COMPOSITIO MATHEMATICA

WALTER BORHO

On the Joseph-small additivity principle for goldie ranks. A study on extensions of noetherian rings with applications to enveloping algebras

Compositio Mathematica, tome 47, nº 1 (1982), p. 3-29

<http://www.numdam.org/item?id=CM_1982__47_1_3_0>

COMPOSITIO MATHEMATICA, Vol. 47, Fasc. 1, 1982, pag. 3–29

ON THE JOSEPH–SMALL ADDITIVITY PRINCIPLE FOR GOLDIE RANKS

A study on extensions of Noetherian rings with applications to enveloping algebras

Walter Borho

Summary

An axiomatic notion of dimension is introduced, unifying the concepts of Gelfand-Kirillov and of Gabriel-Rentschler dimension. A theorem on Orean localizations is proved, unifying and generalizing results of Small [17], Joseph–Small [11], Krause–Lenagan–Stafford [15], and others, on artinian quotientrings, and of Borho–Rentschler [5], and Joseph–Small [11] on localizability of certain ring-extensions important for the study of enveloping algebras. An extension of the whole Joseph–Small theory to the level of abstract noetherian ring theory is given, based on a certain 'restrictedness' condition for extension–rings. For this class of extensions, which seems to be quite natural and useful, a theorem on 'good behaviour' of prime ideals is proved, including statements on equality of dimensions and additivity of Goldie ranks for minimal primes. Some new applications to enveloping algebras are given.

§0. Introduction

Consider an extension of non-commutative noetherian rings $A \subset B$. We are going to study the behaviour of a prime ideal P of the big ring B, when going down to the small ring A. In contrast to commutative algebra, $P \cap A$ will usually not be a prime ideal, but may decompose in an arbitrarily complicated way. Instead of a 'morphism' Spec $B \to$

0010–437X/82070003–29$00.20/0

Spec A, we get only a much more modest correspondence between prime-ideals upstairs resp. downstairs, by relating to P the finite set $P_1, \ldots, P_n$ of minimal primes containing $P \cap A$. However, this correspondence may still involve a considerable amount of 'well-behavedness', as was recently discovered by Joseph and Small [11] in a special situation arising naturally in the study of enveloping algebras of semisimple Lie algebras. Their theory turned out to admit very useful applications to the study of primitive ideals, in particular to the computation of their Goldie ranks, as was demonstrated by extensive work of Joseph [12], [13], [14]. Let us now first make precise what we mean by 'good behaviour' here, and afterwards discuss some rather general, natural conditions on an extension $A \subset B$, which will be proved to imply that all primes of the big ring B are well-behaved over the small ring A.

We assume the ring B noetherian on one side, and A noetherian on both sides. Furthermore, for the purposes of this introduction only, let us assume that A and B are finitely generated k-algebras, and denote $d(M)$ the GK-dimension (Gelfand–Kirillov dimension relative k) for a left A-module M, where k is a (commutative) base-field. Fix a prime P of B, write $\bar{B} = B/P$ resp. $\bar{A} = A/(P \cap A)$ for the residue class rings, and $S \subset \bar{A}$ for the subset of all those elements, which become non-zero-divisors modulo the nilradical of $\bar{A}$. We shall call S the *Small set* of $\bar{A}$, because of its significance in Small's theorem on the existence of an artinian quotient-ring (see [17]). We are now ready to list the "five rules of good behaviour":

(1) *Homogeneity*: $d(\bar{A}b) = d(\bar{A})$ for all $b \in \bar{B}$.

(2) *Regularity*: All elements of S are non-zero-divisors in $\bar{B}$.

(3) *Localizability*: S satisfies the right Ore-condition for $\bar{A}$ and $\bar{B}$.

(4) *Equidimensionality*: All the minimal primes $P_1, \ldots, P_n$ over $P \cap A$ have the same GK-dimension, i.e. $d(A/P_i) = d(\bar{A})$.

