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H. JACQUET
K. F. LAI

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A RELATIVE TRACE FORMULA

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§1. Introduction and notations

(1.1) Let F be a number field, E a quadratic extension of F, M' a division algebra of center F and rank 4. We will make the following assumptions:

(1) the algebra $M'(E) = M' \otimes E$ is a division algebra;

(2) let D be the set of places v of F such that M'_v is a division algebra; then every place v in D splits in E.

We regard the multiplicative group of M' as an algebraic group G' defined over F. Thus:

$$G'(F) = M'^{\times}, \quad G'(E) = M'(E)^{\times},$$

$$G'_v = G'(F_v) \quad \text{if} \quad v \text{ is a place of } F,$$

$$G'_v = G'(E_v) \quad \text{if} \quad v \text{ is a place of } E.$$

Moreover $G'(F_{\mathbf{A}})$ is a closed subset of $G'(E_{\mathbf{A}})$. We will denote by Z' the center of G'. If φ is a continuous function on the quotient

$$Z'(E_{\mathbf{A}})G'(E)\backslash G'(E_{\mathbf{A}})$$

we will set

$$B'(\varphi) = \int \varphi(x)\mathrm{d}x, \quad x \in Z'(F_{\mathbf{A}})G'(F)\backslash G'(F_{\mathbf{A}})$$

and we will say that an automorphic irreducible representation π' of $Z'(E_{\mathbf{A}})\backslash G'(E_{\mathbf{A}})$ is distinguished if there is a smooth function φ in the space of π' such that $B'(\varphi) \neq 0$.

(1.2) Instead of M', we may also consider the algebra $M = M(2, F)$ of two by two matrices and regard its multiplicative group $G = \mathrm{GL}(2)$ as an algebraic group defined over F. It φ is a continuous bounded function on the quotient

$$Z(E_{\mathbf{A}})G(E)\backslash G(E_{\mathbf{A}})$$

we will set

$$B(\varphi)=\int \varphi(x)\mathrm{d}x, \quad x \in Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}}).$$

Let π be an irreducible automorphic cuspidal representation of $Z(E_{\mathbf{A}})\backslash G(E_{\mathbf{A}})$; we will say that π is distinguished if there is a smooth function φ in the space of π such that $B(\varphi)\neq 0$.

(1.3) Our main result is the following theorem:

THEOREM: *Suppose π' is an irreducible automorphic representation of $Z'(E_{\mathbf{A}})\backslash G'(E_{\mathbf{A}})$; suppose π' is infinite dimensional and let π be the corresponding irreducible automorphic cuspidal representation of $Z(E_{\mathbf{A}})\backslash G(E_{\mathbf{A}})$. Then π' is distinguished if and only if π is.*

The motivations for this result can be found in the work of Harder, Langlands and Rapoport on algebraic cycles of certain Shimura varieties. This is not the place for a discussion of their work. Suffices to say that it concerns poles of certain Hasse-Weil zeta functions attached to a Shimura surface. These zeta functions can be computed in terms of automorphic L-functions which have been studied directly by Asai (Cf. [S.A.]) The question arises then of deciding when the L-function attached to an automorphic cuspidal representation π has a pole at $s=1$. Now the L-function has an integral representation. Indeed consider an integral

$$\int \varphi(x)E(x, s)\mathrm{d}x, \quad x \in Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$$

where φ belongs to the space of π and $E(x, s)$ is an Eisenstein series. This integral has a pole at $s=1$ with residue $B(\varphi)$. On the other hand it is equal to the L-function times an elementary factor; in particular the L-function has a pole at $s=1$ if and only if, for at least one φ, the integral has a pole at $s=1$, that is if and only if B is non zero on the space of π, in other words, π is distinguished. In trying to extend the result of Harder, Langlands, Rapoport to the case of a compact Shimura surface one has to replace the group G by the group G' and use the above theorem.

Although this will not play a role in the present paper, we recall the characterization of the distinguished representations: a representation is distinguished if and only if it is the "Base Change" of an automorphic cuspidal representation of $G(F_{\mathbf{A}})$ whose central character is the quadratic idele class character of F attached to E. The proof is relatively complicated since it involves the "Base Change" and the theory of the Asai L-function. It would be interesting to see if this result could be established by using an appropriate relative trace formula, similar to the one we are using here.

(1.4). In order to prove our theorem we need a "relative trace formula": it is stated in section 2. In section 3 we derive our theorem from this trace formula. The remaining sections are devoted to the proof of our trace formula.

(1.5) Finally we thank *R*. Langlands for suggesting this problem to us, for his encouragement and his advice. We thank A. Selberg for his advice on classical results on *L*-functions. We also thank L. Clozel for making his results available to us; although the results are not used in the present paper, they were quite suggestive.

(1.6) We summarize our principal notations.

(1) The group G is the group GL(2), P is the group of upper triangular matrices, A the subgroup of diagonal matrices, N the group of unipotent matrices in P. We also denote by R the algebra of upper triangular matrices.

(2) We write $\mathrm{Mat}[p, q, r, s]$ for the matrix with rows (p, q) and (r, s). We also set:

$$w = \mathrm{Mat}[0, 1, 1, 0], \qquad n(x) = \mathrm{Mat}[1, x, 1, 0],$$

$$\mathrm{diag}(a, b) = \mathrm{Mat}[a, 0, 0, b].$$

(3) So far as the number field F is concerned we follow standard notations. In particular $F_{\mathbf{A}}$ is the ring of adeles and $F_{\mathbf{A}}^{x}$ the group of ideles. We denote by ψ a non trivial character of the quotient $F_{\mathbf{A}}/F$. We denote by $\mathrm{d}x$ the self dual Haar measure on $F_{\mathbf{A}}$ so that $\mathrm{vol}(F_{\mathbf{A}}/F) = 1$ and $\mathrm{d}(ax) = |a|_F \mathrm{d}x$ where $|a|_F$ is the module of the idele a; we drop the subscript F when this does not create confusion. We write $\mathrm{d}x = \Pi \mathrm{d}x_v$, $\psi = \Pi \psi_v$ where $\mathrm{d}x_v$ is the Haar measure on F_v, self dual with respect to ψ_v. We denote by $\mathrm{d}^{\times}x_v$, the Tamagawa measure on $F_v^{\times}$:

$$\mathrm{d}^{\times}x_v = L(1, 1_v)\mathrm{d}x_v/|x_v|.$$

Then we denote by $\mathrm{d}^{\times}x$ the Tamagawa measure on $F_{\mathbf{A}}^{\times}$:

$$\mathrm{d}^{\times}x = \Pi \mathrm{d}^{\times}x_v/[(s-1)L(s, 1)]\,|_{s=1}.$$

We denote by F^1 the group of ideles of norm one and by F_{∞}^{+} the group of ideles whose finite components are one and whose infinite components are all equal to some positive number, the same for all infinite places. Then $F_{\mathbf{A}}^{x}$ is the product of F^1 and F_{∞}^{+} and the Tamagawa measure is the product of a measure on F^1 and the measure $\mathrm{d}t/t$ where $t = |x|$. Recall that $\mathrm{vol}(F^1/F^\times)=1$. In general algebraic groups over F are provided with the Tamagawa measure.

(4) Let v be a place of F. We denote by K_v the standard maximal compact subgroup of G. Thus $K_v = \mathrm{GL}(2, R_v)$ where R_v is the ring of integers if v is finite. If $F_v = \mathbb{R}$, then $K_v = O(2, \mathbb{R})$. If $F_v = \mathbb{C}$, then $K_v = U(2, C)$. We denote by K_F the product of the K_v. The group $G(F_{\mathbf{A}})$ and $Z(F_{\mathbf{A}})$ being provided with the Tamagawa measures and the quotient $Z(F_{\mathbf{A}})\backslash B(F_{\mathbf{A}})$ with the corresponding measure, we have, for a function f on the quotient:

$$\int f(g)\mathrm{d}g = \int f[n(x)\ \mathrm{diag}(a, 1)k]\mathrm{d}x|a|_F^{-1}\mathrm{d}^\times a\mathrm{d}k$$

where $\mathrm{d}k$ is a certain Haar measure on K_F. Note that $\mathrm{vol}(K_F) \neq 1$. Having chosen an invariant differential form of maximum degree ω on G we have for each place v, the measure $|\omega_v|$ and the Tamagawa measure $L(1, 1_v)|\omega_v|$ on G_v. If we give tọ the quotient $Z_v\backslash G_v$ the measure quotient of the Tamagawa measures then we have for a function f on the quotient:

$$\int f(g_v)\mathrm{d}g_v = \int f[n(x_v)\ \mathrm{diag}(a_v, 1)k]\mathrm{d}x_v|a_v|^{-1}\mathrm{d}^\times a_v\mathrm{d}k_v$$

where $\mathrm{d}k_v$ is a certain Haar measure on K_v. Again $\mathrm{vol}(K_v) \neq 1$.

(5) We have also the quadratic extension E of F, with Galois group $\{1, \sigma\}$. Whenever convenient we write $E = F[\sqrt{\tau}]$. We denote by ψ_E non trivial character of $E_{\mathbf{A}}/E$. Usually we assume that is trivial on $F_{\mathbf{A}}$. If u is a place of E we denote by K_u the standard maximal compact subgroup of G_u, by K_E or simply K the product of the K_u.

§2. A relative trace formula

(2.1) We will denote by S a finite set of places of F containing D and all infinite places; the set S will be enlarged as need dictates. For each set T of places of F we will denote by $T^\sim$ the set of places of E which are above a place of T. For each place v of F not in D we choose an isomorphism $M'_v \cong M(2, F_v)$; let α be a basis of M' over F and α_v the R_v-module generated by α in M'_v; we may assume that, for all places v of F not in S, our isomorphism takes α_v to $M(2, R_v)$. Extending the scalars, we have, for each place v of E not in $D^\sim$, an isomorphism $M'(E_v) \cong M(E_v)$. From these, we get for each place v of E (resp. F) not in D (resp. $D^\sim$) an isomorphism $G'_v \cong G_v$; we use it to identify the two groups. For each place v of E (resp. F) we denote by K_v the standard maximal compact subgroup of G_v. We choose, in the usual way, two invariant F differential forms of maximal degree ω and ω' on G and G'; over an algebraic closure of F they are the same. Thus, for each place v

of F the groups G_v and G'_v come equipped with forms ω_v and ω'_v; if v is not in D we may assume the isomorphism of G'_v onto G_v takes $|\omega'_v|$. Similar remarks apply to E.

We denote by f a smooth function of compact support on $Z(E_{\mathbf{A}})\backslash G(E_{\mathbf{A}})$; we assume that f is a product of local components f_v. For all v the local component f_v is smooth, of compact support, bi-K_v-finite; for almost all finite v, it is the characteristic function of $Z_v K_v$. Similarly, we consider a smooth function of compact support f' on $Z'(E_{\mathbf{A}})\backslash G'(E_{\mathbf{A}})$. It is also assume to be a product of local factors f'_v. We make the following assumptions on f and f':

(1) for each v not in $D^{\sim}$, $f_v = f'_v$;

(2) let v be a place in D and $v1$, $v2$ the two places of E above v; we have isomorphisms

$$G'_{v1} \cong G'_{v2} \cong G'_v, \quad G_{v1} \cong G_{v2} \cong G_v;$$

we demand that the convolution products

$$h'_v = f'^{v}_{v1} * f'_{v2}, \quad h_v = f^{v}_{v1} * f_{v2}$$

on G'_v and G_v respectively have the same regular orbital integrals.

In other words, we demand that the hyperbolic orbital integrals of h_v vanish; on the other hand, if T'_v and T_v are isomorphic F_v-subalgebras of rank 2 of M'_v and M_v respectively and if $t' \in T'_v - F_v$ corresponds to $t \in T_v - F_v$, then we demand that

$$\int h(g^{-1}xg)\mathrm{d}g = \int h'(g'^{-1}x'g')\mathrm{d}g', \qquad g \in T_v^{\times}\backslash G_v, \quad g' \in T'^{\times}_v\backslash G'_v.$$

In this formula the Haar measures on G_v and G'_v are the Tamagawa measures attached to ω_v and ω'_v respectively; on the other hand the measures on $T_v^{\times}$ and $T'^{\times}_v$ are the Tamagawa measures attached to differential forms η_v and η'_v which correspond to one another under the isomorphism $T_v \cong T'_v$. Finally the quotients are given the quotient measures.

(2.2) The operator $\rho(f)$ defined by f on the space

$$L^2(Z(E_{\mathbf{A}})G(E)\backslash G(E_{\mathbf{A}}))$$

is defined by a kernel K. We call P the orthogonal projection on the space of cuspidal elements. Then $P\rho(f)P$ is represented by a kernel K_{cusp}; it is a smooth bounded function. Similarly the operator $\rho(f')$ defined by f' on the space

$$L^2(Z'(E_{\mathbf{A}})G'(E)\backslash G'(E_{\mathbf{A}}))$$

is represented by a kernel K'; we call P' the orthogonal projection on the orthogonal complement of the space spanned by the functions of the form $\chi\circ\det$, where χ is a quadratic character of $E^{\times}\backslash E^{\times}_{\mathbf{A}}$. Then $P'\rho(f')P'$ is represented by a kernel K'_{cusp}. It is also smooth and bounded.

(2.3) PROPOSITION: *Suppose that the above assumptions, in particular* (2.1.1) *and* (2.1.2), *are satisfied. Then*:

$$\iint K_{\text{cusp}}(x, y)\mathrm{d}x\mathrm{d}y = \iint K'_{\text{cusp}}(x', y')\mathrm{d}x'\mathrm{d}y',$$

$$x, y \in G(F)Z(F_{\mathbf{A}})\backslash G(F_{\mathbf{A}}), \quad x', y' \in G'(F)Z'(F_{\mathbf{A}})\backslash G'(F_{\mathbf{A}}).$$

In this formula $G(F_{\mathbf{A}})$, $G'(F_{\mathbf{A}})$, $Z'(F_{\mathbf{A}})$ *and* $Z(F_{\mathbf{A}})$ *are given the Tamagawa measures.*

The proof will occupy section 4 and the following sections.

§3. Demonstration of the theorem

(3.1) We will assume Proposition (2.2) and derive Theorem (1.3). Let therefore π' and π be as in (1.3). We will prove that if π is distinguished then π' is distinguished. The proof of the converse assertion is similar and left to the reader. So from now on we assume π is distinguished. We first recall a remark of Langlands. Suppose v is a place of F which splits into $v1$ and $v2$ in the extension E; then we have isomorphisms

$$G_{v1} \cong G_{v2} \cong G_v.$$

The restriction of B to the space of π is a non zero $G(F_{\mathbf{A}})$-invariant form; imbedding the space of $\pi_{v1} \otimes \pi_{v2}$ into the space of π appropriately, we find a non-zero linear form which is invariant under the group $G_{v1} \times G_{v2}$. Therefore π_{v1} and π_{v2} are contragredient to one another. Since they are trivial on the center, we see that the following condition is satisfied:

(1) if v splits into $v1$ and $v2$, then $\pi_{v1} \cong \pi_{v2}$.

Of course the representation π' satisfies the analoguous condition. We set $\pi_v = \pi_{v1} = \pi_{v2}$ and $\pi'_v = \pi'_{v1} = \pi'_{v2}$.

(3.2) Recall that S is a finite set of places of F containing D and the infinite places. Let us set

$$G^{S^{\sim}} = \Pi G_v (\text{restricted product}),$$

$$K^{S^{\sim}} = \Pi K_v, \quad v \notin S^{\sim};$$

$$G_{S^{\sim}} = \Pi G_v, \quad K_{S^{\sim}} = \Pi K_v, \quad v \in S^{\sim};$$

$$f^{S^{\sim}} = \Pi f_v, \quad v \notin S^{\sim}, \qquad f_{S^{\sim}} = \Pi f_v, \quad v \in S^{\sim}.$$

We may choose S so large that Π contains the unit representation of $K^{S^\sim}$. We than take f_v for $v \notin S^\sim$ to be bi-invariant under K_v. In general if σ is an irreducible representation of $Z(E_{\mathbf{A}})\backslash G(E_{\mathbf{A}})$ containing the unit representation of $K^{S^\sim}$, we will denote by e_σ the corresponding character of the Hecke algebra of the group $G^{S^\sim}$. In particular if σ is an automorphic cuspidal representation we will denote by $V(\sigma)$ the space of forms in the space of σ which are invariant under $K^{S^\sim}$. The orthogonal projection onto the space $V(\sigma)$ will be noted P_σ. The space $V(\sigma)$ is invariant under $G_{S^\sim}$ and the corresponding representation will be noted $\pi_{S^\sim}$. Thus for φ in $V(\sigma)$ we have:

$$\sigma(f)\varphi = e_\sigma(f^{S^\sim}) \cdot \pi_{S^\sim}(f_{S^\sim})\varphi. \tag{1}$$

Accordingly we may write K_{cusp} as the sum of the following uniformly convergent series:

$$K_{\text{cusp}}(x, y) = \sum_\sigma e_\sigma(f^{S^\sim}) \cdot K_\sigma(x, y) \tag{2}$$

where K_σ is the kernel attached to the operator $P_\sigma \pi(f_{S^\sim})P_\sigma$ and the sum is over all cuspidal representation σ containing the unit representation of $K^{S^\sim}$. In turn, if φ_j is an orthonormal basis of $V(\sigma)$, then K_σ is equal to the finite sum

$$K_\sigma(x, y) = \sum_j \pi_{S^\sim}(f_{S^\sim})\varphi_j(x) \cdot \bar{\varphi}_j(y).$$

Therefore

$$\iint K_{\text{cusp}}(x, y)\mathrm{d}x\mathrm{d}y = \sum_\sigma e_\sigma(f^{S^\sim})a_\sigma, \text{ with } a_\sigma = \iint K_\sigma(x, y)\mathrm{d}x\mathrm{d}y. \tag{3}$$

We will choose now $f_{S^\sim}$ in such a way that a_π is non zero and there exists a function f' satisfying the conditions (2.1.1) and (2.1.2). To that end let us observe that $\pi_{S^\sim}$ is equivalent to the tensor product representation

$$\otimes \pi_v, \quad v \in S^\sim.$$

We choose therefore an isomorphism

$$\pi_{S^\sim} \cong \pi_{T^\sim} \otimes \pi_{D^\sim}, \quad V(\pi) = V_{T^\sim} \otimes V_{D^\sim}, \qquad \text{where} \quad S = T \cup D;$$

we have set

$$\pi_{T^\sim} = \otimes \pi_v, \quad V_{T^\sim} = \otimes V_v, \quad v \in T^\sim;$$

$$\pi_{D^\sim} = \otimes \pi_v, \quad v_{D^\sim} = \otimes V_v, \quad v \in D^\sim,$$

and V_v is the space of π_v. At the cost of enlarging S we may assume there is a unitary vector u in the space V_{T^-} such that the restriction of B to $u \otimes V_{T^-}$ is non zero. We choose f_{T^-} in such a way that $\pi_{T^-}(f_{T^-})$ is the orthogonal projection on u. Then if v_j is any basis of V_{D^-} we have:

$$\iint K_\pi(x, y)\mathrm{d}x\mathrm{d}y = B\left[u \otimes \pi_{D^-}(f_{D^-})v_j\right] \cdot \overline{B}\left[u \otimes v_j\right]. \tag{4}$$

For each v in D let $v1$ and $v2$ be the two places of E above v. Let C_v be an invariant linear form on the tensor product $V_{v1} \otimes V_{v2}$. Then V_{D^-} is the tensor product of the spaces $V_{v1} \otimes V_{v2}$ and the restriction of B to $u \otimes V_{D^-}$ is, up to a constant factor, the tensor product of the C_v. For each v in D let us choose bases of V_{v1} and V_{v2} dual to one another, a_i and b_j say. Then the above expression is, up to a constant factor, the product over all v in D of the sums:

$$\sum_{i,j} C_v\left[\pi_{v1}(f_{v1})a_i \otimes \pi_{v2}(f_{v2})b_j\right] \cdot \overline{C}_v\left[a_i \otimes b_j\right]. \tag{5}$$

Since $C_v[a_i \otimes b_j] = \delta_{ij}$ this sum reduces to

$$\begin{aligned}
&\sum_i C_v\left[\pi_{v1}(f_{v1})a_i \otimes \pi_{v2}(f_{v2})b_i\right]\\
&= \sum C_v\left[\pi_{v1}(f_{v2}^{v})\pi_{v1}(f_{v1})a_i \otimes b_i\right]\\
&= \sum C_v\left[\pi_{v1}(h_v)a_i \otimes b_i\right]\\
&= \operatorname{tr} \pi_v(h_v),
\end{aligned} \tag{6}$$

where

$$h_v = f_{v2}^{v} * f_{v1}.$$

(3.3) At this point we need a lemma:

LEMMA: *We can choose f_{v1} and f_{v2} in such a way that* $\operatorname{tr} \pi_v(h_v) = 1$ *and* $\operatorname{tr} \sigma(h_v) = 0$ *for every infinite dimensional irreducible representation σ of G_v which is not equivalent to π_v.*

PROOF OF LEMMA: In any case π_v is in the discrete series. Thus if v is finite there is a function h_v with the required traces and it is trivial that it is a convolution product. So we are done in this case. If v is infinite then v is real. So we may identify E to $\mathbb{R}$ and G_v to $\mathrm{GL}(2, \mathbb{R})$. Let K_0 be the group $\mathrm{SO}(2, R)$ of K_v and, for each $n \in Z$, χ_n the character

$$k(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \mapsto \exp(in\theta).$$

Say that π_v is the representation of the discrete series of highest weight $n > 0$. Since π_v is trivial on the center n is necessarily even. Choose smooth functions of compact support a and b on $Z_v \backslash G_v$. We will take then with support in the set of matrices with positive determinant so that we may think of them as being functions on SL(2, R). The function b will be taken bi-invariant under K_0. As for a it will be assumed to transform under $\bar{\chi}_n$ on both sides:

$$a[k(\theta_1)gk(\theta_2)] = \exp[-in(\theta_1 + \theta_2)]a(g).$$

We will take h_v to be $(a-b)*(a+b)$. Then if σ is any irreducible representation of GL(2, $\mathbb{R}$) we have:

$$\operatorname{tr}\sigma(h_v) = \operatorname{tr}\sigma(a*a) - \operatorname{tr}\sigma(b*b) = \operatorname{tr}\sigma(a)^2 - \operatorname{tr}\sigma(b)^2.$$

If furthemore σ is in the discrete series then the second term is zero; the first term is zero as well unless the highest weight m of σ is less than or equal to n, that is unless σ belongs to a certain finite set. The linear independence of characters implies then that we may choose a such that $\operatorname{tr}\pi_v(a) = 1$ and $\operatorname{tr}\sigma(a) = 0$ if σ is another representation of the discrete series. Thus all we have to show is that we can choose b in such a way that

$$\operatorname{tr}\sigma(b) = \operatorname{tr}\sigma(a)$$

if σ is a representation of the principal series trivial on the center. Unraveling this we see that this relation is equivalent to $F_a = F_b$, where we have set, for $x > 0$,

$$F_a(x) = \iint a(k\,\mathrm{Mat}[x, u, 0, x^{-1}]k^{-1})\,\mathrm{d}k\,\mathrm{d}u$$

and F_b is defined similarly. Since F_a is smooth of compact support on $\mathbb{R}_+^+$ and invariant under the substitution $x \mapsto x^{-1}$, there is a function b with the required properly ("Paley Wiener Theorem for SL(2, R)", cf. for instance [S.L.], Th. 3, V, \$2, p. 71). This concludes the proof of the lemma.

