge structures

We recall that  $H = H_2(X_i)$ : intersection numbers on  $X_i$  induce a symmetric bilinear form on  $H$ . We can take a basis of  $H \otimes \mathbb{R}$  for which the matrix of this form is diagonal; then the numbers  $\mu_0$ ,  $\mu_-$  and  $\mu_+$  of zero, negative and positive eigenvalues give further invariants whose sum is  $\mu$ . By a formula of Durfee [13] for complete intersections, extended to all smoothable normal surface singularities by Steenbrink [59],

$$2h = \mu_0 + \mu_+.$$

In our cases  $h = 1$ . Extending slightly the results of Arnol'd [2], it is easy to see that

$$\begin{aligned} \mu_0 = 2, \quad \mu_+ = 0 & \text{ for singularities of type } I_0, \\ \mu_0 = \mu_+ = 1 & \text{ for singularities of type } I_n (n \geq 1), \\ \mu_0 = 0, \quad \mu_+ = 2 & \text{ for all other smoothable elliptic normal surface} \\ & \text{singularities.} \end{aligned}$$

Following Steenbrink [58], we can be yet more precise in the hyper-

surface case. These limiting homology groups have a canonical mixed Hodge structure. To perceive the duality, one separates the unipotent part of the monodromy ( $\lambda = 1$ ) and its orthogonal complement ( $\lambda \neq 1$ ). If we label the Hodge numbers  $h_{\lambda}^{p,q}$  as follows:

$(\lambda = 1)$			$(\lambda \neq 1)$				
$q$	0	1	2	$q$	0	1	2
$p = 0$	0	0	0	$p = 0$	$c$	$d$	$e$
$p = 1$	0	$a$	$b$	$p = 1$	$d$	$f$	$d$
$p = 2$	0	$b$	$a$	$p = 2$	$e$	$d$	$c$

(using the known identities) then, according to Steenbrink [58],

$$\mu_0 = a + 2b, \quad \mu_- = 2d + f,$$

$$\mu_+ = a + 2(c + d + e);$$

thus by Durfee's formula,

$$h = a + b + c + d + e.$$

For the case of genus 1, just one of the numbers  $a, b, c, d$  and  $e$  is 1; the others vanish. We find (rather disappointingly):

$$\begin{aligned} \text{for type } I_0, & \quad b = 1, \\ \text{for type } I_n (n > 0), & \quad a = 1, \\ \text{in all other cases,} & \quad e = 1. \end{aligned}$$

Indeed, whenever there is a good  $\mathbf{C}^*$  action (whether the genus is 1 or higher)  $a = c = d = 0$ .

For an isolated hypersurface singularity, whose equation  $f(\mathbf{x}) = 0$  is homogeneous of degree  $d$  (when  $x_i$  has weight  $w_i$ ,  $w_1 \leq w_2 \leq w_3$ ), the Jacobian algebra  $\mathcal{O}_3/\mathcal{J}_f$  satisfies Poincaré duality, with highest term the Hessian of  $f$ , of weight  $3d - 2(w_1 + w_2 + w_3)$ . The eigenvalues of the residue matrix of  $f$  are obtained [6] from the degrees of basis monomials of  $\mathcal{O}_3/\mathcal{J}_f$  by adding  $w_1 + w_2 + w_3$  and dividing by  $d$ . This yields  $\mu$  rational numbers  $r_1 \leq r_2 \leq \dots \leq r_\mu$ , with  $r_{i+1} + r_{\mu-i} = 3$ , and  $r_1 = d^{-1}(w_1 + w_2 + w_3)$ ,  $r_2 = d^{-1}(2w_1 + w_2 + w_3)$ . Since [57] eigenvalues  $r$  contribute to  $\mu_+$  or  $\mu_-$  according as the integer part  $[r]$  is odd or even (if  $r \in \mathbf{Z}$  there is a contribution to  $\mu_0$ ), we deduce (since  $h = 1$ ) that either  $r_1 = 1$  (which occurs only for type  $I_0$ ) or  $r_1 < 1 < r_2$ . Conversely,  $w_1 + w_2 + w_3 \leq d < 2w_1 + w_2 + w_3$  characterises the weights of weighted homogeneous elliptic hypersurface singularities (which must of course also satisfy Arnold's [3] necessary conditions for the existence of an isolated hyper-

surface singularity). In addition to the cases of Kodaira type there are 15 more, which can be found in [53] or [35].

An analogous discussion applies for complete intersections, with the governing inequality

$$0 \leq (d_1 + d_2) - (w_1 + w_2 + w_3 + w_4) < \min w_i.$$

The other examples will be listed and further discussed in a subsequent paper.

#### 4.4. Discriminant quadratic forms

If  $X$  is a normal surface singularity, we can consider  $X$  as a compact (contractible) neighbourhood of the singular point, whose interior is Stein. Write  $L = \partial X$  for the boundary of  $X$ : the *link* of the singularity. This is a closed 3-manifold, thus there is a well-defined linking pairing

$$b: TH_1(L) \times TH_1(L) \rightarrow \mathbb{Q}/\mathbb{Z},$$

where  $TH_1(L)$  denotes the torsion subgroup of  $H_1(L; \mathbb{Z})$ . This pairing is symmetric, and (if a framing is chosen on  $L$ ) arises from a quadratic map [65,43]

$$q: TH_1(L) \rightarrow \mathbb{Q}/2\mathbb{Z}.$$

More precisely,  $q$  satisfies the following two conditions:

- (i)  $q(rx) = r^2q(x)$ .
- (ii)  $q(x+y) - q(x) - q(y) \equiv 2b(x, y) \pmod{2\mathbb{Z}}$  for  $r \in \mathbb{Z}$  and  $x, y \in TH_1(L)$ .

We shall refer to  $q$  as the *discriminant quadratic form*.

To calculate  $b$  and  $q$  we need an expression for  $L$  as the boundary of a compact 4-manifold. We have two alternative procedures here: we can use a smoothing of  $X$ , or a resolution.

Suppose in either case  $L = \partial M$  and (for simplicity) that  $H_1(M) = H_3(M) = 0$ , so that  $H = H_2(M)$  is free abelian. Suppose also that  $H_1(L)$  is finite (the argument can easily be generalised to allow  $H_1(L)$  infinite. It is also generalised in [39] to the case where  $H_1(M)$  has torsion). Then the exact sequence

$$0 \rightarrow H_2(M) \rightarrow \phi H_2(M, L) \rightarrow \psi H_1(L) \rightarrow 0$$

determines  $H_1(L)$  as the cokernel of  $\phi$ , which is the map from  $H_2(M)$  to its dual induced by the intersection pairing on  $H_2(M)$ . Now for  $x, y \in H_1(L)$  choose  $u, v$  with  $\psi(u) = x$ ,  $\psi(v) = y$ , and  $r \in \mathbb{N}$ ,  $w \in H_2(M)$  with  $\phi(w) = ru$ . Then  $b(x, y) = r^{-1}\langle w, v \rangle \pmod{\mathbb{Z}}$ . However, the natural conventions of [39] give the opposite sign here.

If  $M$  is parallelisable, we also have

$$q(x) = r^{-1}\langle w, u \rangle \pmod{2\mathbb{Z}}.$$

In general this is not well defined and we need a cycle  $z \in H_2(M)$  representing a (Poincaré) dual of the Stiefel class  $w_2(M)$ . If  $M$  has a complex structure (as in the cases above), take  $z$  congruent mod 2 to a representative of a dual of  $c_1(M)$ . The choice of  $z$  determines a framing on  $M - |z|$  and hence on  $L$ . Then

$$q(x) = \langle z, u \rangle + r^{-1}\langle w, u \rangle \pmod{2\mathbb{Z}}.$$

We find that if  $M$  is a smoothing of  $X$ , and  $X$  is Gorenstein,  $M$  is parallelisable (by Theorem 4.1.1), so  $q$  is the quadratic form defined by intersection numbers on  $H_2(M)$ . If  $L$  comes from a resolution this is not the case: we take  $z$  as the fundamental cycle (see section 1.3) or, more conveniently, as the sum of components of odd multiplicity in  $Z$ .

If  $H_1(L)$  has order  $N$ , the Gauss sum

$$\sum \{ e^{\pi i q(x)} : x \in H_1(L) \}$$

is equal to  $\sqrt{N} e^{i\pi s/4}$  for some integer  $s = s(q)$  defined mod 8. When  $M$  is parallelisable,  $s$  is the signature  $\sigma(M)$  of the quadratic form on  $M$ . In general,

$$s(q) = \sigma(M) - (z \cap z) \pmod{8}.$$

This result is essentially due to Van der Blij [62]. Two abstract quadratic maps  $q_1, q_2: G \rightarrow \mathbb{Q}/2\mathbb{Z}$ , where  $G$  is a finite abelian group, satisfying the conditions (i), (ii) at the beginning of this section with isomorphic bilinear forms  $b_1, b_2$  are themselves isomorphic if and only if  $s(q_1) \equiv s(q_2) \pmod{8}$  [47; 1.11.3].

We use this to determine  $s(q)$  for Kodaira singularities. Take  $L$  to be given by a minimal good resolution. Then the intersection form is negative definite, so  $\sigma$  equals minus the rank. For types  $I_n^*$ ,  $II^*$ ,  $III^*$  and  $IV^*$  this rank is  $Eu - 1$ . Thus

$$s(q) \equiv 1 - Eu + D \pmod{8} \equiv 12 - \mu.$$

For types II, III, and IV this rank is 4, and

$$-Z^2 = k_1 + k_2 + k_3,$$

so

$$s(q) \equiv k_1 + k_2 + k_3 - 4 \equiv 20 - \mu.$$

For type  $I_n$  ( $n > 0$ ), the rank of  $H_2$  for a resolution is the length  $n = D^*$  of the sequence. Thus

$$s(q) = D - D^* = 11 - \mu.$$

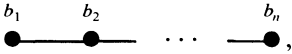
Results including the above have also been obtained by Looijenga and Wahl [39].

We shall also need the following:

LEMMA 4.4.1: *Let  $X$  be a Kodaira singularity. Then  $TH_1(L)$  can be generated by 2 elements, except for type  $I_n^*$  where 3 elements will suffice.*

PROOF: We calculate this group using a resolution  $M$  of  $X$ . If  $C_i$  are the components of the exceptional set, our group is presented by a square  $N \times N$  matrix whose rows and columns correspond to the  $C_i$ . It is enough to find an  $(N - 2) \times (N - 2)$  minor with determinant  $\pm 1$ , as we can then reduce the matrix by elementary operations to the direct sum of the identity matrix  $I_{N-2}$  and a  $2 \times 2$  matrix.

First consider the matrix of a chain

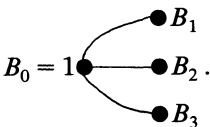


namely,

$$\begin{pmatrix} -b_1 & 1 & & 0 \\ 1 & & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -b_n \end{pmatrix}.$$

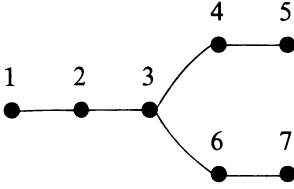
If we delete the first row and the last column, we obtain an upper triangular matrix, with determinant 1.

For a graph of type  $I_n$ ,  $II^*$  or  $III^*$  we can delete one vertex to obtain a chain, so the result follows. For type  $I_n^*$  we can delete two vertices to obtain a chain. For type  $II$ ,  $III$  or  $IV$  a minimal good resolution yields a graph



Here we obtain a nonsingular minor by deleting rows  $B_2$  and  $B_3$  and

columns  $B_1$  and  $B_3$ . For type IV\*,



we delete rows 1 and 4 and columns 5 and 7.  $\square$

Any nonsingular quadratic form on a finite group can [47] be expressed, up to isomorphism, as an (orthogonal) direct sum of forms of the following basic types:

(1)  $G \cong \mathbb{Z}/p^k\mathbb{Z}$  with generator  $x$  ( $p$  prime,  $k \geq 1$ );  $q(x) = up^{-k}$  with  $(u, p) = 1$ .

Notation  $w_{p,k}^\epsilon$ , where

$$\begin{aligned} \epsilon &= \left(\frac{u}{p}\right) \in \{\pm 1\} \text{ (Legendre symbol)} && \text{if } p \text{ is odd,} \\ \epsilon &\equiv u \pmod{8}, && \epsilon \in \{\pm 1, \pm 5\} \quad \text{if } p = 2 \end{aligned}$$

(if  $k = 1$ , this depends only on  $\epsilon \pmod{4}$ ).

(2)  $G \cong \mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^k\mathbb{Z}$  with generators  $x, y$  ( $k \geq 1$ );  $q(x) = q(y)$  and equal to (a) 0 or (b)  $2^{1-k}$ ,  $b(x, y) = 2^{-k}$ .

Notation  $u_k$  in case (a),  $v_k$  in case (b) (see also [7]). We shall use this notation in the tables. Such a decomposition is not unique (even up to isomorphism): a set of relations is given in [47, 1.8.2].

The signature of these types is as follows:

$$\begin{aligned} s(w_{p,k}^\epsilon) &\equiv k^2(1-p) + 2(\epsilon-1)k \pmod{8} \quad (p \text{ odd}), \\ s(w_{2,k}^\epsilon) &\equiv \begin{cases} \pm 1 \pmod{8} & \text{according as } \epsilon \equiv \pm 1 \pmod{4} \text{ (} k \text{ odd),} \\ \epsilon \pmod{8} & \text{(} k \text{ even)} \end{cases} \\ s(u_k) &\equiv 0, \quad s(v_k) \equiv 4k \pmod{8}. \end{aligned}$$

For the systematic description of series we augment this notation by the discriminant quadratic forms of simple singularities:

$$q_{A_k} \cong \mathbb{Z}/(k+1)\mathbb{Z}, \text{ generator } x, q(x) = (k+2)/(k+1),$$

$$q_{D_k} = \begin{cases} u_1, & k \equiv 0 \pmod{8}, \\ v_1, & k \equiv 4 \pmod{8}, \\ w_{2,1}^\epsilon + w_{2,1}^\epsilon, & \epsilon = \pm 1, k \equiv -2\epsilon \pmod{8}, \\ w_{2,2}^\epsilon, & \epsilon = \pm 1 \text{ or } \pm 5, k \equiv -\epsilon \pmod{8}. \end{cases}$$

Thus

$$s(q_{A_i}) \equiv -k \pmod{8},$$

$$s(q_{D_i}) \equiv -k \pmod{8}.$$

Indeed in these cases a smoothing has even, negative definite quadratic form, with rank  $k$ .

#### 4.5. Synthesis of the Milnor lattice

We now need some results from the theory of quadratic forms: these may all be found in [47].