(5) *Additivity*: The Goldie ranks of the P_i, each taken a suitable number $z_i > 0$ of times, add up to that of P, i.e. $\operatorname{rk} B/P = \sum_{i=1}^{n} z_i \operatorname{rk} A/P_i$.

If all this is true, we shall say P is *well-behaved* over A. From the point of view of applications, (4) and (5) deserve the main interest; from a more theoretical point of view, however, (1)–(3) are even more interesting, since they easily imply (4)–(5). Joseph and Small discovered such well-behavedness in the following special situation, involving a generalized Verma module M' for a semisimple Lie-algebra $\mathfrak{g}$: The small ring is $A = U(\mathfrak{g})/\operatorname{Ann} M'$, the big one is $B = L(M', M')$ (ring of $\mathfrak{g}$-finite k-endomorphisms of M'), and the prime is $P = 0$ [11].

Actually, they proved $P = 0$ to behave well over A, whenever (a′) A is a homomorphic image of an enveloping algebra of a finite dimensional k-Lie algebra $\mathfrak{a}$, and

(b′) B is ad $\mathfrak{a}$-finite and finitely generated as A-bimodule.

Since there are examples where additivity (5) definitely fails (Small, unpublished), it is clear that well-behavedness can be expected for an extension $A \subset B$ only if some kind of restrictions are imposed on the extension.

The main goal of the present paper is the following.

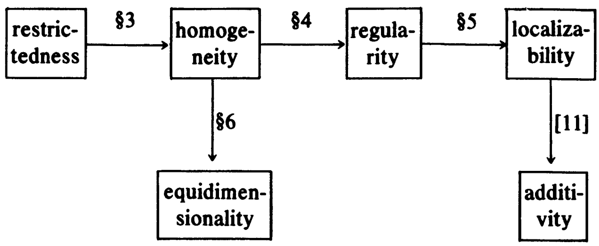
THEOREM: *Assume that* (a) $d(A) < \infty$, *and* (b) AbA *is a r-noetherian A-module for all* $b \in B$. *Then every prime P of B is well-behaved over A.*

Let us call B *r-restricted* over A, if (b) is satisfied. This class of extensions includes on one hand all commutative ones, and on the other hand the Joseph–Small case ((a′) implies (a), and (b′) implies (b)). Furthermore, this class includes also the case $B = U(\mathfrak{b}) \supset A = U(\mathfrak{a})$, where the big ring is an enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{b}$, and the small ring is the envelope of a subalgebra $\mathfrak{a} \subset \mathfrak{b}$. This is perhaps the most natural type of ring-extensions occurring in the theory of enveloping algebras, and so the theorem above may be of some use for the study of prime ideals in enveloping algebras (see §8).

Let us briefly comment on the question, whether our restrictedness-condition (b) is also natural for the problem in some abstract sense. We could replace (b) by a more technical, slightly weaker version of 'restrictedness', which is not only *sufficient* for well-behavedness of primes, but also *necessary*—at least under certain additional assumptions. From the point of view of general abstract ring-theory, that weak restrictedness-condition would be therefore more satisfactory. On the other hand, it is more complicated to state, and to use, and has no additional applications—at least so far. The situation is in some respects analogous to Goldie's theorem on orders in semisimple artinian rings: His weaker noetherian conditions are most interesting from the point of view of abstract ring theory, because they are necessary and sufficient; but from the point of view of applications, it is frequently more natural to work with the usual noetherian condition, which is sufficient only, but covers most applications. For the present paper, I prefer to base the exposition entirely on condition (b) above, and to discuss a necessary and sufficient condition for well-behavedness elsewhere [26].

The organization of the paper reflects to a large extent the organization of the proof for the main theorem as stated above.

Roughly, this is indicated by the following diagram:



However, the relations between the various concepts involved in the theorem are in fact much more delicate. For example, localizability on one side, in connection with primality, is known to imply regularity [2], 2.11; but the proof of localizability on any side needs regularity on the other side (5.1), and the proof of regularity from homogeneity again involves partial knowledge on localizability (4.6, 5.10).