(3.4). Coming back to the situation of (3.2) and choosing h_v as in the lemma we have

$$a_\pi = C\Pi \operatorname{tr}\pi_v(h_v), \quad v \in D, \qquad \text{where } C \text{ is a constant.} \tag{1}$$

Hence $a_\pi = C \neq 0$ as desired.

Now we choose a function f' on $G'(E_{\mathbf{A}})$ so that (2.1.1) and (2.1.2) are satisfied: so for a place v of E not in $D^{\sim}$ we take $f'_v = f_v$. For v in D, the Schur orthogonality relations show there is a convolution product h'_v on G'_v so that tr $\pi'_v(h'_v) = -1$ and tr $\sigma^2(h) = 0$ if σ' is not equivalent to π'_v. Since tr $\pi_v(h_v) = 1$ and tr $\sigma(h_v) = 0$ if σ is not equivalent to π_v we see that h_v and h'_v have the same orbital integrals. We therefore take f'_{v1} and f'_{v2} so that

$$h' = f'^{v}_{v1} * f'_{v2} \tag{2}$$

and we can apply Prop. (2.2). The kernel K'_{cusp} has a decomposition analoguous to (3.2.2). In particular we find

$$\iint K'_{\text{cusp}}(x', y')\mathrm{d}x'\mathrm{d}y' = \sum e_{\sigma'}(f^{S^{\sim}})a_{\sigma'}, \tag{3}$$

where

$$a_{\sigma'} = \iint K'_{\sigma'}(x', y')\mathrm{d}x'\mathrm{d}y'.$$

By Prop. (2.2) expressions (3.2.3) and (3) are equal. We may regard them as infinite linear combinations of characters of the Hecke algebra of $G^{S^{\sim}}$. Since the principle of linear independence of characters apply to these infinite sums ([R.L.] $11) and $e_{\pi} = e_{\pi'}$ we see that $a_{\pi'}$ is non zero. Since $K'_{\pi'}(x, y)$ is a sum of terms of the form $\varphi(x)\bar{\varphi}'(y)$ where φ and φ' belong to the space of π' we see that the restriction of B' to the space of π' is non zero. This concludes the proof of Theorem (1.3).

§4. Double cosets

(4.1) In order to prove our trace formula we need to study the double cosets of $G'(F)Z'(E)$ in $G'(E)$ and the double cosets of $G(F)Z(E)$ in $G(E)$. An element of $G'(F)Z'(E)$ or $G(F)Z(E)$ will be termed singular. An element not in this set will be termed regular.

We start with G. We let Q be a set of representatives for the $G(F)$ conjugacy classes in the set of subfields of $M(2, F)$ of rank 2 over F. We let R be the subalgebra of $M(2, F)$ formed of the upper triangular matrices and P the subgroup of $G(F)$ formed of the upper triangular matrices.

(1) Lemma: *If T is in Q then any singular element of $(T \otimes E)^{\times}$ is in $T^{x} \cdot E^{\times}$. Similarly any singular element of $P(E)$ is in $P(F)Z(E)$.*

Proof of (1): Suppose x is singular in $(T \otimes E)^{\times}$. Then there is c in $E^{\times} = Z(E)$ such that $x = cy$ with y in $G(F)$. Then y is in the intersection of the algebras $T \otimes E$ and $M(2, F)$, that is in T. Since it is invertible, it is in $T^{\times}$ and we are done. The proof of the second assertion is similar, using the algebra R instead of the algebra T.

Recall that the Galois group of E over F is $\{1, \sigma\}$.

(2) LEMMA: *Suppose x is in $G(E)$. Then x is regular (in our sense) if and only if $x^{\sigma}x^{-1}$ is regular in the usual sense.*

PROOF OF (2): Suppose x is singular in our sense. Then $x = gc$ with g in $G(F)$ and c in $E^{\times}$. Then $x^{\sigma}x^{-1} = c^{\sigma-1}$ is singular. Conversely, suppose $x^{\sigma}x^{-1} = c$ where c is in $E^{\times}$. Then $c^{\sigma}c = 1$ so $c = z^{\sigma-1}$ with z in $E^{\times}$. This gives $(xz^{-1})^{\sigma} = xz^{-1}$. Hence xz^{-1} is in $G(F)$ and x is singular in our sense.

(3) LEMMA: *If x is in $G(E)$ then either x is singular, or x is in a double coset $G(F)pG(F)$ with p in $P(E) - P(F)$, or x is in a double coset $G(F)tG(F)$, where t is in $(T \otimes E)^{\times} - T^{\times} \cdot E^{\times}$ and T is in Q. Moreover the three possibilities are exclusive of one another.*

PROOF OF (3): Let us prove the first assertion. Let E be the extension $F[\sqrt{\tau}]$. Then any x in $M(2, E)$ can be written uniquely in the form:

$$x = a + b\sqrt{\tau}, \quad \text{with } a \text{ and } b \text{ in } M(2, F).$$

Suppose x is invertible and not singular; recall that in the present context this means that x is not in $Z(E)G(F)$. Then a is not zero, otherwise x would be in $G(F)E^{\times}$, hence would be singular. Suppose a is non zero but non invertible. We claim there is an element c of $E^{\times}$ such that $x' = xc$ has the form

$$x' = a' + b'\sqrt{2} \quad \text{with } a' \text{ in } G(F) \text{ and } b' \text{ in } M(2, F).$$

Indeed a has the following form:

$$a = g \operatorname{diag}(u, 0) g^{-1}, \quad \text{with } g \text{ in } G(F) \text{ and } u \text{ in } F^{\times}.$$

Let us write also:

$$b = g \operatorname{Mat}[p, q, r, s] g^{-1}.$$

We claim that for some appropriate z in F the matrix $za + b$ has a non zero determinant. Suppose not. Then

$$(zu + p)s - rq = 0$$

for all z in F. This implies $s=0$ and then $rq=0$. Say $r=0$. Then the matrix x has the form

$$g \operatorname{Mat}\left[1+p\sqrt{\tau}\,,\ q\sqrt{\tau}\,,0,0\right] g^{-1}$$

and is thus not invertible, a contradiction. Let therefore z be an element of F such that $za+b$ is invertible. Set $c=z+1/\sqrt{\tau}$. Then c is in $E^{\times}$ and xc has the form $a'+b'\sqrt{\tau}$ where a' is invertible. Thus we may assume that a is invertible. Replacing x by xa^{-1} we may even assume that a is 1. If b is not invertible or if it is invertible but not elliptic (in the usual sense) in $G(F)$, there is a g in $G(F)$ such that $b=gpg^{-1}$ where p is in $R(F)$. Then:

$$x=gqg^{-1} \quad \text{with } q=1+p\sqrt{\tau}\,.$$

Since q is in $R(E)$ we are done in this case. If b is an elliptic element of $G(F)$ then there is a T in Q and a t in $T^{\times}$ such that $b=gtg^{-1}$. Then:

$$x=gyg^{-1} \quad \text{with } y=1+t\sqrt{\tau}\,.$$

Since y is in $T\otimes E$ we are again done.

It remains to prove the second assertion of the lemma (3). Because of lemma (1), all we have to prove is that for T in Q an element x of $(T\otimes E)^{\times}-T^{\times}\cdot E^{\times}$ cannot be of the form g_1pg_2, with p in $P(E)$ and g_1 and g_2 in $G(F)$. Suppose it is. After a simple computation we find:

$$x^{\sigma}x^{-1}=g_1qg_1^{-1}, \quad \text{with } q=p^{\sigma}p^{-1}.$$

Suppose the F algebra T is not isomorphic to E. Then $T\otimes E$ is a field, in fact a quadratic extension of E and $x^{\sigma}x^{-1}$ is an element of that field which is not an elliptic element of $G(E)$ (in the usual sense). Thus $x^{\sigma}x^{-1}$ is singular in the usual sense and by lemma (4.1.2) x is singular in our sense, a contradiction. Suppose now that T is isomorphic to E; in passing note that there is exactly one element of Q with that property. Then $T\otimes E$ is not a field; it is a sum of two copies of E. In particular the minimal polynomial of x over F has degree one or two and the same is true of q. This is possible only if the two eigenvalues of g are in F. Since they are elements of E whose F norm is 1, they are $+1$ or -1. In any case:

$$(a-1)(a+1)=0, \quad \text{where } a=x^{\sigma}x^{-1}.$$

This gives $a^2=1$ and $a^{\sigma}a=1$. SInce x is regular $a\neq \pm 1$ (lemma (4.1.2)). Let us identify $T\otimes E$ with the sum of two copies of E; then

$$(x,\ y)^{\sigma}=(y,\ x).$$

From $a^2 = 1$ and $a \neq \pm 1$ we get $a = (1, -1)$ or $(-1, 1)$. Then $a^\sigma a = -1$ a contradiction.

We will say that an element of $G(E)$ is elliptic if it is not in the set $G(F)P(E)G(F)$.

(4.2) We will now classify all elliptic elements.

(1) LEMMA: *Suppose T and T' are in Q. Suppose that x and x' are regular elements of $T \otimes E$ and $T' \otimes E$ respectively but have the same double class modulo $Z(E)G(F)$. Then $T = T'$.*

PROOF OF (1): Since E is contained in $(T \otimes E)^\times$ and $(T \otimes E)^\times$ we may as well assume that x and x' have the same double class modulo $G(F)$:

$$x = cx'c' \quad \text{with } c \text{ and } c' \text{ in } G(F).$$

Then $x^{\sigma-1} = cx'^{\sigma-1}c^{-1}$. Morever $x^{\sigma-1}$ is a regular element of $(T \otimes E)^\times$ in the usual sense (lemma (4.1.2)). Hence $c(T \otimes E)c^{-1} = T' \otimes E$. Taking the intersections with $M(2, F)$ we obtain $cTc^{-1} = T'$, hence $T = T'$.

(2) LEMMA: *Suppose that c and c' are in $G(F)$, T is in Q, x and x' are regular elements of $(T \otimes E)^\times$ and z is in $E^\times$. Suppose that*

$$x = cx'c'z.$$

Then c' and c normalize T and cc' is in T.

PROOF OF (2): Again we have $x^\sigma x^{-1} = cx'^\sigma x'^{-1} z^{\sigma-1} c^{-1}$ and $x^{\sigma-1}$ is a regular element in the usual sense. It follows that c normalizes $T \otimes E$, hence also T. Finally $cc' = cx'^{-1}c^{-1}z^{-1}x$ is in $T \otimes E$, hence in T.

(4.3) We now deal with the remaining elements of $G(E)$.

(1) LEMMA: *Every element x of $P(E)$ is in the same double class modulo $G(F)Z(E)$ as one of the following element:*

$$\text{(i) } e, \quad \text{(ii) } n(\sqrt{\tau}), \quad \text{(iii) } \operatorname{diag}(a, 1), \quad a \in E^\times - F^\times.$$

PROOF OF (1): Let us write x in the form

$$x = \operatorname{Mat}[a, b, 0, c].$$

If a/c is in F then after multiplying by an element of $Z(E)A(F)$ we may assume that $a = c = 1$. If b is in F then x is in $N(F)$ and we are in case (i). If b is not in F then after multiplying by a matrix in $N(F)$ we may assume that $b = u\sqrt{\tau}$ with u in F. Then

$$\mathrm{diag}(u, 1) \times \mathrm{diag}(u, 1)^{-1}$$

is the matrix (ii). If a/c is not in F then, after multiplying by an element of $Z(E)$, we may assume $c = 1$. Since a is not in F we may write b in the form

$$b = u + av \quad \text{with } u \text{ and } v \text{ in } F.$$

Then the matrix

$$n(-u) \times n(-v)$$

has the form (iii). This concludes the proof of the lemma.

(2) LEMMA: *In lemma* (1) *cases* (i), (ii), (iii) *are exclusive of one another.*

PROOF OF LEMMA (2): In view of lemma (4.1.1), it suffices to show that case (ii) and (iii) are not compatible. Suppose they were. Then there would be g and g in $G(F)$ and a matrix d in $A(E)$, where A is the group of diagonal matrices, such that:

$$n(\sqrt{\tau}) = g_1 d g_2$$

Then we would find:

$$n(2\sqrt{\tau}) = n(\sqrt{\tau})^{\sigma - 1} = g_1 d^{\sigma - 1} g_2 .$$

Since $n(2\sqrt{\tau})$ is unipotent and $d^{\sigma - J}$ is a diagonal matrix this is a contradiction.

(3) LEMMA: *Suppose c and c' are in $G(F)$, x and x' are regular elements of $A(E)$ and z is in E. Suppose that*

$$x = cx'c'z .$$

Then c and c' normalize A and cc' is in $A(F)$.

The proof is the same as the proof of lemma (4.2.2).

(4) LEMMA: *Let $n = n(\sqrt{\tau})$. Suppose g and g' are in $G(F)$, z in E. Suppose that*

$$gng' = zn .$$

Then z is in F, $gg' = z$ and g and g' are in $N(F)Z(E)$, where N denotes the group of triangular matrices with unit diagonal.

PROOF OF (4): Once more we find $gn^{\sigma-1}g^{-1} = z^{\sigma-1}n^{\sigma-1}$. Since $n^{\sigma-1}$ is a regular unipotent element, this implies that $z^{\sigma-1} = 1$ and g centralizes $n^{\sigma-1}$; in turn this implies that g is in $Z(E)N(E)$ hence in fact in $Z(F)N(F)$ and z is in F. Similarly g' is in $Z(F)N(F)$. Since $Z(E)N(E)$ is a commutative group $gg' = z$ and we are done.

(4.4) Altough not necessary for the description of the double classes the two following lemmas will be useful latter on.

(1) LEMMA: *Suppose ξ is in $G(E)$ and $\xi^\sigma\xi^{-1}$ is in $P(E)$. Then ξ is in $G(F)P(E)$.*

PROOF OF (1): Set $p = \xi^\sigma\xi^{-1}$. By assumption p is in $P(E)$. On the other hand $p^\sigma p = 1$. We claim there is a q in $P(E)$ such that $p = q^\sigma q^1$. This will prove the lemma because then ξq^{-1} is invariant under σ hence in $G(F)$. To prove our claim we write

$$p = \mathrm{Mat}[a, x, b, 0], \; q = \mathrm{Mat}[u, y, v, 0].$$

The condition that $p^\sigma p = 1$ gives:

$$\text{(i) } a^\sigma a = 1, \quad \text{(ii) } b^\sigma b = 1, \quad \text{(iii) } zx^\sigma + x = 0 \quad \text{where} \quad z = a/b. \tag{2}$$

The condition $q^\sigma q^{-1} = p$ gives:

$$\text{(i) } a = u^\sigma u^{-1}, \quad \text{(ii) } b = v^\sigma v^{-1}, \quad \text{(iii) } x = -y^\sigma z + y. \tag{3}$$

Because of (2.i) and (2.ii) it is possible to solve (3.i) and (3.ii). Next regard (3.iii) as a system of 2 linear equations for r and s where $z = r + s\sqrt{\tau}$. The condition $z^\sigma z = 1$ means that the determinant of the system is zero and the condition (3.iii) is then the compatibility condition for the two equations. Our claim follows.

Say that an element of $G(E)$ is semi-simple regular if it is elliptic or in the double class modulo $G(F)Z(E)$ of an element diag$(a, 1)$ with $a \notin F$. Say it is unipotent regular if it is in the double class of an element of the form $n(y)$ with y not in F.

(4) LEMMA: *An element x of $G(E)$ is singular (resp. semi-simple regular, resp. unipotent regular) if and only if $x^{\sigma-1}$ is singular (resp. semisimple, resp. unipotent regular) in the usual sense.*

REMARK: In this lemma an element of the form $zn(y)$ with z in Z is regarded as unipotent.

PROOF OF (4): We may already assume x regular (lemma ((4.1.2)). Then we know that $x = gyg'$ where g and g' are in $G(F)$ and y is either in $(T \otimes E)^\times - T^\times E^\times$ or of the form diag(a, b) with $a/b \notin F$, or of the form $zn(u)$ with $u \notin F$ and z in $Z(E)$. Then $x^\sigma x^{-1} = gy^\sigma y^{-1} g^{-1}$. In the first case $y^\sigma y^{-1}$ is an element of $T \otimes E$ hence a semi-simple one. In the second case $y^\sigma y^{-1}$ is hyperbolic and in the last case it is unipotent. In any case it is regular (loc. cit.).

(4.5) The previous results apply mutatis mutandis to G'. Every element is either singular or elliptic. The set Q is replaced by a set Q' of conjugacy classes for the subalgebras of M' of rank 2. Of course, any element of Q' is isomorphic to exactly one element of Q.

§5. Terms attached to the elliptic and singular elements

(5.1) To prove our trace formula we now write down K_{cusp} more explicitly:

$$K_{\text{cusp}} = K - K_{\text{eis}} - K_{\text{sp}} \tag{1}$$

where:

$$K(x, y) = \sum f(x^{-1}\xi y),\ \xi \in Z(E)\backslash G(E). \tag{2}$$

A formula for the two other kernels will be recalled below. In turn we write (2) as the sum of three more kernels:

$$K = K_{\text{s}} + K_{\text{e}} + K_{\text{r}} \quad \text{where:}$$

$$K_{\text{s}}(x, y) = \sum f(x^{-1}\xi y), \quad \xi \text{ singular}; \tag{4}$$

$$K_{\text{e}}(x, y) = \sum f(x^{-1}\xi y), \quad \xi \text{ elliptic}; \tag{5}$$

$$K_{\text{r}}(x, y) = \sum f(x^{-1}\xi y), \quad \xi \text{ regular, but not elliptic}. \tag{6}$$

Similarly, we write K'_{cusp} as

$$K'_{\text{cusp}} = K' - K'_{\text{sp}} \tag{7}$$

where

$$K'(x, y) = \sum f'(x^{-1}\xi y), \quad \xi \in Z(E)\backslash G(E). \tag{8}$$

We write (8) as the sum of two other kernels:

$$K' = K'_{\text{s}} + K'_{\text{e}}. \tag{9}$$

where

$$K'_s(x, y) = \sum f'(x^{-1}\xi y), \quad \xi \text{ singular}; \tag{9}$$

$$K'_e(x, y) = \sum f'(x^{-1}\xi y), \quad \xi \text{ elliptic}; \tag{10}$$

In this section we will prove that, under the assumptions of Proposition (2.2), the kernels K_e and K'_e, K_s and K'_s, K_{sp} and K'_{sp} respectively, have the same integrals.

(5.2) We first deal with K_e and prove that K_e is indeed integrable over the product of the quotient $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ by itself. To that end we first prove a lemma. We let K_E be the standard maximal compact subgroup of $g(E_{\mathbf{A}})$ and for g in $G(E_{\mathbf{A}})$ define the height $H(g)$ of g to be:

$$H(g) = |a/b|_E,$$

$$\text{if} \quad g = ndk,\ n \in N(E_{\mathbf{A}}),\ d = \operatorname{diag}(a, b),\ k \in K_E. \tag{1}$$

We first recall without proof a standard lemma (Cf. [J.A.]):

(2) LEMMA: *Let Ω be a compact subset of $G(E_{\mathbf{A}})$. Then there is a number $d > 0$ such that the relations*:

$$h'^{-1}\xi h \in \Omega Z(E_{\mathbf{A}}), \quad \xi \in G(E), \quad H(h) > d, \quad H(h') > d$$

imply ξ is in $P(E)$.

We derive from this lemma the following result:

(3) LEMMA: *Suppose Ω is a compact subset of $G(E)$. Then there is a number $d > 0$ such that the relations*

$$x^{-1}\xi h \in \Omega Z(E_{\mathbf{A}}), \quad \xi \in G(E), \quad x \in G(F_{\mathbf{A}}), \quad H(h) > d$$

imply that ξ is in $G(F)P(E)$.

PROOF OF (3): We have also $h^{-\sigma}\xi^{-\sigma} \in \Omega^{-\sigma}$. After multiplication we get $h^{-\sigma}\xi^{-\sigma}\xi h \in \Omega^{-\sigma}\Omega$. Applying the previous lemma we see that if d is sufficient large and $H(h) > d$ then $\xi^{-\sigma}\xi \in P(E)$. The conclusion follows from lemma (4.4.1).

Recall that

$$K_e(x, y) = \sum f(x^{-1}\xi y), \quad \xi \text{ elliptic}.$$

If x and y are in $G(F_{\mathbf{A}})$ then the previous lemma shows that the relation $f(x^{-1}\xi y)=0$ implies first that x and y are in sets compact modulo $G(F)Z(E_{\mathbf{A}})$; in turn this implies that ξ stays in a set finite modulo $E^{\times}$. Hence K_{e} is indeed integrable. Furthemore, by taking f positive in what follows we can justify our formal manipulations.