If we are given a free abelian group  $H$  and a symmetric bilinear pairing of  $H$  with itself to  $\mathbb{Z}$ ,  $H$  is called a lattice. If, for each  $x \in H$ ,  $(x, x)$  is even, then  $H$  is an even lattice. We can construct invariants of lattices as follows:

First, tensor with  $\mathbb{R}$ . The form can then be diagonalised. Write  $t_-$ ,  $t_0$ ,  $t_+$  for the numbers of diagonal terms which are  $< 0$  (resp.  $= 0$ ,  $> 0$ ). As  $t_0$  does not interact with the other invariants, we usually suppose it zero.

Next let  $H^\# \subset H \otimes \mathbb{Q}$  denote the subgroup dual to  $H$ . Then  $G = H^\# / H$  is a finite group, and the pairing induces a non-singular symmetric bilinear map  $G \times G \rightarrow \mathbb{Q}/\mathbb{Z}$ . Moreover, if  $H$  is even, the composite  $G \leftarrow H^\# \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/2\mathbb{Z}$  is also well defined, so we have a discriminant quadratic form  $q_H$ . (This is essentially a special case of the construction in section 4.4).

For  $G$  a finite abelian group, we denote by  $l(G)$  the minimum number of generators of  $G$ .

We now enquire to what extent these invariants determine  $H$ . We have:

(A) ([47,1.10.2]) An even lattice  $H$  with invariants  $t_+$ ,  $t_-$  and  $(G, q)$  exists provided that  $s(q) \equiv t_+ - t_- \pmod{8}$ ,  $t_+ \geq 0$ ,  $t_- \geq 0$ ,  $t_+ + t_- > l(G)$ .

(B) ([47, 1.9.4]) The invariants  $t_+$ ,  $t_-$  and  $(G, q)$  determine the genus of  $H$ .

(C) ([47,1.13.3]) The invariants determine  $H$  up to isomorphism provided that  $t_+ \geq 1$ ,  $t_- \geq 1$  and  $t_+ + t_- \geq 2 + l(G)$ .

Now according to section 4.1, for any Kodaira singularity with  $D \leq 9$  we have smoothings  $\{X_i\}$ , each determining a Milnor lattice, which is unique for  $D \leq 5$ . Moreover, we saw in section 4.2 that we can define the rank  $\mu$  of  $H$  in all cases (even for  $D > 9$ ) and in section 4.3 how to determine  $\mu_+$  and  $\mu_0$ , and hence  $\mu_-$ . In section 4.4 we have shown how to calculate the discriminant quadratic form  $(G, q)$ , in the case when  $H_1(X_i; \mathbb{Z}) = 0$  (which holds for  $D \leq 5$ ). We now apply these results to reconstruct  $H$ .

Recall that: for type  $I_0$ ,  $\mu_+ = 0$ ,  $\mu_0 = 2$ ; for type  $I_n$  ( $n > 0$ ),  $\mu_+ = \mu_0 =$



1; otherwise,  $\mu_+ = 2$ ,  $\mu_0 = 0$  (so in all cases  $\mu_- = \mu - 2$ ). Let  $\bar{H}$  be the nondegenerate lattice obtained from  $H$  by factoring out the radical (= subspace orthogonal to  $H$ ): then  $\bar{H}$  determines  $H$ , as  $\mu_0$  is known. Now

$$\mu_+ - \mu_- = \begin{cases} 2 - \mu & (\text{type } I_0), \\ 3 - \mu & (\text{type } I_n, n > 0), \\ 4 - \mu & (\text{otherwise}). \end{cases}$$

Our calculation of  $s(q)$  in section 4.4 now shows that, in all cases,

$$s(q) \equiv \mu_+ - \mu_- \pmod{8}.$$

To apply (C), we recall that by section 4.4,  $l(G) \leq 2$  (type  $I_n^*$ :  $\leq 3$ ). Thus  $\bar{H}$  (and hence  $H$ ) is determined up to isomorphism provided  $\mu_+ \geq 1$  (true in all cases except type  $I_0$ ),  $\mu_- \geq 1$ , and  $\mu_+ + \mu_- \geq 4$  (5) (which as  $\mu_+ \leq 2$ , implies the condition  $\mu_- \geq 2$  (3)).

Next, we seek to show that  $H \cong K \oplus U$ , where  $U$  is a hyperbolic plane, with matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . If this is so, the invariants for  $K$  are obtained from those of  $H$  by diminishing each of  $t_+$  and  $t_-$  by 1. Now applying (A) we see that a suitable lattice  $K$  exists provided that

$$\mu_+ \geq 1, \quad \mu_- \geq 1, \quad \mu_+ + \mu_- \geq l(G) + 2,$$

giving the same conditions again. Moreover, since  $K \oplus U$  and  $H$  have the same invariants, they are isomorphic by (C).

Further, the lattice  $K$  is unique if

$$\mu_+ \geq 2, \quad \mu_- \geq 2, \quad \mu_+ + \mu_- \geq l(G) + 4.$$

Let us summarise these results.

**PROPOSITION 4.5.1:** *Exclude type  $I_0$ . Then for a Kodaira singularity with  $\mu - \mu_0 \geq 4$  ( $I_n^*$ :  $\geq 5$ ), the parameters  $\mu_+$ ,  $\mu_0$ ,  $\mu_-$  and  $(G, q)$  determine a lattice  $H$  unique up to isomorphism.  $H$  is of the form  $K \oplus U$  for some lattice  $K$ .*

*If we exclude also type  $I_n$ , and suppose  $\mu \geq 6$  (7), then  $K$  is unique up to isomorphism.*

The lattice  $H$  determined above is isomorphic to the lattice  $H_2(X_t)$  for any smoothing with  $H_1(X_t; \mathbb{Z}) = 0$ .

As  $\mu = 11 - D + \text{Eu}$ , all these results apply to all cases  $D \leq 7$ , since then  $\mu \geq 4 + \text{Eu}$ : if we exclude type  $I_0$ ,  $\text{Eu} \geq 1$  and  $\mu_0 \leq 1$ ; if we exclude all  $I_n$  then  $\text{Eu} \geq 2$  ( $I_n^*$ :  $\text{Eu} \geq 3$ ).

#### 4.6. Obstructions to smoothing

We observed in section 4.1 that all Kodaira singularities with  $D \leq 9$  are known to be smoothable, and this includes those in which we are principally interested.

As to the rest, by Theorem 4.1.1 any smoothing  $X_t$  has  $H_1(X_t; \mathbb{R}) = 0$ , and the numbers  $\mu_0$ ,  $\mu_-$  and  $\mu_+$  of the intersection form on  $H_2(X_t; \mathbb{R})$  are calculated in sections 4.2 and 4.3. This leads at once to the principal necessary condition for existence of a smoothing: namely, that  $\mu_- \geq 0$ . In the hyperbolic case, this yields  $9 + \sum k_r^* \geq \sum k_r$ ; and in the other cases, it reduces to  $9 + \text{Eu} \geq D$ .

One might expect to obtain stronger necessary conditions from lattice considerations. For smoothings which satisfy the additional condition  $H_1(X_t; \mathbb{Z}) = 0$ , the Milnor lattice must be an even lattice  $H$  with invariants  $\mu_+$ ,  $\mu_-$  and  $(G, q)$ . It thus follows that

$$s(q) \equiv \mu_+ - \mu_- \pmod{8},$$

$$\mu_+ \geq 0, \quad \mu_- \geq 0 \quad \text{and} \quad \mu_+ + \mu_- \geq l(G).$$

However, the first and second conditions are always satisfied, and in the cases  $\mu_+ = 2$ , the third implies the fourth (except for those cases of type  $I_n^*$  where  $l(G) = 3$ ). Thus in most cases, we obtain no more than the simple condition  $\mu_- \geq 0$  above; even in the others, we can only strengthen it to  $\mu_- \geq 1$ . A lattice-theoretic analysis applying to the general case is given in [39].

The singularities of types II, III and IV are often called triangle singularities. It was shown by Looijenga [38] and Pinkham [51] that all cases with  $D \leq 9 + \text{Eu}$  admit simply-connected smoothings, except for

$$D_{2,6,14} = \text{III}(2, 10),$$

$$D_{6,6,10} = \text{IV}(3, 3, 7),$$

$$D_{2,10,10} = \text{III}(6, 6).$$

Of these, only the last has no smoothings at all.

#### 4.7. Notes on the calculation of discriminant quadratic forms

As we observed in section 4.4 these can be easily calculated from the resolution for all Kodaira singularities. However, some special features can be observed without doing the numerical work.

For hyperbolic singularities, the resolution forms a cycle; this contributes an element of infinite order in  $\mathcal{G} = H_1(L)$ . However, the torsion subgroup of  $H_1(L)$  can be obtained from the quadratic form defined by

the exceptional divisor, or alternatively from the matrix  $A$  given earlier: either leads to an inductive formula for the order of this group. Since dual singularities can be “attached” to form an Inoue–Hirzebruch surface, the links  $L$  and  $L^*$  must be diffeomorphic. Hence there is an isomorphism of  $\mathcal{G}$  on  $\mathcal{G}^*$  taking  $q$  to  $-q^*$ .

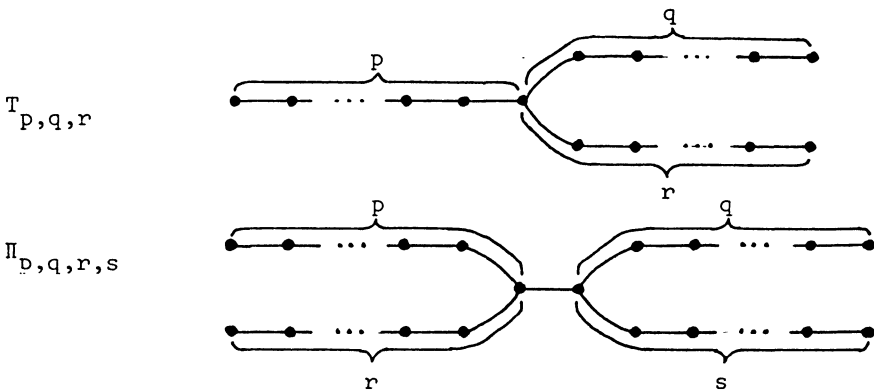
For a triangle singularity  $X$  we can use the resolution of  $X$  given by three smooth rational curves with a single common point. Now this has the same parameters as a resolution for the hyperbolic singularity  $\beta X$  (with exceptional set a triangle), except that the components instead of forming a cycle have a single common point. This does not affect the calculation of the discriminant quadratic form, which is thus the same for  $X$  and  $\beta X$ .

We have already given in section 1.2 the order of  $G$  in most cases; there remains the case  $I_n^*(k_1, k_2, k_3, k_4)$ . The order here is a linear function of  $n$ ; the constant term is given by the value at  $n=0$ ; the coefficient of  $n$  turns out to be

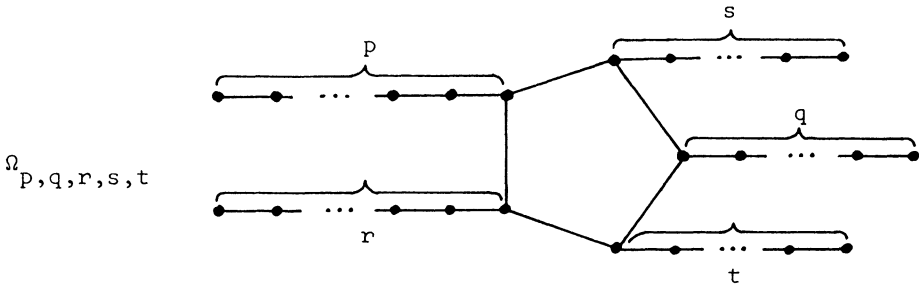
$$(k_1 k_3 + k_1 + k_3)(k_2 k_4 + k_2 + k_4).$$

We shall also need the discriminant quadratic forms corresponding to the lattices with the graphs of Schemas 4,5. As usual, such a graph defines a lattice as follows. The vertices correspond to the elements of a basis of the lattice and the matrix of the bilinear form with respect to this basis is determined by the graph: the diagonal terms are  $-2$  and the remaining terms 0 unless the corresponding nodes are joined by an edge, when we have 1.

Here we use a device first proposed by Brieskorn [7]. We augment the graph by joining each vertex of valence 1 to a new vertex, which will give a new basis vector with diagonal entry  $-1$  in the extended matrix. We



Schema 4.



Schema 5.

can now see inductively that the extended quadratic forms are (odd) unimodular, essentially using the fact, that “blowing down  $(-1)$ -vertices” does not alter the discriminant. The orthogonal complements of the original lattices have bases  $\{a_1, \dots, a_l\}$ , where  $l = 3, 4, 5$  respectively, as follows. We describe  $a_l$  by assigning to the vertex of the extended graph the coefficient of the corresponding basis vector, writing  $a_l$  as a linear combination in the extended basis. Then the vector  $a_1$  is described in the diagrams of Schemas 6,7. The other basis vectors are defined analogously, where the arms of the graphs interchange their rôles. The bilinear form with respect to these bases is described by the following matrices:

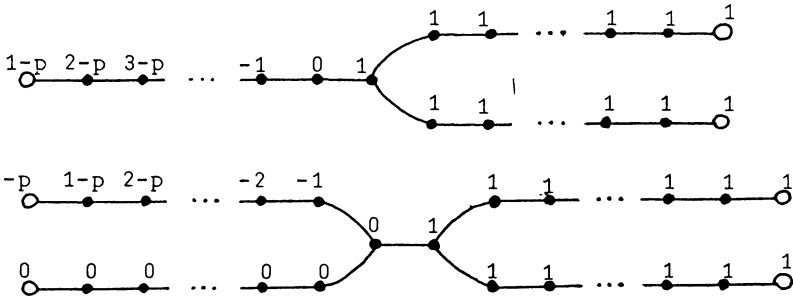
$$\begin{pmatrix} 1-p & 1 & 1 \\ 1 & 1-q & 1 \\ 1 & 1 & 1-r \end{pmatrix}, \quad \begin{pmatrix} -p & 1 & 0 & 1 \\ 1 & -q & 1 & 0 \\ 0 & 1 & -r & 1 \\ 1 & 0 & 1 & -s \end{pmatrix},$$

$$\begin{pmatrix} -1-p & 1 & 0 & 0 & 1 \\ 1 & -1-q & 1 & 0 & 0 \\ 0 & 1 & -1-r & 1 & 0 \\ 0 & 0 & 1 & -1-s & 1 \\ 1 & 0 & 0 & 1 & -1-t \end{pmatrix},$$

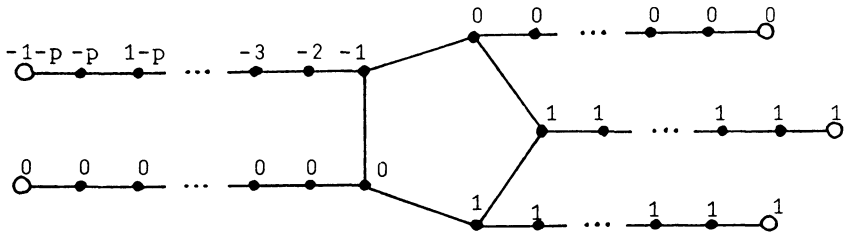
which are the intersection matrices of the (minimal good) resolution graphs of the hyperbolic singularities

- $I_1(r-4)$   $(p=2, q=3, r>6),$
- $I_2(q-2, r-2)$   $(p=2, r \geq q \geq 4, r>4),$
- $I_3(p-1, q-1, r-1)$   $(r \geq q \geq p \geq 3, r>3),$
- $I_4(p, q, r, s)$   $(s \geq r \geq q \geq p \geq 2, s>2),$
- $I_5(p+1, q+1, r+1, s+1, t+1)$   $(t \geq s \geq r \geq q \geq p \geq 1, t>1).$

If we have a primitive embedding of a lattice  $H$  in an (odd) unimodular



Schema 6.



