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BERTINI THEOREMS FOR WEAK NORMALITY

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Introduction

In this paper we study the general hyperplane sections of weakly normal and seminormal algebraic varieties.

These varieties were introduced by Andreotti–Bombieri [3] and Traverso [16] respectively, as the algebraic counterpart of the notion of weakly normal (or maximal) complex analytic space given by Andreotti–Norguet [4] (the definitions, which agree in characteristic zero, are recalled in section 0).

The first ones to study the hyperplane sections of these varieties were Adkins, Andreotti and Leahy, who proved a Bertini theorem for a particular class of projective weakly normal algebraic varieties over **C**, called "optimal" (see [1] Th. 6.20 and Cor. 6.23); and Adkins–Leahy, who proved the same for all weakly normal projective varieties over **C** of dimension = 3 ([2] Th. 4.7).

Here we use a different point of view. Our main result, proved in section 1, is a local Bertini theorem for weak normality in characteristic zero (see Th. 1.8) which follows by elementary algebraic methods from the analogous results of Flenner [7] for "reduced" and "normal" and an algebraic characterization of weak normality given in [12].

From the local statement we deduce, by standard techniques, a global statement (Th. 2.4), which implies that the general hyperplane section of a weakly normal projective variety over an algebraically closed field of characteristic zero is again weakly normal (Cor. 2.5). This generalizes to any dimension the above mentioned result of Adkins–Leahy. The same

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methods together with some results of Ooishi [13, 14] allow us also to prove that the weak normalization and the seminormalization of a projective variety are again projective (Prop. 2.7).

The situation in positive characteristic is not so clear. First of all we note that there is a seminormal surface in $\mathbf{P}_k^3$, where k is algebraically closed of characteristic 2, with no seminormal hyperplane section (see 2.6). On the other hand we can give a positive result for the particular class of those projective weakly normal varieties which are S_2 and such that the normalization morphism is unramified outside a closed subset of codimension at least 2. These varieties, studied in [6], resemble the optimal spaces of [1] and include all weakly normal Gorenstein varieties. Our statement is a global one (Th. 3.7) and can be deduced from the local analytic description of such varieties given in [6].

The general problem in positive characteristic remains open. The main results of the first two section were announced in [5].

Conventions

All rings are assumed to be commutative and noetherian.

By "general hyperplane H of the projective space $\mathbf{P}_k^n$" we mean that H is a hyperplane describing a suitable non empty open subset of the dual projective space $(\mathbf{P}_k^n)^*$.

0. We recall some basic facts about weak normality and seminormality. For more details see [3], [16], [8] and [12].

Let B be an integral overring of a ring A. Consider the subrings A' of B containing A and such that $\operatorname{spec}(A') \to \operatorname{spec}(A)$ is a universal homeomorphism, (see [10] I, 3.5.8), (resp. a universal homeomorphism with trivial residue field extensions). The greatest such subring is called the weak normalization (resp. the seminormalization) of A in B and is denoted by ${}^*_BA$ (resp. ${}^+_BA$). A is said to be weakly normal (WN) (resp. seminormal (SN)) in B if $A = {}^*_BA$ (resp. $A = {}^+_BA$). It is clear that the two notations coincide if A contains a field of characteristic zero, but in general they are different (see [12], §0).

A ring A is said to be WN (resp. SN) if: (i) A is reduced and its integral closure $\bar{A}$ in its total ring of fractions is finite over A and (ii) A is WN (resp. SN) in $\bar{A}$. The weak normalization (resp. the seminormalization) of A in $\bar{A}$ is denoted by ${}^*A$ (resp. ${}^+A$).

For each ring we put:
$WN(A) := \{\mathfrak{p} \in \operatorname{spec}(A) \mid A_\mathfrak{p} \text{ is } WN\}$; $SN(A) := \{\mathfrak{p} \in \operatorname{spec}(A) \mid A_\mathfrak{p} \text{ is } SN\}$; these are open subsets of $\operatorname{spec}(A)$, if $\overline{A_{\mathrm{red}}}$ is finite over A.

An algebraic variety X over a field k (i.e. a k-scheme of finite type) is said to be WN (resp. SN) if $\mathcal{O}_{X,x}$ is WN (resp. SN) for each $x \in X$. This is equivalent to say that X can be covered by affine open subsets whose rings are WN (resp. SN).