It is another main point of this paper, to study some of these relations in some detail and generality. Therefore, the proof of the main theorem will appear here eventually in §7 as a mere combination of general theorems established before, which may be also of independent interest. The reader just interested in applying the main theorem may easily extract a stream-lined, shorter, more direct proof from our exposition.

Let us mention here, for example the following localization-theorem (5.1):

THEOREM: *Let M be a r-restricted bimodule over a r-noetherian ring A with Small set S. Then S l-regular for M implies S r-orean for M.*

Note that the essential part of Small's theorem is a corollary (take $M = A$); in fact, we even obtain a slightly generalized version, assuming regularity on one side only. The formulation in terms of bimodules here has the advantage of simplifying proofs by induction, and to admit a single, unified proof (5.2–5.6) for several formerly separated results on Orean localization, including Small's theorem. But let us mention that, on the other hand, if one wants to *assume* both Small's theorem *and* regularity (3), then there is a much shorter proof for localizability, see 7.3.

Finally we note that the results of the present paper do not only hold for GK-dimension, but as well for GR-dimension (Krull-dimension in the sense of Gabriel Rentschler), provided that a certain symmetry-property is satisfied (§2). In fact, our exposition is based on an axiomatic notion of dimension, for which GK- and GR-dimension provide examples. Instead of our 'axiom of symmetry' (see 2.4), again another condition, related to Stafford's 'ideal-invariance' [21], could be employed to develop the whole theory on a different base. This would fit better for incorporating more of the results of [15], and for generalizing from two-sided to one-sided noetherian assumptions. Again, I prefer to discuss this alternative approach elsewhere [26].

Acknowledgements

A preliminary version of this work was presented at Bielefeld University on a conference organized by C.M. Ringel in June 1980, and it had benefit from the conversations stimulated by this conference. I would like to thank especially C.M. Ringel for his suggestions concerning axiomatization of dimension, L. Small for his communication of counter-examples to the additivity principle in general, and him and M. Lorenz for comments on group rings and GR-dimension. The final version of this work was reported at the Séminaire d'Algèbre at Paris in December 1980. Thanks are also due to G. Bergman, R. Rentschler, R.B. Warfield for useful hints, and to A. Joseph for showing me his unpublished notes on a course about dimension in non-commutative algebra held at Paris in 1979/1980. Finally, I am most grateful to the IHES at Bures-sur-Yvette for its kind hospitality while writing the paper.

§1. Dimension

1.1. *Conventions*

In this paper, a 'ring' is an associative but not necessarily commutative ring with unit element 1, and a '*subring*' has the same unit element as the big ring. We shall always denote by k some commutative field, and by A a ring, which will usually be assumed right noetherian (r-noetherian, for short), and sometimes also a k-algebra. All definitions and statements about A-modules M will be applied without further comments to the case where $M = B$ is an extension-ring of A, that is a ring containing A as a subring (resp. as a k-subalgebra). We shall indicate by the prefix l- resp. r-, whether the left or right A-module structure is referred to. The term 'ideal' will mean two-sided ideal. All modules are unitary.

1.2. *Axiomatic notion of dimension*

Suppose we have a function d(or d_A), attaching to each finitely generated left A-module M a real number or $\pm\infty$, such that the following three properties hold:

(d1) $d(M) \geq 0$ *for* $M \neq 0$, *and* $d(0) = -\infty$.

(d2) $d(M) \geq \sup(d(N), d(M/N))$ *for all submodules* $N \subset M$, *with equality for a direct summand* N.

(d3) $d(M/Ms) \leq d(M) - 1$ *for each monomorphism* $s : M \to M$.
Then we may extend this function uniquely to arbitrary left A-modules M, by asking that

(d4) $d(\bigcup_i M_i) = \sup_i d(M_i)$ *for each inductive system of submodules* $M_i \subset M$.