(5.3) The results of the sub-section (4.2) allow us to write:

$$K_{\mathrm{e}}(x, y)=\sum f(x^{-1}\gamma\xi\delta y), \tag{1}$$

$$T\in Q, \quad \gamma\in G(F)/T^{\times}, \quad \delta\in N(T)\backslash G(F),$$

$$\xi \text{ regular in } (T\otimes E)^{\times}/E^{\times};$$

we have noted $N(T)$ the normalizer of T in $G(F)$. Since T has index 2 in $N(T)$ this can also be written as:

$$K(x, y)=1/2\sum f(x^{-1}\gamma\xi\delta y), \tag{2}$$

$$T\in Q, \gamma\in G(F)/Z(F), \quad \delta\in T^{\times}\backslash G(F),$$

$$\xi \text{ regular in } T^{\times}\backslash(T\otimes E)^{\times}/E^{\times}.$$

If we integrate formally over the product of the quotient $G(F)Z(E_{\mathbf{A}})\backslash G(F_{\mathbf{A}})$ by itself we find:

$$\iint K_{\mathrm{e}}(x, y)\mathrm{d}x\mathrm{d}y=1/2\sum v_T\iint f(x\xi y)\mathrm{d}x\mathrm{d}y; \tag{3}$$

the sum is over all T in Q and then ξ regular in $T^{\times}\backslash(T\otimes E)^{\times}/E^{\times}$; the integral in x is over $G(F_{\mathbf{A}})/Z(F_{\mathbf{A}})$ and the integral in y over the quotient $T_{\mathbf{A}}^{\times}\backslash G(F_{\mathbf{A}})$; finally v_T is the volume of the quotient $F_{\mathbf{A}}^{\times}T^{\times}\backslash T_{\mathbf{A}}^{\times}$. In turn, each integral in (3), apart from a constant factor, can be written as a product over all places v of F of local ones.

If v is a place of F which does not splits in E and u is the corresponding place of E then E is a quadratic extension of F and G is a subgroup of G. The corresponding local factor is nothing but:

$$\iint f_u(x\xi y)\mathrm{d}x\mathrm{d}y, \quad x\in G_v/Z_v, \quad y\in T_v^{x}\backslash G_v. \tag{4}$$

The convergence of this integral can be established as follows: if the integrand is not zero then $x\xi y$ belongs to a set $Z_u\Omega$ where Ω is compact. Then $y^{-1}\xi^{1-\sigma}y$ is in a compact set. Since $\xi^{1-\sigma}$ is semi-simple regular (lemma (4.4.4)) y must be in a set compact $\mathrm{mod}(T_v\otimes E_v)^{\times}$. Now

$(T_v \otimes E_v)^\times$ is compact mod $T_v^\times$. So y is in a set $T_v^\times \Omega'$ where Ω' is a compact set of G_v. For y in Ω' the functions $x \mapsto f(x\xi y)$ have support in a fixed compact set of $G(F_v)$ and are uniformly bounded. Thus the double integral (4) converges.

For almost all such v, the integral (4) can be described as follows: f_u is the characteristic function of $Z_u K_u$, E_u is the unramified quadratic extension of F_v and ξ is in $K_u Z_u$. If $f(x\xi y) = 0$ then $x\xi y$ is in $K_u Z_u$ hence $y^{-1}\xi^{1-\sigma}y$ is in K_u. Since $\xi^{1-\sigma}$ is in K_u and is semi-simple regular, this implies that y is in $(T_v \otimes E_u)^\times \cdot K_u \cap G_v$. Because E_u is unramified this is $T_v^\times K_v$. Taking then y in K_v we find that x must be in $Z_v K_v$. So the factor in this case is equal to

$$\mathrm{vol}(Z_v \backslash Z_v K_v) \cdot \mathrm{vol}(T_v \backslash T_v K_v). \tag{5}$$

If on the contrary v is a place of E which splits into $v1$ and $v2$ in E then we have isomorphisms $G_{v1} \cong G_v$ and $G_{v2} \cong G_v$; let μ_1 and μ_2 be the images of $\mu \in G(E)$ under these isomorphisms. Then G_v may be identified with the diagonal of the product $G_{v1} \times G_{v2}$ and our local factor is

$$\iint f_{v1}(x\xi_1 y) \cdot f_{v2}(x\xi_2 y) \mathrm{d}x \mathrm{d}y, \quad x \in G_v/Z_v, \quad y \in T_v^\times \backslash G_v. \tag{6}$$

After a change of variables, (6) can be written also as:

$$\iint f_{v1}(x_{v1}) \cdot f_{v2}\big(xy^{-1}\xi_1^{-1}\xi_2 y\big) \mathrm{d}x \mathrm{d}y \tag{7}$$

or

$$\int h_v\big(y^{-1}\eta_v y\big) \mathrm{d}y, \quad \text{with } \eta_v = \xi_1^{-1}\xi_2 \text{ and } h_v = f_{v1}^v * f_{v2}. \tag{8}$$

Now $\eta_v = (\xi_2)^{1-\sigma}$. Therefore this is a regular semi-simple element of $G(F_v)$ (lemma (4.4.4)) and this is an orbital integral for the function h_v; in particular it converges.

Again let us see what happens to the integral at almost all v. Then f_{vi} is the characteristic function of $Z_v K_v$ and ξ_1 and ξ_2 are in $Z_v K_v$. If the integrand is non zero we find that $x\xi_1 y$ and $x\xi_2 y$ are in $Z_v K_v$. Hence $y^{-1}\eta_v^{-1}y$ is in the same set. Since η_v is in K and is semisimple regular, this implies that y is in $T_v K_v$. Taking y in K_v we find that x is in $Z_v K_v$ and the integral is again equal to (5). Since the product of the factors (5) is finite our assertion is justified: each one of the local integral converges and their product converges too.

(5.4) Very much the same considerations apply to G' and the kernel K'. It is now time to prove the equality of the integral of K_e and K_e' under the assumptions of Prop. (2.2). To that end, consider a term in (5.3.3). Suppose that T is an algebra which does not imbed into H. Then there is at least one v in D such that T does not imbed into H, that is such that T is not a field. Then with the previous notations η_v is an hyperbolic regular element and the factor corresponding to v is an hyperbolic orbital integral of h_v hence zero by our choice of h_v. Suppose that now that T does imbed into H. It is therefore isomorphic to exactly one element T' of Q'. Then $v_T = v_{T'}$ and all we have to check is the equality of the integrals:

$$\iint f(x\xi y)\mathrm{d}x\mathrm{d}y = \iint f'(x'\xi' y')\mathrm{d}x'\mathrm{d}y'$$

if ξ corresponds to ξ'. Once more we decompose each global integral into a product of local ones and a constant factor, the same on both sides. Suppose v is not in D. Then $G_v = G'_v$ and $f_v = f'_v$. So the factor attached to v is trivially the same in both sides of the equality. Suppose v is in D. Then the factor attached to v in the left hand side is the integral of h_v on the orbit of η_v. Similarly the factor attached to v in the right hand side is the integral of h'_v on the orbit of η'_v, where η'_v corresponds to η_v in the isomorphism of T_v with T'_v. Again, by assumption, these factors must be the same and we have proved the required equality.

(5.5) We quickly dispose of the singular terms. Recall that

$$K_s(x, y) = \sum f(x^{-1}\xi y), \quad \xi \in G(F)/Z(F).$$

Then:

$$\iint K_s(x, y)\mathrm{d}x\mathrm{d}y = \iint f(xy)\mathrm{d}x\mathrm{d}y,$$

$$\text{with } x \in G(F_{\mathbf{A}})/Z(F_{\mathbf{A}}), \quad y \in G(F)Z(F_{\mathbf{A}})\backslash G(F_{\mathbf{A}}).$$

If we integrate with respect to x first the resulting integral does not depend on y so we find also:

$$\iint K_s(x, y)\mathrm{d}x\mathrm{d}y = \mathrm{vol}\int f(x)\mathrm{d}x, \quad x \in Z(F_{\mathbf{A}})\backslash G(F_{\mathbf{A}}), \tag{1}$$

where vol stands for the volume of $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$.

Similarly we find:

$$\iint K'_s(x, y)\mathrm{d}x\mathrm{d}y = \mathrm{vol}'\int f'(x)\mathrm{d}x, \quad x \in Z'(F_{\mathbf{A}})\backslash G'(F_{\mathbf{A}}), \tag{2}$$

where vol′ stands now for the volume of $Z'(F_{\mathbf{A}})G'(F)\backslash G'(F_{\mathbf{A}})$. We want to show that (1) and (2) are equal. To begin with vol = vol′. Next we write the integrals on the right hand sides of (1) and (2) as products of local integrals (and a constant factor). Again the factor attached to a place v of F which is not in D is trivially the same in both integrals. To see that the factors attached to a place v in D are the same we note that in the right hand side of (1) for instance this factor is nothing but the following integral:

$$\int f_{v1}(x)f_{v2}(x)\mathrm{d}x, \quad x \in G_v/Z_v.$$

This is in fact $h_v(e)$. Since h_v and h'_v have the same orbital integrals they have also the same value at e and we are done.

(5.6) We also dispose of the kernels K_{sp} and K'_{sp}. For K_{sp} we have the following expression:

$$K_{\mathrm{sp}}(x, y) = 1/\mathrm{VOL}\sum \int f(g)\chi(\det g)\mathrm{d}g \cdot \chi(\det x)\cdot \bar{\chi}(\det y),$$

where VOL denotes the volume of the quotient $G(E)Z(E_{\mathbf{A}})\backslash G(E_{\mathbf{A}})$, the sum is over all quadratic characters χ of the idele class group of E and the integral is over $Z(E_{\mathbf{A}})\backslash G(E_{\mathbf{A}})$. The integral of the term attached to χ is zero unless the restriction of χ to $F_{\mathbf{A}}^{\times}$ is trivial. Thus we find:

$$\iint K_{\mathrm{sp}}(x, y)\mathrm{d}x\mathrm{d}y = (\mathrm{vol})^2/\mathrm{VOL}\sum \int f(g)\chi(\det g)\mathrm{d}g, \tag{1}$$

the sum being over all such χ. Similarly, we have:

$$\iint K'_{\mathrm{sp}}(x, y)\mathrm{d}x\mathrm{d}y = (\mathrm{vol}')^2/\mathrm{VOL}'\sum \int f'(g')\chi(\det g')\mathrm{d}g' \tag{2}$$

where VOL′ denotes the volume of $G'(E)Z'(E)\backslash G'(E)$ and we have written det for the reduced norm. To prove the equality of the integrals (1) and (2), it suffices therefore to check that

$$\int f(g)\chi(\det g)\mathrm{d}g = \int f'(g')\chi(\det g')\mathrm{d}g'. \tag{3}$$

We decompose each integral into a product over all places of E of analoguous local integrals (and a constant). It is again clear that the factor attached to a place not in $D^{\sim}$ is the same on both sides. If v is a place of D and $v1$ and $v2$ the two corresponding places of E then in the left hand side we have the factor:

$$\int f_{v1}(g_{v1})\chi_{v1}(\det g_{v1})\mathrm{d}g_{v1}\int f_{v2}(g_{v2})\chi_{v2}(\det g_{v2})\mathrm{d}g_{v2}. \tag{4}$$

Since the restriction of χ to $F_{\mathbf{A}}^{\times}$ is trivial we have $\chi_{v1}=\bar{\chi}_{v2}=\chi_v$ say and this can be written also as

$$\int h_v \chi_v(\det g_v)\mathrm{d}g_v. \tag{5}$$

So what we have to prove is that this is also

$$\int h'_v \chi_v(\det g'_v)\mathrm{d}g'_v. \tag{6}$$

Call σ the special representation of G. Then (6) is minus tr $\sigma\otimes\chi_v(h_v)$. But because h_v has zero trace in any principal series this is also equal to (5) and we are done.

§6. Truncation of K

(6.1) In order to finish the proof of Proposition (2.2) we need only show that the integral of the difference $K_{\mathrm{r}}-K_{\mathrm{eis}}$ over the product of $\mathrm{z}(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ by itself is zero. This will be proved under the following assumption:

(1) There is a set X of places of F with at least two elements and the following property: every v in X splits into $v1$ and $v2$ in E and the hyperbolic integrals of $h_v=f_{v1}^{v}*f_{v2}$ vanish.

Since K_{cusp}, K_{e}, K_{s} and K_{sp} are integrable over the product, it is clear that the difference $K_{\mathrm{r}}-K_{\mathrm{eis}}$ is indeed integrable. However, just as in the classical case, each term alone is not integrable and therefore we must use what one could call a "restriction truncation operator". It is defined as follows: let h be a continuous function on $Z(E_{\mathbf{A}})G(E)\backslash G(E_{\mathbf{A}})$, let c be a positive number and let χ_c be the characteristic function of $]c,+\infty[$; then we will denote by $T^c h$ the function on $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ defined by:

$$T^c h(g)=h(g)-\sum h_N(\gamma g)\chi_c(H(\gamma g)),\quad \gamma\in P(F)\backslash G(F); \tag{1}$$

we have denoted by h_N the constant term of f along N. Recall that it is the function on $G(E_{\mathbf{A}})$ defined by:

$$h_N(g)=\int h(ng)\mathrm{d}n,\quad n\in N(E)\backslash N(E_{\mathbf{A}}). \tag{2}$$

Recall also that $H(g)$ is the height: $H(g)=|a/b|_E$ if $g=n(y)$ diag$(a/b)k$. The following lemma is standard (Cf. [J.A]):

(4) LEMMA: *If γ is in $G(E)$ and there is a g in $G(E_{\mathbf{A}})$ such that $H(g) > 1$ and $H(\gamma g) > 1$ then γ belongs to $P(E)$.*

This lemma implies that the sum in (1) has at most one term and also that if $H(g) > c > 1$ then $T^c h(g) = h(g) - h_N(g)$. Of course if h is a cuspidal function on $Z(E_{\mathbf{A}})G(E)\backslash G(E_{\mathbf{A}})$, then $T^c h = h$ for any c. Now $K_{\mathrm{cusp}}(x, y)$ is a cuspidal function of x and a cuspidal function of y. We will denote by T_1^c the truncation operator with respect to the first variable and by T_2^c the truncation operator with respect to the second variable. We will take two numbers c_1 and c_2 with $c_1 > c_2 > 0$. Then $T_1^{c_1} T_2^{c_2} K_{\mathrm{cusp}} = K_{\mathrm{cusp}}$. Thus:

$$K_{\mathrm{cusp}}(x, y) = T_1^{c_1} T_2^{c_2} K(x, y) - T_1^{c_1} T_2^{c_2} K_{\mathrm{eis}}(x, y) - T_1^{c_1} T_2^{c_2} K_{\mathrm{sp}}(x, y), \tag{5}$$

for x and y in $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$. The function f being fixed we will take c_1 and c_2 as large as need dictates. The number c_2 being fixed, we will see that if c_1 is large enough then every term in (5) is integrable over the product of $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ by itself; furthemore the integrals have limits as c_1 tends to infinity so that we will be able to write:

$$\begin{aligned}\iint K_{\mathrm{cusp}}(x, y)\mathrm{d}x\mathrm{d}y = {} & \lim \iint T_1^{c_1} T_2^{c_2} K(x, y)\mathrm{d}x\mathrm{d}y \\ & - \lim \iint T_1^{c_1} T_2^{c_2} K_{\mathrm{eis}}(x, y)\mathrm{d}x\mathrm{d}y \\ & - \lim \iint T_1^{c_1} T_2^{c_2} K_{\mathrm{sp}}(x, y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

At this point, each term will be a function of c_2. We will perform an asymptotic evaluation of each term for c_2 large: in a precise way we will write each term as the sum of a linear function of $\log c_2$ plus a function tending to 0 as c_2 tends to infinity. The integral of K_{cusp} will then be evaluated by evaluating each linear function at zero. This will give the expression we need: see section (10).

(6.2) In section (6) we study the truncation of K. By definition we have:

$$T_1^{c_1} T_2^{c_2} K(x, y) = \tag{1}$$

$$K(x, y) \tag{1.i}$$

$$-\sum_{\gamma_2}\int\sum_{\xi} f(x^{-1}\xi n\gamma_2 y)\mathrm{d}n\chi_{c2}(H(\gamma_2 y)) \tag{1.ii}$$

$$-\sum_{\gamma_1}\int\sum_{\xi} f(x^{-1}\gamma_1^{-1}n\xi y)\mathrm{d}n\chi_{c_1}(H(\gamma_1 x)) \tag{1.iii}$$

$$+\sum_{\gamma_1,\gamma_2}\iint\sum_{\xi} f(x^{-1}\gamma_1^{-1}n_1\xi n_2\gamma_2 y)\mathrm{d}n_1\mathrm{d}n_2$$

$$\chi_{c_1}(H(\gamma_1 x))\chi_{c_2}(H(\gamma_2 y)); \tag{1.iv}$$

in this expression the sums for γ_1 and γ_2 are over $P(F)\backslash G(F)$; the sum for ξ is over $Z(E)\backslash G(E)$; the integrals for n, n_1 and n_2 are over $N(E)\backslash N(E_{\mathbf{A}})$. We first prove a lemma:

(2) LEMMA: *Given f there is a number d with the following property: suppose $c > d$; then if c is sufficiently large and x, y are in $G(F_{\mathbf{A}})$ the terms* (1.iii) *and* (1.iv) *cancell while in* (1.ii) *ξ is in $G(F)P(E)$.*

PROOF OF (2): By lemma (5.2.2) there is $d > 0$ such that for $c_1 > d$, $c_2 > d$ in (1.iv) ξ is actually in $P(E)$. Then the integration for n_2 is superfluous. By lemma (5.2.3) we may assume c_1 so large that in (1.iii) ξ is actually in the set $P(E)G(F)$. Then (1.iii) and (1.iv) do cancel. Furthermore we may assume d so large that, for $c_2 > d$, ξ in (ii) is actually in $G(F)P(E)$. This concludes the proof of the lemma.

We may further write the expression in (1.ii) as the sum of two terms written below as (3.i) and (3.ii):

$$\sum_{\gamma_2,\gamma_2}\int\sum_{\xi} f'(x^{-1}\gamma_1^{-1}\xi n\gamma_2 y)\mathrm{d}n\chi_{c_2}(H(\gamma_2 y)),$$

$$\xi\in N(E),\quad n\in N(E)\backslash N(E_{\mathbf{A}}),\quad \gamma_1\in Z(F)N(F)\backslash G(F),$$

$$\gamma_2\in P(F)\backslash G(F); \tag{3.i}$$

$$\sum_{\gamma_1,\gamma_2}\int\sum_{\xi} f(x^{-1}\gamma_1^{-1}\xi n\gamma_2 y)\mathrm{d}n\chi_{c_2}(H(\gamma_2 y)),$$

$$\xi\in Z(E)\backslash[P(E)-P(F)N(E)],\quad n\in N(E)\backslash N(E_{\mathbf{A}}),$$

$$\gamma_2\in P(F)\backslash N(F),\quad \gamma_1\in P(F)\backslash G(F). \tag{3.ii}$$

In each term we may recombine an integration on $N(E)\backslash N(E_{\mathbf{A}})$ with a sum on $N(E)$ to get:

$$\sum_{\gamma_1,\gamma_2} \int f(x^{-1}\gamma_1^{-1}n\gamma_2 y)\mathrm{d}n\chi_{c_2}(H(\gamma_2 y)),$$

$$n \in N(E_{\mathbf{A}}),\quad \gamma_1 \in Z(F)N(F)\backslash G(F),\quad \gamma_2 \in P(F)\backslash G(F); \tag{4.i}$$

$$\sum_{\gamma_1,\gamma_2} \int \sum_{\alpha} f(x^{-1}\gamma_1^{-1}\,\mathrm{diag}(\alpha, 1)n\gamma_2 y)\mathrm{d}n\chi_{c_2}(H(\gamma_2 y)),$$

$$n \in N(E_{\mathbf{A}}),\quad \alpha \in E^{\times} - F^{\times},\quad \gamma_2 \in P(F)\backslash G(F),$$

$$\gamma_1 \in P(F)\backslash G(F). \tag{4.ii}$$

Similarly we write K as the sum

$$K = K_{\mathrm{s}} + K_{\mathrm{e}} + K_{\mathrm{r}} \tag{5}$$

and use the results of section 4 to break up K_{r} into a sum of two terms written below as (5.i) and (5.ii):

$$\sum f(x^{-1}\gamma_1^{-1}n(v)\gamma_2 y),$$

$$v \in E - F,\ \gamma_1 \in Z(F)N(F)\backslash G(F),\quad \gamma_2 \in P(F)\backslash G(F); \tag{5.i}$$

$$\tfrac{1}{2}\sum f(x^{-1}\gamma_1^{-1}\,\mathrm{diag}(d, 1)\gamma_2 y),$$

$$\alpha \in E^{\times} - F^{\times},\quad \gamma_1 \in A(F)\backslash G(F),\quad \gamma_2 \in A(F)\backslash G(F). \tag{5.ii}$$

Note that in (5.ii) we have used the fact that $A(F)$ has index two in its normalizer. We have thus written $T_1^{c_1}T_2^{c_2}K$ as the sum of K_{s}, K_{e}, the difference (6.2.5.i)–(6.2.4.i) and the difference (6.2.5.ii)-(6.2.4.ii).

(6.3) We now deal with the difference (6.2.5.i)–(6.2.4.i).