It is clear what does it mean $WN(X)$ or $SN(X)$.

A weak normalization (resp. a seminormalization) of a reduced variety X is a couple $({}^*X, p)$ (resp. $({}^+X, p)$) where ${}^*X$ (resp. ${}^+X$) is a WN (resp. SN) algebraic variety and $p: {}^*X \to X$ (resp. $p: {}^+X \to X$) is a finite birational morphism which is a homeomorphism (resp. a homeomorphism with trivial residue field extensions) of the underlying topological spaces.

1. In this section we prove a local Bertini type theorem for weak normality, (th. 1.8). As a corollary it follows that if $X \subseteq \mathbf{C}^n$ is a germ of complex space at the origin 0 which is weakly normal (according to [4]) and depth $(\mathcal{O}_{X,0}) \geqq 3$, then the germ $X \cap H$ is weakly normal for any general hyperplane through 0.

We need some preliminary results.

1.1. Lemma: *Let A be a ring and let*

$$E: 0 \to E_0 \to E_1 \xrightarrow{\alpha} E_2$$

be a complex of finitely generated A-modules. Put $I = \mathrm{Ass}(E_1/E_0) \cup \mathrm{Ass}(E_2/\alpha(E_1)) \cup \mathrm{Ass}(A)$ and let $\mathfrak{p} \in \mathrm{spec}(A)$ be such the $E_\mathfrak{p}$ is exact. Then if $t \in \mathfrak{p}$ and $t \notin \mathfrak{q}$, for all $\mathfrak{q} \in I$ and $\mathfrak{q} \subseteq \mathfrak{p}$, the complex $E_\mathfrak{p}/tE_\mathfrak{p}$ is exact.

Proof: Since the associated primes do localize, we may assume that A is local with maximal ideal $\mathfrak{p}$ and that E is exact.

Put $\bar{E}_1 = E_1/E_0$, $\bar{E}_2 = E_2/\alpha(E_1)$. Then t is regular for $\bar{E}_1$ and $\bar{E}_2$ and hence $\mathrm{Tor}_1^A(A/t, \bar{E}_1) = \mathrm{Tor}_1^A(A/t, \bar{E}_2) = 0$. Then if we tensor by A/tA the exact complexes

$$0 \to E_0 \to E_1 \to \bar{E}_1 \to 0$$

$$0 \to \bar{E}_1 \to E_2 \to \bar{E}_2 \to 0$$

we get exact complexes. This easily implies our claim.

1.2. Lemma: *Let B be a ring and I and ideal of B. Then*

$$(B/I)_{\mathrm{red}} = (B_{\mathrm{red}}/IB_{\mathrm{red}})_{\mathrm{red}}$$

Proof: Straightforward.

1.3. Theorem (Flenner): *Let k be a field of characteristic* 0 *and let B be a semilocal excellent k-algebra. Let $y_1,\dots,y_n \in \mathrm{rad}(B)$; for $\lambda = (\lambda_1,\dots,\lambda_n) \in k^n$, put $y_\lambda = \sum \lambda_i y_i$ and $V_\lambda = (\mathrm{spec}(B) - V(y_1,\dots,y_n)) \cap V(y_\lambda)$. Assume that B is reduced (resp. normal). Then there is a non empty open set $U \subseteq k^n$ such that for all $\lambda \in U$ and all $\mathfrak{p} \in V_\lambda$, the ring $B_\mathfrak{p}/y_\lambda B_\mathfrak{p}$ is reduced (resp. normal).*

Proof: If B is local this is just [7] 4.8. The general case follows easily, as "reduced" and "normal" are local properties.