A function d with these properties will be called a *dimension* for left A-modules. We note that (d1), (d2) will extend to infinitely generated modules, but (d3) will generally not. In the sequel, *we shall always assume that a specific dimension for left A-modules has been chosen, and denote it by d.*

1.3. *Example GK-dimension*

Let A be a finitely generated k-algebra. For an A-module M, let $d(M)$ denote the Gelfand–Kirillov dimension with respect to k, or GK-dimension for short.

PROPOSITION: *Then d is a dimension in the sense of* 1.2.

We refer to Gelfand–Kirillov [8] for the first definition of this notion for algebras, to Bernstein [1] for its extension to modules, to Borho–Kraft [4] for a reformulation and a systematic study of GK-dimension for algebras, and to Joseph–Small [11] for an extension of this study to the case of modules. The proof of (d3) below is similar to [4], 3.4, cf. also [11], 2.5(i).

PROOF: Choose finite-dimensional generating k-subspaces V of A resp. E of M, as usual with $1 \in V$. Let us prove (d3). Let $\bar{E}$ denote the image of E in $\bar{M} = M/Ms$. For each $n \geq 0$, let D_n denote a k-subspace of V^nE complementary to $Ms \cap V^nE$. Then D_n maps isomorphically onto $V^n\bar{E}$. We claim that the sum of subspaces $D_n + D_ns + \cdots + D_ns^j$ is direct for all j. Proceeding by induction on j, let us assume $D_n + \cdots + D_ns^{j-1}$ direct. Then s maps this sum isomorphically onto $D_ns + \cdots + D_ns^j$, which is therefore also direct. Since $D_n \cap Ms = 0$, this gives that $D_n + D_ns + \cdots + D_ns^j$ is direct. Now the spaces V^nE

exhaust M, so $Es \subset V^m E$ for some m. By induction, $V^{mn}E \supset Es^n$ for all n. We conclude that $D_n + D_n s + \cdots + D_n s^n \subset V^n E + V^n Es + \cdots + V^n Es^n \subset V^n E + V^{n+m} E + \cdots + V^{n+mn} E = V^{(m+1)n} E$. This gives $\dim V^{(m+1)n} E \geq n \dim D_n = n \dim \; V^n \bar{E}$, and this inequality proves $d(M) \geq 1 + d(\bar{M})$. We leave the proof of (d2) to the reader (cf. also [11], 2.1(i)). Q.e.d.

1.4. *Example GR-dimension*

Next assume A l-noetherian, and denote by $d(M)$ the 'Krull-dimension in the sense of Gabriel–Rentschler' of a left A-module M, or the GR-dimension for short.

PROPOSITION: *Again, d is a dimension in the sense of* 1.2.

There exists an enormous amount of literature about this notion of dimension. Let us here just refer to the original definition by Gabriel–Rentschler [7], and to Gordon–Robson [10] for a systematic study.

PROOF: For property (d2), we refer to [10], lemma 1.1. For property (d3), observe that the monomorphism s of M induces isomorphisms $M/Ms \xrightarrow{\sim} Ms/Ms^2 \xrightarrow{\sim} Ms^2/Ms^3 \ldots$. Considering the chain of submodules $M \supset Ms \supset Ms^2 \supset \ldots$, (d3) becomes obvious from the definition of GR-dimension. (Cf. also [10], proof of proposition 6.1.) Q.e.d.

REMARK: Let us call a dimension d *partitive* (Joseph [23]), if in (d2) equality holds for all submodules N. The reference above gives even that GR-dimension is partitive. However, it is an open problem, whether GK-dimension is partitive.[1])

1.5. *Elementary properties*

From now on, d is an arbitrary dimension, unless otherwise stated. We list here a few trivial consequences of axioms (d1) to (d4), for matters of easy reference. Let M be a left A-module.

(1) $d(M) \geq d(N)$ for a submodule N of M.

(2) $d(M) \geq d(N)$ for a homomorphic image N of M.