(1) LEMMA: *The difference* (6.2.5.i)–(6.2.4.i) *is integrable over the product of* $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ *by itself.*

PROOF OF (1): This difference can be written as

$$\sum F(\gamma_1 x, \gamma_2 y),\quad \gamma_1 \in Z(F)N(F)\backslash G(F),\quad \gamma_2 \in P(F)\backslash G(F)$$

where we have set (here $c_2 = c$):

$$F(x, y) = \sum f[x^{-1}n(\nu)y] - \int f(x^{-1}ny)\mathrm{d}n\chi_c(H(y)),$$

$$\nu \in E - F,\quad n \in N(E_{\mathbf{A}}). \tag{2}$$

Thus it suffices to show that F is integrable over the product

$$N(F)Z(F_{\mathbf{A}})G(F_{\mathbf{A}})\times P(F)Z(F_{\mathbf{A}})\backslash G(F_{\mathbf{A}}).$$

To see that we will use the Iwasawa decomposition. Treating the variables in the maximal compact as parameters and then ignoring these parameters we see that what we have to show is the following: let H be the function on $E_{\mathbf{A}}^{x}\times E_{\mathbf{A}}$ defined by

$$H(a, b)=f[\mathrm{diag}(a, 1)n(b)]; \tag{3}$$

then the following expression is finite:

$$\int\left|\sum H[a^{-1}b, b^{-1}(v+n_1+n_2)]-\chi_c(|b|_E)\int H(a^{-1}b, b^{-1}x)\mathrm{d}x\right|$$

$$|ba|_F^{-1}\mathrm{d}n_1\mathrm{d}n_2\mathrm{d}^{x}a\mathrm{d}^{x}b. \tag{4}$$

In this expression ν is summed over $E-F$, n_1 and n_2 are integrated over the quotient $N(F)\backslash N(F_{\mathbf{A}})$, x is integrated over $E_{\mathbf{A}}$, a is integrated over the idele group of F and b over the idele class group of F. Clearly one of the integrations over $N(F)\backslash N(F_{\mathbf{A}})$ is superfluous. Moreover we may change a into ab. Then a varies in a compact set so we can treat it as a parameter that we ignore. Finally what we have to prove is the following: let Φ be a smooth function of compact support on $E_{\mathbf{A}}$. Then the following expression is finite:

$$\int\left|\sum\Phi[b^{-1}(v+n)]-\chi_c(|b|)\int\Phi(b^{-1}x)\mathrm{d}x\right||b|_F^{-2}\mathrm{d}n\mathrm{d}^{x}b; \tag{3}$$

the summation is for ν in $E-F$ and the integration for n is over $F\backslash F_{\mathbf{A}}$ and the integration for b is over the idele class group of F. In (3) we first integrate for $|b|_F<c^{1/2}$. We identify E to the sum of 2 copies of F. Then we get:

$$\iint\left|\sum_{\alpha,\beta}\Phi(b^{-1}\alpha, b^{-1}\beta+b^{-1}n)\right|\mathrm{d}n|b|_F^{-2}\mathrm{d}^{x}b, \quad \beta\in F, \quad \alpha\in F^{\times}. \tag{4}$$

This is less than the following expression:

$$\iint\sum_{\alpha,\beta}|\Phi(b^{-1}\alpha, b^{-1}\beta+b^{-1}n)||b|_F^{-2}\mathrm{d}^{x}b\mathrm{d}n.$$

If we combine the sum on β with the integration on n and then change n to bn we get:

$$\int\sum_{\alpha}|\Phi(b^{-1}\alpha, n)|\mathrm{d}n|b|_F^{-1}\mathrm{d}^{\times}b, \quad \alpha\in F^{\times}, \quad n\in F_{\mathbf{A}}, \quad |b|_F<c^{1/2}$$

which is clearly finite. Now we integrate over $|b|_F > c^{1/2}$ in (3). We get:

$$\int \left| \sum_v \Phi[b^{-1}(v+n)] - \int \Phi(b^{-1}x)\mathrm{d}x \right| \mathrm{d}n |b|_F^{-2} \mathrm{d}^\times b. \tag{5}$$

But this is majorized by the sum of the following 2 terms:

$$\int \left| \sum_v \Phi[b^{-1}(\nu+n)] - \int \Phi(b^{-1}x)\mathrm{d}x \right| \mathrm{d}n |b|_F^{-2} \mathrm{d}^\times b, \quad v \in E; \tag{5.i}$$

$$\iint \left| \sum_v \Phi[b^{-1}(\nu+n)] \right| \mathrm{d}n |b|_F^{-2} \mathrm{d}^\times b, \quad v \in F. \tag{5.ii}$$

The term (5.ii) is itself majorized by

$$\int \sum_v |\Phi|[b^{-1}(\nu+n)]\mathrm{d}n|b|_F^{-2}\mathrm{d}^\times b = \int |b|_F^{-2}\mathrm{d}^\times b \int |\Phi|(b^{-1}n)\mathrm{d}n;$$

in the right hand side n is integrated over $F_{\mathbf{A}}$. After changing n to bn we finally get

$$\int |b|_F^{-1} \mathrm{d}^\times b \int |\Phi|(n)\mathrm{d}n, \quad |b|_F > c^{1/2},$$

which is clearly finite. As for (5.i) by Poisson summation formula it is equal to

$$\int \left| \sum \Phi\hat{\ }(b\gamma)\psi_E(\nu n) \right| \mathrm{d}n \mathrm{d}^\times b, \quad \nu \in E^\times.$$

This in turn is less than

$$\int \sum |\Phi\hat{\ }|(bv)\mathrm{d}^x b, \quad v \in E^x, \quad |b|_E > c,$$

which is clearly finite. This concludes the proof of lemma (1).

We now compute the integral of the difference (6.2.5.i)–(6.2.4.i). We proceed formally. We let H be the function on $E_{\mathbf{A}}^\times \times E_{\mathbf{A}}$ defined by:

$$H[a, b] = \iint f[k_1 \operatorname{diag}(a, 1)n(b)k_2]\mathrm{d}k_1\mathrm{d}k_2. \tag{6}$$

Then the integral of the difference (6.2.5.i)–(6.2.4.i) is equal to:

$$\iint\Big\{\sum_{v} H\left[a^{-1}b,\, b^{-1}(v+n_1+n_2)\right] -$$

$$\chi_{c_2}(|b|)\int H(a^{-1}b,\, b^{-1}x)\mathrm{d}x\Big\}\mathrm{d}n_1\mathrm{d}n_2|ab|_F^{-1}\mathrm{d}^{x}a\mathrm{d}^{x}b. \tag{7}$$

The set in which the variables are is the same as before; in particular ν is in $E-F$. We change a into ab and then, for convenience, b into b^{-1}. We get:

$$\int\Big\{\int\sum_{\nu}\Phi[b(\nu+n)]\mathrm{d}n-\int\Phi(bx)\mathrm{d}x\chi_{c_2}\big(|b|_E^{-1}\big)\Big\}|b|_F^{2}\mathrm{d}^{\times}b, \tag{8}$$

where we have set:

$$\Phi(b)=\int H(a^{-1},\, b)|a|_F^{-1}\mathrm{d}^{\times}a. \tag{9}$$

In (8) we break up the integral over b into two pieces: $|b|>1$ and $|b|<1$. For the piece $|b|<1$ we remark that $|b|_F^{-1}>C_2^{-1/2}>1$ implies $|b|_F<1$. We also use a simple consequence of the Poisson summation formula:

$$\int\sum_{\nu}\Phi[b(\nu+n)]\mathrm{d}n-\int\Phi(bx)\mathrm{d}x$$

$$=\sum\Phi\hat{\ }(b^{-1}\nu)|b|_F^{-2}-\int\Phi(bn)\mathrm{d}n, \tag{10}$$

$\nu\in E-F$, $x\in E_{\mathbf{A}}$, $n\in F\backslash F_{\mathbf{A}}$ on the left; $\nu\in F-\emptyset$, $n\in F_{\mathbf{A}}$ on the right. This supposes that ψ_E is so chosen that its restriction to $F_{\mathbf{A}}$ is trivial. Thus (8) can be expressed as a sum:

$$\int_{|b|\geqslant 1}\Big\{\sum_{\nu\in E-F}\Phi[b(\nu+n)]\mathrm{d}n\Big\}|b|_F^{2}\mathrm{d}^{\times}b$$

$$+\Big\{\int_{|b|\leqslant 1}\big(1-\chi_{c_2}\big(|b|_E^{-1}\big)\mathrm{d}^{\times}b\Big\}\Phi\hat{\ }(0).$$

$$+\int_{|b|\leqslant 1}\sum_{v\in F^{\times}}\Phi\hat{\ }(b^{-1}v)\mathrm{d}^{\times}b-\iint\Phi(n)\mathrm{d}n|b|_F\mathrm{d}^{\times}b. \tag{11}$$

Now we write $E=F[\sqrt{\tau}\,]$ and take $\psi_E(z)=\psi_F((z-z)/2\sqrt{\tau}\,)$. We set

$$\Psi(v)=\int\Phi(u+v\sqrt{\tau}\,)\mathrm{d}u \tag{12}$$

and let $\Psi\hat{}$ denote the Fourier transform of Ψ as a Schwartz-Bruhat function on $F_{\mathbf{A}}$. We see at last that the integral of the difference (6.2.5.i)–(6.2.4.i) is given by the following expression:

$$\int_{|b|\geqslant 1}\Big\{\sum_{v\in F^\times}\Psi(bv)\Big\}|b|_F\mathrm{d}^\times b+\int_{|b|\geqslant 1}\Big\{\sum_{v\in F^\times}\Psi\hat{}(bv)\Big\}\mathrm{d}^\times b$$

$$+\Psi\hat{}(0)\int_{|b|\leqslant 1}\Big\{1-\chi_{c_1}\big(|b|_E^{-1}\big)\Big\}\mathrm{d}^\times b-\Psi(0)\int_{|b|\leqslant 1}|b|_F\mathrm{d}^\times b. \qquad (13)$$

Recall that the Tate integral, or rather its analytic continuation, has a pole at $s=1$:

$$\int\Psi(b)|b|_F^s\mathrm{d}^\times b=c_{-1}(\Psi)/s-1+c_0(\Psi)+\ \ldots .$$

Furthemore recall that $c_0(\Psi)$ vanishes if Ψ is a product of local components Ψ_v and there are at least 2 places v of F such that

$$\int\Psi_v(b_v)\mathrm{d}b_v=0.$$

The expression (13) is nothing but

$$c_0(\Psi)+C\log(c_2),\quad C \text{ a constant.}$$

We are now within reach of our goal in this subsection:

(14) Lemma: *The integral of the difference* (6.2.5.*i*)–(6.2.4.*i*) *is a linear function of* $\log c_2$. *If condition* (6.1.1)) *is satisfied the constant term vanishes.*

Proof of (14): The function Ψ introduced in (12) is a product of local functions over all places of F and we need only verify that for v in X the integral of the local component vanishes. Call again $v1$ and $v2$ the two places of E above v. Then, with obvious notations:

$$\int\Psi_v(b_v)\mathrm{d}b_v$$

$$=\int H_{v1}\big[(a^{-1},\,n+b\sqrt{\tau})\big]\,H\big[(a^{-1},\,n-b\sqrt{\tau})\big]\mathrm{d}n\mathrm{d}b\mathrm{d}^\times a,$$

$$n\in F_v,\quad b\in F_v,\quad a\in F_v^\times.$$

Recalling the definition of H (cf. (6) above) we see this is the integral over k_1, k_2 in K_v, a in $F_v^\times$, x and b in F_v of:

$$f_{v1}\left[k_1^{-1}\operatorname{diag}(a,1)n(x+b\sqrt{\tau})k_2\right]f_{v2}\left[k_1^{-1}\operatorname{diag}(a,1)n(x-b\sqrt{\tau})k_2\right].$$

This is also the following integral, where g is integrated over G_v:

$$\int f_{v1}\left[gn(b\sqrt{\tau})k\right]f_{v2}\left[gn(-b\sqrt{\tau})k\right]\mathrm{d}g\mathrm{d}b\mathrm{d}k.$$

Recall that h_v is the convolution product of f_{v1}^{v} and f_{v2}. This integral is nothing but:

$$\iint h_v\left[k^{-1}n(-2b\sqrt{\tau})k\right]\mathrm{d}b\mathrm{d}k.$$

This is a unipotent orbital integral of h_v. Since the hyperbolic integrals of h_v vanish the above integral vanish too. This concludes the proof of lemma (13).

(6.4) We now study the difference (6.2.5.ii)–(6.2.4.ii). We start with a remark. Consider the following function of $y \in G(F_{\mathbf{A}})$:

$$\sum f\left[x^{-1}\gamma_1^{-1}\operatorname{diag}(\alpha,1)y\right],\quad \gamma_1 \in A(F)\backslash G(F),\quad \alpha \in E^\times - F^\times.$$

It is invariant on the left under the normalizer of A. In particular call w the following element of this normalizer:

$$w = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

Then we have:

$$\sum f\left[x^{-1}\gamma_1^{-1}\operatorname{diag}(\alpha,1)\gamma_2 y\right]\chi_{c_2}(H(\gamma_2 y))$$
$$=\sum f\left[x^{-1}\gamma_1^{-1}\operatorname{diag}(\alpha,1)\gamma_2 y\right]\chi_{c_2}(H(\omega\gamma_2 y)),$$
$$\alpha \in E^\times - F^\times,\quad \gamma_1,\gamma_2 A(F)\backslash G(F).$$

In particular the difference (6.2.5.ii)–(6.2.4.ii) can be written as the sum of two terms written below as (1.i) and (1.ii):

$$\frac{1}{2}\sum_{\gamma_1,\gamma_2}\left\{\sum_{\alpha} f\left[x^{-1}\gamma_1^{-1}\operatorname{diag}(\alpha,1)\gamma_2 y\right]\right.$$
$$\left.\left[1-\chi_{c_2}(H(\gamma_2 y)-\chi_{c_2}(H(\omega\gamma_1 y))\right]\right\},$$
$$\gamma_1,\gamma_2 \in A(F)\backslash G(F),\quad \alpha \in E^\times - F^\times; \tag{1.i}$$

$$\sum_{\gamma_1, \gamma_2} \left\{ f\left[x^{-1}\gamma_1^{-1} \operatorname{diag}(\alpha, 1)\gamma_2 y\right] \chi_{c_2}(H(\gamma_2 y)), \right.$$

$$- \sum_{\gamma_1, \gamma_2} \int \sum_{\alpha} f\left[x^{-1}\gamma_1^{-1} \operatorname{diag}(\alpha, 1) n \gamma_2 y\right] \mathrm{d}n \chi_{c_2}(H(\gamma_2 y)), \qquad (1.\text{ii})$$

in the first term $\gamma_1, \gamma_2 \in A(F)\backslash G(F)$, in the second term $\gamma_1, \gamma_2 \in P(F)\backslash G(F)$, $n \in N(E_{\mathbf{A}})$; in both terms $\alpha \in E^{x} - F^{x}$.

We discuss the integral of each term separately.

(2) LEMMA: *The term* (1.i) *is integrable. Its integral is a linear function of* $\log c_2$. *Under the assumptions of* (6.1.1) *the constant term is zero.*

PROOF OF (2): We forego the verification of integrability. It does not differ substantially from the formal computation of the integral that we now present. When we integrate (1.i) on the product of the quotient $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ by itself we can combine the summations on $A(F)\backslash G(F)$ with the integration to obtain an integral on the product of $Z(F_{\mathbf{A}})A(F)\backslash G(F_{\mathbf{A}})$ by itself:

$$\iint \sum_{\alpha} f\left[x^{-1} \operatorname{diag}(\alpha, 1) y\right]\left(1 - \chi_{c_2}(H(y)) - \chi_{c_2}(H(wy))\right) \mathrm{d}x \mathrm{d}y,$$

$$\alpha \in E^{\times} - F^{\times}, \quad x, y \in Z(F_{\mathbf{A}})A(F)\backslash G(F_{\mathbf{A}}).$$

We then use the Iwasawa decomposition to compute the integrals in x and y. We get:

$$\int \sum_{\alpha} f\left[k_1^{-1} \operatorname{diag}(a^{-1}\alpha b, 1) n(\alpha^{-1} b^{-1} x + y) k_2\right]$$

$$\left[1 - \chi_{c_2}(|b|) - \chi_{c_2}(|b|_E^{-1} H(wn))\right] |a|_F^{-1} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}x \mathrm{d}y \mathrm{d}^{\times}a \mathrm{d}^{\times}b;$$

here α is summed over $E^{\times} - F^{\times}$, a and b are integrated over the idele class group of F, x and y over the adeles of F; furthermore

$$n = n(y).$$

In the above integral we can change a to ab and y to yb. The integral

$$\int \sum_{a} f\left[k_1^{-1} \operatorname{diag}(a^{-1}\alpha, 1) n(\alpha^{-1}x + y) k\right] \mathrm{d}x \mathrm{d}y \mathrm{d}k_1 \mathrm{d}k_2 |a|_F^{-1} \mathrm{d}^{\times}a$$

$$\left\{1 - \chi_{a_2}(|b|) - \chi_{c_2}(|b|^{-1} H(wn))\right\} \mathrm{d}^{\times}b.$$

In the inner integral we have $H(wn)<1$. We may assume $c_2>1$ and then the range of b is defined by the inequalities $c_2^{-1}<H(wN)<|b|_E<c_2$. Thus after evaluation the inner integral gives us a factor $C'\log_2+C''\log H(wn)$ where C' and C'' are constants. Thus the integral of (1.1) can be written as

$$C\log c_2+C''\int\sum_{\alpha}f\left[k_1^{-1}\operatorname{diag}(a^{-1}\alpha,1)n(\alpha^{-1}x+y)k\right]$$

$$\log H(wn)|a|_F^{-1}d^{\times}a\,dx\,dy\,dk_1dk_2.$$

We can break up the sum on α into a sum on β in $F^{\times}$ followed by a sum on α in $(E^{\times}-F^{\times})/F^{\times}$; furthemore we can change x into $x\beta$ and then combine the integration on a with the summation on β to get an integral on the idele group of F. After a simple formal manipulation we get:

$$C\log c+C''\sum_{\alpha}\int f\left[k_1^{-1}\operatorname{diag}(a^{-1},1)n(x)\operatorname{diag}(\alpha,1)n(y)k_2\right]$$

$$\log H(wn)dx\,dy\,dk_1dk_2|a|_F^{-1}d^{\times}a.$$

Using again the Iwasawa decomposition we recombine the integrals on a, x and k into an integral on $Z(F_{\mathbf{A}})\backslash G(F_{\mathbf{A}})$. So we see that the integral of (1.i) can be written also as:

$$C\log c_2+C''\sum_{\alpha}\int f\left[g\operatorname{diag}(\alpha,1)nk\right]\log H(wn)dg\,dn\,dk,$$

$$\alpha\in F^{\times}\backslash(E^{\times}-F^{\times}),\quad n\in N(F_{\mathbf{A}}),\quad g\in G(F_{\mathbf{A}}),\quad k\in K_F.$$

We claim that under the assumption of (6.1.1) the second term vanishes. Indeed, it can alos be written as a double (actually finite) sum over α and all places u of F of the following integral:

$$\int f\left[g\operatorname{diag}(\alpha,1)nk\right]\log H(wn)dg\,dn\,dk. \tag{4}$$

This integral is a product of local integrals and a constant. Since X has two elements, there is at least one v in X different of u. The corresponding local integral is a factor of (4). We claim that this local integral is actually zero; our claim will then prove lemma (2). Let $v1$ and $v2$ be the two places of E above x; let also α_1 and α_2 be the images of α under the isomorphisms $E_{v1}\cong F_v$ and $E_{v2}\cong F_v$. Then our local integral is:

$$\int f_{v1}\left[g\operatorname{diag}(\alpha_1,1)nk\right]f_{v2}\left[g\operatorname{diag}(\alpha_2,1)nk\right]dg\,dn\,dk.$$

After a change of variables, this can be written as:

$$\int f_{v1}[g] f_{v2}[gk^{-1}n^{-1} \operatorname{diag}(\alpha_1^{-1}\alpha_2, 1)nk] \mathrm{d}g\mathrm{d}n\mathrm{d}k.$$

Recall the convolution product h_v of f_{v1}^{v} and f_{v2}. This integral is nothing but

$$\int h_v[k^{-1}n^{-1} \operatorname{diag}(\alpha_1^{-1}\alpha_2, 1)nk] \mathrm{d}n\mathrm{d}k;$$

since α is not in F, $\alpha_1^{-1}\alpha_2 \neq 1$ and this is an hyperbolic orbital integral of h_v. Thus it is zero by assumption.

We now pass to the term (1.ii).

(5) LEMMA: *The term* (1.ii) *is integrable over the product of the quotient* $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ *by itself. Its integral is zero.*

PROOF OF (5): Consider the first term in the difference (1.ii). It can also be written as follows:

$$\sum f[x^{-1}\gamma_1^{-1} \operatorname{diag}(\alpha, 1)n(\alpha^{-1}\nu_1 + \nu_2)\gamma_2 y] \chi_{c_2}(H(\gamma_2 y)),$$

$$\alpha \in E^{\times} - F^{\times}, \quad \gamma_1, \gamma_2 \in P(F)\backslash G(F), \quad \nu_1, \nu_2 \in F.$$

Since α is not in F every element of E can be written as a sum $\alpha^{-1}\nu_1 + \nu_2$, where v_1 and v_2 are in F. Thus the first term in the difference (1.ii) can also be written in the form:

$$\sum f[x^{-1}\gamma_1^{-1} \operatorname{diag}(\alpha, 1)n(\mu)\gamma_2 y] \chi_{c_2}(H(\gamma_2 y)),$$

$$\alpha \in E^{\times} - F^{\times}, \quad \gamma_1, \gamma_2 \in P(F)\backslash G(F), \quad \mu \in E.$$

As a consequence the difference (1.ii) can now be written as a double sum

$$\sum F(\gamma_1 x, \gamma_2 y), \gamma_1, \gamma_2 \in P(F)\backslash G(F),$$

where we have set:

$$F(x, y) = \sum_{\alpha,\mu} f[x^{-1} \operatorname{diag}(\alpha, 1)n(\mu)y] \chi_{c_2}(H(y))$$

$$- \sum_{\alpha} f[x^{-1} \operatorname{diag}(\alpha, 1)n(z)y] \mathrm{d}x \chi_{c_2}(H(y)) \mathrm{d}z. \qquad (6)$$

To show that (1.ii) is integrable it is therefore enough to show that F is integrable over the product of $Z(F_{\mathbb{A}})P(F)\backslash G(F_{\mathbb{A}})$ by itself. We use the Iwasawa decomposition. To show integrability we treat the variables in the maximal compact as parameters that we ignore. Setting

$$H(a, x) = f[\mathrm{diag}(a, 1)n(x)] \tag{7}$$

we see that we have to prove the following expression is finite:

$$\int \Big| \sum_{\alpha,\mu} H\big[a^{-1}\alpha b,\ b^{-1}\big(\alpha^{-1}x_1 + \mu + x_2\big)\big] - \int H[a^{-1}\alpha b,\ b^{-1}z]\mathrm{d}z \Big| |a|_F^{-1} |b|_F^{-1} \chi_{c_2}(|b|)\mathrm{d}^\times b\,\mathrm{d}^\times a\,\mathrm{d}x_1\mathrm{d}x_2. \tag{8}$$

We may change a to ab to get

$$\int \Big| \sum_{\alpha,\mu} H\big[a^{-1}\alpha,\ b\big(\alpha^{-1}x_1 + \mu + x_2\big)\big] - \int H[a^{-1}\alpha,\ b^{-1}z]\mathrm{d}z \Big| |a|_F^{-1} |b|_F^{-1} \chi_{c_2}(|b|)\mathrm{d}^\times b\,\mathrm{d}^\times a\,\mathrm{d}x_1\mathrm{d}x_2. \tag{9}$$

Now $a^{-1}\alpha$ stays in a fixed compact set. Hence the module of a stays in a compact set of $\mathbb{R}^\times$. This implies that a stays in a relatively compact set; therefore α stays in a compact set, hence in fact in a finite set. Thus our assertion on convergence will be proved if we show that the following expression is finite:

$$\int \sum \Big| \Phi\big[b^{-1}\big(\alpha^{-1}x_1 + x_2 + \mu\big)\big] - \int \Phi[b^{-1}z]\mathrm{d}z \Big| \mathrm{d}x_1\mathrm{d}x_2 |b|_F^{-2} \chi_{c_2}(|b|)\mathrm{d}^\times b; \tag{10}$$

In this expression α is some element of $E - F$, μ is summed over E, z is integrated over E and x_1, x_2 over $F\backslash F_{\mathbb{A}}$. By Poisson summation formula this is also:

$$\int \Big| \sum_{\mu} \Phi\hat{\ }(b)\psi_E\big(\alpha^{-1}x_1 + x_2\big)\mu\big) \Big| \chi_{c_2}(|b|_E)\mathrm{d}^\times b\,\mathrm{d}x_1\mathrm{d}x_2,$$

where μ is now in $E^\times$. This is majorized by

$$\int \sum |\Phi\hat{\ }(b\mu)|\mathrm{d}^\times b, \quad \mu \in E^\times, \quad |b|_E > c^{1/2}.$$

This is clearly finite.