Schema 7.

lattice, then [47] the discriminant bilinear form of the orthogonal complement  $H^\perp$  of  $H$  is the negative of the discriminant bilinear form of  $H$ . Thus we can compute the discriminant bilinear forms of the original lattices from the above matrices. By the results of sections 4.4 and 4.5 we can thus also determine the discriminant quadratic forms.

From the results of section 4.5 and the identification of the discriminant quadratic forms of  $X$  and  $\beta X$  we deduce:

**PROPOSITION 4.7.1.:** *If  $X$  is a triangle singularity with  $D \leq 5$  then there is an even lattice  $K$ , unique up to isomorphism, such that the Milnor lattice of  $X$  is  $K \oplus U$  and that of  $\beta X$  is  $K \oplus (0)$ .*

As  $t_+ = 1$ ,  $t_0 = 0$  for  $K$ , we shall refer to  $K$  as the associated hyperbolic lattice. As  $K$  is determined by  $t_+$ ,  $t_-$  and the discriminant quadratic form, we can identify  $K$  as follows. We have

Dual to:	Hyperbolic lattice $K$ :
$I_5(b_1, b_2, b_3, b_4, b_5)$	$\Omega_{b_1-1, b_2-1, b_3-1, b_4-1, b_5-1}$ ,
$I_4(b_1, b_2, b_3, b_4)$	$\Pi_{b_1, b_2, b_3, b_4}$ ,
$I_3(b_1, b_2, b_3)$	$T_{b_1+1, b_2+1, b_3+1}$ ,
$I_2(b_1, b_2)$	$T_{2, b_1+2, b_2+2}$ ,
$I_1(b_1)$	$T_{2, 3, b_1+4}$ .

A more direct construction of the graphs giving these lattices is given by Looijenga [37].

The parameters on the right-hand side defining the graphs coincide with the parameters  $p_i^*$  associated to the hyperbolic singularity  $\beta X$  in sections 2.2 and 3.2. If  $X$  is a triangle singularity with  $D \leq 3$ , they are often known as Gabrielov numbers (see below).

Indeed, one can interpret the above imbedding of the lattice  $K$  in a unimodular lattice  $N$  geometrically using Looijenga's construction. Namely following [37], the Milnor fibre of the hyperbolic singularity  $\beta X$  is homeomorphic to a rational surface with homology lattice  $N$ , where a cycle of rational curves with one of the above intersection matrices is removed.

## 5. The Milnor lattice

### 5.1. Dynkin diagrams

We have seen in sections 4.1 and 4.5 how to associate a Milnor lattice to each Kodaira singularity with  $D \leq 7$ , and that each such lattice is uniquely of the form  $K \oplus (0)$  (hyperbolic cases) or  $K \oplus U$  (otherwise), where  $K$  has signature  $(1, \mu - 2)$  or  $(1, \mu - 3)$  and is called the hyperbolic sublattice.

Now it turns out that if  $D \leq 5$  each such lattice  $K$  has a basis  $\{e_i\}$  such that, for each  $i, j$ ,

$$(e_i, e_i) = -2, \quad (e_i, e_j) = 0 \text{ or } 1 \quad \text{if } i \neq j.$$

As is customary, we represent this by a graph, called the Dynkin diagram, with a vertex  $v_i$  for each basis vector  $e_i$ , and an edge joining  $v_i, v_j$  whenever  $(e_i, e_j) = 1$ . The basis can be taken such that this diagram is connected and contains no vertex of valence  $> 3$ . Thus the matrix of the bilinear form with respect to this basis, multiplied by  $-1$ , is an indecomposable symmetric Cartan matrix of negative type in the sense of Kac [24], such that the corresponding Dynkin diagram has no multiple edges, and no vertices of valence  $> 3$ .

For the hyperbolic and triangle singularities, an appropriate diagram was constructed in the preceding section. Alternatives for some of these, and diagrams for the remaining cases, are displayed in Table 4.

In the hypersurface case, each singularity also determines a class of distinguished bases of vanishing cycles in the lattice. In these cases the diagrams of Table 4 can be extended to Dynkin diagrams of distinguished bases of vanishing cycles as follows: For the hyperbolic singularities let  $v_\rho$  be the vertex of valence 3 and join  $v_\rho$  and the adjacent vertices with a new vertex  $v_{\rho+1}$  as in Fig. 1. For the corresponding hypersurface singularities of type II, III, IV one has to join  $v_{\rho+1}$  with

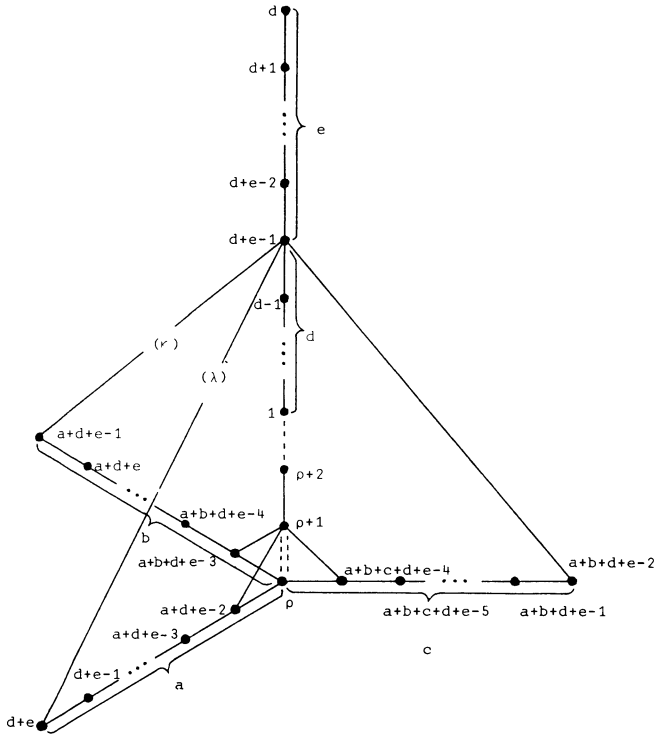


FIGURE 1.

another vertex  $v_{\rho+2}$  as in Fig. 1. This yields the diagrams of Gabrielov [20a]. Recall that these diagrams led to the original discovery of the strange duality. In the other hypersurface cases, let  $v_\rho$  be the left vertex of valence 3 and apply the same construction. But now  $v_{\rho+2}$  has to be joined by a dotted edge to another vertex  $v_1$ , such that the resulting graph fits into the pattern of Fig. 1. Here  $a, b, c \geq 2, d, e \geq 1, \kappa, \lambda \in \{0, 1\}$  and  $\kappa = 0$  ( $1$ ) means that there is no edge (is an edge) between  $v_{d+e-1}$  and  $v_{a+d+e-1}$  (cf. [18]).

In the nonhypersurface but nonhyperbolic complete intersection case we claim that the vertices of the diagrams of Table 4 still correspond to vanishing cycles in the Milnor lattice. Here we argue as follows. Pinkham [50] has shown, that the monodromy groups of the complete intersection singularities of triangular type II, III, IV can be characterized arithmetically. The first author has extended Pinkham's result to large classes of hypersurface singularities, including all nonhyperbolic Kodaira singularities with  $D \leq 3$  [16,17], and very recently to all complete intersection singularities, which deform into one of the above singularities [17, Note added in proof; 19], thus to all nonhyperbolic Kodaira singularities with  $D \leq 4$  (see section 5.3). But from the description of the monodromy

group one can derive that the set of vanishing cycles can also be described in purely lattice-theoretic terms: The vanishing cycles are exactly the minimal vectors  $x$  in  $H$  with  $(x, x) = -2$ , such that there exists a  $y \in H$  with  $(x, y) = 1$ . But these conditions are satisfied for the elements corresponding to the vertices of the diagrams of Table 4. Moreover, if one extends the diagrams as for the hypersurface singularities of type II, III, IV, the corresponding bases are bases of vanishing cycles, such that (by [19, Theorem 3.2]) the monodromy groups are generated by the corresponding reflections. We mention that for the 8 complete intersection singularities of triangular type, the diagrams of Table 4 were already given by Pinkham [50].

### 5.2. The strange duality

We observed in section 4.2 that extending the strange duality from  $D \leq 3$  to  $D \leq 4$  involved a “virtual” singularity  $n = -1$  in each series  $I_n^*(k_1, k_2, k_3, k_4)$ . To this we can associate a lattice in the following way.

Let  $H^{(n)}$  ( $n \in \mathbb{N}$ ) be the Milnor lattice associated to  $I_n^*(k_1, k_2, k_3, k_4)$  with  $\sum k_i \leq 5$ . Then  $H^{(n)}$  has rank  $\mu^{(0)} + n$  and discriminant  $\Delta^{(0)} + bn$  for certain integers  $\mu^{(0)}$ ,  $\Delta^{(0)}$ ,  $b$  which are listed above. Moreover, the resolution graphs determine a primitive embedding of  $H^{(n)}$  in  $H^{(n+1)}$  for each  $n$ . Let  $J^{(n)}$  be the orthogonal complement, which has rank 1 and is therefore determined by its discriminant disc  $J^{(n)}$ . But

$$\text{disc } J^{(n)} = p(n)$$

for a polynomial  $p$  which depends on the series. Let  $J^{(-1)}$  be the lattice of rank 1 with disc  $J^{(-1)} = p(-1)$ . One can show in each case, using [47, 1.15.1], that there is a unique primitive sublattice  $H^{(-1)}$  of  $H^{(0)}$  of rank  $\mu^{(0)} - 1$  and discriminant  $\Delta^{(0)} - b$  and  $(t_0, t_+) = (0, 2)$ , such that the orthogonal complement is equal to  $J^{(-1)}$ . The finite quadratic form corresponding to  $H^{(-1)}$  is given by setting  $n = -1$  in Table 3. Again we can uniquely decompose

$$H^{(-1)} = K^{(-1)} \oplus U,$$

and the hyperbolic lattice  $K^{(-1)}$  has again a basis as above. One gets a Dynkin diagram with respect to such a basis, if one sets  $n = -1$  in Table 4 in the corresponding entry. In general one can take any sequence of diagrams of  $\{K^{(n)}\}_{n=0,1,\dots}$ , which induces the right embeddings of  $H^{(n)}$  in  $H^{(n+1)}$ , and take the diagram for  $n = -1$ , if it is defined. But now these lattices  $K^{(-1)}$  coincide with the hyperbolic lattices associated with  $\gamma_n^{-1}X = I_4(k_1 + 2, k_2 + 2, k_3 + 2, k_4 + 2)$ . Thus the lattice duality  $(\delta)$  already established for the hyperbolic singularities  $D \leq 4$ ,  $D^* \leq 4$  yields a duality  $(\delta_2)$  here, with the newly constructed lattices  $K^{(-1)}$  added to the list of lattices of the triangle singularities. Let us define the Dolgachev



numbers of the “virtual” singularity  $n = -1$  in a series  $I_n^*(k_1, k_2, k_3, k_4)$  to be  $p_i = 2 + k_i$ ,  $1 \leq i \leq 4$ , and the Gabrielov numbers to be the Gabrielov numbers  $p_i^*$  of the lattice  $K^{(-1)}$  (cf. section 4.7). Then, as in the original strange duality, one makes the following observation (see Tables 2 and 4):

**THEOREM 5.2.1:** *With respect to the extended duality  $(\delta_2)$ , the Dolgachev numbers of a singularity coincide with the Gabrielov numbers of the dual singularity.*

Moreover, one has the following fact: Let  $K$  be the hyperbolic sublattice associated to the Milnor lattice of a singularity of type  $I_n$ , II, III, IV or to  $H^{(-1)}$  for a series of singularities of type  $I_n^*$  and let  $K^*$  be the corresponding dual lattice. Then by [47,1.14.4] there exists a unique primitive embedding of  $K \oplus U$  into the unimodular lattice

$$E_8 \oplus E_8 \oplus U \oplus U \oplus U,$$

which is the homology lattice of a K-3 surface, and the orthogonal complement is just  $K^*$ . For the exceptional unimodular singularities, Pinkham [43] has given a geometric realization of this embedding and conjectured the uniqueness.

Another feature of the original strange duality observed by Arnold is that the Coxeter numbers of dual singularities coincide. The Coxeter number of a singularity was originally defined by Arnold using integrals of a corresponding oscillating function [2,4]. For the 14 hypersurface triangle singularities it is a negative integer  $-N$ ; its values can be found in [4]. In this case there are also two other interpretations of the absolute value  $N$  of this number. It coincides with the degree  $d$  of a weighted homogeneous equation defining the singularity (see section 3.4). In section 4.3 we defined  $\mu$  rational numbers  $r_i$ ,  $1 \leq i \leq \mu$ , such that  $\lambda_i = \exp(2\pi\sqrt{-1}r_i)$  are the eigenvalues of the classical monodromy operator. Then  $N$  is also the least common multiple of the denominators of these numbers  $r_i$ .

We use these two interpretations to extend the definition of the Coxeter number to complete intersection triangle singularities on one hand, and to the “virtual” singularities of series of bimodal hypersurface singularities on the other hand.

For a complete intersection singularity which is given by weighted homogeneous equations  $f_1$  and  $f_2$  of degree  $d_1$  and  $d_2$ , we define  $N$  to be the least common multiple of  $d_1$  and  $d_2$ .