1.4. Corollary: *Let k be a field of characteristic* 0, *$(A, \mathfrak{m})$ a local excellent k-algebra, B a finite reduced (resp. normal) A-algebra. Let $x_1,\dots,x_n \in \mathfrak{m}$; for $\lambda \in k^n$, put $x_\lambda = \sum \lambda_i x_i$ and $T_\lambda = (\mathrm{spec}(A) - V(x_1,\dots,x_n)) \cap V(x_\lambda)$. Then there is a non empty open set $U \subseteq k^n$ such that, if $\lambda \in U$ and $\mathfrak{p} \in T_\lambda$, $B \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/x_\lambda A_\mathfrak{p}$ is reduced (resp. normal).*

Proof: Let y_i be the image of x_i in B, let V_λ be as in 1.3 and let $\mathfrak{p} \in T_\lambda$. Let $\mathfrak{P} \in \mathrm{spec}(B)$ be a prime lying over $\mathfrak{p}$. Then $\mathfrak{P} \in V_\lambda$, hence $B_\mathfrak{P}/x_\lambda B_\mathfrak{P}$ is reduced (resp. normal). Moreover the maximal ideals of $B_\mathfrak{p}$ correspond to the primes of B which lie over $\mathfrak{p}$ and hence $B_\mathfrak{p}/x_\lambda B_\mathfrak{p}$ is reduced (resp. normal).

1.5. Corollary: *Let k, (A, m), $x_1,\dots,x_n$ be as in* 1.4 *and let C be a finite A-algebra. Then there is a non empty open set $U \subseteq k^n$ such that*

$$(C_\mathfrak{p})_{\mathrm{red}}/x_\lambda (C_\mathfrak{p})_{\mathrm{red}} = (C_\mathfrak{p}/x_\lambda C_\mathfrak{p})_{\mathrm{red}}$$

for all $\lambda \in U$ and all $\mathfrak{p} \in T_\lambda$.

Proof: It is an easy consequence of 1.2 and of 1.4 applied to C_{red}.

1.6. Lemma: *Let A be a reduced ring and let $\bar{A}$ be the integral closure of A. Assume that $\bar{A}$ is finite over A. Let $t \in A$ be a regular element and assume that $t \notin \mathfrak{q}$, for all $\mathfrak{q} \in \mathrm{Ass}(\bar{A}/A)$. Then $A/tA \hookrightarrow \bar{A}/t\bar{A}$ and every regular element of A/tA is regular in $\bar{A}/t\bar{A}$.*

Proof: The first assertion follows from 1.1. Now, since $\bar{A}$ is S_2, every associated prime of $\bar{A}/t\bar{A}$ is of the form $\mathfrak{P}/t\bar{A}$, where $\mathfrak{P}$ is a prime ideal of height 1. Let $\mathfrak{p} = \mathfrak{P} \cap A$; it is sufficient to prove that $ht(\mathfrak{p}) = 1$ since then $\mathfrak{p}/tA \in \mathrm{Ass}(A/tA)$. Let $\mathfrak{b}$ be the conductor of A. If $\mathfrak{P} \not\supseteq \mathfrak{b}$ then $A_\mathfrak{p} = \bar{A}_\mathfrak{P}$ and hence $ht(\mathfrak{p}) = ht(\mathfrak{P}) = 1$. If $\mathfrak{P} \supseteq \mathfrak{b}$ then $\mathfrak{P} \in \mathrm{Ass}_{\bar{A}}(\bar{A}/\mathfrak{b})$, being $ht(\mathfrak{P}) = 1$; it follows that there exists $x \in \bar{A}/\mathfrak{b}$ such that $\mathfrak{P} = \mathrm{Ann}_{\bar{A}}(x)$; but $\mathrm{Ann}_A(x) = \mathrm{Ann}_{\bar{A}}(x) \cap A = \mathfrak{p}$, hence $\mathfrak{p} \in \mathrm{Ass}_A(\bar{A}/\mathfrak{b})$. Moreover by the exact sequence

$$0 \to A/\mathfrak{b} \to \bar{A}/\mathfrak{b} \to (\bar{A}/\mathfrak{b})/(A/\mathfrak{b}) \cong \bar{A}/A$$

we have $\mathrm{Ass}_A(\bar{A}/\mathfrak{b}) \subseteq \mathrm{Ass}_A(\bar{A}/A) \cup \mathrm{Ass}_A(A/\mathfrak{b})$; but we have $\mathrm{Ass}_A(A/\mathfrak{b}) \subseteq \mathrm{Ass}_A(\bar{A}/A)$, hence we get a contradiction, being $\mathfrak{p} \in \mathrm{Ass}_A(\bar{A}/A)$ and $t \in \mathfrak{p}$.