It remains to prove that the integral of (1.ii) over the product of $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$ by itself is zero. We compute it as the integral of (6) over the product of $Z(F_{\mathbb{A}})P(F)\backslash G(F_{\mathbb{A}})$ by itself. Setting now instead of (7)

$$H(a, x)=\iint f[k_1 \operatorname{diag}(a, 1)n(x)k_2]\mathrm{d}k_1\mathrm{d}k_2 \tag{11}$$

we get for our integral an analogue of (8) and then (9):

$$\int\Big\{\sum_{\alpha,\mu} H\big[a^{-1}\alpha,\ b^{-1}\big(\alpha^{-1}x_1+\psi+x_2\big)\big]$$
$$-\sum\int H[a^{-1}\alpha,\ b^{-1}z]\mathrm{d}z\Big\}|a|_F^{-1}|b|_F^{-2}\chi_{c_2}(|b|)\mathrm{d}^\times b\mathrm{d}^\times a\mathrm{d}x_1\mathrm{d}x_2. \tag{12}$$

However this is also:

$$\int\Big\{\sum_{\alpha,\mu} H\big[a^{-1}\alpha,\ b^{-1}\big(\alpha^{-1}x_1+\mu+x_2\big)\big]\mathrm{d}x_1\mathrm{d}x_2$$
$$-\int H[a^{-1}\alpha,\ b^{-1}z]\mathrm{d}z\Big\}|a|_F^{-1}|b|_F^{-2}\chi_{c_2}(|b|)\mathrm{d}^\times b\mathrm{d}^\times a. \tag{13}$$

Since α is not in F the map $(x_1, x_2)\mapsto\alpha^{-1}x_1+x_2$ is a bijection of F^2 onto E. So the sum on μ in E and the integrations for x_1 and x_2 can be combined to give an integration on $E_{\mathbb{A}}$ and this expression is zero.

(6.5) We may summarize the results of this section as follows:

PROPOSITION: *If c_1 is sufficiently large then $T_1^{c_1}T_2^{c_2}K$ is independent of c_1; it is integrable over the product of $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$ by itself. Furthermore its integral is a linear function in* $\log c_2$ *whose constant term vanishes if* (6.1.1) *is satisfied, plus the integral of K_{e} plus the integral of K_A.*

Indeed this integral is the sum of the integral of K_{e}, K_{s}, the difference (6.2.5.i)–(6.2.4.i), the term (6.4.1.i) and the term (6.4.1.ii). Our assertion follows then from lemmas (6.3.1), (6.3.14), (6.4.2), (6.4.5) and (6.2.6).

7. Local preliminaries

(7.1) In order to study the truncation of the kernel K_{eis} we need some information on local "intertwining operators" and certain local integrals, not all of which is available. This information is reviewed or presented in this section. Accordingly, in this section the field F is local and G is the group GL(2) regarded as an F-group.

(7.2) Let χ be a character of (module one) of $F^{\times}$. We denote by $\mathbb{H}(s, \chi)$ the space of functions h on $G(F)$ such that:

$$h[n(x) \operatorname{diag}(a, b)g] = |a/b|^{s+1/2} \chi(a/b) h(g) \tag{1}$$

which are square integrable on K the standard maximal compact subgroup of $G(F)$. We also denote by $\rho(s, \chi)$ the representation of $G(F)$ on $\mathbb{H}(s, \chi)$ by right shifts. We may regard the collection of the spaces $\mathbb{H}(s, \chi)$ as a fiber bundle of base $\mathbb{C}$. We set $\mathbb{H}(\chi) = \mathbb{H}(-1/2, \chi)$. If h is in that space we define a section of our fiber bundle by:

$$h(g, s) = h(g) H(g)^{s+1/2},$$

$$\text{where } H(g) = |a/b| \text{ if } g = n \operatorname{diag}(a, b)k. \tag{2}$$

The fiber bundle is trivial and sections obtained in this way will be called constant. We may identify $\mathbb{H}(s, \chi)$ with $\mathbb{H}(\chi)$ and regard $\rho(s, \chi)$ as operating on $\mathbb{H}(\chi)$. On the product $\mathbb{H}(\chi) \times \mathbb{H}(\chi)$ we define a bilinear form:

$$\langle h, h' \rangle = \int h(k) h'(k) \mathrm{d}k; \tag{3}$$

then:

$$\langle \rho(s, \chi)(g)h, \rho(-s, \bar{\chi})(g)h' \rangle = \langle h, h' \rangle.$$

Similarly on the product $\mathbb{H}(\chi) \times \mathbb{H}(\chi)$ we define a sesquilinear form:

$$(h, h') = \int h(k) \bar{h}'(k) \mathrm{d}k; \tag{4}$$

then:

$$(\rho(s, \chi)(g)h, \rho(-s, \chi)(g)h') = (h, h').$$

In particular when s is purely imaginary the representation $\rho(s, \chi)$ is unitary. As usual we define the intertwining operator $M(s, \chi)$ as an operator from $\mathbb{H}(\chi)$ to $\mathbb{H}(\chi)$ by

$$[M(s, \chi)h](g) H(g)^{-s+1/2} = \int h(wng) H(wng)^{s+1/2} \mathrm{d}n, \quad n \in N(F); \tag{5}$$

the integral converges for Re $s > 0$ but extends meromorphically to the whole complex plane. We may also regard $M(s, \chi)$ as an operator from $\mathbb{H}(s, \chi)$ to $\mathbb{H}(-s, \bar{\chi})$. We define also a "normalized intertwining operator" $R(s, \chi)$ by

$$M(s, \chi) = \left[L(2s, \chi^2)/L(2s+1, \chi^2)\varepsilon(2s, \chi^2, \psi)\right] \cdot R(s, \chi). \quad (6)$$

For our purposes it is convenient to use a description of these operators in terms of the Whittaker modele (Cf. [F.S.]): accordingly let $W(s, \chi)$ be the map from $\mathbb{H}(\chi)$ to the Whittaker modele of the representation $\rho(s, \chi)$; it is defined by

$$W(s, \chi)h(g) = \int h[wn(x)g, s] \cdot \psi_F(-x)\mathrm{d}x, \ x \in F. \quad (7)$$

The integral converges for Re $s > 0$ but extends holomorphically to the whole complex plane. In particular we have:

$$[W(s, \chi)h](\mathrm{diag}(a, 1)) = \int h[wn(x)g, s] \cdot \psi_F(-ax)\mathrm{d}x\bar{\chi}(a)|a|^{-s+1/2}. \quad (8)$$

If Re $s = 0$ the integral on the right hand side can be interpreted as the Fourier transform of the square integrable function $x \mapsto f[wn(x)]$. In particular the scalar product (3) can be computed as

$$\langle h, h' \rangle = c \int W(\mathrm{diag}(a, 1))W'(\mathrm{diag}(-a, 1))\mathrm{d}^\times a, \quad (9)$$

where W and W' are the images of h and h' under $W(s, \chi)$ and $W(-s, \bar{\chi})$ respectively and c is a constant. Similarly, the scalar product (4) can be computed as

$$(h, h') = c \int W(\mathrm{diag}(a, 1))\overline{W'}(\mathrm{diag}(a, 1))\mathrm{d}a, \quad (10)$$

where W and W' are the images of h and h' under $W(s, \chi)$. This being so we have the following result:

(11) LEMMA: *With the previous notatons for h in $\mathbb{H}(s, \chi)$ we have*:

$$W(-s, \chi)M(s, \chi)h = \left[L(2s, \chi^2)/L(1-2s, \bar{\chi}^2)\varepsilon(2s, \chi^2, \psi_F)\right] \times W(s, \chi)h.$$

PROOF OF (11): It is easy to see that a constant section can be represented by an integral:

$$h(g, s)=\int \Phi[(0, t)g]|t|^{2s+1}\chi^2(t)\mathrm{d}^x t\chi(\det g)|\det g|^{s+1/2},$$

where Φ is a Schwartz-Bruhat function with compact support contained in the orbit $(0, 1)G(F)$. Then

$$W(s, \chi)h(e)=\iint \Phi[(t, tx)]\chi^2(t)|t|^{2s+1}\mathrm{d}^x t\psi(-x)\mathrm{d}x\chi(-1).$$

After a change of variables this can be written as

$$W(s, \chi)h(e)=\int \Phi_1(t, t^{-1})\chi^2(t)|t|^{2s}\mathrm{d}^\times t\chi(-1)$$

where

$$\Phi_1(x, y)=\int \Phi(x, v)\psi(-vy)\mathrm{d}v.$$

On the other hand if we apply the intertwining operator to $h(g, s)$ we obtain:

$$\int \Phi[(t, tx)g]|t|^{2s+1}\chi^2(t)\mathrm{d}^\times t\chi(\det g)|\det g|^{s+1/2}.$$

We can change x to xt^{-1}; if we introduce the Fourier transform of Φ defined by:

$$\Phi\hat{\ }(x, y)=\iint \Phi(u, v)\psi(xv-yu)\mathrm{d}u\mathrm{d}v$$

and use the local functional equation of the Tate integral we get:

$$C\int \Phi\hat{\ }[(0, t)g]|t|^{-2s+1}\chi^{-2}(t)\mathrm{d}^\times t\tilde{\chi}(\det g)|\det g|^{-s+1/2},$$

where C is the factor in the right hand side of the lemma's formula. Next we find:

$$W(-s, \bar{\chi})M(s, \chi)h(e)=C\chi(-1)\int \hat{\Phi}_1(t, t)|t|^{-2s}\chi^2(t)\mathrm{d}^\times t$$

where

$$\hat{\Phi}_1(x, y)=\int \Phi\hat{\ }(x, v)\psi(-vy)\mathrm{d}v.$$

If we change t to t^{-1} and apply Fourier inversion formula our result follows.

(7.3) If now we substitute R to M in lemma (7.2.11) we find:

$$W(-s,\bar{\chi})R(s,\chi)h=\left[L(1+2s,\chi^2)/L(1-2s,\bar{\chi}^2)\right]W(s,\chi)h \tag{1}$$

The properties of the normalized intertwining operator are now easily established:

$$R(-s,\bar{\chi})R(s,\chi)=\mathrm{Id}; \tag{2}$$

$$\text{If } \chi^2=1 \text{ then } R(0,\chi)=\mathrm{Id}; \tag{3}$$

$$\text{For } h\in\mathbb{H}(\gamma),\ h'\in\mathbb{H}(\chi),\ \langle R(s,\chi)h,\ R(-s,\chi)h'\rangle=\langle h,h'\rangle; \tag{4}$$

$$\text{For } \operatorname{Re} s=0 \text{ and } h,h'\in\mathbb{H}(\chi),\ (R(s,\chi)h,\ R(s,\chi)h')=(h,h'). \tag{5}$$

We will need also the dependence of $R(s,\chi)h$ on s, for h in $\mathbb{H}(\chi)$.

(6) Lemma: *Suppose h is a K-finite element of $\mathbb{H}(\chi)$. Then for k in K, $R(s,\chi)h(k)$ is an elementary function of s, without singularity on the line* $\operatorname{Re} s=0$.

Sketch of Proof of (6): By an elementary function of s we mean a sum of rational functions of s times an exponential if F is Archimedean and a rational fraction in q^{-s} if F is non Archimedean with a residual field of q elements. Now a constant K-finite section can also be represented by an integral:

$$h(g,s)=Q(s)/L(1+2s,\chi^2)$$
$$\times\int\Phi[(0,t)g]\chi^2(t)|t|^{2\lambda+1}\chi(\det g)|\det g|^{s+1/2} \tag{7}$$

where Q is an elementary function of s. Here, if F is Archimedean the function is a standard Schwartz function: that is, if $F=\mathbb{R}$, it has the form $\Phi(x,y)=P(x,y)\exp(-\pi(x^2+y^2))$, and if $F=\mathbb{C}$, it has the form $\Phi(x,y)=P(x,\bar{x},y,\bar{y})\exp(-2\pi(x\bar{x}+y\bar{y}))$, where P is a polynomial.

Applying the normalized intertwining operator we find

$$R(s,\chi)h(g,-s)=Q(s)/L(1-2s,\chi^2)$$
$$\times\int\Phi\hat{}\,[(0,t)g]\,\bar{\chi}^2(t)|t|^{-2s+1}\mathrm{d}^\times t$$
$$\times\bar{\chi}(\det g)|\det g|^{-s+1/2}. \qquad (8)$$

If F is Archimedean Φ is still a standard Schwartz function and our assertion follows from the properties of the Tate integral.

We also recall the following improvement on Lemma (6) for functions invariants under K:

(9) LEMMA: *Suppose that $h(k)=1$ for all k in K. Suppose also that F is non Archimedean, the order ψ of is zero and χ is unramified. Then $R(s,\chi)h(k)=1$ for all k.*

The proof is similar to the proof of lemma (6).

(7.4) We now change our notations: E is a quadratic extension of F and χ a character of $E^\times$, accordingly $\rho(s,\chi)$ is now a representation of $G(E)$. We will be interested in linear forms on the space of smooth vectors in $\mathbb{H}(\chi)$ (or $\mathbb{H}(s,\chi)$) which are invariant under $G(F)$ operating via the representation $\rho(s,\chi)$. Suppose first the restriction of χ to $F^\times$ is trivial and $s=0$. Then the restriction of a smooth function in $\mathbb{H}(s,\chi)$ to $G(F)$ transforms on the left according to the module of the group $P(F)$; therefore the integral of a such function on the group K defines a linear form with this invariant property. What we want to establish in this section is the following result:

(1) LEMMA: *With the above notations, for any smooth vector h in $\mathbb{H}(\chi)$ we have:*

$$\int R(0,\chi)h(k)\mathrm{d}k=\int h(k)\mathrm{d}k,\quad k\in K_F(=K\cap G(F)).$$

PROOF OF (1): Once more it is convenient to use the Whittaker modele. As usual ψ_F is a non trivial additive character of F; we choose a non trivial additive character of E whose restriction to F is trivial. For instance we write $E=F[\sqrt{\tau}]$ and then we take $\psi_E(x)=\psi_F[(x^\sigma-x)/2\sqrt{2}]$. We first establish the existence of a constant c such that, for any h,

$$\int h(k)\mathrm{d}k=c\int W[\mathrm{diag}(a,1)]\mathrm{d}^\times a,\quad k\in K,\quad a\in F^\times. \qquad (2)$$

where W is the image of h under $W(0,\ )$. There is a constant c such that:

$$\int h(k)\mathrm{d}k = c\int h[wn(x), 0]\mathrm{d}x, \quad x \in F. \tag{3}$$

On the other hand we have:

$$W[(\mathrm{diag}(a, 1)] = \int h[wn(z), 0]\,\psi_F(-az)\mathrm{d}z|a|_E^{1/2}\bar{\chi}(a);$$

integrating over F we find:

$$\int W[(\mathrm{diag}(a, 1)]\mathrm{d}^\times a = \int |a|_F d^\times a$$

$$\times \int\int h\left[wn\left(x+\sqrt{\tau}\,y\right)\right]\psi(-ay)\mathrm{d}y\mathrm{d}x.$$

Our assertion follows now from Fourier inversion formula. In view of formula (6.3.1) we have now only to establish the following lemma:

(4) LEMMA: *Suppose χ is a character of $E^\times$ trivial on $F^\times$. We have*:

$$L(s, \chi^2)/L(s, \bar{\chi}^2) = 1.$$

PROOF OF (4): If $F = \mathbb{R}$, $E = \mathbb{C}$, the character χ has the form

$$\chi(z) = z^n(z\bar{z})^{-n/2}$$

and the formula is checked at once. Suppose F is non Archimedean. If χ^2 is ramified both L-factors are equal to one and our assertion is trivial. Suppose χ^2 is unramified. If x_F is a uniformizer for F we have $\chi(x_F) = 1$. Now either x_F is a uniformizer for E or the square of a uniformizer. Thus if χ_E is a uniformizer for E we have $\chi^2(x_E) = 1$. In any case χ^2 is trivial and our assertion follows.

(7.5) We continue with the notations of (7.4). This time we assume that χ is a character of $E^\times$ invariant under σ where $\mathrm{Gal}(E/F) = \{1, \sigma\}$. We first need a description of the double coset space $P(E)\backslash G(E)/G(F)$.

(1) LEMMA: *We have a disjoint union*

$$G(E) = P(E)G(F) \cup P(E)\lambda G(F).$$

Furthemore the algebra $L = \gamma^{-1}R(E)\gamma \cap M(2, F)$ is an algebra of degree 2 on F which is F-isomorphic to E. Finally if $e(l)$ and $e'(l)$ are the two *eigenvalues of* $\gamma l \gamma^{-1}$ *then* $l \mapsto e(l)$ *and* $l \mapsto e'(l)$ *are the two isomorphisms of* L *onto* E.

PROOF OF (1): Recall that R is the algebra of triangular matrices. We will prove the corresponding assertion for the cosets $G(F)\backslash G(E)/P(E)$. We let G operates on the right on the vector space of column vectors of dimension 2. We denote by $\{e, e'\}$ the canonical basis. The group $P(E)$ is the fixator of the line Ee. We let μ be any element of $E - F$ and then set

$$\lambda = \mathrm{Mat}[1, 0, \mu\, 1].$$

Let h be an element of $G(E)$ and D the line hEe, c its "slope". If c is finite this means that D contains the vector $e + ce'$. Suppose c is in F (resp. infinite). Then there is g in $G(F)$ such that $e + ce' = ge$ (resp. $e' = ge$) and then $D = gEe$. Suppose c is not in F. Then $c = p + q\mu$ for some p and q in F and

$$e + ce' = g(e + \mu e'), \qquad \text{where} \quad g = \mathrm{Mat}[1, 0, p, q]$$

Hence $D = g\lambda Ee$. This already proves the first assertion of the lemma. To prove the second assertion we remark that if g is in L then $g\lambda e = m(g)\lambda e$, with $m(g)$ in E. Furthemore $ge = (p + q\mu)e + (r + s\mu)e'$ if $g = \mathrm{Mat}[p, q, r, s]$ so $m(g)$ cannot be zero unless g is. Hence $g \mapsto m(g)$ is an injective morphism from L to E. Since L is not reduced to F it is indeed isomorphic to E. Finally it is clear that e and e' are F-morphisms of L into E. Since L has some non scalar elements e and e' are distinct and we are done.

(7.6) Coming back to our goal we let χ be a character of E^x invariant under σ. If h is a smooth function of $\mathbb{H}(\chi)$ then the function $g \mapsto h(\lambda g, s)$ on $G(F)$ is invariant on the left under $L^\times$. Thus we are led to set:

$$I(s, \chi)h = \int h(\lambda g, s)\mathrm{d}g, \quad g \in L^\times\backslash G(F). \tag{1}$$

The integral converges only if Re $s > 0$ and we need to show that it has analytic continuation. One way is to use once more the Whittaker modele; we sketch a proof. We are going to show there is a constant $c(s)$ such that

$$I(s, \chi)h = c(s)\int W[\mathrm{diag}(a, 1)]\,\mathrm{d}^\times a, \quad a \in F^\times, \tag{2}$$

if W corresponds to h. Recall that W is related to h by the following formula:

$$W[\mathrm{diag}(a,1)] = \iint h\left[wn(x+y\sqrt{\tau})\right]$$

$$\times \psi_F(-ay)\mathrm{d}y\mathrm{d}x|a|_E^{-s+1/2}\bar{\chi}(a).$$

Now χ has the form $\mu o N(E/F)$ where μ is a character of $F^\times$. Hence the restriction of χ to $F^\times$ is μ^2. Integrating over $F^\times$ we find:

$$\int W[\mathrm{diag}(a,1)]\mathrm{d}a = \int h\left[wn(x+y\sqrt{\tau}),\, s\right]$$

$$\times \psi_F(-ay)\mathrm{d}y\mathrm{d}x|a|_F^{-2s+1}\bar{\mu}^2(a)\mathrm{d}^\times a$$

Using the Tate functional equation we find this is

$$\left[\epsilon(2s,\mu^2,\psi_F)L(1-2s,\bar{\mu}^2)/L(2s,\mu^2)\right]$$

$$\times \int h\left[(w(x+a\sqrt{\tau}),\, s\right]|a|_F^{2s}\mu^2(a)\mathrm{d}^\times a\mathrm{d}x.$$

Of course we should check that the functions are sufficiently smooth to allow this. But this is not hard. Now the integral on the right can be recognized as being

$$\iint h[\lambda\, \mathrm{diag}(a,1)n(x),\, s]|a|\mathrm{d}^\times a\mathrm{d}x$$

Since $G(F)=L^\times P(F)$ with $P(F)\cap L^\times = F^\times$, this integral is proportional to the integral of $h(\lambda g, s)$ over $L^\times\backslash G(F)$ and we are done with:

$$c(s) = cL(2s,\mu^2)/L(1-2s,\bar{\mu}^2)\varepsilon(2s,\mu^2,\psi_F). \tag{3}$$

This gives the analytic properties of $I(s,\chi)$. There is another way to obtain these properties. We take:

$$\mu = wn(\sqrt{\tau});\qquad \text{then}\quad L=\{\mathrm{Mat}[a,b,b,a]\}. \tag{4}$$

Next we use once more the device of representing $h(g,s)$ in the form

$$h(g,s) = \left[Q(s)/L(2s+1,\chi^2)\right]$$

$$\times \int \Phi[(0,t)g]\chi^2(t)|t|^{2s+1}\chi(\det g)|\det g|_E^{s+1/2}, \tag{5}$$

where Φ is a Schwartz-Bruhat function in two variables and Q an elementary function. If $t = a + b\sqrt{\tau}$ then

$$(0, t)\lambda = (1, \sqrt{\tau})1 \qquad \text{with} \quad 1 = \mathrm{Mat}[a, b\tau, b, a] \tag{6}$$

$$|t|_E^{2s+1} = |\det(1)|_E^{2s+1}, \quad \chi(t) = \mu o N(E/F)(t) = \mu(\det(1)),$$

$$\chi(\det g) = \mu(\det g \det g^{\sigma}) = \mu^2(\det g), \qquad \text{for} \quad g \in G(F)$$

and, for g in $G(F)$, the above integral for $h(\lambda g, s)$ can be written as an integral on L:

$$h(\lambda g, s) = \int \Phi[(1, \sqrt{\tau})lg]\,\mu^2(\det lg)|\det lg|_F^{2s+1}\,\mathrm{d}^{\times}l \tag{7}$$

Therefore the integral on $L^{\times}\backslash G(F)$ can combined with this integral to give an integral on $G(F)$:

$$\int h(\lambda g, s)\mathrm{d}g = [Q(s)/L(2s+1, \chi^2)]$$

$$\times \int \Phi[(1, \sqrt{\tau})g]\,\mu^2(\det g)|\det g|_F^{2s+1}\,\mathrm{d}g. \tag{8}$$

We can use the Iwasawa decomposition on $G(F)$ to compute this integral. We find:

$$[Q(s)/L(2s+1, \chi^2)]$$

$$\times \int \Phi[a, b\sqrt{\tau} + x)k]\,\mathrm{d}x\mu^2(a)|a|_E^{2s}\mu^2(b)|b|_F^{2s+1}\,\mathrm{d}^{\times}a\,\mathrm{d}^{\times}b\,\mathrm{d}k. \tag{9}$$

After integrating over x and k the resulting function of a and b is Schwartz Bruhat and the integral is a double Tate integral. In particular we see that for $\mathrm{Re}\, s > 0$ all our integrals converge. Furthemore, as a function of s, this integral has the form:

$$Q'(s)\cdot L(2s, \mu^2)L(2s+1, \mu^2)/L(2s+1, \chi^2)$$

$$= Q'(s)L(2s, \mu^2)/L(2s+1, \mu^2\eta), \tag{11}$$

where Q' is another elementary function of s and η the quadratic character attached to the extension E of F. We are led to define a "normalized version" of the linear form $I(s, \chi)$. It will be noted $J(s, \chi)$. It is defined by:

$$I(s, \chi) = L(2s, \mu^2)/L(2s+1, \mu^2\eta)J(s, \chi). \tag{12}$$

The advantage of J is that it is defined for all s with $\operatorname{Re} s = 0$. In contrast if the restriction of χ to $F^\times$ is trivial then $\mu^2 = 1$ and $I(s, \chi)$ has a pole at $s = 0$. Furthemore the analytic dependence of $J(s, \chi)$ on s is simple:

(13) LEMMA: *Suppose h is a constant section. Then, if h is K-finite, $J(s, \chi)h$ is an elementary function of s. In particular if $h = 1$ on K, F is non Archimedean, E is the quadratic unramified extension of F, χ is unramified and ψ_F has order* 0 *then $J(s, \chi)h = \operatorname{vol}(K_F)$ for all s.*

This follows at once from the above computations.