(3) $d\left(\sum_{m \in F} Am\right) = \sup_{m \in F} d(Am)$ for a finite set $F \subset M$.

[1] Added in proof: Three months after this paper was distributed, G. Bergman [24] found that GK-dimension is definitely *not* partitive. He has examples such that $d(N) = d(M/N) = 1$, but $d(M)$ is arbitrarily large.

(4) $d(M) = \sup_{m \in G} d(Am)$ for a generating set G of M.

(5) $d(M) \leq d(A)$ for A considered as a left A-module.

Here (1) and (2) are just special cases of (d2), whereas $\sup_{m\in F} d(Am) \leq d(\Sigma_{m\in F} Am) \leq d(\oplus_{m\in F} Am) = \sup_{m\in F} d(Am)$ by (1), (2), (d2) gives (3). Property (4) is proved by considering an inductive system of finitely generated submodules, and applying first (d4), then (3). Noting that the cyclic modules Am occuring in (4) are all homomorphic images of the left module A, we finally derive (5) from (2) and (4). Similarly, we find even

(6) $d(M) \leq d(A/\operatorname{Ann} M)$, where $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$.

1.6. *Significance of axiom* (d3)

LEMMA: *Assume A l-noetherian. Let $P \subset Q$ be two ideals of A, with P prime. Then $d(A/Q) \leq d(A/P) - 1$.*

PROOF: As a prime noetherian ring, A/P contains a non-zero-divisor in each of its nonzero ideals (Goldie). So there exists $s \in Q$, such that right multiplication by s induces a monomorphism of the left A-module $A/P = M$. Hence $d(M/Ms) \leq d(M) - 1$ by (d3). Observing that $M/Ms \simeq A/(P + As)$ maps homomorphically onto A/Q, we obtain from 1.5(2) that

$$d(A/Q) \leq d(M/Ms) \leq d(M) - 1 = d(A/P) - 1. \qquad \text{Q.e.d.}$$

COMMENTS: This property is well-known for GR- and GK-dimension (see [10], Corollary 7.2 resp. [4], Korollar 3.5), and seems to be basic for any reasonable notion of dimension. As a consequence of this property, any chain of primes of A has length $\leq d(A)$.

1.7. *Irrelevancy of nilradicals*

Assuming A l- or r-noetherian, let $\sqrt{0}$ denote its nilradical. Assume that d is partitive (1.4, remark).

PROPOSITION: $d(A) = d(A/\sqrt{0}) = d(A/P)$ *for some prime P of A.*

PROOF: Since $\sqrt{0} \subset P$, the inequalities $d(A) \geq d(A/\sqrt{0}) \geq d(A/P)$ follow from 1.5(2), for any prime P. Since $\sqrt{0}$ is a finite intersection of minimal primes $P_1, \ldots, P_r$, so $A/\sqrt{0}$ embeds in $A/P_1 \times \cdots \times A/P_r$, the second equality $d(A/\sqrt{0}) = d(A/P)$ follows from (d2), for $P =$ some P_i. For the first equality, consider the chain of ideals

$$0 = \sqrt{0}^r \subset \sqrt{0}^{r-1} \subset \cdots \subset \sqrt{0}^2 \subset \sqrt{0} \subset A,$$

where r is the index of nilpotency of $\sqrt{0}$. Then each factor $F_i = \sqrt{0}^i/\sqrt{0}^{i+1}$ in the chain is annihilated by $\sqrt{0}$, so has dimension $d(F_i) \leq d(A/\sqrt{0})$ by 1.5(6). Now $d(A) = d(A/\sqrt{0})$ follows by iterated application of partitivity. Q.e.d.

REMARK: We shall avoid using 1.7 in the sequel, since it depends on partitivity (cf. 1.4).