1.7. LEMMA: *Let k be a field, $(A, \mathfrak{m})$ a lokal k-algebra and $x_1, \ldots, x_n \in \mathfrak{m}$. Put $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $x_\lambda = \sum \lambda_i x_i$; let $\mathfrak{q} \in \mathrm{spec}(A) - V(x_1, \ldots, x_n)$. Then $U = \{\lambda \in k^n \mid x_\lambda \notin \mathfrak{q}\}$ is open and dense in k^n.*

PROOF: The map of k-vector spaces $\Phi: k^n \to A$ given by $\Phi(\lambda) = x_\lambda$ is linear and $\Phi^{-1}(\mathfrak{q}) \neq k^n$ as $\mathfrak{q} \notin V(x_1, \ldots, x_n)$. Thus U is the complement of a linear subspace of k^n, different from k^n.

Now we can prove the main result of this section.

1,8. THEOREM: *Let k be a field of characteristic 0, $(A, \mathfrak{m})$ a local excellent k-algebra and $x_1, \ldots, x_n \in \mathfrak{m}$; put $\lambda = (\lambda_1, \ldots, \lambda_n)$, $x_\lambda = \sum \lambda_i x_i$ and $Z_\lambda = (\mathrm{spec}(A) - V(x_1, \ldots, x_n)) \cap WN(A) \cap V(x_\lambda)$. Then there is a non empty open set $U \subseteq k^n$ such that if $\lambda \in U$ and $\mathfrak{p} \in Z_\lambda$, then $A_\mathfrak{p}/x_\lambda A_\mathfrak{p}$ is WN.*

PROOF: We shall use the following characterization of WN rings (see [12], th. 1.6): let $R \subseteq S$ be rings, S integral over R, then R is WN in S if and only if the sequence of R-modules

$$R \to S \xrightarrow{\sigma} (S \otimes_R S)_{\mathrm{red}}$$

is exact, where σ maps b to $b \otimes 1 - 1 \otimes b \bmod N(S \otimes_R S)$, being $N(S \otimes_R S)$ the nilradical of $S \otimes_R S$. Consider now the complex of A-modules

$$E: A \to B \to C$$

where $B = \overline{(A_{\mathrm{red}})}$ and $C = (B \otimes_A B)_{\mathrm{red}}$. If $A_\mathfrak{p}$ is reduced we have $B_\mathfrak{p} = \overline{A_\mathfrak{p}}$; hence if $\mathfrak{p} \in WN(A)$, we have

$$E_\mathfrak{p}: A_\mathfrak{p} \to \overline{A_\mathfrak{p}} \to (\overline{A_\mathfrak{p}} \otimes_{A_\mathfrak{p}} \overline{A_\mathfrak{p}})_{\mathrm{red}}$$

and this complex is exact. Therefore, by 1.1 and 1.7, there is $U \subseteq k^n$ open and non empty, such that, whenever $\lambda \in U$ and $\mathfrak{p} \in Z_\lambda$, the complex

$E_\mathfrak{p}/x_\lambda E_\mathfrak{p}$ is exact. Now by 1.4 we may assume that $E_\mathfrak{p}/x_\lambda E_\mathfrak{p}$ coincides with

$$R \to S \to (S \otimes_R S)_{\text{red}}$$

where $R = A_\mathfrak{p}/x_\lambda A_\mathfrak{p}$, $S = \overline{A_\mathfrak{p}}/x_\lambda \overline{A_\mathfrak{p}}$. Hence R is WN in S by the above mentioned result. Moreover, by 1.4 and 1.6 respectively, we may assume that S is normal and $R \subseteq \bar{R} \subseteq S$, whence R is WN.

1.9. Corollary: *Let k, $(A, \mathfrak{m})$, $x_1, \ldots, x_n$ be as in* 1.8, *let moreover A be WN and* $\mathfrak{m} = \text{rad}(x_1, \ldots, x_n)$. *Then there is a non empty open set $U \subseteq k^n$ such that for all $\lambda \in U$*

(i) $\text{spec}(A/x_\lambda A) - \{\mathfrak{m}/x_\lambda A\}$ *is WN;*

(ii) *if moreover* $\text{depth}(A) \geq 3$, *then $A/x_\lambda A$ is WN.*

Proof: (i) follows by 1.8; (ii) follows by (i) and by [12] IV.4.