On the other hand, consider a series of singularities  $I_n^*(k_1, k_2, k_3, k_4)$  with  $D \leq 3$ . We associated above to the “virtual” singularity  $n = -1$  certain Dynkin diagrams of the hyperbolic sublattice by setting  $n = -1$  in a sequence of diagrams for  $n \geq 0$ . Now extend these diagrams to

Dynkin diagrams of distinguished bases of vanishing cycles as explained in section 5.1. Let  $C^{(n)}$  be the product of the reflections corresponding to the basis elements in the given order in a diagram for  $I_n^*(k_1, k_2, k_3, k_4)$ . Note that, for  $n \geq 0$ ,  $C^{(n)}$  is just the monodromy operator. Now set again  $n = -1$  and let  $\lambda_1, \dots, \lambda_{\mu^{(-1)}}$  be the eigenvalues of  $C^{(-1)}$ , which can be computed using the general formula for the characteristic polynomial given in [18]. They are again of the form  $\lambda_i = \exp(2\pi\sqrt{-1}r_i)$  for certain rational numbers  $r_i$  determined modulo  $\mathbb{Z}$ . Then define  $N$  to be the least common multiple of the denominators of these numbers  $r_i$ . By the remark following Theorem 5.4.1. below,  $N$  does not depend on the chosen Dynkin diagram of an entry of Table 4; one can also take the diagram defined by the Gabrielov numbers and extend it as in the hypersurface case. For each series, the value of  $N$  can be found in the tables of [20b], where one has to set  $p = -1$  in the column corresponding to  $N$  (but note that in [20b]  $N$  is defined to be some common multiple of the denominators, not necessarily the least common multiple).

Now we observe the following fact:

**THEOREM 5.2.2:** *Given any of the 8 triangle singularities with  $D = 4$ , its Coxeter number  $N$  coincides with the Coxeter number  $N$  of the corresponding dual “virtual” singularity of a bimodal series under our extended duality ( $\delta_2$ ).*

Using the notation of Table 2, we tabulate for each triangle singularity with  $D = 4$  the values of  $d_1, d_2$ , and  $N$ , and its dual:

Notation	$J'_9$	$J'_{10}$	$J'_{11}$	$L_{10}$	$L_{11}$	$K'_{10}$	$K'_{11}$	$M_{11}$
$d_1, d_2$	16,18	12,14	10,12	11,12	9,10	10,12	8,10	8,9
$N$	144	84	60	132	90	60	40	72
Dual	$E_{3,-1}$	$Z_{1,-1}$	$Q_{2,-1}$	$W_{1,-1}^\#$	$S_{1,-1}^\#$	$W_{1,-1}$	$S_{1,-1}$	$U_{1,-1}$

Now let us regard the duality  $\delta_3$  defined in section 4.2 which concerns the right part of the tables. For  $g = 1$  we have a symmetry around the diagonal starting in the upper right corner of the table. Here the Milnor lattices of dual singularities are not complementary but isomorphic. (Concerning column  $I_5$  and the last row, this statement is only true for the hyperbolic sublattices.) For  $g = 2$ , we have to replace the right column of the double column  $I_1^*$  by the corresponding column for  $I_2^*$  (this does not change the discriminant). Then one has a symmetry of part of the diagram, but not in the principal diagonal. Again the quadratic forms of dual singularities are isomorphic. In particular we get examples of singularities of grade 4 and 5 which have the same quadratic forms as

hypersurface singularities. But whenever defined, the Coxeter numbers of dual singularities are not equal.

In Section 5.4 we shall construct an extension of the table for  $g = 2$  on the right by two additional columns (for lattices corresponding to “virtual” singularities). In the new larger table (Table 5) the symmetry described above can be extended. One can prolong the axis of symmetry to the first row to get on both sides isomorphic lattices, because up to the last element the first new column corresponds exactly to the right part of the first row. Moreover, one can extend the axis to the last row. Then the Milnor numbers and discriminants of the symmetric pairs of the last row and the corresponding column satisfy

$$\mu - \mu^* = d - d^* - 1,$$

with no replacement in the columns  $I_1^*$ .

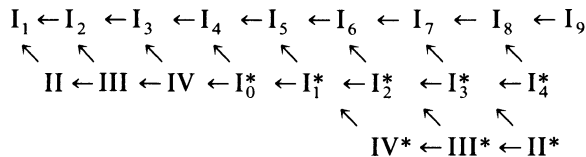
### 5.3. Adjacencies

We first note a slight extension of a result standard for hypersurfaces.

**PROPOSITION 5.3.1:** *Let  $(X, x)$  and  $(Y, y)$  be Kodaira singularities, and suppose  $(X, x)$  deforms into  $(Y, y)$ . Then there exists a primitive embedding of the Milnor lattice of  $(Y, y)$  into that of  $(X, x)$ .*

**PROOF:** In the semiuniversal deformation of  $(X, x)$ , choose a small ball around  $y \in Y$  which intersects a smooth fibre  $X'$  in a Milnor fibre  $Y'$  of  $(Y, y)$ . The proposition then follows from the exact homology sequence of the pair  $(X', Y')$ , using  $H_i(X', Y') = H_i(Y, y)$ ,  $H_1(Y') = 0$  and the fact that  $Y$  can be chosen to be Stein.  $\square$

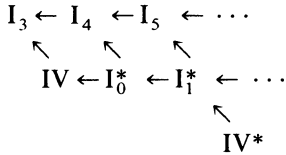
Now according to Laufer [36, Theorem 4.13] we have, for each series of singularities, adjacency relations with constant grade forming (a subdiagram of) the following diagram:



More precisely: for each  $k_1 \geq 1$ , we have the entire diagram (other  $k_i$ , where needed, are taken as 0;  $k_1^* = n$  is determined uniquely for each  $I_n$ ). For  $k_1 \geq k_2 \geq 1$  (and other  $k_i = 0$ ), we have to omit the entries  $I_1$ ,  $II$  and  $II^*$ . There are two choices for  $I_4$ ,  $I_5$  and  $I_n^*$  ( $n \geq 1$ ):  $III^*$  (resp.  $IV^*$ ) deforms to each of the alternative  $I_2^*$  (resp.  $I_1^*$ );  $I_1^*(k, k', 0, 0)$  deforms to

each of the  $I_5$  while  $I_1^*(k, 0, k', 0)$  deforms only to that with  $k^* = (4, 1)$  (but not to the one with  $k^* = (3, 2)$ ), and specialisations of the  $I_n$  proceed by diminishing one of the  $k_i^*$  by 1.

For  $g = 3$  the diagram reduces to

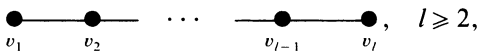


but the cases corresponding to the entries can be more numerous.

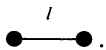
We next consider these adjacencies from a different viewpoint: this will show that the singularities of type  $I_5$  play for lattices of types  $IV^*$ ,  $III^*$ ,  $II^*$  a role analogous to that of  $I_4$  for lattices of type  $I_n^*$ .

#### 5.4. Transforms

Let  $S$  be the Dynkin diagram corresponding to an indecomposable symmetric Cartan matrix. We assume that  $S$  has no multiple edges and no vertices of valence  $> 3$ . We associate a weighted graph  $S'$  to  $S$  as follows: Replace each subgraph



where  $v_i$  is a vertex of valence 2 for  $i \neq 1, l$  and of valence  $\neq 2$  for  $i = 1, l$ , by



We call the graph  $S'$ , which can now have loops and multiple edges, the *scheme* of  $S$ , the weights of  $S'$  the *weights of  $S$*  and the underlying unweighted graph the *shape* of  $S$ .

We call a weight of  $S'$  an *outer weight*, if one of the vertices of the corresponding edge of  $S'$  has valence 1, otherwise it is called an *inner weight*. Let  $S_1$  and  $S_2$  be two such graphs of the same shape with weights  $w_1^{(1)}, \dots, w_r^{(1)}$  respectively  $w_1^{(2)}, \dots, w_r^{(2)}$  and let  $I$  be a subset of the index set  $\{1, \dots, r\}$ . We define

$$S_1 \leq S_2 \quad (\text{with respect to } I)$$

if and only if  $w_i^{(1)} = w_i^{(2)}$  for  $i \notin I$ ,  $w_i^{(1)} \leq w_i^{(2)}$  for  $i \in I$ . The number

$$d(S_1, S_2) = \sum (w_i^{(2)} - w_i^{(1)})$$

is called the *distance* of  $S_1$  and  $S_2$ .

We define an operation on graphs, which produces new graphs defining the same quadratic form. Let  $S$  be a graph as above. Let  $Q$  be a subgraph of  $S$ , which is an extended Dynkin diagram in the classical sense, that is, an extended  $A_n$ -,  $D_n$ -,  $E_6$ -,  $E_7$ - or  $E_8$ -diagram and let  $v_k$  be a vertex of  $S$  which is neither contained in  $Q$  nor connected by an edge to a vertex of  $Q$ . We shall define a transformation  $\tau_{Q,k}$  depending on the choice of  $Q$  and  $v_k$  and only defined for special choices of  $Q$  and  $v_k$  as follows. The lattice corresponding to  $Q$  contains a distinguished isotropic vector  $w$ ;  $w$  is the sum of the longest root of the corresponding finite root system and the additional vector [8]. Thus the coefficients of  $w$  are exactly the coefficients of the fundamental cycle associated to a resolution graph, which corresponds to the extended Dynkin diagram  $Q$  (cf. section 1.3). Let

$$e_k^{(1)} = w - e_k, \quad e_i^{(1)} = e_i \quad \text{for } i \neq k.$$

Then the new basis  $\{e_i^{(1)}\}$  satisfies again  $(e_i^{(1)}, e_i^{(1)}) = -2$ , since  $(w, e_k) = (w, w) = 0$ . If  $(e_k^{(1)}, e_i^{(1)}) \in \{0, 1\}$  for  $i \neq k$  and the new graph  $S^{(1)}$  corresponding to  $\{e_i^{(1)}\}$  is connected, then we define

$$\tau_{Q,k}(S) = S^{(1)}.$$

If  $S^{(1)}$  is not connected or if there exists an  $i \neq k$  with  $|(e_k^{(1)}, e_i^{(1)})| > 1$ , then  $\tau_{Q,k}$  is not defined. If finally for some  $i$   $(e_k^{(1)}, e_i^{(1)}) = -1$ , we shall continue transforming as follows. Choose  $j_1$  with  $(e_k^{(1)}, e_{j_1}^{(1)}) = -1$  and let

$$e_k^{(2)} = e_k^{(1)} - e_{j_1}^{(1)}, \quad e_i^{(2)} = e_i^{(1)} \quad \text{for } i \neq k.$$

If now  $|(e_k^{(2)}, e_i^{(2)})| > 1$  for some  $i$  or  $(e_k^{(2)}, e_i^{(2)}) = -1$  for  $e_i^{(2)} \in Q$ , then  $\tau_{Q,k}$  is again not defined. If  $(e_k^{(2)}, e_{j_2}^{(2)}) = -1$  for  $e_{j_2} \notin Q$  we set again

$$e_k^{(3)} = e_k^{(2)} - e_{j_2}^{(2)}, \quad e_i^{(3)} = e_i^{(2)} \quad \text{for } i \neq k,$$

and continue in this way until we reach at step  $N$  one of the following situations:

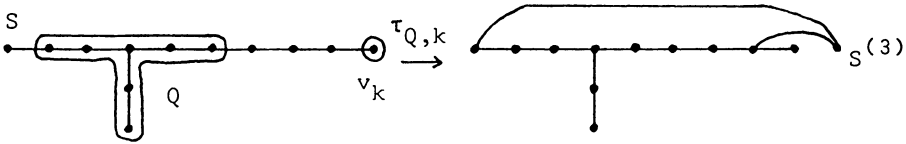
- (a)  $|(e_k^{(N)}, e_i^{(N)})| > 1$  for some  $i \neq k$ ,
- (b)  $(e_k^{(N)}, e_i^{(N)}) = -1$  for some  $i \neq k$  with  $e_i^{(N)} \in Q$ ,
- (c)  $(e_k^{(N)}, e_i^{(N)}) \in \{0, 1\}$  for all  $i \neq k$ .

In cases (a) and (b)  $\tau_{Q,k}$  is not defined. In case (c) the matrix corresponding to  $\{e_i^{(N)}\}$  is again a Cartanmatrix and the corresponding graph  $S^{(N)}$  has no multiple edges. We define in this case

$$\tau_{Q,k}(S) = S^{(N)}.$$

One easily checks that there is such an  $N$ , at which the process stops, and that  $\tau_{Q,k}$  does not depend on the choice of the sequence  $j_1, j_2, \dots, j_N$ , if it is defined.

EXAMPLE:



Schema 8.

The graphs obtained from  $S$  by these transformations or sequences of these transformations with same  $Q$  are called the *transforms* of  $S$ . The transforms of  $S$  which are different from  $S$  are called the *proper transforms*.

Using these definitions we have the following characterisation:

**THEOREM 5.4.1:** *The hyperbolic sublattices of the singularities of type II, III and IV with  $D \leq 5$  are exactly those given by the graphs of the types  $T_{p,q,r}$ ,  $\Pi_{p,q,r,s}$  and  $\Omega_{p,q,r,s,t}$ , which define hyperbolic lattices and have no proper transforms.*

Let  $\mathcal{S}$  be the set of these graphs. Let  $\mathcal{B}^{(d)}$  be the set of all graphs  $R$  of the types  $T_{2,3,r}$ ,  $T_{2,q,r}$  ( $q, r \geq 4$ ),  $T_{p,q,r}$  ( $p, q, r \geq 3$ ),  $\Pi_{p,q,r,s}$ ,  $\Omega_{p,q,r,s,t}$  such that  $S \leq R$  with respect to  $p, q, r, s, t$  for all  $S \in \mathcal{S}$  and

$$d = \min\{d(S, R) \mid S \in \mathcal{S}\}.$$

The graphs of  $\mathcal{B}^{(1)}$  are listed in column  $I_4 \leftrightarrow I_n^*$  in Table 4. They correspond to the hyperbolic sublattices of the singularities of type  $I_4$  resp. to the lattices  $K^{(-1)}$ . The graphs of  $\mathcal{B}^{(2)}$  are listed in the middle columns. They correspond to the hyperbolic sublattices of the singularities of type  $I_5$ . All these graphs do have transforms. They are listed in Table 4 in the same entries (setting  $n = -1$  in column  $I_4 \leftrightarrow I_n^*$ ).

**REMARK:** There are certain groups of transformations of Dynkin diagrams, which preserve the properties of being the diagram of a (weakly) distinguished basis. Let  $Z^0$  resp.  $Z^* \subset Z^0$  be this group with respect to weakly distinguished resp. distinguished bases as defined in [18]. Then in general  $\tau_{Q,k} \notin Z^0$ , since  $Z^0$  leaves the group generated by the reflections corresponding to the basis elements invariant. But if one extends in the hypersurface case the diagrams of column  $I_4 \leftrightarrow I_n^*$  ( $n = -1$ ) and  $I_5$  by two additional vertices to diagrams in the canonical form with respect to

distinguished bases as described in section 5.1, then the corresponding diagrams in the same entry are equivalent under  $Z^*$ , as can be shown.