Finally we remark that if the restriction of χ to $F^\times$ is trivial then there exists a constant c such that:

$$J(0, \chi)h = c\int h(k)\mathrm{d}k.$$

This follows at once from formulas (7.4.2) and (7.6.2).

§8. Truncation of an Eisenstein series

(8.1) We will need to study the truncation of the kernel K_{eis} and the integral of its truncation over the product of $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$ by itself. To that end we first recall a few facts on Eisenstein series; we study their truncation and the integral of their truncation over $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$.

As usual we shall consider functions on $G(E_{\mathbb{A}})$ invariant under the centre $Z(E_{\mathbb{A}})$. For a character χ of the idele class group of E and a complex number s we let $\mathbb{H}(s, \chi)$ be the space of functions h on $G(E_{\mathbb{A}})$ such that

$$h[n(x)\operatorname{diag}(a, b)g] = |a/b|^{s+1/2}\chi(a/b)h(g) \tag{1}$$

and whose restriction to K is square integrable. We also denote by $\rho(s, \chi)$ the representation of $G(E_{\mathbb{A}})$ on $\mathbb{H}(s, \gamma)$. The union of the sets $\mathbb{H}(s, \gamma)$ over s is thus a holomorphic fiber bundle of base $\mathbb{C}$. This fiber bundle is trivial; just as in the local case we define $\mathbb{H}(\gamma)$ and for h in $H(\chi)$ the section $h(g, s)$ whose restriction to K is independent of s. If h is any section of this bundle we form an Eisenstein series:

$$E(g, h, s) = \sum h(\gamma g, s), \quad \gamma \in P(E)\backslash G(E). \tag{2}$$

This series converges for $\operatorname{Re} s > 1/2$ and has analytic continuation to the whole complew plane. If h is in $H(\chi)$ the constant term of E along $N(E_{\mathbb{A}})$ has the form

$$E_N(g, h, s) = h(g, s) + [M(s, \chi)h](g, -s) \tag{3}$$

where $M(s, \chi)$ is the operator from $\mathbb{H}(\chi)$ to $\mathbb{H}(\bar{\chi})$ defined by

$$[M(s, \chi)h](g, -s) = \int h(wng, s)\mathrm{d}n, \qquad \text{if} \quad h \text{ is in } \mathbb{H}(\chi).$$

The integral converges also for $\mathrm{Re}(s) > 1/2$ but $M(s, \chi)$ extends to the whole complex plane. Moreover we have

$$(M(s, \chi)h, h') = (h, M(-s, \bar{\chi})h') \quad \text{if} \quad h \in \mathbb{H}(\chi),\ h' \in \mathbb{H}(\chi), \tag{4}$$

$$M(-s, \bar{\chi})M(s, \chi) = \mathrm{Id}. \tag{5}$$

In particular $M(s, \chi)$ is a unitary operator on the line $\mathrm{Re}\ s = 0$. We also introduce a "normalized intertwining operator":

$$\begin{aligned} R(s, \chi) &= M(s, \chi)L(1 + 2s, \chi)\varepsilon(2s, \chi)/L(2s, \chi) \\ &= M(s, \chi)L(1 + 2s, \chi)/L(1 - 2s, \chi). \end{aligned}$$

We need some information on $M(0, \chi)$. Suppose $\chi^2 = 1$. Then $M(0, \chi)$ is an operator commuting with the representation $\rho(0, \chi)$ on $\mathbb{H}(\chi)$. Since this representation is irreducible the operator is a scalar. By (5) this scalar is $+1$ or -1. In fact:

(6) Lemma: *If* $\chi^2 = 1$ *then* $M(0, \chi) = -\mathrm{Id}$.

This is standard (Cf. [R.L.] for instance). Indeed, since $L(s, 1)$ has a pole at $s = 1$, the ratio $L(1 + 2s, \chi^2)/L(1 - 2s, \chi^2)$ takes the value -1 at $s = 0$. Thus it suffices to prove that $R(0, \chi) = 1$. This follows from the fact that $R(0, \chi_v) = 1$ for all v (Cf. (7.3.3)). If $\chi^2 \neq 1$ then $M(0, \chi)$ is no longer an operator intertwinning a representation with itself. We have however the following result:

(7) Lemma: *If* $\chi^2 \neq 1$ *but the restriction of* χ *to* $F_{\mathbf{A}}^\times$ *is trivial we have for any element h of* $\mathbb{H}(\chi)$:

$$\int [M(0, \chi)h](k)\mathrm{d}k = \int h(k)\mathrm{d}k, \tag{8}$$

both integral are over $K_F(= K \cap G(F_{\mathbf{A}}))$.

Proof of (7) Since χ^2 is not trivial $L(1 + 2s, \chi^2)$ is holomorphic at $s = 0$. We claim that $L(1, \chi^2) = L(1, \bar{\chi}^2)$. As a matter of fact we claim that

$$L(s, \chi^2) = L(s, \bar{\chi}^2) \qquad \text{for all} \quad s.$$

Indeed it suffices to prove this for Re $s \gg 0$. Then both sides are product over all places v of F of local factors. If v is a place of F which does not split and u the corresponding place of E the local factors on both sides are $L(s, \chi_u^2)$ and $L(s, \bar{\chi}_u^2)$ respectively. They are equal because the restriction of χ to $F_v^\times$ is trivial (Lemma (7.4.4)). On the other hand if v splits into $v1$ and $v2$ the corresponding factors are

$$L(s, \chi_{v1}^2)L(s, \chi_{v2}^2) \quad \text{and} \quad L(s, \bar{\chi}_{v1}^2)L(s, \bar{\chi}_{v2}^2)$$

and they are trivially equal because $\chi_{v1} = \bar{\chi}_{v2}$. Hence $M(0, \chi) = R(0, \chi)$. Now $R(s, \chi)$ is a product over all places u of E of local "normalized intertwining operators" described in section 7. If we assume that h is the product of local components h_u, the integrals on both sides of (8) are product over all places v of F of local analogous integrals and a constant factor the same on both sides. So it suffices to prove the local analogue of (8) for all places v of F. If v is a place of F which splits into $v1$ and $v2$ in E then the local analogue is:

$$\int [R(0, \chi_{v1})h_{v1}](k_v) \cdot [R(0, \chi_{v2})h_{v2}](k_v)\mathrm{d}k_v = \int h_{v1}(k_v)h_{v2}(k_v)\mathrm{d}k_v.$$

Since the restriction of χ to $F_{\mathbf{A}}^\times$ is trivial the character χ_{v1} is actually the inverse of the character χ_{v2} and this follows from (7.3.4). If v does not split and u is the unique place of E above v then the local equality to be proved reads:

$$\int [R(0, \chi_u)h_u](k_v)\mathrm{d}k_v = \int h_u(k_v)\mathrm{d}k_v, \quad k \in K_v.$$

Now the restriction of χ_v to $F_v^\times$ is trivial and this has been proved in (7.4.1). This concludes the proof of lemma (7).

(8.2) We need some estimates on the Eisenstein series. Denote by $\|g\|$ a "norm function" on the group $G(E_{\mathbf{A}})$: it is the product of the norms of the local components g_v divided by $|\det g|$. It is known that, if Ω is a compact set of $\mathbb{C}$ which does not contain any pole of $E(g, s, h)$ and X an element of the envelopping algebra at infinity, then there are constants C and N such that $|\rho(X)E(g, s, h)| < C\|g\|^N$, for s in Ω. We have denoted by $\rho(X)$ the left invariant operator associated to X. It follows from this that the difference of E and its constant term is bounded (in fact rapidly decreasing) on any Siegel set. Consider now our truncated Eisenstein series $T^cE(g, s, h)$. There is a Siegel set S of $G(F_{\mathbf{A}})$ on which it is equal to the difference between the Eisenstein series and its constant term (cf. Lemma (6.1.4)). Moreover the quotient $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ is the union of a compact set and the image of S. We conclude that for a given h and s in a given compact set of $\mathbb{C}$ we have a uniform estimate $|T^c E(g, s, h)| < C$. In fact we could obtain much better estimates but this will not be needed. We will need however majorizations uniform with respect to s when s is purely imaginary:

(1) LEMMA: *Given h there is a polynomial $P(t)$ such that*

$$|T^c E(g, it, h)| < P(t), \qquad \text{for all real } t.$$

PROOF OF LEMMA (1): Just as before it suffices to establish majorizations of the form

$$|\rho(X) E(g, it, h)| < P(t) \|g\|^N$$

where P and N may depend on X. To see that let us introduce for any Schwartz-Bruhat function Φ in two variables the action of our fibre bundle defined by:

$$F_\Phi(g, s) = \int \Phi[(0, t)g] \chi^2(t) |t|^{2s+1} \mathrm{d}^\times t \chi(\det g) |\det g|^{s+1/2}.$$

On the right we have a Tate integral or rather its analytic continuation. There is in fact a finite set T of places of E containing all places at infinity and a function such that

$$h(g, s) = F_\Phi(g, s)/L^T(1 + 2s, \chi^2).$$

where $L^T(s, \chi^2)$ stands for the product of the $L(s, \chi_u)$ over all u not in T. It is classical fact that $1/L^T(1 + 2s, \chi^2)$ is at most of polynomial growth on the line $\mathrm{Re}(s) = 0$. Thus it suffices to find majorizations for the Eisenstein series $E(g, it, F_\Phi)$ and its left invariant derivatives on the group $G(E_{\mathbb{A}})$. However we may also write this Eisenstein series as the following expression:

$$\int \sum \Phi(t\xi) |t|^{2s+1} \chi^2(t) \mathrm{d}t \chi(\det g) |\det g|^{s+1/2}, \quad \xi \in E - 0, \quad t \in E_{\mathbb{A}}^\times / E^\times.$$

Poisson summation formula gives the analytic continuation of this expression and, at the same time, the required majorizations.

(8.3) We now study the integral of a truncated Eisenstein series over the quotient $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$. The description of the double coset space $P(E)\backslash G(E)/G(F)$ given in Lemma (7.5.1) applies: we have a disjoint union

$$G(E) = P(E)G(F) \cup P(E)\lambda G(F).$$

Furthemore the algebra $L=\lambda^{-1}R(E)\lambda\cap M(2, F)$ is an algebra of degree 2 on F which is F-isomorphic to E.

Consider now the integral

$$\int T^c E(g, s, h)\mathrm{d}g, \quad g\in Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}}).$$

From the majorizations obtained in (2) we see that the integral converges and represents a meromorphic function of s, without pole on the line $\operatorname{Re} s=0$. We are interested in this integral for s purely imaginary; however in order to compute it we assume $\operatorname{Re}(s)>1/2$ and then analytically continue our answer. We have

$$T^c E(g, s, h)=\sum h(\xi g, s)-\sum E_N(\gamma g, s, h)\chi_c(H(\gamma g)). \tag{1}$$

The first sum is over $P(E)\backslash G(E)$, the second sum over $P(F)\backslash G(F)$. Using the description of $P(E)\backslash G(E)/G(F)$ we may rewrite the first sum as a sum on $L^{\times}\backslash G(F)$ and a sum on $P(F)\backslash G(F)$. In a precise way we get:

$$T^c E(g, s, h)=\sum h(\lambda\xi g, s)$$
$$+\sum\{h(\gamma g, s)-E_N(\gamma g, s, h)\chi_c(H(\gamma g))\}. \tag{2}$$

In the first sum, ξ varies in $L^{\times}\backslash G(F)$; in the second sum γ varies over $P(F)\backslash G(F)$. Taking in account the formula for E_N we may rewrite this as

$$T^c E(g, s, h)= \tag{3}$$

(i) $\sum h(\lambda\xi g, s)$

(ii) $+\sum h(\gamma g, s)[1-\chi_c(H(\gamma g))]$

(iii) $-\sum[M(s, \chi)h](\gamma g, -s)\chi_c(H(\gamma g),$

Combining summation and integration in the usual way we get:

$$\int T^c E(g, s, h)\mathrm{d}g= \tag{4}$$

(i) $\int h(\lambda x, s)\mathrm{d}x$

(ii) $+\int h(y, s)[1-\chi_c(H(y))]\mathrm{d}y$

(iii) $-\int [M(s, \chi)h](y, -s)\chi_c(H(y))\mathrm{d}y,$

the integrals over x in $L^\times Z(F_{\mathbf{A}})\backslash G(F_{\mathbf{A}})$ and y in $Z(F_{\mathbf{A}})P(F)\backslash G(F_{\mathbf{A}})$. Using the Iwasawa decomposition we compute integral (4.ii) as

$$\int \chi(a)(1-\chi_c(|a|_E))|a|_E^{s+1/2}|a|_F^{-1}\mathrm{d}^\times a\int h(k)\mathrm{d}k.$$

As usual we denote by F^1 the group of ideles of norm 1 and by F_∞^+ the group of ideles whose finite components are 1 and whose infinite components are equal to some fixed positive number. Then $F_{\mathbf{A}}^\times$ is the direct product of F^1 and F_∞^+. Similarly $F_{\mathbf{A}}^x$ is the product of E^1 and E_∞^+ and when we imbed $F_{\mathbf{A}}^\times$ into $E_{\mathbf{A}}^\times F_\infty^+$ goes to E_∞^+. We choose the character χ to be trivial on E_∞^+ so that χ is trivial on F^2 if and only if it is trivial on $F_{\mathbf{A}}^\times$. Our Haar measure on $F_{\mathbf{A}}^\times$ is the product of a Haar measure on F^1 and the measure $\mathrm{d}t/t$ where $t=|a|$. Then the above integral for a is zero unless the restriction of χ to $F_{\mathbf{A}}^\times$ is trivial in which case its value is:

$$\mathrm{vol}(F^1/F^\times)\int_0^{c^{1/2}} t^{2s}\mathrm{d}t = c^s/2s\ \mathrm{vol}(F^1/F^\times).$$

If we set $\delta(\chi)=\mathrm{vol}(F^1/F^\times)$ if the restriction of χ to $F_{\mathbf{A}}^\times$ is trivial and zero otherwise we see that (8.3.4.ii) is equal to

$$\delta(\chi)\frac{c^s}{2s}\int h(k)\mathrm{d}k. \tag{4.i}$$

Similarly, using the fact that $M(s, \chi)h(g, -s)$ lies in $\mathbb{H}(-s, \bar{\chi})$ we calculate that (8.3.4.iii) is equal to:

$$-\delta(\chi)\frac{c^{-s}}{2s}\int [M(s, \chi)h](k)\mathrm{d}k. \tag{4.ii}$$

Actually $\delta(\chi)=1$ but we will not have to use this fact. We write (8.3.4.i) as

$$\int h(\lambda lg, s)d^x l\mathrm{d}g, \quad l\in F_{\mathbf{A}}^\times L^\times\backslash L_{\mathbf{A}}^\times, \quad g\in L_{\mathbf{A}}^\times\backslash G(F_{\mathbf{A}}).$$

Next we recall that $\lambda^{-1}1\lambda$ is a triangular matrix whose eigenvalues are $e(1)$ and $e'(1)$ where $1\mapsto e(1)$ and $1\mapsto e'(1)$ are the two isomorphisms of L onto E. Then in the above integral the inner integral takes the following form:

$$\int h(\lambda g, s)|t^{\tau-1}|^{s+1/2}\chi(t^{\sigma-1})d^\times t, \quad t\in F^\times E_{\mathbf{A}}^\times\backslash E_{\mathbf{A}}^\times.$$

This is zero unless $\chi^{\sigma-1}=1$, that is unless χ has the form $\mu o N(E/F)$ for some character μ of $F_{\mathbf{A}}^\times$. If we let $\varepsilon(\chi)=\mathrm{vol}(F_{\mathbf{A}}^\times E^\times\backslash E_{\mathbf{A}}^\times)$ if $\chi^{\sigma-1}=1$ and zero otherwise then (8.3.4.i) is

$$\varepsilon(\chi)\int h(\lambda g, s)dg, \quad g\in L_{\mathbf{A}}^\times\backslash G(F_{\mathbf{A}}).$$

In fact $\varepsilon(\chi)=2$. We need to study the analytic properties of this function of s. We proceed as in the local case (Cf. (7.6)). We take:

$$\lambda = wn(\sqrt{\tau}); \quad \text{then} \quad L=\{\mathrm{Mat}[a, b\tau, b, a]\}.$$

Next we use the standard device of representing $h(g, s)$ in the form

$$h(g, s)=[Q(s)/L(2s+1, \chi^2)]$$
$$\times\int\Phi[(0, t)g])\chi^2(t)|t|_E^{2s+1}\chi(\det g)|\det g|^{s+1/2},$$

where Φ is a Schwartz-Bruhat function in two variables and Q an elementary function, that is a linear combinations of products of rational and exponential functions of s; then

$$(0, t)\lambda=(1, \sqrt{\tau})l \qquad \text{with} \quad 1=\mathrm{Mat}[a, b\tau, b, a],$$

and the above integral for g in $G(F)$ can be written as an integral on $L_{\mathbf{A}}^x$:

$$\int\Phi[(1, \sqrt{\tau})1g]\mu^2(\det 1g)|\det 1g|_F^{2s+1}d^x1.$$

Therefore the integral on $L_{\mathbf{A}}^\times\backslash G(F_{\mathbf{A}})$ can be combined with this integral to give an integral on $G(F_{\mathbf{A}})$:

$$\int h(\lambda g, s)dg=[Q(s)/L(2s+1, \chi^2)]$$
$$\times\int\Phi[(1, \sqrt{\tau})g]\mu^2(\det g)|\det g|_F^{2s+1}dg.$$

We can use the Iwasawa decomposition on $G(F_{\mathbf{A}})$ to compute this integral. We find:

$$[Q(s)/L(2s+1,\chi^2)]$$
$$\times\int\Phi[(a, b\sqrt{\tau}+x)k]\mathrm{d}x\mu^2(a)|a|_F^{2s}\mu^2(b)|b|_F^{2s+1}\mathrm{d}^\times a\mathrm{d}^\times b\mathrm{d}k.$$

After integrating over x and k the resulting function of a and b is Schwartz-Bruhat and the integral is a double Tate integral. In particular we see that as a function of s this has the form:

$$Q'(s)\cdot L(2s,\mu^2)L(2s+1,\mu^2)/L(2s+1,\chi^2)$$
$$=Q'(s)L(2s,\mu^2)/L(2s+1,\mu^2\eta)$$

where Q' is another elementary function of s and η the quadratic character attached to the extension E of F. This also shows that the original integral converges for Re $s>1/2$. Furthemore, just as in the local case, this suggests introducing a normalized version of the integral. In a precise way if h is in $H(\chi)$ and $\chi^{s-1}=1$ we will set

$$I(s,\chi)h=\int h(\lambda g,s)\mathrm{d}g,\quad g\in L_{\mathbf{A}}^\times\backslash G(F_{\mathbf{A}}),$$

$$I(s,\chi)h=L(2s,\mu^2)/L(2s+1,\mu^2\eta)\cdot J(s,\chi),$$

where $\chi=\mu o N(E/F)$. It is clear that $J(s,\chi)h$, for h K-finite is an elementary function of s without singularity on the line Re $s=0$.
(8.4) We summarize the results obtained so far:

PROPOSITION: *The integral of $E(g,h,s)$ over $Z(F)G(F)\backslash G(F)$ is the sum of the following*:

$$\delta(\chi)\frac{c^s}{2s}\int h(k)\mathrm{d}k, \tag{1}$$

$$-\delta(\chi)\frac{c^{-s}}{2s}\int[M(s,\chi)h](k)\mathrm{d}k, \tag{2}$$

$$\varepsilon(\chi)I(s,\chi)h. \tag{3}$$

Furthemore

$$I(s,\chi)=L(2s,\mu^2)/L(2s+1,\mu^2\eta)J(s,\chi)h$$

and $J(s,\chi)h$ is an elementary function of s if h is K-finite.