1.10. Corollary: *Let $X \subseteq \mathbb{C}^n$ be a germ of analytic complex space at the origin and assume that X is WN (according to* [4]) *and that* $\text{depth}(\mathcal{O}_{X,0}) \geq 3$. *Then if H is a general hyperplane through* 0, *we have that the germ $X \cap H$ is WN.*

Proof: By [8] 6.12 we have that X is WN if and only if $\mathcal{O}_{X,0}$ is a WN ring. Then the conclusion follows from 1.9.

1.11. Remarks: (i) We do not know whether 1.8 is true if k is a field of positive characteristic, not even if k is algebraically closed and $\mathfrak{m} = (x_1, \ldots, x_n)$: these assumptions are stronger that the ones used by Flenner in proving his local Bertini theorems for reduced and normal rings when the residue field has positive characteristic; but they are not enough to apply our methods of proof, because we need them to be verified in the local rings of $\bar{A}$ and $(\bar{A} \otimes_A \bar{A})_{\text{red}}$ and this fact, in general, does not happen.

2. In this section, by reducing to the local case, we prove that the general hyperplane section of a WN projective variety over a field of characteristic zero is again WN; we give then a counterexample for the case of SN varieties in positive characteristic. We end by showing that the weak normalization and the seminormalization of a projective variety are still projective (whence there are significant examples where to apply our results).

2.1. PROPOSITION (see also [7] 5.1): *Let k be a field, $X \subseteq \mathbb{P}_k^n$ a projective variety and $Y \subseteq X$ a closed subset of X. Let $Y^+ \subseteq X^+ \subseteq \mathbb{A}_k^{n+1}$ be the corrisponding affine cones; put $R = \mathcal{O}_{X^+,v}$ (where v is the vertex) and let I be the ideal of Y^+ in R. Let P be a local property which is preserved by polynomials and fractions and which descends by faithful flatness. Then the following are equivalent:*

(i) *$X - Y$ is P;*

(ii) *$X^+ - Y^+$ is P (if $Y = \emptyset$ we agree that $Y^+ = \{v\}$);*

(iii) spec$(R) - V(I)$ *is P.*

PROOF: Recall that if $A = k[x_0,\dots,x_n]$ is the graded ring of X (and the coordinate ring of X^+), there are canonical isomorphisms

$$A_{\{x_i\}}[T, T^{-1}] \to A_{x_i} \qquad \text{(see [EGA] II, 2.2.1)} \qquad (^\circ)$$

(i) $\Rightarrow$ (ii). Let $V_i = \operatorname{spec}(A_{\{x_i\}})$, $V_i^+ = \operatorname{spec}(A_{x_i})$ and let $\phi_i: V_i^+ \to V_i$ be the canonical morphism. Put $U = X - Y$. Then by $(^\circ)$ we have that $\phi_i^{-1}(U \cap V_i)$ has the property P and since $X^+ - Y^+ = \bigcup_{i=0}^{n} \phi_i^{-1}(U \cap V_i)$ the conclusion follows.

(ii) $\Rightarrow$ (iii) is clear, because the local rings of spec$(R) - V(I)$ are localizations of the local rings of the closed points of $X^+ - Y^+$.

(iii) $\Rightarrow$ (i). If $x \in X$ and $\mathfrak{P} \in \operatorname{spec}(R)$ is the corresponding prime ideal, the canonical homomorphism $\mathcal{O}_{X,x} \to R_{\mathfrak{P}}$ is faithfully flat by $(^\circ)$; whence the conclusion.

2.2. COROLLARY: *Let X, R and P be as in* 2.1. *Then X is P if and only if* spec$(R) - \{\mathfrak{m}\}$ *is P, where $\mathfrak{m}$ is the maximal ideal of R.*

2.3. COROLLARY: (i) 2.1 *and* 2.2 *hold if either $P = WN$ or $P = SN$. In particular if X and R are as in* 2.2, *we have:*

(ii) *X is WN if and only if $R_{\mathfrak{q}}$ is WN for all* $\mathfrak{q} \neq \mathfrak{m}$;

(iii) *if* depth$(R) \geq 2$, *then X is WN if and only if R is WN.*

PROOF: $P = WN$ and $P = SN$ verify the assumptions of 2.1 (see [12] II.1 for WN and [8] 1.6, 2.1, 5.2 for SN). Thus we have (i).