The graphs of type  $\Pi$  and  $\Omega$  have a distinguished  $\tilde{D}_5$ - resp.  $\tilde{A}_4$ -subgraph. We now consider the order relation  $\leq$  between graphs with respect to those inner weights which in the cases  $\Pi$  and  $\Omega$  do not belong to these subgraphs. These are the underlined weights in Table 4. We then have the following characterisation of the remaining lattices:

**THEOREM 5.4.2:** (a) *The hyperbolic sublattices of the singularities of type  $I_n^*$  are exactly those given by graphs  $R > S$ , where  $S$  is a transform of the graph of  $\mathcal{B}^{(1)}$  associated to the singularity under the bijection  $\gamma_n$  of section 2.3.*

(b) *The hyperbolic sublattices of the singularities of type  $I_1^*$ ,  $IV^*$ ,  $III^*$  and  $II^*$  of a row of Table 4 are those given by the graphs  $R$  satisfying the following conditions:*

- (i)  *$R > S$ , where  $S$  is a transform of a graph of  $\mathcal{B}^{(2)}$  of the same row.*
- (ii)  *$R$  defines a hyperbolic lattice.*
- (iii) *For each chain  $R_1 \leq R_2 \leq \dots \leq R_k = R$  with  $d(R_i, R_{i+1}) = 1$  and  $R_1$  minimal, there exists an  $i \neq k$  such that  $R_i$  is a proper transform of a graph of  $\mathcal{B}^{(2)}$  of the same row.*
- (iv) *A transform of each graph of  $\mathcal{B}^{(2)}$  of the corresponding row is reached by (iii).*

We have listed all such graphs in Table 4. To each relation  $R_1 \leq R_2$  between such graphs corresponds an adjacency (achieved over the simultaneous-blow-down parameter space [36]) between the corresponding singularities. There are still some classes of graphs which satisfy only the conditions (i), (ii) and (iii) and not (iv) of Theorem 5.4.2. In each row of Table 4 also the graphs which satisfy these conditions and belong to singularities of the same row are given. But in the table for  $g = 2$  these are not all the graphs satisfying (i) to (iii): There are additional graphs and one can extend Table 4 ( $g = 2$ ) according to these additional graphs. The result is shown in Table 5(c). It turns out that these graphs define lattices, which can be identified with the hyperbolic sublattices of certain Kodaira singularities. The names of these singularities, Milnor numbers and discriminants are tabulated in Table 5(a), the discriminant quadratic forms in Table 5(b). Table 5(a) resp. 5(b) are the corresponding extensions of Table 2 resp. 3. Concerning these singularities, there are now relations  $R_1 \leq R_2$  in Table 5(c), which do not correspond to adjacency relations.

### 5.5. Fundamental vectors

Here we give another characterisation of the hyperbolic sublattices of the singularities of type II, III and IV with  $D \leq 5$ .

Let  $K$  be an even, nondegenerate, indefinite lattice with a basis  $B = \{e_1, \dots, e_\rho\}$ , such that  $A = -(e_i, e_j)$  is an indecomposable symmetric Cartan matrix. Let  $S$  be the corresponding graph.

DEFINITION: A *fundamental vector with respect to  $B$*  is a vector  $x = \sum \xi_i e_i \in K$  such that

- (i)  $\forall i \xi_i > 0,$
- (ii)  $\forall i (x, e_i) \geq 0.$

So a fundamental vector with respect to  $B$  is thus a positive integral vector which lies on the boundary of the fundamental chamber with respect to  $B$  (cf. [26]).

LEMMA 5.5.1: *Let  $x$  be a fundamental vector with respect to  $B$ . Let  $J = \{j \in \{1, \dots, \rho\} | (x, e_j) = 0\}$ . Then  $M = \sum_{j \in J} \mathbb{Z} e_j$  is a negative definite lattice (and hence a sum of  $A_n, D_n, E_6, E_7$  or  $E_8$ ).*

PROOF: Since  $K$  is nondegenerate,  $J \neq \{1, \dots, \rho\}$ . Let  $J = J_1 \cup \dots \cup J_k$  be the partition of  $J$  corresponding to the decomposition of the subgraph of  $S$  with respect to  $\{e_j | j \in J\}$  into connected components. Let  $x^{(l)} = \sum_{j \in J_l} \xi_j e_j$  for  $1 \leq l \leq k$ . Now since  $(x, e_j) = 0$  for  $j \in J_l$  and since there exist numbers  $i \notin J_l, j_0 \in J_l$ , such that  $(e_i, e_{j_0}) > 0$ , it follows that

$$(x^{(l)}, e_j) \leq 0$$

for all  $j \in J_l$  and  $< 0$  for at least one  $j$ . Therefore  $(x^{(l)}, x^{(l)}) < 0$  and by [5, Proposition 2]  $K$  is negative definite.  $\square$

DEFINITION: A *small fundamental vector* (with respect to  $B$ ) is a fundamental vector  $x = \sum \xi_i e_i$ , satisfying the following additional conditions:

- (i)  $(x, x) \leq 4,$
- (ii)  $\forall i (x, e_i) \leq 2, (x, e_i) = 2$  for at most one  $i, (x, e_i) = 2 \Rightarrow \xi_i = 1.$

If  $x$  is a small fundamental vector, the coefficients  $\xi_i$  give a valuation  $\xi: S \rightarrow \mathbb{N} - \{0\}$  of the vertices of  $S$ . We call such a valuation a *small fundamental valuation*.

THEOREM 5.5.2: *The hyperbolic sublattices of the singularities of type II, III and IV are exactly those given by the graphs of the following type, which possess a small fundamental valuation:*

- $D = 1: T_{2,3,r},$
- $D = 2: T_{2,q,r}, \quad q, r \geq 4,$
- $D = 3: T_{p,q,r}, \quad p, q, r \geq 3,$
- $D = 4: \Pi_{p,q,r,s},$
- $D = 5: \Omega_{p,q,r,s,t}.$



The graphs together with a small fundamental valuation corresponding to a small fundamental vector of minimal length are listed in Table 6.

All the singularities of type  $I_n^*$ ,  $IV^*$ ,  $III^*$ ,  $II^*$  also have a diagram with a small fundamental valuation; in each case one of the listed diagrams in Table 4 satisfies this condition.

## 6. Tables

### *Notations and remarks*

The numbers in parentheses refer to the section, where the notations are defined.

*Table 1: Normal degrees and sequences  $k^*$ ,  $k$*

This table defines the scheme of all the tables. The upper row of an entry of this table corresponds to an alternating sequence  $k^*$ ,  $k$ ,

$$k_1^*, k_1, k_2^*, k_2, \dots, k_g$$

(section 2.1), the lower row to the corresponding sequence

$$b = \underbrace{2, \dots, 2}_{k_1^* - 1}, k_1 + 2, \underbrace{2, \dots, 2}_{k_2^* - 1}, k_2 + 2, \dots, k_g + 2.$$

On the left-hand side we have listed the 92 cases of such sequences up to cyclic order with  $D$ ,  $D^* \leq 5$  (section 2.1). We have made up a subtable for each value of  $g$ ,  $1 \leq g \leq 5$ . For a fixed value of  $g$ , we have written in a row all alternating sequences  $k^*$ ,  $k$  with the same  $k$ , in a column all those with the same  $k^*$ . The order among the rows and columns is given by the order of the sequences  $k$  resp.  $k^*$  as listed in section 2.1. If the cyclic order of the alternating sequence  $k^*$ ,  $k$  is not unique, we divide the corresponding entry into two parts separated by a dotted line.

Each column is labelled by the type  $I_n$  to which the sequence corresponds. The correspondences  $\beta$  and  $\gamma_n$  (section 2.3) are also indicated. Each row is labelled by  $D$ , which is the grade of the singularity (section 1.3). The duality  $\delta$  (section 2.1) is now given by reflection in the principal diagonals starting in the upper left corners.

Now we reflect the tables at the middle columns, interchanging the columns for  $I_4 \leftrightarrow I_n^*$ , if there are two of them. This gives the complete table. To the right-hand side we associate the singularities of type  $I_1^*$ ,  $IV^*$ ,  $III^*$ , and  $II^*$  according to the correspondences  $\gamma_1$  and  $\epsilon \circ \beta^{-1}$ . The corresponding columns are labelled by these types.

TABLE 1: Normal degrees and sequences  $k^*$ ,  $k$  :  $g = 1$ .

	$I_1 \leftrightarrow II$	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_4^*$	$I_5$	$I_1^\dagger$	$IV^*$	$III^*$	$II^*$
$D = 1$	1,1	2,1	3,1	4,1	5,1	22223	3,1	2,1	1,1
	3	23	223	223	2223	4,1	2223	2,1	23
$D = 2$	1,2	2,2	3,2	4,2	5,2	22224	3,2	2,2	1,2
	4	24	224	224	2224	4,2	2224	2,2	24
$D = 3$	1,3	2,3	3,3	4,3	5,3	22225	3,3	2,3	1,3
	5	25	225	225	2225	4,3	2225	2,3	25
$D = 4$	1,4	2,4	3,4	4,4	5,4	22226	3,4	2,4	1,4
	6	26	226	226	2226	4,4	2226	2,4	26
$D = 5$	1,5	2,5	3,5	4,5	5,5	22227	3,5	2,5	1,5
	7	27	227	227	2227	4,5	2227	2,5	27
								2,5	27
								2,5	27

TABLE 1: Normal degrees and sequences  $k^*$ ,  $k$ :  $g = 2$ .

	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$					$I_1^*$					$IV^*$					$III^*$				
$D = 2$	1,1,1,1 33	2,1,1,1 233	3,1,1,1 2233	2,1,2,1 2323	4,1,1,1 22233	3,1,2,1 22323	3,1,1,1 22333	2,1,2,1 2233	3,1,1,1 2233	2,1,2,1 2323	2,1,2,1 2323	2,1,1,1 233	2,1,2,1 2323	2,1,1,1 233	1,1,1,1 33								
$D = 3$	1,1,1,2 34	2,1,1,2 234	3,1,1,2 2234	2,1,2,2 2324	4,1,1,2 22234	3,1,2,2 22324	3,1,1,2 2234	2,1,2,2 2324	2,1,2,2 2324	2,1,2,2 2324	2,1,2,2 2324	2,1,1,2 234	2,1,2,2 2324	2,1,1,2 234	1,1,1,2 34								
$D = 4$	1,1,1,3 35	2,1,1,3 235	3,1,1,3 2235	2,1,2,3 2325	4,1,1,3 22235	3,1,2,3 22325	3,1,1,3 2235	2,1,2,3 2325	2,1,2,3 2325	2,1,2,3 2325	2,1,2,3 2325	2,1,1,3 235	2,1,2,3 2325	2,1,1,3 235	1,1,1,3 35								
$D = 5$	1,2,1,2 44	2,2,1,2 244	3,2,1,2 2244	2,2,2,2 2424	4,2,1,2 22244	3,2,2,2 22424	3,2,1,2 2244	2,2,2,2 2424	2,2,2,2 2424	2,2,2,2 2424	2,2,2,2 2424	2,2,1,2 244	2,2,2,2 2424	2,2,1,2 244	1,2,1,2 44								
	1,1,1,4 36	2,1,1,4 236	3,1,1,4 2236	2,1,2,4 2326	4,1,1,4 22236	3,1,2,4 22326	3,1,1,4 2236	2,1,2,4 2326	2,1,2,4 2326	2,1,2,4 2326	2,1,2,4 2326	2,1,1,4 236	2,1,2,4 2326	2,1,1,4 236	1,1,1,4 36								
	1,2,1,3 45	2,2,1,3 245	3,2,1,3 2245	2,2,2,3 2425	4,2,1,3 22245	3,2,2,3 22425	3,2,1,3 2245	2,2,2,3 2425	2,2,2,3 2425	2,2,2,3 2425	2,2,2,3 2425	2,2,1,3 245	2,2,2,3 2425	2,2,1,3 245	1,2,1,3 45								

TABLE 1: Normal degrees and sequences  $k^*$ ,  $k$ :  $g = 3$ .

	$I_3 \leftrightarrow IV$		$I_4 \leftrightarrow I_n^*$		$I_5$		$I_1^*$		$IV^*$	
$D = 3$	1,1,1,1,1,1 333	2,1,1,1,1,1	2,1,1,1,1,1	2,1,1,1,1,1	2,1,2,1,1,1	2,1,2,1,1,1	2,1,1,1,1,1	2,1,1,1,1,1	2,1,1,1,1,1	2,1,1,1,1,1 333
$D = 4$	1,1,1,1,1,2 334	2,1,1,1,1,2 2334	2,1,1,2,1,1 2334	3,1,1,1,1,2 22334	2,1,2,1,1,1 23234	2,1,2,1,1,1 23234	2,1,1,1,1,1 22334	2,1,1,1,1,2 2334	2,1,1,1,1,2 2334	2,1,1,1,1,2 2343
$D = 5$	1,1,1,1,1,3 335	2,1,1,1,1,3 2335	2,1,1,3,1,1 2353	3,1,1,1,1,3 22335	2,1,2,1,1,3 23235	2,1,2,1,1,3 23235	3,1,1,3,1,1 22353	2,1,1,1,1,3 2335	2,1,1,3,1,1 2353	1,1,1,1,1,3 335
	1,1,1,2,1,2 344	2,1,1,2,1,2 2344	2,2,1,1,1,2 2434	3,1,1,2,1,2 22344	2,1,2,2,1,2 23244	2,1,2,2,1,2 23244	3,2,1,1,1,2 22434	2,1,1,1,1,2 2344	2,2,1,1,1,2 2434	1,1,1,2,1,2 344

TABLE 1: Normal degrees and sequences  $k^*$ ,  $k$ :  $g = 4$

	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$
$D = 4$	1,1,1,1,1,1,1,1 3333	2,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1 3333
$D = 5$	1,1,1,1,1,1,1,2 3334	2,1,1,1,1,1,1,2 23334	1,1,1,1,1,1,1,2 23343 3334

TABLE 1: Normal degrees and sequences  $k^*$ ,  $k$ :  $g = 5$ .

	$I_5$
$D = 5$	1,1,1,1,1,1,1,1,1 33333

Table 2: Names, Milnor numbers and discriminants

We have indicated in an entry:

*Upper left corner:* Notation of Arnold–Wall (section 3.1). In the case of the correspondences  $\beta$  and  $\gamma_n$  we refer to the singularities of type II, III, IV and  $I_n^*$ .

*Upper right corner:* Sequence  $\mathbf{p}$  defined for type II, III, IV,  $I_0^*$  in section 1.2, for  $I_n$ ,  $1 \leq n \leq 4$ , in section 2.2, for  $I_5$  in section 3.2.