Because the Eisenstein series has no pole on the line Re $s=0$ and its truncation is integrable the sum of (1), (2) and (3) is actually holomorphic at $s=0$. It will be useful to make this statement more precise. To begin with we make the following elementary remark about the character χ:

(4) LEMMA: *Consider the following conditions*:

(i) $\chi^{\sigma-1}=1$;

(ii) *the restriction of* χ *to* $F_{\mathbf{A}}^{\times}$ *is trivial*;

(iii) *the restriction of* χ *to* $F_{\mathbf{A}}^{\times}$ *is the quadratic character* η *attached to the extension* E;

(iv) $\chi^2=1$.

Then: (iv) $\Leftrightarrow$ (i) *and* [(ii) *or* (iii)]

PROOF OF (4): Suppose $\chi^2=1$. If χ is actually the trivial character then (i) and (ii) are satisfied. Suppose not and let D be the quadratic extension of E corresponding to χ. Then D is a Galois extension of F of degree 4 hence Abelian. By class field theory we may view χ as a character of Gal(D/E). Assertion (i) amounts to say that if σ' is an element of Gal(D/F) whose image in Gal(E/F) is σ then

$$\chi(\sigma' g \sigma'^{-1})=\chi(g).$$

Since Gal(D/F) is Abelian this is clear. This being so let us write χ in the form $\mu o N(E/F)$. By (iv) we see that μ^2 is trivial on the image of $N(E/F)$. Thus μ^2 is either the character trivial or the character η. Since the restriction of χ to $F_{\mathbf{A}}^{\times}$ is μ^2, we see that either (ii) or (iii) holds. Now suppose (i) is satisfied. Write therefore χ in the form $\mu o N(E/F)$. Again the restriction of χ to $F_{\mathbf{A}}^{\times}$ is μ^2. If (ii) is satisfied we find $\mu^2=1$. A fortiori $\chi^2=\mu^2 o N(E/F)=1$. If (iii) is satisfied we find $\mu^2=\eta$. A fortiori $\chi^2=\mu o N(E/F)=1$. So in any case $\chi^2=1$.

Let us go back now to the sum of the three terms in Proposition (8.4). Suppose first that the restriction of χ to $F_{\mathbf{A}}^{\times}$ is trivial but $\chi^{\sigma-1}\neq 1$. Then the term (3) vanishes. Furthemore $\chi^2\neq 1$ otherwise by the lemma $\chi^{\sigma-1}=1$. Accordingly by (8.1.7):

$$\int M(0,\chi)h(k)\mathrm{d}k=\int h(k)\mathrm{d}k \tag{5}$$

and the residues of the terms (1) and (2) at $s=0$ do cancell. Suppose now that $\chi^{\sigma-1}=1$ but the restriction of χ to $F_{\mathbf{A}}^{\times}$ is not trivial. Then the terms (1) and (2) vanish. Let us write χ in the form $\chi=\mu o N(E/F)$ so that the restriction of χ to $F_{\mathbf{A}}^{\times}$ is μ^2. Now μ^2 cannot be trivial otherwise the restriction of χ to $F_{\mathbf{A}}^{\times}$ would be trivial. Thus $L(2s,\mu^2)$ is holomorphic at

$s=0$ and (3) is too. Note that if $\chi^2=1$ then, by the lemma, $\mu^2\eta=1$ and $1/L(2s+1, \mu^2\eta)$ has a zero at $s=0$; hence (3) vanishes at $s=0$. Finally suppose that the restriction of χ to $F_{\mathbb{A}}^{\times}$ is trivial and $\chi^{\sigma-1}=1$. Then $\chi^2=1$ and writing $\chi=\mu o N(E/F)$ we have $\mu^2=1$. Then $M(0, \chi)=-1$ and (1) and (2) have the same residue at $s=0$. Doubling this residue we find then

$$\delta(\chi)\int h(k)\mathrm{d}k+\varepsilon(\chi)\lim_{s\to 0} sI(s, \chi)h=0 \tag{6}$$

or

$$\delta(\chi)\int h(k)\mathrm{d}k+\varepsilon(\chi)2c_0/L(1, \mu^2\eta)J(0, \chi)h=0 \tag{7}$$

where c is the residue of $L(s, 1_F)$ at 0. Of course this relation could be proved by local means as well (See below).

(8.5) For Re $s>1/2$ the integral $\int h(\lambda g, s)\mathrm{d}g$ converges absolutely. Assume that h is a product over all places u of E of local components h_u belonging to $\mathbb{H}(\chi_u)$; of course for almost all place u, $h_u(k_u)=1$ for all k_u. Then, for Re $s>1/2$, the integral can be written as a product of local integrals and a constant factor; the product is over all places of F. We now describe the factor attached to a place v of F. Suppose first that v does not split in the extension and let u be the unique place of E above v. Then the local component is just the local integral

$$\int h_v(\lambda g_v, s)\mathrm{d}g_v=L(2s, \mu_v^2)/L(2s+1, \mu_v^2\eta_v)J(s, \chi_u)h_u.$$

Suppose now that v splits into $v1$ and $v2$ in the quadratic extension. Thus we have isomorphisms $G_{v1}\cong G_v$ and $G_{v2}\cong G_v$. Let λ_1 and λ_2 be the images of λ under these two isomorphisms. The local component is the following integral over $L_v^{\times}\setminus G_v$:

$$\int h_{v1}(\lambda_1 g_v)\cdot h_{v2}(\lambda_2 g_v)\mathrm{d}g_v.$$

It will be convenient to take

$$\lambda_i=wn(\pm\sqrt{\tau}), \quad \text{with } + \text{ for } i=1 \text{ and } - \text{ for } i=2.$$

Note that h_{v1} and h_{v2} are invariant under $N(F_v)$. Thus the local integral may also be written as:

$$\int h_{v1}(\mu_1 g)\cdot h_{v2}(\mu_2 g_v)\mathrm{d}g_v.$$

where

$$\mu_\iota = n(\mp 1/2\sqrt{\tau})wn(\pm\sqrt{\tau}).$$

Observe that

for $1 = \mathrm{Mat}[a, b\tau, b, a]$ in L we have

$$\mu_1^{-1} 1 \mu_2 = \mathrm{diag}(a + b\sqrt{\tau}, a - b\sqrt{\tau}).$$

Hence $\mu_1^{-1} L \mu_2 = A$. After changing f to $\mu_2^{-1} g$ we get therefore an integral over $A_v \backslash G_v$:

$$\int h_{v1}(\mu_1 \mu_2^{-1} g) h_{v2}(g) \mathrm{d}g$$

$$= c_v |1/4\tau|_v^{2s+1} \chi_v(-1/4\tau) \int h_{v1}(wg) h_{v2}(g) \mathrm{d}g$$

because $\mu_1 \mu_2^{-1} = \mathrm{diag}(-1/2\sqrt{\tau}, +2\sqrt{\tau})w$. Here c_v is a constant. There is a question of which measures we are using. We regard $L^\times$ and A as algebraic groups over F and choose invariant differential forms on them. Then for any place v of F we inherit the Tamagawa measure on L and A. They are so defined that that for almost all v the measure of the maximal compact subgroup is one. Thus, for almost all v, the isomorphism $1 \mapsto \mu_2^{-1} 1 \mu_2$ is compatible with our choice of measures, that is, $c_v = 1$.

Now because $\chi^{\sigma - 1} = 1$ we have $\chi_{v1} = \chi_{v2} = \mu_v$ and

$$\int h_{v1}(wg) h_{v2}(g) \mathrm{d}g = \int h_{v1}(wn(x_v)k_v) h_{v2}(n(x_v)k_v) \mathrm{d}x_v \mathrm{d}k_v$$

$$= \langle M(s, \mu_v) h_{v1}, h_{v2} \rangle.$$

In turn, apart from an ε factor, this is:

$$L(2s, \mu_v^2)/L(2s+1, \mu_v^2) \langle R(s, \mu_v) h_{v1}, h_{v2} \rangle.$$

We conclude that apart from a factor CA^s, $J(s, \chi)h$ is the product over all places v of F of a factor which is either $J(s, \chi_u)h_u$ or $\langle R(s, \mu_v) h_{v1}, h_{v2} \rangle$.

§9. Truncation of the Eisenstein kernel

(9.1) In this section we deal with the integral of the truncation of $K_{\mathrm{eis}}(x, y)$. We shall prove that the following limit exists:

$$\lim_{c_1 \to \infty} \iint T_1^{c_1} T_2^{c_2} K_{\mathrm{eis}}(x, y) \mathrm{d}x \mathrm{d}y, \quad x, y \in Z(F_{\mathbf{A}}) G(F) \backslash G(F_{\mathbf{A}}).$$

The resulting function of c_2 has an asymptotic expansion of the form $A \log c_2 + B + o(1)$ as $c_2 \to +\infty$, where A, B are constants. We shall prove that, under the assumptions of (6.1.1), $B = 0$.

We recall the formula for K_{eis} (cf. [J.A.] for instance). For each character χ of $E_{\mathbf{A}}^{\times}/E^{\times}$ trivial on E_{∞}^{+}, we choose an orthonormal basis h_α of $\mathbb{H}\chi$). Then:

$$K_{\text{eis}}(x, y) = \sum_{\chi} K(x, y, \chi), \text{ where}$$

$$K(x, y, \chi) = 1/4\pi \sum_{\beta,\alpha} \int_{-i\infty}^{i\infty} \big(\rho(s, \chi)(f)h_\beta, h_\alpha\big) \times E(x, h_\alpha, s)\bar{E}(y, h_\beta, s)|ds|. \tag{1}$$

The first sum is over all χ; however for a given f almost all terms are zero. Similarly, the second sum is for all pairs (α, β); however for a given f almost all terms are zero. The integral converges uniformly over compacts set of $G(E_{\mathbf{A}})$. In particular to compute the truncation of our kernel, we may "bring the truncation operator under the integral sign":

$$T_1^{c_1}T_2^{c_2}K_{\text{eis}}(x, y) = \sum T_1^{c_1}T_2^{c_2}K(x, y, \chi),$$

$$T_1^{c_1}T_2^{c_2}K(x, y, \chi) = 1/4\pi \sum \int \big(\rho(s, \chi)h_\beta, h_\alpha\big)T_1^{c_1}E(x, h_\alpha, s) \cdot T_2^{c_2}E(y, h_\beta, s)\big)|ds|. \tag{2}$$

Since $K_{\text{cusp}}(x, y)$ is integrable and $T_1^{c_1}T_2^{c_2}K$, $T_1^{c_1}T_2^{c_2}K_{\text{sp}}$ are integrable, it is clear that $T_1^{c_1}T_2^{c_2}K_{\text{eis}}$ is also integrable. Moreover $|T^cE(x, h, s)|$ is bounded independently of x by a polynomial in $\text{Im}(s)$ on the line $\text{Re}(s) = 0$ (Cf. (8.2)). In particular we may interchange the integral in x and y with the summations and the integration in s:

$$\iint T_1^{c_1}T_2^{c_2}K_{\text{eis}}(x, y)dxdy = \sum \iint T_1^{c_1}T_2^{c_2}K(x, y, \chi)dxdy,$$

$$\iint T_1^{c_1}T_2^{c_2}K(x, y, \chi)dxdy = \sum 1/4\pi \int \big(\rho(s, \chi)h_\beta, h_\alpha\big) \int T_1^{c_1}E(x, h_\alpha, s)dx \cdot \overline{\int T_2^{c_2}E(y, h_\beta, s)dy}|ds|. \tag{3}$$

(9.2) To evaluate the limit of this as c_1 tends to infinity we will use the following elementary lemmas which are recalled without proof:

(1) LEMMA: *Syppose F is a Schwartz function on* $\mathbb{R}$. *Then*:

$$\lim_{x\to\infty}\int F(y)\exp(\pm 2\mathrm{i}\pi xy)\mathrm{d}y/y = \pm\mathrm{i}\pi F(0)$$

We remark that the integral is actually an improper integral. Furthemore this function of x tends rapidly toward its limit. Thus:

$$x\int F(y)\exp(\pm 2\mathrm{i}\pi xy)\mathrm{d}y/y = \pm\mathrm{i}\pi xF(0)+o(1). \tag{2}$$

(3) LEMMA: *If F is a Schwartz function on R and* $F(0)=0$ *then*

$$\int_{-\infty}^{\infty}(F(y)/y^2)\mathrm{d}y = \int_{-\infty}^{\infty}F'(y)\mathrm{d}y/y.$$

Again this is an equality of improper integrals.

We will need to show that the functions we are dealing with are indeed Schwartz functions. We proceed to that end via a series of lemmas.

(4) LEMMA: *For a fixed and* $|t|\mapsto +\infty$ *we have*:

$$\Gamma^{(n)}(a+\mathrm{i}t)=\Gamma(a+\mathrm{i}t)O(\log|t|^n).$$

PROOF OF (4): For $n=1$ this is a classical estimate on the function $\psi(z)=\Gamma'(z)/\Gamma(z)$. Suppose $n>1$; then the n-th derivative of ψ can be computed as the sum of $\Gamma^{(m)}/\Gamma$ and products of the form $\pm\Pi\Gamma^{(i)}/\Gamma$ where the orders i are stricly less than n and their sum is n. Our assertion follows then by induction on n from the classical estimate $\psi^{(n)}(a+\mathrm{i}t)=O(1)$.

Next we recall without proof two estimates on L-functions.

(5) LEMMA: *Let* χ *be an idele class character of a number field. Denote by* $L_\infty(s,\chi)$ *the product of the* $L(s,\chi_v)$ *for v infinite and by* $L^\infty(s,\chi)$ *the product of the* $L(s,\chi_v)$ *for v finite. Then, for* $|t|$ *large, we have*

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}L^\infty(1+\mathrm{i}t,\chi)=O\left(\log|t|^{n+1}\right).$$

On the other hand, there is some $N>0$ such that for $|t|$ large:

$$1/L^\infty(1+\mathrm{i}t)=O\left(|t|^N\right)$$

We are now in position to state the results that we need:

(6) LEMMA: *Let* χ *and* η *be idele class characters of a number field F. Assume* η *is trivial or quadratic. Then any derivative of the ratio*

$$L(2s, \chi)/L(1+2s, \chi\eta)$$

is at most of polynomial growth on the line Re $s=0$.

PROOF OF (6): Our ratio is the product of an exponential factor,

$$L^{\infty}(1-2s, \bar{\chi})/L^{\infty}(1+2s, \chi\eta) \quad \text{and}$$

$$L_{\infty}(1-2s, \bar{\chi})/L_{\infty}(1+2s, \chi\eta).$$

It follows from lemma (5) that any derivative of the first ratio is at most of polynomial growth on the line Re $s=0$. So it suffices to prove the second ratio has the same property. In turn it is a product of factors of the form:

$$\Gamma_1(s)/\Gamma_2(s) \quad \text{with} \quad \Gamma_1(s)=\Gamma(a(1-2s)+b), \quad \Gamma_2(s)=\Gamma(a(1+2s)+b+c)$$

where a and c are real. All we have to show therefore is that any derivative of a function Γ_1/Γ_2 is at most of polynomial growth on the line Re $s=0$. By Stirling formula this is true of the function itself. On the other hand any derivative of Γ_1/Γ_2 is a sum of products: in each product we have a factor ± 1, the factor Γ_1/Γ_2 and factors of the form $\Gamma_i^{(j)}/\Gamma_i$. Our assertion follows then from Lemma (4).

(9.3) We set $c_i = \exp(2\pi u_i)$, $i=1, 2$. We also set:

$$\Phi(\alpha, u, t) = 2\mathrm{i}t\int T^c E(x, h_\alpha, \mathrm{i}t)\mathrm{d}x, \qquad \text{if} \quad c=\exp(2\pi u).$$

Then, by Proposition (8.4):

$$\Phi(\alpha, u, t) = \delta(\chi)\exp(2\mathrm{i}\pi ut)\int h_\alpha(k)\mathrm{d}k$$

$$-\delta(\chi)\exp(-2\mathrm{i}\pi ut)\int[M(\mathrm{i}t, \chi)h_\alpha](k)\mathrm{d}k$$

$$+\varepsilon(\chi)2\mathrm{i}tI(\mathrm{i}t, \chi)h_\alpha. \tag{2}$$

The integral is for k in $K_F = K \cap G(F_{\mathbb{A}})$. We remind ourselves that when $\delta(\chi)$ and $\varepsilon(\chi)$ are not zero $I(\mathrm{i}t, \chi)$ has a pole at $t=0$ and

$$\lim_{t\to 0}\mathrm{i}tI(\mathrm{i}t, \chi)h_\alpha = -\int h_\alpha(k)\mathrm{d}k \tag{3}$$

(Cf. section (8.5)). Furthemore in any case $\Phi(\alpha, u, 0) = 0$ (loc. cit.). With this notation we have:

$$\iint T_1^{c_1} T_2^{c_2} K_{\text{eis}}(x, y) \mathrm{d}x \mathrm{d}y = \sum_{\chi, \alpha, \beta} G(u_1, u_2, \chi, \alpha, \beta) \tag{4.i}$$

where

$$G(u_1, u_2, \chi, \alpha, \beta) = \int_{-\infty}^{\infty} F_{\beta\alpha}(t) \Phi(\alpha, u_1, t) \overline{\Phi}(\beta, u_2, t) \mathrm{d}t/t^2. \tag{4.ii}$$

$$F_{\beta\alpha}(t) = 1/16\pi \big(\rho(s, \chi)(f) h_\beta, h_\alpha\big). \tag{4.iii}$$

To continue in the expression for G we replace $\Phi(\alpha, u_1, t)$ by its expression (2):

$$G(u_1, u_2, \chi, \alpha, \beta) = \tag{5}$$

(i) $$\int F_{\beta\alpha}(t) \frac{\overline{\Phi}(\beta, u_2, t)}{t} \delta(\chi) \int h_\alpha(k) \mathrm{d}k \exp(2\mathrm{i}\pi u_1 t) \frac{\mathrm{d}t}{t}$$

(ii) $$\int F_{\beta\alpha}(t) \frac{\overline{\Phi}(\beta, u_2, t)}{t} \delta(\chi)$$

$$\int (-M(\mathrm{i}t, \chi) h_\alpha)(k) \mathrm{d}k \exp(-2\mathrm{i}\pi u_1 t) \frac{\mathrm{d}t}{t}$$

(iii) $$\int F_{\beta\alpha}(t) \overline{\Phi}(\beta, u_2, t) \cdot 2\mathrm{i}t\varepsilon(\chi) I(\mathrm{i}t, \chi) h_\alpha \frac{\mathrm{d}t}{t^2}.$$

We may apply lemma (9.2.1) to (i) and (ii). Indeed $F_{\beta\alpha}(t)$ is a linear combination of products of the form $\exp(ait)\hat{F}(t)$ where F is a smooth function of compact support on $\mathbb{R}$. On the other hand

$$\int [M(\mathrm{i}t, \chi)h](k) \mathrm{d}k = L(2\mathrm{i}t, \chi^2)/L(1 + 2\mathrm{i}t, \chi^2)$$

$$\times \varepsilon(2\mathrm{i}t, \chi^2) \int R(\mathrm{i}t, \chi) h(k) \mathrm{d}k$$

$$= L(2\mathrm{i}t, \chi^2)/L(1 + 2\mathrm{i}t, \chi^2) Q(\mathrm{i}t)$$

where Q is an elementary function of t (Lemma (7.3.6)). Similarly:

$$I(\mathrm{i}t, \chi)h = Q'(\mathrm{i}t) L(2\mathrm{i}t, \mu^2)/L(1 + 2\mathrm{i}t, \mu^2\eta)$$

where Q' is an elementary functions of s, μ an idele class character of F and η is the quadratic character attached to E (Cf. Prop (8.4)). It follows from lemma (6) that the derivatives of these functions of t are at most of polynomial growth. Taking (2) in account we see that the same is true for the derivatives of $\Phi(\alpha, u_2, t)$. We conclude that indeed the functions of t under the integral sign in (5.i) and (5.ii) are Schwartz functions. Applying then (9.2.1) we find that the limit of (i) + (ii) as u tends to plus infinity is

$$\mathrm{i}\pi F_{\beta\alpha}(0)\cdot \left.\frac{\overline{\Phi}(\beta, u_2, t)}{t}\right|_{t=0}\cdot\delta(\chi)$$

$$\cdot\left[\int h_\alpha(k)\mathrm{d}k+\int M(O, \chi)h_\alpha(k)\mathrm{d}k\right].$$

Of course this is zero unless the restriction of χ to $F_{\mathbf{A}}^{\times}$ is trivial. Assuming this to be the case, if $\chi^2=1$ then $M(0, \chi)=-\mathrm{Id}$ and this vanishes. If, on the contrary, $\chi^2\neq 1$ then both terms are equal (Cf. lemma (8.1.6) and (8.1.7)). Set $\theta(\chi)=1$ if $\chi^2\neq 1$ and zero otherwise. Then we see that the limit of (i) + (ii) as u_1 tends to infinity is:

$$2\mathrm{i}\pi F_{\beta\alpha}(0)\cdot \left.\frac{\overline{\Phi}(\beta, u_2, t)}{t}\right|_{t=0}\int h_\alpha(k)\mathrm{d}k\delta(\chi)\theta(\chi). \tag{7}$$

Hence we see that the limit of $G(u_1, u_2, \chi, \alpha, \beta)$ as u_1 tends to infinity is the sum of (7) and (5.iii). Recall that $\Phi(\beta, u_2, 0)=0$. So $\Phi(\beta, u_1, t)/t|_{t=0}$ is the derivative $\mathrm{d}\Phi(\beta, u_2, t)/\mathrm{d}t$ at $t=0$. To compute it we use (2). However if (7) is non zero the restriction of χ to $F_{\mathbf{A}}^{\times}$ is trivial but $\chi^2\neq 1$. By lemma (8.4.4) $\varepsilon(\chi)=0$ and the last term in (2) vanishes. Thus our derivative is actually:

$$\delta(\chi)2\pi\mathrm{i}u_2\left[\int h_\beta(k)\mathrm{d}k+\int\big(M(0, \chi)h_\beta\big)(k)\mathrm{d}k\right]$$

$$-\delta(\chi)\,\frac{\mathrm{d}}{\mathrm{d}t}\left[\int\left[M(\mathrm{i}t, \chi)h_\beta\right](k)\mathrm{d}k\right]\Bigg|_{t=0}.$$

On the other hand, since $\chi^2\neq 1$, we have by lemma (8.1.7)

$$\int M(0, \chi)h_\beta(k)\mathrm{d}k=\int h_\beta(k)\mathrm{d}k.$$

So (7) under the conditions $\delta(\chi)\neq 0$ and $\theta(\chi)=1$ is the sum of

$$-\theta\delta^2(\chi)8\pi^2F_{\beta\alpha}(0)\cdot u_\alpha\int h_\alpha(k)\mathrm{d}k\cdot\overline{\int h_\beta(k)\mathrm{d}k}. \tag{8}$$

$$-2\pi i\theta\delta^2(\chi)F_{\beta\alpha}(0)\cdot\int h_\alpha(k)\mathrm{d}k\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k\right]\bigg|_{t=0}. \tag{9}$$

At this point we have evaluated our limit as c_1 tends to infinity. We state this as a lemma:

(10) LEMMA: *The limit of the integral of* $T_1^{c_1}T_2^{c_2}K_{\mathrm{eis}}$ *as* c_1 *tends to infinity exists. It is the sum over all* χ, α, β *of*:

(i) $$-2\pi i\delta^2\theta(\chi)F_{\beta\alpha}(0)\cdot\int h_\alpha(k)\mathrm{d}k\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\overline{\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k}\right]\bigg|_{t=0}$$

(ii) $$+\int_{-\infty}^{\infty}F_{\beta\alpha}(t)\overline{\Phi}(\beta,u_2,t)\cdot 2\mathrm{i}t\varepsilon(\chi)I(\mathrm{i}t,\chi)h_\alpha\frac{\mathrm{d}t}{t^2},$$

and a term $A\log c_2$.