(ii) is clear, while (iii) follows from [12] IV.4.

In the next theorem, which is the main result of this section, the intersections of algebraic varieties are algebraic (not only set theoretical).

2.4. THEOREM: *Let k be a field of characteristic* 0 *and let X be a closed* *subvariety of the projective space* $\mathbb{P}^r = \mathbb{P}^r_k$. *Let* $F_0, \ldots, F_n$ *be forms of the same degree in* $k[X_0, \ldots, X_r]$; *put* $Y = V(F_0, \ldots, F_n) \cap X$ *and* $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$; *let* F_λ *be the hypersurface* $\sum \lambda_i F_i = 0$. *Then there is a non empty open set* $U \subseteq k^{n+1}$ *such that* $(WN(X - Y)) \cap F_\lambda \subseteq WN(X \cap F_\lambda)$, *for all* $\lambda \in U$.

PROOF: It is an easy consequence of 2.3 (ii) and 1.8.

2.5. COROLLARY: *The general hyperplane section of a WN projective variety over a field of characteristic* 0 *is WN.*

2.6. REMARK: 2.5 holds by substituting *WN* with *SN*, since in characteristic zero the two concepts coincides. We do not know whether the general hyperplane section of a *WN* projective variety over a field of positive characteristic is *WN*. This is however false for *SN* varieties.

For example, let k be an algebraically closed field of characteristic 2 and let $X \subseteq \mathbb{P}^3_k$ be the surface $X_0X_1^2 + X_3X_2^2 = 0$. X is *SN*, but not *WN* (e.g. [12] §0) and no plane section of X is *SN*. Consider indeed in the chart $U_3 : X_3 = 1$ a plane of the form $X + bY + cZ + d = 0$. Then by using the change of coordinates $x = X + bY + cZ + d$, $y = Y$, $z = Z$ we see that $H \cap X \cap U_3$ is the curve of the plane $x = 0$ having equation $(by + cz)y^2 + (\delta y + z)^2 = 0$, where $\delta^2 = d$. Hence $H \cap X$ has a cusp and is not *SN* (e.g. by [8] §8). In the same way one can see that $H \cap X$ is either non reduced or non *SN* for all other planes H.

We conclude this section by proving the following:

2.7. PROPOSITION: *Let* $X \subseteq \mathbb{P}^n_k$ *be a reduced projective variety and let* A *be its graded ring. Then* X *has a weak normalization* $^*X$ (*resp. a seminormalization* ^+X), *which can be embedded in some* $\mathbb{P}^N_k$ *with graded ring* $(^*A)^{(d)}$ $(:= \bigoplus_{n \in \mathbb{N}} (^*A)_{nd})$(*resp.* $(^+A)^{(d)}$) *for d sufficiently large.*

PROOF: By [14] Cor. 9 and Cor. 6, $^*A$ and ^+A are graded rings. Moreover by [14] Cor. 10 and Cor. 7, for each $d \geq 0$, $(^*A)^{(d)}$ is *WN* and $(^+A)^{(d)}$ is *SN*.

In the following we will examine only the case of weak normalization, as the seminormal one is the same. Although $^*A$ and $(^*A)^{(d)}$ are not isomorphic in general, however, for each $d \geq 0$, $\mathrm{Proj}(^*A) \cong \mathrm{Proj}((^*A)^{(d)})$ and, by an argument quite similar to the one of [15] p. 26, whenever d is sufficiently large, $(^*A)^{(d)}$ is generated, as $(^*A)$-algebra, by its forms of degree 1, i.e. it is the graded ring of a projective variety $^*X := \mathrm{Proj}((^*A)^{(d)})$, which is *WN* by 2.3 (i).

The inclusion $A \hookrightarrow {}^*A$ induces a projective morphism

$$\text{Proj}({}^*A) \cong \text{Proj}(({}^*A)^{(d)}) = {}^*X \xrightarrow{p} \text{Proj}(A) = X$$

which, by an argument similar to [15] p. 27, is finite and birational.