*Lower left corner:* Discriminant = order of  $TH_1(L)$  (section 4.4).

*Lower right corner:* Milnor number (section 4.2). In the case of the correspondences  $\beta$  and  $\gamma_n$  we make the same convention as for the upper left corner. Compare section 4.2 for the relations between the Milnor numbers under the correspondences.

Table 3: Discriminant quadratic forms

The notations for the discriminant quadratic forms are explained in section 4.4. In order to give a uniform description of the discriminant quadratic forms of the series  $I_n^*$ , it is necessary to use also the following notation:  $(k/m)$ , where  $k, m \in \mathbb{N}$ , denotes the finite quadratic form  $(G, q)$ , where  $G$  is a cyclic group of order  $m$  and  $q$  takes the value  $(k/m) \pmod{2\mathbb{Z}}$  on a generator of  $G$ .

Two cases need an extra explanation. The symbols  $q_1^{(n)}$  resp.  $q_2^{(n)}$  stand for the finite quadratic forms defined on the finite abelian groups  $G_1^{(n)}$  resp.  $G_2^{(n)}$  by the following matrices with respect to a standard set of generators (where the diagonal entries have to be taken mod  $2\mathbb{Z}$ , the other entries mod  $\mathbb{Z}$ ):

$$(G_1^{(n)}, q_1^{(n)}) := \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/(18+2n)\mathbf{Z}, \begin{pmatrix} 1+\frac{3}{4}n & \frac{1}{2} \\ \frac{1}{2} & \frac{10+n}{9+n} \end{pmatrix}, \\ n \equiv 0 \pmod{2}, \\ \mathbf{Z}/(36+4n)\mathbf{Z}, \begin{pmatrix} 1-n(9+n) \\ 4(9+n) \end{pmatrix}, & n \equiv 1 \pmod{2}; \end{cases}$$

$$(G_2^{(n)}, q_2^{(n)}) := \begin{cases} \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/(32+4n)\mathbf{Z}, \begin{pmatrix} \frac{1}{4}(6-n) & \frac{n}{8} \\ \frac{n}{8} & \frac{4-(2+3n)(8+n)}{16(8+n)} \end{pmatrix}, \\ n \equiv 0 \pmod{4}, \\ \mathbf{Z}/(64+8n)\mathbf{Z}, \begin{pmatrix} 1-(8+3n)(8+n) \\ 16(8+n) \end{pmatrix}, & n \equiv 1 \pmod{4}, \\ \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/(16+2n)\mathbf{Z}, \begin{pmatrix} \frac{5}{16}(2+n) & \frac{1}{4} \\ \frac{1}{4} & \frac{9+n}{8+n} \end{pmatrix}, & n \equiv 2 \pmod{4}, \\ \mathbf{Z}/(64+8n)\mathbf{Z}, \begin{pmatrix} 1-(4+3n)(8+n) \\ 16(8+n) \end{pmatrix}, & n \equiv 3 \pmod{4}. \end{cases}$$

TABLE 2: Names, Milnor numbers and discriminants:  $g = 1$ .

	$I_1 \leftrightarrow II$	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$	$IV^*$	$III^*$	$II^*$								
$D = 1$	$E_{12}$ 1	$2,3,7$ 12	$E_{13}$ 2	$2,4,5$ 13	$E_{14}$ 3	$3,3,4$ 14	$E_{3,n}$ 4	$2,2,2,3$ $16+n$	$1,1,1,1,2$ 15	$E_{3,1}$ 4	$17$	$E_{18}$ 3	$18$	$E_{19}$ 2	$19$	$E_{20}$ 1	$20$
$D = 2$	$Z_{11}$ 2	$2,3,8$ 11	$Z_{12}$ 4	$2,4,6$ 12	$Z_{13}$ 6	$3,3,5$ 13	$Z_{1,n}$ 8	$2,2,2,4$ $15+n$	$1,1,1,1,3$ 14	$Z_{1,1}$ 8	$16$	$Z_{17}$ 6	$17$	$Z_{18}$ 4	$18$	$Z_{19}$ 2	$19$
$D = 3$	$Q_{10}$ 3	$2,3,9$ 10	$Q_{11}$ 6	$2,4,7$ 11	$Q_{12}$ 9	$3,3,6$ 12	$Q_{2,n}$ 12	$2,2,2,5$ $14+n$	$1,1,1,1,4$ 13	$Q_{2,1}$ 12	$15$	$Q_{16}$ 9	$16$	$Q_{17}$ 6	$17$	$Q_{18}$ 3	$18$
$D = 4$	$J_6$ 4	$2,3,10$ 9	$J_{10}$ 8	$2,4,8$ 10	$J_{11}$ 12	$3,3,7$ 11	$J_{2,n}$ 16	$2,2,2,6$ $13+n$	$1,1,1,1,5$ 12	$J_{2,1}$ 16	$20$	$J_{15}$ 12	$15$	$J_{16}$ 8	$16$	$J_{17}$ 4	$17$
$D = 5$		$2,3,11$ 8		$2,4,9$ 9		$3,3,8$ 10		$2,2,2,7$ $12+n$	$1,1,1,1,6$ 11		$25$	$13$	$14$	$10$	$15$	$5$	$16$

TABLE 2: Names, Milnor numbers and discriminants:  $g = 2$ .

$D = 2$	$I_2 \leftrightarrow III$		$I_3 \leftrightarrow IV$		$I_4 \leftrightarrow I_n^*$		$I_5$			$I_7^*$			$IV^*$			$III^*$		
	$W_{12}$ 5	$2,5,5$ 12	$W_{13}$ 8	$3,4,4$ 13	$W_{1,n}^{\#}$ $12+n$	$2,2,3,3$ $15+n$	$W_{1,n}$ 12	$2,3,2,3$ $15+n$	14	$1,1,1,2,2$ 14	16	$1,1,2,1,2$ 14	$W_{1,1}^{\#}$ 13	$W_{1,1}$ 12	16	$W_{1,1}$ 16	17	$W_{18}$ 7
$D = 3$	$S_{11}$ 8	$2,5,6$ 11	$S_{12}$ 13	$3,4,5$ 12	$S_{1,n}^{\#}$ $20+2n$	$2,2,3,4$ $14+n$	$S_{1,n}$ 20	$2,3,2,4$ $14+n$	23	$1,1,1,2,3$ 13	27	$S_{1,1}^{\#}$ 27	$S_{1,1}$ 22	15	$S_{16}$ 17	16	$S_{17}$ 12	17
	$L_{10}$ 11	$2,5,7$ 10	$L_{11}$ 18	$3,4,6$ 11	$L_{1,n}^{\#}$ $28+3n$	$2,2,3,5$ $13+n$	$L_{1,n}$ 28	$2,3,2,5$ $13+n$	32	$1,1,1,2,4$ 12	38	$L_{1,1}^{\#}$ 31	$L_{1,1}$ 28	14	$L_{15}$ 24	15	$L_{16}$ 17	16
$D = 5$	$K_{10}^*$ 12	$2,6,6$ 10	$K_{11}^*$ 20	$3,5,5$ 11	$K_{1,n}^{\#}$ $32+4n$	$2,2,4,4$ $13+n$	$K_{1,n}^*$ 32	$2,4,2,4$ $13+n$	36	$1,1,1,3,3$ 12	44	$K_{1,1}^{\#}$ 36	$K_{1,1}^*$ 32	14	$K_{15}^*$ 28	15	$K_{16}^*$ 20	16
	14	$2,5,8$ 9	23	$3,4,7$ 10	$36+4n$ $12+n$	$2,2,3,6$ $12+n$	36	$2,3,2,6$ $12+n$	41	$1,1,1,2,5$ 11	49	40	36	13	31	14	22	15
	16	$2,6,7$ 9	27	$3,5,6$ 10	$44+6n$ $12+n$	$2,2,4,5$ $12+n$	44	$2,4,2,5$ $12+n$	49	$1,1,1,3,4$ 11	61	50	44	13	39	14	28	15



TABLE 2: Names, Milnor numbers and discriminants:  $g = 3$ .

$D = 3$	$I_3 \leftrightarrow IV$		$I_4 \leftrightarrow I_n^*$		$I_5$		$I_6^*$		$IV^*$	
	$U_{12}$	$4,4,4$	$U_{1,n}$	$2,3,3,3$	$1,1,2,2,2$	$1,2,1,2,2$	$U_{1,1}$	$U_{16}$	$U_{12}$	$U_{16}$
	16	12	$27 + 3n$	$14 + n$	13	13	35	15	15	16
$D = 4$	$M_{11}$	$4,4,5$	$M_{1,n}^\#$	$2,3,3,4$	$1,1,2,3,2$	$1,2,1,2,3$	$M_{1,1}^\#$	$M_{15}$	$M_{1,1}$	$M_{15}$
	24	11	$42 + 5n$	$13 + n$	12	12	54	14	14	15
$D = 5$	$4,4,6$	$2,3,3,5$	$M_{1,n}$	$2,3,5,3$	$1,1,2,4,2$	$1,2,1,2,4$	$M_{1,1}$	$M_{1,1}$	$M_{1,1}$	$M_{1,1}$
	32	10	$57 + 7n$	$12 + n$	11	11	73	13	13	14
	$4,5,5$	$2,3,4,4$	$M_{1,n}$	$2,4,3,4$	$1,1,3,2,3$	$1,2,1,3,3$	$M_{1,1}$	$M_{1,1}$	$M_{1,1}$	$M_{1,1}$
	35	10	$64 + 10n$	$12 + n$	11	11	81	13	13	14

TABLE 2: Names, Milnor numbers and discriminants:  $g = 4$ .

$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$
$D = 4$	$3,3,3,3$ $54 + 9n$	$I_{1,1}$ 63
	$13 + n$	12
$D = 5$	$3,3,3,4$ $81 + 15n$	$1,2,2,2,2$ 96
	$12 + n$	11
		13

TABLE 2: Names, Milnor numbers and discriminants:  $g = 5$ .

$I_5$
$D = 5$
121
$2,2,2,2,2$ 11

TABLE 3: Discriminant quadratic forms:  $g = 1$ .

	$I_1 \leftrightarrow II$	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$	$IV^*$	$III^*$	$II^*$
$D = 1$	0	$w_{2,1}^{-1}$	$w_{3,1}^1$		$w_{5,1}^1$	$w_{2,2}^{-5}$	$w_{3,1}^{-1}$	$w_{2,1}^1$	0
$D = 2$	$w_{2,1}^1$	$w_{2,1}^1 + w_{2,1}^{-1}$	$w_{2,1}^1 + w_{3,1}^1$	$q_{D_{4+n}}$	$w_{2,1}^1 + w_{5,1}^1$	$w_{2,2}^{-5} + w_{2,1}^1$	$w_{2,1}^1 + w_{3,1}^{-1}$	$w_{2,1}^1 + w_{2,1}^1$	$w_{2,1}^1$
$D = 3$	$w_{3,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^{-1}$	$w_{3,1}^1 + w_{3,1}^{-1}$	$q_{D_{4+n}} + w_{3,1}^1$	$w_{3,1}^{-1} + w_{5,1}^1$	$w_{2,2}^{-5} + w_{3,1}^1$	$w_{3,1}^1 + w_{3,1}^{-1}$	$w_{2,1}^1 + w_{3,1}^1$	$w_{3,1}^{-1}$
$D = 4$	$w_{2,2}^{-5}$	$w_{2,2}^{-5} + w_{2,1}^{-1}$	$w_{2,2}^{-5} + w_{3,1}^1$	$q_{D_{4+n}} + w_{2,2}^{-5}$	$w_{2,2}^{-5} + w_{5,1}^1$	$w_{2,2}^{-5} + w_{2,2}^5$	$w_{2,2}^{-5} + w_{3,1}^{-1}$	$w_{2,2}^{-5} + w_{2,1}^1$	$w_{2,2}^{-5}$
$D = 5$	$w_{5,1}^1$	$w_{2,1}^{-1} + w_{5,1}^1$	$w_{3,1}^1 + w_{5,1}^1$	$q_{D_{4+n}} + w_{5,1}^1$	$w_{5,1}^1 + w_{5,1}^1$	$w_{2,2}^{-5} + w_{5,1}^1$	$w_{3,1}^{-1} + w_{5,1}^1$	$w_{2,1}^1 + w_{5,1}^1$	$w_{5,1}^1$

TABLE 3: Discriminant quadratic forms:  $g = 2$ .

	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$	$IV^*$	$III^*$
$D = 2$	$w_{5,1}^{-1}$	$w_{2,3}^{-5}$	$q_{A_{11+n}}$	$q_{D_{1+n}} + w_{3,1}^1$	$w_{2,1}^{-1} + w_{7,1}^{-1}$	$w_{2,1}^{-1} + w_{5,1}^{-1}$	$w_{7,1}^1$
$D = 3$	$w_{2,3}^5$	$w_{13,1}^{-1}$	$q_{A_{9+n}} + w_{2,1}^{-1}$	$w_{23,1}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^1$	$w_{2,2}^5 + w_{5,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^1$
$D = 4$	$w_{11,1}^{-1}$	$w_{2,1}^1 + w_{3,2}^{-1}$	$\left( \frac{31+3n}{28+3n} \right) q_{A_{7+n}} + w_{7,1}^{-1}$	$w_{2,5}^{-1}$	$w_{31,1}^{-1}$	$w_{2,3}^{-5} + w_{3,1}^1$	$w_{17,1}^{-1}$
	$u_1 + w_{3,1}^{-1}$	$w_{2,2}^1 + w_{5,1}^1$	$q_{A_{7+n}} + w_{2,1}^{-1} + w_{2,1}^{-1}$	$w_{2,2}^{-1} + w_{3,2}^{-1}$	$w_{2,1}^{-1} + w_{2,1}^{-1} + w_{3,2}^1$	$w_{2,2}^5 + w_{2,3}^5$	$u_1 + w_{5,1}^{-1}$
$D = 5$	$w_{2,1}^1 + w_{7,1}^1$	$w_{23,1}^1$	$q_1^{(n)}$	$w_{41,1}^1$	$w_{2,3}^{-5} + w_{5,1}^{-1}$	$w_{2,2}^{-1} + w_{3,2}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^1$
	$w_{2,4}^5$	$w_{3,3}^1$	$w_{2,1}^1 + \left( \frac{25+3n}{22+3n} \right) q_{D_{2+n}} + w_{11,1}^{-1}$	$w_{7,2}^1$	$w_{2,1}^{-1} + w_{5,2}^{-1}$	$w_{3,1}^1 + w_{13,1}^{-1}$	$w_{2,2}^{-1} + w_{7,1}^{-1}$

TABLE 3: Discriminant quadratic forms:  $g = 3$ .