(9.4) We now obtain an asymptotic expansion for c_2 large. We remark that in (9.3.10.ii) the factor $\overline{\Phi}(\beta, u_2, t)$ has a zero at $t=0$ while the factor $tI(it,\chi)h_\alpha$ has no pole at $t=0$. Furthemore the product of these functions by $F_{\beta\alpha}(t)$ is a Schwartz function. Thus we may use the integration by parts formula given in Lemma (9.2.2). We replace $\Phi(\beta, u_2, t)$ by its expression from (9.3.2). We obtain then for (9.3.10.ii):

$$-\varepsilon(\chi)4\int\frac{\mathrm{d}}{\mathrm{d}t}\left[F_{\beta\alpha}(t)\cdot tI(\mathrm{i}t,\chi)h_\alpha\cdot\overline{tI(\mathrm{i}t,\chi)h_\beta}\right]\frac{\mathrm{d}t}{t}, \tag{1}$$

$$\delta\varepsilon(\chi)\int\frac{\mathrm{d}}{\mathrm{d}t}\left[F_{\beta\alpha}(t)\cdot 2\mathrm{i}tI(\mathrm{i}t,\chi)h_\alpha\right]\cdot\int\overline{h}_\beta(k)\mathrm{d}k e^{-2\mathrm{i}\pi u_2 t}\frac{\mathrm{d}t}{t}, \tag{2}$$

$$-2\mathrm{i}\pi u_2\delta\varepsilon(\chi)\int F_{\beta\alpha}(t)\cdot 2\mathrm{i}tI(\mathrm{i}t,\chi)h_\alpha\cdot\int\overline{h}_\beta(k)\mathrm{d}k e^{2\mathrm{i}\pi u_2 t}\frac{\mathrm{d}t}{t}, \tag{3}$$

$$-\delta\varepsilon(\chi)\int\frac{\mathrm{d}}{\mathrm{d}t}\left[F_{\beta\alpha}(t)\cdot 2\mathrm{i}tI(\mathrm{i}t,\chi)h_\alpha\right]$$

$$\overline{\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k}e^{2\mathrm{i}\pi u_2 t}\frac{\mathrm{d}t}{t}, \tag{4}$$

$$-\delta\varepsilon(\chi)\int F_{\beta\alpha}(T)\cdot 2\mathrm{i}tI(\mathrm{i}t,\chi)h_\alpha$$

$$\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\overline{\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k}\right]e^{2\mathrm{i}\pi u_2 t}\frac{\mathrm{d}t}{t}, \tag{5}$$

$$-2\pi i u_2 \delta\varepsilon(\chi)\int F_{\beta\alpha}(t)\cdot 2itI(it,\chi)h_\alpha$$

$$\overline{\int M(it,\chi)h_\beta(k)\mathrm{d}k}\, e^{2i\pi u_2 t}\frac{\mathrm{d}t}{t}. \tag{6}$$

We can apply lemma (9.2.1) to (3) and (6). We get that each term has the form $A\log c_2 + o(1)$. So we can ignore these terms. On the other hand we can use the same lemma to find the limit of (2), (4) and (5). We get:

$$-\delta\varepsilon(\chi)\pi i\,\frac{\mathrm{d}}{\mathrm{d}t}\left[F_{\beta\alpha}(t)\cdot 2itI(it,\chi)h_\alpha\right]\Big|_{t=0}\cdot\int\bar{h}_\beta(k)\mathrm{d}k, \tag{7}$$

$$-\delta\varepsilon(\chi)\pi i\,\frac{\mathrm{d}}{\mathrm{d}t}\left[F_{\beta\alpha}(t)\cdot 2itI(it,\chi)h_\alpha\right]\Big|_{t=0}\cdot\overline{\int M(0,\chi)h_\beta(k)\mathrm{d}k}, \tag{8}$$

$$-\delta\varepsilon(\chi)\pi i\Big\{\left[F_{\beta\alpha}(t)\cdot 2itI(it,\chi)h_\alpha\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left[\int M(it,\chi)h_\beta(k)\mathrm{d}k\right]\right\}\Big|_{t=0}. \tag{9}$$

Again (7) and (8) are zero unless $\chi^{\sigma-1}=1$ and the restriction of χ to $F_{\mathbb{A}}^{\times}$ is trivial. Then $\chi^2=1$ by lemma (8.4.4) and their sum is zero because $M(0,\chi)=-\mathrm{Id}$ (lemma (8.1.6)). Thus (7) + (8) is zero in any case. Let us look at (9). Again it is zero unless $\chi^{\sigma-1}=1$ and the restriction of χ to $F_{\mathbb{A}}^{\times}$ is trivial. Then we find by (8.4.6):

$$\lim_{t\to 0}\varepsilon(\chi)itI(it,\chi)h_\alpha = -\delta(\chi)\int h_\alpha(k)\mathrm{d}k$$

and (9) becomes

$$\pi i\delta^2(\chi)F_{\beta a}(0)\cdot\int h_\alpha(k)\mathrm{d}k\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\overline{\int M(it,\chi)h_\beta(k)\mathrm{d}k}. \tag{10}$$

(9.4) We summarize our results in a proposition:

PROPOSITION: *The integral of* $T_1^{c_1}T_2^{c_2}K_{\mathrm{eis}}$ *over the product of* $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$ *by itself has a limit as* c_1 *tends to infinity. The resulting function of* c_2 *is the sum of a term* $A\log c_2$, *a term* $o(1)$ *and the sum over* χ, α, β *of the following terms*:

$$\varepsilon(\chi)\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left[F_{\beta\alpha}(T)\cdot tI(it,\chi)h_\alpha\cdot\overline{tI(it,\chi)h_\beta}\right]\frac{\mathrm{d}t}{t} \tag{1}$$

$$-2\pi i\delta^2(\chi)\chi(\chi)F_{\beta\alpha}(0)\int h_\alpha(k)\mathrm{d}k\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left[\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k\right] \tag{2}$$

$$\pi i\delta^2\varepsilon(\chi)F_{\beta\alpha}(0)\int h_\alpha(k)\mathrm{d}k\,\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left[\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k\right]. \tag{3}$$

We remark that in (2) and (3) the notation is somewhat misleading: both terms vanish unless $\delta(\chi)=0$. If $\delta(\chi)\neq 0$ then by lemma (8.4.4) either $O(\chi)=0$ or $\varepsilon(\chi)=0$ (exclusive or). Thus either (2) or (3) or both vanish.

(9.5) We now prove that, under the assumptions of (6.6.1), given χ, the sum over α, β of (1) (resp. (2), (3)) in Prop. (9.4) is zero. We start with (2) and (3). In view of the above remarks what we have to prove is this: given χ whose restriction to $F_{\mathbb{A}}^\times$ is trivial the following function of t has a zero of order 2 at $t=0$:

$$\sum_{\alpha,\beta}\big(\rho(\mathrm{i}t,\chi)(f)h_\beta,h_\alpha\big)\int h_\alpha(k)\mathrm{d}k\cdot\overline{\int M(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k}. \tag{1}$$

Since $M(\mathrm{i}t,\gamma)=m(t)R(\mathrm{i}t,\chi)$ where m is a scalar function which is regular at $t=0$, it suffices to prove this assertion for the following function:

$$\sum\big(\rho(\mathrm{i}t,\chi)(f)h_\beta,h_\alpha\big)\int h_\alpha(k)\mathrm{d}k\cdot\overline{\int R(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k}. \tag{2}$$

Of course we have a finite expansion:

$$\rho(\mathrm{i}t,\chi)(f)h_\beta=\sum_\alpha\big(\rho(\mathrm{i}t,\chi)(f)h_\beta,h_\alpha\big)h_\alpha.$$

Apllying the linear form defined by integration over K_F to this identity we see that our function can be written as:

$$\sum_\beta\int\rho(\mathrm{i}t,\chi)(f)h_\beta(k)\mathrm{d}k\overline{\int R(\mathrm{i}t,\chi)h_\beta(k)\mathrm{d}k}. \tag{3}$$

The space $\mathbb{H}(\chi)$ may be regarded as the tensor product over all places u of E of the local analogue spaces $\mathbb{H}(\chi_u)$. Similarly the operator $R(s,\chi)$ may be identified with the tensor product of the local normalized intertwining operators $R(s,\chi_u)$. Furthemore we may pick up an orthonormal basis of $\mathbb{H}(\chi_u)$ for each u and take for basis of $\mathbb{H}(\chi)$ the tensor products of the elements of the local bases. Accordingly the expression

(3) will be, apart from a constant factor, a product over all places v of F of local analogue expressions. What we have to see is that the factor attached to a place v of X is zero at $t=0$. The expression for this factor is the following:

$$\int [\rho(\mathrm{i}t, \chi_{v1})(f_{v1})]\, h_{\beta 1}(k_v)[\rho(\mathrm{i}t, \chi_{v2})(f_{v2})]\, h_{\beta 2}(k_v)\mathrm{d}k_v$$

$$\int R(\mathrm{i}t, \chi_{v1})h_{\beta 1}(k_v)R(\mathrm{i}t, \chi_{v2})h_{\beta 2}(k_v)\mathrm{d}k_v.$$

Since the restriction of χ to $F_{\mathbb{A}}^{\times}$ is trivial we have $\chi_{v1}=\bar{\chi}_{v2}$. By (7.3.4) for $t=0$ the above expression can be written as:

$$\sum \langle \rho(0, \chi_{v1})(f_{v1})h_{\beta 1}, \rho(0, \chi_{v2})(f_{v2})h_{\beta 2}\rangle\langle h_{\beta 1}, h_{\beta 2}\rangle.$$

Here $h_{\beta 1}$ denotes an orthonormal basis of $\mathbb{H}(\chi_{v1})$ and h an orthonormal basis of $\mathbb{H}(\chi_{v2})$. We remark that $h \mapsto \bar{h}$ is an antilinear isometric map of $\mathbb{H}(\chi_{v1})$ onto $\mathbb{H}(\chi_{v2})$. Thus we may assume that the bases are indexed by the same set: $h_{1\alpha}$ and $h_{2\alpha}$ will be the bases. More to the point we may assume that $\bar{h}_{1\alpha}=h_{2\alpha}$. Since $\langle h, h'\rangle = (h, \bar{h}')$ we have then:

$$\langle h_{1\beta}, h_{2\alpha}\rangle = \delta_{\beta\alpha}.$$

Thus our sum reduces to

$$\sum_{\beta} \langle \rho(0, \chi_{v1})(f_{v1})h_{1\beta}, \rho(0, \chi_{v2})(f_{v2})h_{2\beta}\rangle.$$

Since the map $h \mapsto \bar{h}$ transforms the representation $\rho(0, \chi_{v1})$ into $\rho(0, \chi_{v2})$ we get:

$$\sum_{\beta} \big(\rho(0, \chi_2)(f_{v1})h_{1\beta}, \rho(0, \chi_{v1})(\bar{f}_{v2})h_{1\beta}\big)$$

$$=\sum \big(\rho(0, \chi_{v1})(f_{v2}^{v})\cdot\rho(0, \chi_{v1})(f_{v1})h_{1\beta}, h_{1\beta}\big)$$

$$=\operatorname{tr}\rho(0, \chi_{v1})(h_v)$$

where as usual h_v is the convolution of f_{v2}^{v} and f_{v1}. Since the hyperbolic integrals of h_v vanish this vanishes too.

(9.6) We now examine the term (1) in Proposition (9.4). We want to prove that given χ such tht $\varepsilon(\chi)\neq 0$ the following function of t is zero, provided the conditions of (6.1.1) are satisfied:

$$\sum_{\alpha,\beta} \big(\rho(\mathrm{i}t, \chi)(f)h_{\beta}, h_{\alpha}\big)\cdot tI(\mathrm{i}t, \chi)h_{\alpha}\cdot\overline{tI(\mathrm{i}t, \chi)h_{\beta}}.$$

We may assume $t = 0$ and then ignore the factor t and replace $I(\mathrm{i}t, \chi)$ by its normalized version $J(\mathrm{i}t, \chi)$. Furthemore we have the following finite expansion:

$$\big(\rho(\mathrm{i}t, \chi)(f)h_\beta, h_\alpha\big)h_\alpha = \rho(\mathrm{i}t, \chi)(f)h_\beta.$$

If we apply the linear form $J(\mathrm{i}t, \chi)$ to this we find that the function we have to deal with can be written as:

$$\sum_\beta J(\mathrm{i}t, \chi)\big[\rho(\mathrm{i}t, \chi)(f)h_\beta\big]\cdot\overline{J(\mathrm{i}t, \chi)h_\beta}.$$

We can choose the basis h_β as before and then this function will be the product over all places v of F of factors themselves functions of t (Cf. (8.5)). It will be enough to show that the factor attached to a place v of X is zero. Recall that $\chi = \mu o N(E/F)$ and $\chi_{v1} = \chi_{v2} = \mu_v$. We may therefore assume that the bases for $\mathbb{H}(\chi_{v1})$ and $\mathbb{H}(\chi_{v2})$ are the same. By (8.5) this factor is a multiple of the following function:

$$\sum_{\alpha,\beta}\langle R(\mathrm{i}t, \mu_v)\rho(\mathrm{i}t, \mu_v)(f'_{v1})h_\beta, \rho(\mathrm{i}t, \mu_v)(f_{v2})h_\alpha\rangle$$
$$\overline{\langle R(\mathrm{i}t, \mu_v)h_\beta, h_\alpha\rangle}.$$

This is also:

$$\sum_{\alpha,\beta}\langle\rho(-\mathrm{i}t, \bar{\mu}_v)(f^v_{v2})R(\mathrm{i}t, \mu_v)\rho(\mathrm{i}t, \mu_v)(f_{v1})h_\beta, h_\alpha\rangle$$
$$\overline{\langle R(\mathrm{i}t, \mu_v)h_\beta, h_\alpha\rangle}.$$

Now we appeal to the following summation formula where a and b are elements of $\mathbb{H}(\bar{\mu}_v)$:

$$(a, b) = \sum_\alpha\langle a, h_\alpha\rangle\overline{\langle b, h_\alpha\rangle}.$$

We find for our factor:

$$\sum\big(\rho(-\mathrm{i}t, \bar{\mu}_v)(f^v_{v2})R(\mathrm{i}t, \mu_v)\rho(\mathrm{i}t, \mu_v)(f_{v1})h_\beta, R(\mathrm{i}t, \mu_v)h_\beta\big).$$

Because $R(\mathrm{i}t, \mu_v)$ is an intertwining unitary operator this is also:

$$\sum\big(\rho(\mathrm{i}t, \mu_v)(f_{v2}), \rho(\mathrm{i}t, \mu_v)(f_{v1})h_\beta, h_\alpha\big)$$
$$= \sum_\beta\big(p(\mathrm{i}t, \mu_v)(h_v)h_\beta, h_\beta\big) = \operatorname{tr}\rho(\mathrm{i}t, \mu_v)(h_v)$$

and this vanishes because the hyperbolic orbital integrals of h_v are zero.

(9.5) We summarize our results:

PROPOSITION: *The limit of the integral* $T_1^{c_1}T_2^{c_2}K_{\text{eis}}$ *as* c_1 *tends to infinity exists. The resulting function of* c_2 *has the form*

$$A \log c_2 + B + o(1).$$

Furthermore $B = 0$ *under the assumptions of* (6.1.1).

§10. Summing up

(10.1) We deal first with the truncation of K_{sp} then we summarize our results. Recall that

$$K = V^{-1}\sum \chi(\det x)\bar{\chi}(\det y)\int f(g)\chi(\det g)\mathrm{d}g,$$

the sum over all quadratic characters χ of the idele class group of E. The truncation of K_{sp} is therefore:

$$\begin{aligned} T_1^{c_1}T_2^{c_2}K_{\text{sp}}(x, y) &= K_{\text{sp}}(x, y)\left[1 - \sum \chi c_1(H(\gamma x))\right] \\ &\quad \times \left[1 - \sum \chi_{c2}(H(\gamma y))\right] \\ &= K_{\text{sp}}(x, y) - \sum K_{\text{sp}}(x, \gamma_1 y)\chi_{c_1}(H(\gamma_1 x)) \\ &\quad - \sum K_{\text{sp}}(x, \gamma_2 y)\chi_{c_2}(H(\gamma_2 y)) \\ &\quad + \sum K_{\text{sp}}(\gamma, x, \gamma_2 y)\chi_{c_1}(H(\gamma_1 x))\chi_{c_2}(H(\gamma_2 y)), \end{aligned}$$

where all summations are over $P(F)\backslash G(F)$. When we integrate over the product of $Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}})$ by itself we find a sum of 4 terms:

$$\iint K_{\text{sp}}(x, y)\mathrm{d}x\mathrm{d}y, \quad x, y \in Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}}); \tag{1}$$

$$-\iint K_{\text{sp}}(x, y)\chi_{c_1}(H(x))\mathrm{d}x\mathrm{d}y, \quad x \in Z(F_{\mathbf{A}})P(F)\backslash G(F_{\mathbf{A}}),$$

$$y \in Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}}); \tag{2}$$

$$-\iint K_{\text{sp}}(x, y)\chi_{c_2}(H(y))\mathrm{d}x\mathrm{d}y, \quad y \in Z(F_{\mathbf{A}})P(F)\backslash G(F_{\mathbf{A}}),$$

$$x \in Z(F_{\mathbf{A}})G(F)\backslash G(F_{\mathbf{A}}); \tag{3}$$

$$+\iint K_{sp}(x, y)\chi_{c_1}(H(x))\chi_{c_2}(H(y))\mathrm{d}x\mathrm{d}y,$$

$$x, y \in Z(F_{\mathbf{A}})P(F)\backslash G(F_{\mathbf{A}}). \tag{4}$$

We claim that c_2 being fixed (2) and (3) tend to 0 as c_1 tends to infinity. Let us prove it for (2): (2) is a finite linear combination of integrals of the forms:

$$\int \bar{\chi}(\det y)\mathrm{d}y \int \chi(\det x)\chi_{c_1}(H(x))\mathrm{d}x.$$

This is zero unless the restriction of χ to $F_{\mathbf{A}}^{\times}$ is trivial. Then it can be computed by using the Iwasava decomposition. We find it is a constant times

$$\int_0^{c_2^{1/2}} t^{-1}\mathrm{d}t = c_1^{-1/2}.$$

Hence the limit of the integral of $T_1^{c_1}T_2^{c_2}K$ as c_1 tends to infinity exists: it is equal to (1) + (2). Similarly as c_2 tends to infinity the term (2) tends to 0; that is, for c_2 large we have the following asymptotic expansion:

$$\lim_{c_1 \to 0} \iint T_1^{c_1}T_2^{c_2}K_{sp}\mathrm{d}x\mathrm{d}y = \iint K_{sp}\mathrm{d}x\mathrm{d}y + o(1).$$

(10.2) It is now time to sum up. We have:

$$\iint K_{cusp}(x, y)\mathrm{d}x\mathrm{d}y =$$

$$\iint K_e(x, y)\mathrm{d}x\mathrm{d}y \tag{1}$$

$$+\iint K_s(x, y)\mathrm{d}x\mathrm{d}y \tag{2}$$

$$+\iint (K_r(x, y) - K_{eis}(x, y))\mathrm{d}x\mathrm{d}y \tag{3}$$

$$-\iint K_{sp}(x, y)\mathrm{d}x\mathrm{d}y. \tag{4}$$

The third term is integrable because all others are. On the other hand since K_{cusp} is equal to its truncation we have:

$$\iint K_{\text{cusp}}(x, y)\mathrm{d}x\mathrm{d}y =$$

$$+\iint T_1^{c_1} T_2^{c_2} K(x, y)\mathrm{d}x\mathrm{d}y \tag{5}$$

$$-\iint T_1^{c_1} T_2^{c_2} K_{\text{eis}}(x, y)\mathrm{d}x\mathrm{d}y \tag{6}$$

$$-\iint T_1^{c_1} T_2^{c_2} K_{\text{sp}}(x, y)\mathrm{d}x\mathrm{d}y. \tag{7}$$

We now let c_1 tends to infinity. Under the assumptions of (6.6.1) the functions of c_2 obtained from the limit of (5) and (6) have the form $(1)+(2)+A \log c_2$ and $A' \log c_2 + o(1)$ respectively. Similarly the limit of (7) has the form $\iint K_{\text{sp}}\mathrm{d}x\mathrm{d}y + o(1)$. Comparing the two expressions for the integral of K_{cusp} we conclude that (3) vanishes. So we have established the result which was our goal:

PROPOSITION: *Let X be a set of places of F which split in E. Suppose that X has at least two elements. For v in X let $v1$ and $v2$ be the two corresponding places of E and h_v the convolution product $f_{v1}^{v} * f_{v2}$. Suppose that for each v in X the hyperbolic orbital integral of h_v vanish. Then the integral of the difference $K_r - K_{\text{eis}}$ over the product of $Z(F_{\mathbb{A}})G(F)\backslash G(F_{\mathbb{A}})$ by itself vanishes.*

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Department of Mathematics
Columbia University
New York, NY 10027
USA