Moreover it is clear that p is a homeomorphism between the underlying topological spaces, hence $({}^*X, p)$ is a weak normalization of X.

3. There is a nice class of WN varieties, called $WN1$, which includes all Gorenstein WN varieties (see [6]). In this section we show that if $X \subseteq \mathbb{P}^n_k$ is $WN1$ and S_2, then the general hyperplane sections of X are again $WN1$ and S_2 (Th. 3.7). It follows that if X is WN and Gorenstein, then the same is true for all general hyperplane sections of X (Cor. 3.8).

In the following we consider algebraic varieties over an algebraically closed field of arbitrary characteristic.

First of all we recall some facts from [6].

3.1. Definition (see [6] 2.2, 2.3): Let X be an algebraic variety and let Y be an irreducible subvariety of codimension 1 (i.e. $\dim(\mathcal{O}_{X,y}) = 1$, where y is the generic point of Y). We say that Y is of *general type* if there is a non empty open set $U \subseteq Y$ such for any singular closed point $x \in U$:

$$\hat{\mathcal{O}}_{X,x} = k[[X_1, \ldots, X_{n+s-1}]]/(\ldots, X_iX_j, \ldots)$$

where n, s are independent of $x \in U$ and $i, j \in \{1, \ldots, s\}$, $i \neq j$.

3.2. Definition: An algebraic variety X is said to be $WN1$ if it is WN and all its singular subvarieties of codimension 1 are of general type.

Thus, in order to study the general hyperplane sections of $WN1$ varieties, it is sufficient to see the behaviour of the singular subvarieties of codimension 1 (see Th. 3.5). For this we need several lemmas.

3.3. Lemma: *Let X be an algebraic variety, let $Y \subseteq X$ be a singular* 1-*codimensional subvariety of general type and let U be as in* 3.1. *Then we have:*

(i) *for any closed point $x \in U$ the ideal corresponding to Y in $\hat{\mathcal{O}}_{X,x}$ is* $(x_1, \ldots, x_s)\hat{\mathcal{O}}_{X,x}$;

(ii) $U \subseteq \text{Reg}(Y)$.

PROOF: (i) Let $x \in U$ be a closed point, let $A := \mathcal{O}_{X,x}$ and let $\mathfrak{p}$, $\hat{\mathfrak{p}} := \mathfrak{p}\hat{A}$ be the ideals of Y in A and $\hat{A}$ respectively. Let $I := (x_1, \ldots, x_s)\hat{A}$. Since $\hat{A}$ is Cohen–Macaulay one can use [6] 2.3 (proof of (v) $\Leftrightarrow$ (vii)) to show that I is a prime ideal of height 1 and is the conductor of $\hat{A}$ (in its integral closure). As A is excellent, we have $I = \mathfrak{b}\hat{A}$, where $\mathfrak{b}$ is the conductor of A. Since $A_{\mathfrak{p}}$ is not normal, we have $\mathfrak{p} \supseteq \mathfrak{b}$, which implies $\hat{\mathfrak{p}} \supseteq I$. But $\mathfrak{p}$ is prime of height 1, hence by flatness every minimal prime of $\hat{\mathfrak{p}}$ has height 1; this easily implies that $\hat{\mathfrak{p}} = I$, so (i) is proved.

(ii) follows by the fact that $\hat{A}/\hat{\mathfrak{p}} = (A/\mathfrak{p})\hat{}$ and $\hat{A}/\hat{\mathfrak{p}}$ is clearly regular.

3.4. LEMMA: *Let $Y \subseteq \mathbb{P}^n_k$ be an irreducible projective variety and let $U \subseteq Y$ be non empty open. Then if H is a general hyperplane, $U \cap H$ is dense in $Y \cap H$.*

PROOF: Let $Z_1, \ldots, Z_t$ be the irreducible components of $Y - U$. Then a general hyperplane H does not contain any of the Z_i's and each non embedded irreducible component of $Y \cap H$ has codimension 1. Moreover, by looking at the dimensions, we see that every non embedded component of $Y \cap H$ intersects $U \cap H$, which implies our claim.