	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_6^*$	$IV^*$	
$D = 3$	$v_2$	$q_{4_{8+n}} + w_{3,1}^1$	$w_{2,1}^{-1} + w_{2,4}^{-1}$	$w_{5,1}^{-1} + w_{7,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^1 + w_{5,1}^{-1}$	$w_{5,1}^1 + w_{5,1}^{-1}$
$D = 4$	$w_{2,3}^{-1} + w_{3,1}^{-1}$	$\begin{pmatrix} 47+5n \\ 42+5n \end{pmatrix} \cdot q_{4_{6+n}} + w_{2,1}^{-1} + w_{3,1}^1$	$w_{2,1}^{-1} + w_{5,2}^1 \cdot w_{2,4}^5 + w_{3,1}^{-1}$	$w_{2,1}^1 + w_{3,3}^1 \cdot w_{2,3}^5 + w_{7,1}^{-1}$	$w_{47,1}^{-1} \cdot w_{2,1}^{-1} + w_{2,3}^1 + w_{3,1}^1$	$w_{2,3}^5 + w_{5,1}^1$
$D = 5$	$w_{2,1}^1 + w_{2,4}^1$	$w_{3,1}^1 + \begin{pmatrix} 64+7n \\ 57+7n \end{pmatrix} \cdot w_{3,1}^1 + \begin{pmatrix} 22+3n \\ 19+3n \end{pmatrix}$	$w_{2,1}^1 + w_{2,1}^1 \cdot u_3 + w_{17,1}^1$	$w_{73,1}^1 \cdot w_{7,1}^{-1} + w_{11,1}^{-1}$	$w_{2,6}^{-1} \cdot w_{2,1}^{-1} + w_{3,1}^1 + w_{11,1}^{-1}$	$w_{5,1}^1 + w_{11,1}^{-1}$
	$w_{5,1}^{-1} + w_{7,1}^{-1}$	$w_{2,1}^{-1} \cdot w_{2,1}^{(n)} + \begin{pmatrix} 37+5n \\ 32+5n \end{pmatrix}$	$w_{73,1}^1 \cdot w_{7,1}^1 + w_{11,1}^1$	$w_{3,2}^1 + w_{3,2}^{-1} \cdot w_{5,1}^{-1} + w_{17,1}^1$	$w_{2,1}^{-1} + w_{37,1}^{-1} \cdot w_{2,3}^{-1} + w_{3,2}^1$	$w_{3,2}^1 + w_{7,1}^{-1}$

TABLE 3: Discriminant quadratic forms:  $g = 4$

	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$
$D = 4$	$q_{A_{3+n}} + w_{3,1}^1 + w_{3,1}^1$	$w_{2,1}^{-1} + w_{3,1}^{-1} + w_{1,1,1}^1$	$w_{3,1}^1 + w_{3,1}^1 + w_{7,1}^1$
$D = 5$	$w_{3,1}^1 + \left( \frac{32+5n}{27+5n} \right)$	$w_{2,1}^1 + w_{2,1}^{-1} + w_{5,2}^1$	$w_{3,1}^1 + w_{2,5}^5$

TABLE 3: Discriminant quadratic forms:  $g = 5$ .

	$I_5$
$D = 5$	$w_{1,1,1}^1 + w_{1,1,1}^{-1}$

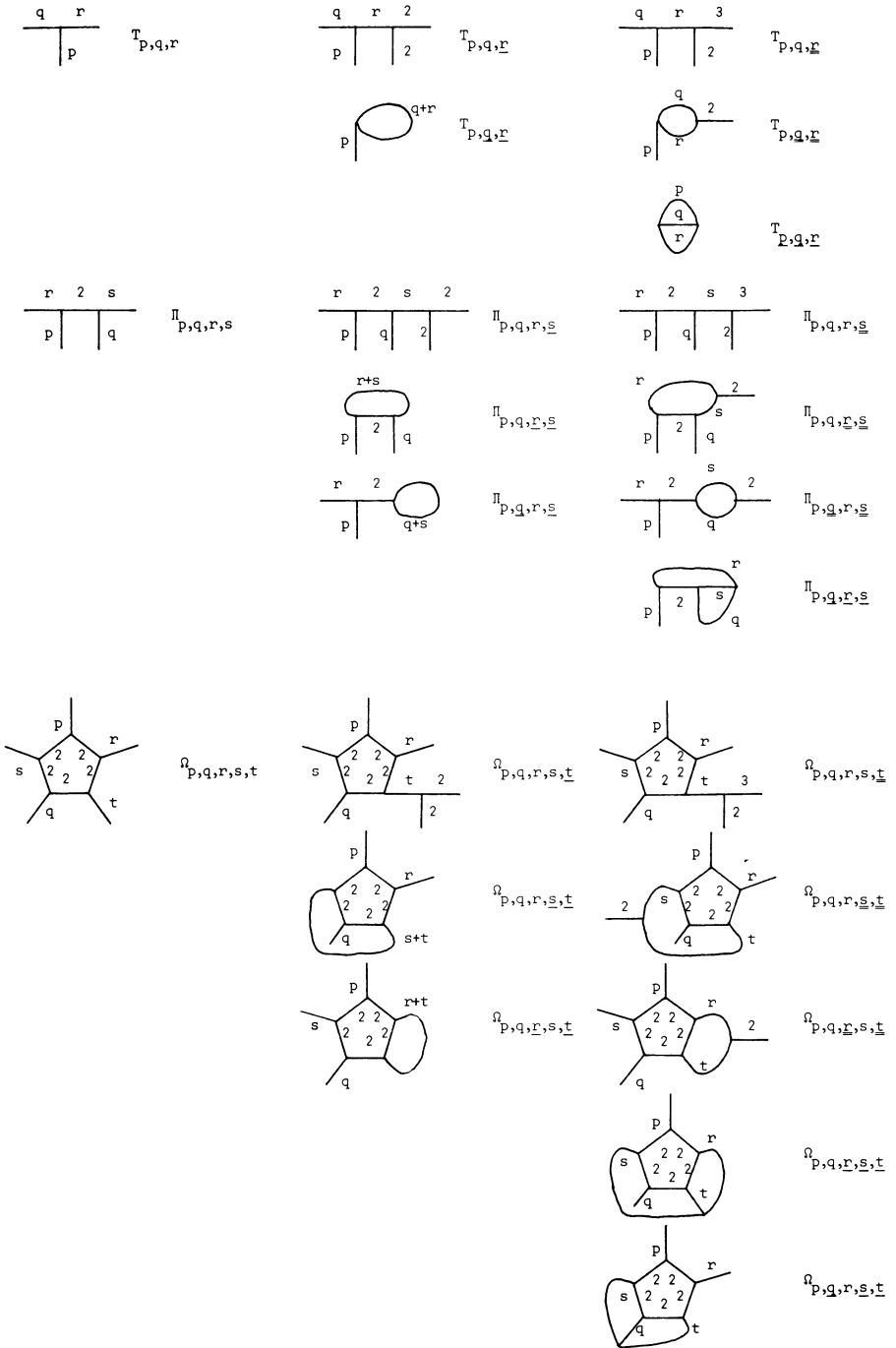


FIGURE 2.

TABLE 4: Dynkin diagrams:  $g = 1$ .

	$I_1 \leftrightarrow II$	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$	$IV^*$	$III^*$	$II^*$
$D = 1$	$T_{2,3,7}$	$T_{2,3,8}$	$T_{2,3,9}$	$T_{2,3,10}$ $T_{2,3,9+n}$	$T_{2,3,11}$ $T_{2,3,8}$	$T_{2,3,9}$	$T_{2,3,10}$	$T_{2,3,11}$	$T_{2,3,12}$
$D = 2$	$T_{2,4,5}$	$T_{2,4,6}$	$T_{2,4,7}$	$T_{2,4,8}$ $T_{2,4,7+n}$	$T_{2,4,9}$ $T_{2,4,6}$	$T_{2,4,7}$	$T_{2,4,8}$	$T_{2,4,9}$	$T_{2,4,10}$
$D = 3$	$T_{3,3,4}$	$T_{3,3,5}$	$T_{3,3,6}$	$T_{3,3,7}$ $T_{3,3,6+n}$	$T_{3,3,8}$ $T_{3,3,5}$	$T_{3,3,6}$	$T_{3,3,7}$	$T_{3,3,8}$	$T_{3,3,9}$
$D = 4$	$\Pi_{2,2,2,3}$	$\Pi_{2,2,2,4}$	$\Pi_{2,2,2,5}$	$\Pi_{2,2,2,6}$ $\Pi_{2,2,2,5+n}$	$\Pi_{2,2,2,7}$ $\Pi_{2,2,2,4}$	$\Pi_{2,2,2,5}$	$\Pi_{2,2,2,6}$	$\Pi_{2,2,2,7}$	$\Pi_{2,2,2,8}$
$D = 5$	$\mathfrak{G}_{1,1,1,1,2}$	$\mathfrak{G}_{1,1,1,1,3}$	$\mathfrak{G}_{1,1,1,1,4}$	$\mathfrak{G}_{1,1,1,1,5}$ $\mathfrak{G}_{1,1,1,1,4+n}$	$\mathfrak{G}_{1,1,1,1,6}$ $\mathfrak{G}_{1,1,1,1,3}$	$\mathfrak{G}_{1,1,1,1,4}$	$\mathfrak{G}_{1,1,1,1,5}$	$\mathfrak{G}_{1,1,1,1,6}$	$\mathfrak{G}_{1,1,1,1,7}$



TABLE 4: Dynkin diagrams:  $g = 2$ .

	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_4^*$	$I_5$	$I_1^*$	$I_4^*$	$III^*$		
$D = 2$	$T_{2,5,5}$	$T_{2,5,6}$	$T_{2,5,7}$ $T_{2,6,7+n}$	$T_{2,6,6}$ $T_{2,5,6+n}$	$T_{2,5,8}$ $T_{2,6,7}$	$T_{2,6,7}$ $T_{2,5,5}$ $T_{2,5,8}$	$T_{2,5,6}$ $T_{2,6,8}$	$T_{2,5,7}$ $T_{2,7,8}$	$T_{2,5,8}$ $T_{2,8,8}$
$D = 3$	$T_{3,4,4}$	$T_{3,4,5}$	$T_{3,4,6}$ $T_{3,5,6+n}$	$T_{3,5,5}$ $T_{3,4,5+n}$	$T_{3,4,7}$ $T_{3,5,6}$	$T_{3,5,6}$ $T_{3,4,4}$ $T_{3,4,7}$	$T_{3,4,5}$ $T_{3,5,7}$	$T_{3,4,6}$ $T_{3,6,7}$	$T_{3,4,7}$ $T_{3,7,7}$
$D = 4$	$\Pi_{2,2,3,3}$	$\Pi_{2,2,3,4}$	$\Pi_{2,2,3,5}$ $\Pi_{2,2,4,5+n}$	$\Pi_{2,2,3,6}$ $\Pi_{2,2,4,5}$	$\Pi_{2,2,4,5}$ $\Pi_{2,2,3,3}$ $\Pi_{2,2,3,6}$	$\Pi_{2,2,3,4}$ $\Pi_{2,2,4,6}$	$\Pi_{2,2,3,4}$ $\Pi_{2,2,4,6}$	$\Pi_{2,2,3,5}$ $\Pi_{2,2,5,6}$	$\Pi_{2,2,3,6}$ $\Pi_{2,2,6,6}$
$D = 5$	$\Pi_{2,3,2,3}$	$\Pi_{2,3,2,4}$	$\Pi_{2,3,2,5}$ $\Pi_{2,4,2,5+n}$	$\Pi_{2,4,2,4}$ $\Pi_{2,3,2,4+n}$	$\Pi_{2,4,2,5}$ $\Pi_{2,4,2,5}$ $\Pi_{2,3,2,6}$	$\Pi_{2,4,2,5}$ $\Pi_{2,3,2,3}$ $\Pi_{2,3,2,6}$	$\Pi_{2,3,2,4}$ $\Pi_{2,4,2,6}$	$\Pi_{2,3,2,5}$ $\Pi_{2,5,2,6}$	$\Pi_{2,3,2,6}$ $\Pi_{2,6,2,6}$
$D = 5$	$\Omega_{1,1,1,2,2}$	$\Omega_{1,1,1,2,3}$	$\Omega_{1,1,1,2,4}$ $\Omega_{1,1,1,3,4+n}$	$\Omega_{1,1,1,3,3}$ $\Omega_{1,1,1,2,3+n}$	$\Omega_{1,1,1,2,5}$ $\Omega_{1,1,1,3,4}$	$\Omega_{1,1,1,3,4}$ $\Omega_{1,1,1,2,2}$ $\Omega_{1,1,1,2,5}$	$\Omega_{1,1,1,2,3}$ $\Omega_{1,1,1,3,5}$	$\Omega_{1,1,1,2,4}$ $\Omega_{1,1,1,4,5}$	$\Omega_{1,1,1,2,5}$ $\Omega_{1,1,1,5,5}$
$D = 5$	$\Omega_{1,1,2,1,2}$	$\Omega_{1,1,2,1,3}$	$\Omega_{1,1,2,1,4}$ $\Omega_{1,1,3,1,4+n}$	$\Omega_{1,1,3,1,3}$ $\Omega_{1,1,2,1,3+n}$	$\Omega_{1,1,2,1,5}$ $\Omega_{1,1,3,1,4}$	$\Omega_{1,1,3,1,4}$ $\Omega_{1,1,2,1,2}$ $\Omega_{1,1,2,1,5}$	$\Omega_{1,1,2,1,3}$ $\Omega_{1,1,3,1,5}$	$\Omega_{1,1,2,1,4}$ $\Omega_{1,1,4,1,5}$	$\Omega_{1,1,2,1,5}$ $\Omega_{1,1,5,1,5}$

TABLE 4: Dynkin diagrams:  $g = 3$ .