3.5. THEOREM: *Let $X \subseteq \mathbb{P}^n_k$ be a projective algebraic variety, let $Y \subseteq X$ be a singular 1-codimensional subvariety of general type. Then if H is a general hyperplane, every non embedded irreducible component of $Y \cap H$ is of general type.*

PROOF: Let $U \subseteq Y$ be an open set as in 3.1. Then by 3.3 $U \subseteq \mathrm{Reg}(Y)$ and hence if H is a general hyperplane we have $U \cap H \subseteq \mathrm{Reg}(Y \cap H)$ (see [7] 5.2). Now let Z be an irreducible non embedded component of $Y \cap H$. Then Z has codimension 1 and $Z \cap H \cap U$ is non empty by 3.4. Let then $x \in Z \cap H \cap U$ be a closed point and let $\mathrm{B} := \hat{\mathcal{O}}_{X,x} \cong k[[X_1, \ldots, X_{n+s-1}]]/(\ldots, X_iX_j, \ldots)$, $i \neq j$, $i, j \in \{1, \ldots, s\}$; let $\hat{\mathfrak{p}}$ and hB respectively be the ideals of Y and H in B. Then $\hat{\mathfrak{p}} = (x_1, \ldots, x_s)$ by 3.3 and $B/(\hat{\mathfrak{p}}, h)$ is regular. Hence we may assume that h has a lifting $f \in k[[X_1, \ldots, X_{n+s-1}]]$ of the form $f(X_1, \ldots, X_{n+s-1}) = \sum_{i=1}^{n+s-1} a_iX_i \bmod (X_1, \ldots, X_{n+s-1})^2$, where $a_{n+s-1} \neq 0$.

Then by using the substitutions:

$$X_i \to X_i \qquad \text{for } i \leq n+s-2$$

$$X_{n+s-1} \to f$$

we have

$$\hat{\mathcal{O}}_{Z,x} = B/hB = (k[[X_1,\dots,X_{n+s-1}]]/(\dots,X_iX_j,\dots))/f = \\ = k[[X_1,\dots,X_{n+s-2}]]/(\dots,X_iX_j,\dots), i \neq j,\ i,j \in \{1,\dots,s-1\}.$$

and hence the closed points of the open non empty set $U \cap H \cap Z$ of Z verify the definition.

We shall need the following:

3.6. Proposition (see [6] 2.5): *Let X be a reduced S_2 algebraic variety. Assume that every singular subvariety of codimension 1 of X is of general type. Then X is $WN1$.*

3.7. Theorem: *Let $X \subseteq \mathbb{P}^n_k$ be a $WN1$ S_2 algebraic variety. Then the general hyperplane section of X is $WN1$ and S_2.*

Proof: If H is a general hyperplane, we have by [7] 5.2 and by 3.5:

(i) $H \cap X$ is reduced;

(ii) $H \cap X$ is S_2;

(iii) $H \cap \mathrm{Reg}(X) \subseteq \mathrm{Reg}(H \cap X)$;

(iv) if $Y \subseteq X$ is a singular 1-codimensional subvariety of general type, every component of $H \cap Y$ is 1-codimensional and of general type.

Then it is sufficient to show that if H verifies (i), (ii), (iii), (iv) and $Z \subseteq H \cap X$ is singular of codimension 1, then Z is of general type, (by (i), (ii) and 3.6). But by (iii) we have $Z \subseteq H \cap \mathrm{Sing}(X)$ and hence Z must be an irreducible component of $H \cap Y$, where Y is a singular subvariety of codimension 1 of X. Then the conclusion follows by (iv).

3.8 Corollary: *Let $X \subseteq \mathbb{P}^n_k$ be a WN Gorenstein algebraic variety. Then the general hyperplane section of X is Gorenstein and WN.*

Proof: By well known properties of Gorenstein rings (see [11] Ch. I, §1), we have that $H \cap X$ is Gorenstein whenever H does not contain any irreducible component of X. Moreover X is S_2 and is $WN1$ by [6] 3.8. The conclusion follows then by the preceeding theorem.

Note added in proof

Recently M. Vitulli gave analogous results about hyperplane sections of weakly normal varieties (see M.A. Vitulli, "The hyperplane sections of weakly normal varieties", preprint).

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