	$I_3 \leftrightarrow IV$	$I_4 \leftrightarrow I_n^*$	$I_5$	$I_1^*$	$IV^*$
$D = 3$	$T_{4,4,4}$	$T_{4,4,5}$ $T_{4,5,5+n}$	$T_{4,4,6}$	$T_{4,5,5}$ $T_{4,6,6}$	$T_{4,5,6}$ $T_{6,6,6}$
	$\Pi_{2,3,3,3}$	$T_{4,5,5}$ $\Pi_{2,3,4,3}$ $\Pi_{2,3,4,4+n}$	$T_{4,4,6}$ $T_{5,5,6}$ $\Pi_{2,3,3,5}$ $\Pi_{2,3,4,4}$ $\Pi_{2,4,5,4}$	$T_{4,5,5}$ $T_{4,6,6}$ $\Pi_{2,3,4,4}$ $\Pi_{2,3,3,5}$ $\Pi_{2,4,3,5}$ $\Pi_{2,5,3,5}$	$T_{4,5,6}$ $T_{6,6,6}$ $\Pi_{2,3,5,4}$ $\Pi_{2,5,4,5}$
$D = 4$	$\Pi_{2,3,3,3}$	$T_{4,5,5}$ $\Pi_{2,3,4,3}$ $\Pi_{2,3,4,4+n}$	$T_{4,4,6}$ $T_{5,5,6}$ $\Pi_{2,3,3,5}$ $\Pi_{2,3,4,4}$ $\Pi_{2,4,5,4}$	$T_{4,5,5}$ $T_{4,6,6}$ $\Pi_{2,3,4,4}$ $\Pi_{2,3,3,5}$ $\Pi_{2,4,3,5}$ $\Pi_{2,5,3,5}$	$T_{4,5,6}$ $T_{6,6,6}$ $\Pi_{2,3,5,4}$ $\Pi_{2,5,4,5}$
	$\Omega_{1,1,2,2,2}$	$\Omega_{1,1,2,3}$ $\Omega_{1,1,2,3,3+n}$	$\Omega_{1,1,2,2,4}$ $\Omega_{1,1,2,3,3}$ $\Omega_{1,1,3,3,4}$	$\Omega_{1,1,2,3,3}$ $\Omega_{1,1,2,2,4}$ $\Omega_{1,1,2,2,4}$ $\Omega_{1,1,2,4,4}$	$\Omega_{1,1,2,4,4}$ $\Omega_{1,1,2,4,4}$ $\Omega_{1,1,4,4,4}$
$D = 5$	$\Omega_{1,2,1,2,2}$	$\Omega_{1,2,1,2,3}$ $\Omega_{1,2,3,1,3+n}$	$\Omega_{1,2,1,2,4}$ $\Omega_{1,2,3,1,3}$ $\Omega_{1,3,1,3,4}$	$\Omega_{1,2,1,3,3}$ $\Omega_{1,2,2,1,4}$ $\Omega_{1,2,1,4,4}$	$\Omega_{1,2,3,1,4}$ $\Omega_{1,3,1,4,4}$ $\Omega_{1,4,1,4,4}$
	$\Omega_{1,2,1,2,2}$	$\Omega_{1,2,1,2,3}$ $\Omega_{1,2,3,1,3+n}$	$\Omega_{1,2,1,2,4}$ $\Omega_{1,2,3,1,3}$ $\Omega_{1,3,1,3,4}$	$\Omega_{1,2,1,3,3}$ $\Omega_{1,2,2,1,4}$ $\Omega_{1,2,1,4,4}$	$\Omega_{1,2,3,1,4}$ $\Omega_{1,3,1,4,4}$ $\Omega_{1,4,1,4,4}$

TABLE 4: Dynkin diagrams:  $g = 4$ .

	$I_4 \leftrightarrow I_4^*$	$I_5$	$I_4^*$
$D = 4$	$\Pi_{3,3,3,3}$ $\Pi_{3,3,3,4+n}$	$\Pi_{3,3,3,4}$ $\Pi_{3,3,3,4}$ $\Pi_{3,4,4,4}$	$\Pi_{3,4,3,4}$ $\Pi_{3,4,4,5}$
$D = 5$	$\Omega_{1,2,2,2,2}$ $\Omega_{1,2,2,2,3+n}$	$\Omega_{1,2,2,2,3}$ $\Omega_{1,2,3,2,2}$ $\Omega_{1,2,3,3,3}$ $\Omega_{1,3,2,2,4}$	$\Omega_{1,2,2,3,2}$ $\Omega_{1,2,2,2,3}$ $\Omega_{1,2,2,4,3}$ $\Omega_{1,3,2,3,3}$

TABLE 4: Dynkin diagrams:  $g = 5$ .

	$I_5$
$D = 5$	$\Omega_{2,2,2,2,2}$ $\Omega_{2,2,2,3,3}$

TABLE 5: Extension of the Tables 2–4 for  $g = 2$ . (a) Table 2 extended.

$D$	$I_5$			$I_1^*$			$I_2^*$			$IV^*$			$III^*$			$E_{20}$
	14	14	14	16	14	14	$W_{1,1}^\#$	16	$W_{1,2}$	17	17	$W_{17}$	18	$E_{3,3}$	19	
$D = 2$	14	14	14	16	14	14	$W_{1,1}^\#$	16	$W_{1,2}$	17	17	$W_{17}$	18	$E_{3,3}$	19	20
$D = 3$	23	13	13	27	13	13	$S_{1,1}^\#$	15	$S_{1,2}$	16	16	$S_{16}$	17	$W_{18}$	18	$Z_{19}$
							$L_{1,1}^\#$	31	$L_{1,2}$	28	15	$L_{15}$	14	$L_{16}$	17	$W_{17}$
$D = 4$	36	12	12	44	12	12	$K_{1,1}^\#$	14	$K_{1,2}$	32	15	$K_{15}$	16	$W_{1,2}$	17	$Z_{18}$
							$L_{1,1}^\#$	31	$L_{1,2}$	28	15	$L_{15}$	14	$L_{16}$	17	$W_{17}$
$D = 5$	41	11	11	49	11	11	$K_{1,1}^\#$	40	$K_{1,2}$	36	14	31	22	$W_{1,2}$	13	$J_{17}$
							$L_{1,1}^\#$	40	$L_{1,2}$	36	14	31	22	$L_{16}$	17	$W_{17}$
$D = 5$	49	11	11	61	11	11	$K_{1,1}^\#$	50	$K_{1,2}$	44	14	39	28	$L_{16}$	17	$Q_{17}$
							$L_{1,1}^\#$	50	$L_{1,2}$	44	14	39	28	$L_{16}$	17	$W_{17}$

TABLE 5: Extension of the Tables 2-4 for  $g = 2$ . (b) Table 3 extended.

$D$	$I_5$					$I_5^*$	$I_5^\dagger$	$I_5^\ddagger$	$IV^*$	$III^*$	0
	$w_{2,1}^{-1} + w_{7,1}^{-1}$	$w_{2,4}^5$	$w_{1,3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$						
$D = 2$	$w_{2,1}^{-1} + w_{7,1}^{-1}$	$w_{2,4}^5$	$w_{1,3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{7,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^{-1}$	$w_{7,1}^{-1}$	$w_{2,2}^5$
$D = 3$	$w_{2,3,1}^{-1}$	$w_{3,3}^1$	$w_{2,1}^{-1} + w_{11,1}^{-1}$	$w_{2,1}^{-1} + w_{5,1}^{-1}$	$w_{2,1}^{-1} + w_{5,1}^{-1}$	$w_{17,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{7,1}^{-1}$
$D = 4$	$w_{2,5}^{-1}$	$w_{2,1}^{-1} + w_{19,1}^{-1}$	$w_{3,1}^{-1}$	$w_{2,2}^5 + w_{7,1}^{-1}$	$w_{2,2}^5 + w_{7,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{17,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^{-1}$
	$w_{2,2}^{-1} + w_{3,2}^{-1}$	$w_{2,2}^1 + w_{11,1}^{-1}$	$w_{2,1}^{-1} + w_{2,1}^{-1} + w_{3,2}^{-1}$	$w_{2,1}^{-1} + w_{2,3}^5$	$w_{2,1}^{-1} + w_{7,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$	$w_{2,2}^5 + w_{3,1}^{-1}$
$D = 5$	$w_{4,1}^1$	$w_{7,2}^{-1}$	$w_{2,3}^{-5} + w_{5,1}^{-1}$	$w_{2,1}^{-1} + w_{2,1}^{-1} + w_{3,2}^{-1}$	$w_{2,1}^{-1} + w_{2,1}^{-1} + w_{3,2}^{-1}$	$w_{31,1}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^{-1}$	$w_{13,1}^{-1}$	$w_{2,2}^{-5}$
	$w_{7,2}^1$	$w_{6,1,1}^{-1}$	$w_{2,1}^{-1} + w_{5,2}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^{-1}$	$w_{2,1}^{-1} + w_{11,1}^{-1}$	$w_{3,1}^{-1} + w_{13,1}^{-1}$	$w_{3,1}^{-1} + w_{7,1}^{-1}$	$w_{2,2}^{-1} + w_{7,1}^{-1}$	$w_{2,2}^{-1} + w_{7,1}^{-1}$	$w_{17,1}^{-1}$	$w_{2,1}^{-1} + w_{3,1}^{-1}$

TABLE 5: Extension of the Tables 2-4 for  $g = 2$ . (c) Table 4 extended.

$I_5$	$I^*$	$I_2^*$	$IV^*$	$III^*$			
$D = 2$	$T_{2,5,8}$	$T_{2,6,7}$		$T_{2,5,7}$	$T_{2,5,9}$	$T_{2,5,8}$ $T_{2,8,8}$	$T_{2,5,10}$
	$T_{2,6,7}$	$T_{2,5,5}$ $T_{2,5,8}$	$T_{2,5,6}$ $T_{2,6,8}$		$T_{2,7,8}$		
	$T_{3,4,7}$ $T_{3,5,6}$	$T_{3,5,6}$ $T_{3,4,4}$ $T_{3,4,7}$	$T_{3,4,5}$ $T_{3,5,7}$		$T_{3,4,6}$ $T_{3,6,7}$	$T_{3,4,7}$ $T_{3,7,7}$	$T_{3,4,9}$
$D = 3$	$\Pi_{2,2,3,6}$ $\Pi_{2,2,4,5}$	$\Pi_{2,2,4,5}$ $\Pi_{2,2,3,3}$ $\Pi_{2,2,3,6}$	$\Pi_{2,2,3,4}$ $\Pi_{2,2,4,6}$		$\Pi_{2,2,3,5}$ $\Pi_{2,2,5,6}$	$\Pi_{2,2,3,6}$ $\Pi_{2,2,6,6}$	$\Pi_{2,2,3,8}$
	$\Pi_{2,3,2,6}$ $\Pi_{2,4,2,5}$	$\Pi_{2,4,2,5}$ $\Pi_{2,3,2,3}$ $\Pi_{2,3,2,6}$	$\Pi_{2,3,2,4}$ $\Pi_{2,4,2,6}$		$\Pi_{2,3,2,5}$ $\Pi_{2,5,2,6}$	$\Pi_{2,3,2,6}$ $\Pi_{2,6,2,6}$	$\Pi_{2,3,2,8}$
	$\Omega_{1,1,1,2,5}$ $\Omega_{1,1,1,3,4}$	$\Omega_{1,1,1,3,4}$ $\Omega_{1,1,1,2,2}$ $\Omega_{1,1,1,2,5}$	$\Omega_{1,1,1,2,3}$ $\Omega_{1,1,1,3,5}$		$\Omega_{1,1,1,2,4}$ $\Omega_{1,1,1,4,5}$	$\Omega_{1,1,1,2,5}$ $\Omega_{1,1,1,5,5}$	$\Omega_{1,1,1,2,7}$
$D = 5$	$\Omega_{1,1,2,1,5}$ $\Omega_{1,1,3,1,4}$	$\Omega_{1,1,3,1,4}$ $\Omega_{1,1,2,1,2}$ $\Omega_{1,1,2,1,5}$	$\Omega_{1,1,2,1,3}$ $\Omega_{1,1,3,1,5}$		$\Omega_{1,1,2,1,4}$ $\Omega_{1,1,4,1,5}$	$\Omega_{1,1,2,1,5}$ $\Omega_{1,1,5,1,5}$	$\Omega_{1,1,2,1,7}$

TABLE 6: Small fundamental valuations:  $g=1$

	$I_1 \leftrightarrow II$	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$
$D=1$	<p>4 8 12 10 8 6 4 2 1</p> <p>2 47</p>	<p>2 5 8 7 6 5 4 3 2 1</p> <p>2 58</p>	<p>3 6 9 8 7 6 5 4 3 2 1</p> <p>4</p>
2	<p>2 4 6 8 6 4 2 1</p> <p>2 31</p>	<p>1 2 4 6 5 4 3 2 1</p> <p>2 40</p>	<p>1 3 5 7 6 5 4 3 2 1</p> <p>4</p>
3	<p>2 4 6 4 2 1</p> <p>2 23</p>	<p>1 3 5 4 3 2 1</p> <p>2 31</p>	<p>2 4 6 5 4 3 2 1</p> <p>4</p>
4	<p>2 4 4 2 1</p> <p>2 17</p>	<p>1 3 4 3 2 1</p> <p>2 24</p>	<p>1 4 5 4 3 2 1</p> <p>4</p>
5	<p>2 2 2 1</p> <p>2 12</p>	<p>2 1 3 2 1</p> <p>2 18</p>	<p>2 4 3 2 1</p> <p>4</p>

TABLE 6: Small fundamental valuations:  $g = 2, 3$

$g = 2:$	$I_2 \leftrightarrow III$	$I_3 \leftrightarrow IV$
$D = 2$	<p>27</p>	<p>34</p>
3	<p>19</p>	<p>25</p>
4	<p>14</p>	<p>19</p>
5	<p>13</p>	<p>18</p>
5	<p>10</p>	<p>14</p>
9	<p>9</p>	<p>13</p>
$g = 3:$		<p><math>I_3 \leftrightarrow IV</math></p> <p>22</p>
$D = 3$		<p>16</p>
4		<p>12</p>
5		<p>11</p>



*Table 4: Dynkin diagrams*

Here we give Dynkin diagrams of the hyperbolic sublattices (section 5.1). Below (in Figure 2) is an explanation of the symbols used. We draw the schemes of the corresponding graphs (section 5.4). Note that the first diagram in each entry of column  $I_4 \leftrightarrow I_n^*$  is a diagram of  $I_4$ . For more information we refer to section 5.

*Table 5: Extension of the Tables 2–4 for  $g = 2$*

This table presents the extension of the Tables 2–4 for  $g = 2$  at the right-hand side defined in section 5.4. It shows the strange duality considered at the end of section 5.2. Note that the right column of the original column  $I_1^*$  is replaced by the corresponding column for  $I_2^*$ . The notation is as above.

*Table 6: Small fundamental valuations*

This table shows the small fundamental valuations (section 5.5), which exist according to Theorem 5.5.2. A small fundamental valuation  $\xi: S \rightarrow \mathbb{N} - \{0\}$  corresponds to a small fundamental vector  $\mathbf{x} = \sum \xi_i \mathbf{e}_i$ . We have additionally indicated in an entry: a vertex  $v_i$  with  $(\mathbf{x}, \mathbf{e}_i) = 1$  resp. 2 by

$$\odot \quad \text{resp.} \quad \odot \oplus,$$

the number  $\sum \xi_i$  in the lower left corner, and  $(\mathbf{x}, \mathbf{x})$  in the lower right corner.

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