§2. Symmetry

2.1. *Bimodules*

An A-bimodule M is a left A-module which is also endowed with a right A-module-structure such that $(am)b = a(mb)$ for all $m \in M$ and $a, b \in A$. Statements on bimodules depending on sides are formulated for one side, and tacitly used for the other side too. A bimodule M having property x as a left module is called $\ell - x$. A bimodule which is both $\ell - x$ and $r - x$ is called bi-x for short. Let us give an example for this convention on terminology. For a left A-module M, an element $s \in A$ will be called *regular*, if $am = 0$ implies $m = 0$ for all $m \in M$. In case of a bimodule M, this property will be called ℓ-regular. And s is called *bi-regular* for M, if it is both ℓ-regular and r-regular for M. A subset of A is called (ℓ-, r-, *bi*) regular for M, if all its elements are. For any subset $T \subset M$, the *ℓ-annihilator* $\{a \in A \mid aT = 0\}$ is denoted ℓ-Ann T or $\ell\text{-Ann}_A\, T$.

2.2. *Dimension-lemma*

LEMMA: *For an A-bimodule finitely generated on the right, we have* $d(A/\ell\text{-Ann}\, M) = d(M)$.

PROOF: Let $m_1, \ldots, m_r \in M$ be r-generators, that is $M = m_1A + \cdots + m_rA$. Then set $I = \ell\text{-Ann}\, M$, and observe that this is just the intersection of the $\ell\text{-Ann}\, m_i$. But this is also the kernel of the canonical homomorphism from A into $Am_1 \oplus \cdots \oplus Am_r$ (the 'diagonal map'). Hence this induces an embedding of ℓ-modules $A/I \hookrightarrow \bigoplus_{i=1}^r Am_i$ and we obtain

$$d(A/I) \leq d\left(\bigoplus_{1\leq i\leq r} Am_i\right) = \sup_{1\leq i\leq r} d(Am_i) \leq \sup_{m\in M} d(Am) = d(M),$$

using 1.5 (1), (3), (4). The converse inequality $d(M) \leq d(A/I)$, is trivial (and was noted already in 1.5(6)). Q.e.d.

COMMENTS: This equality is a remarkable property of bimodules, which does not hold for one-sided modules. For instance, the standard module of the Weyl algebra, which is faithful (ℓ-Ann $M = 0$), has only half the GK-dimension of the algebra. In the commutative case of course, all modules are bimodules, so that the lemma always applies.

2.3. *Symmetry-lemma*

In the study of extension-rings, or more generally of bimodules, it is important to consider the dimension on both sides. Let us adopt the convenient notation, introduced by Joseph–Small [11] in the case of GK-dimension, to denote dimension of left A-modules by d, and dimension of right A-modules by d'. It is obvious from the definition of GK-dimension that $d(A) = d'(A)$, but one can easily construct bimodules M such that $d(M) \neq d'(M)$. However, GK-dimension has the following remarkable property.

LEMMA: *Let A be a finitely generated k-algebra, and M a bimodule finitely generated as a r-module over A. Then we have*

$$d(M) \leq d'(M)$$

for the GK-dimensions on the left resp. right.

PROOF: Let A resp. M be generated by finite-dimensional k-vectorspaces $V \subset A$ resp. $E \subset M$. As usual choose a V containing 1. By our assumptions, the ascending chain of finite-dimensional vector spaces $EV^m (m = 1, 2, 3, \ldots)$ exhausts our module $M = EA$. We have to consider an arbitrary finite-dimensional k-subspace $F \subset M$, which we may (and do) assume to contain E. From what has just been said, there must exist some $m \in \mathbb{N}$ such that $VF \subset EV^m \subset FV^m$. By induction, this implies $V^nF \subset FV^{mn}$, for all n. By definition of GK-dimension, this gives $d(AF) \leq d'(FA)$. By passing to suprema, the lemma follows. Q.e.d.

2.4. *Axiom of symmetry*

In the sequel, we shall always assume that two dimensions d, d' are given, d for left and d' for right A-modules, both satisfying the axioms (d1)–(d4). Sometimes we shall have to assume that they are related in the following way:

(d5) $d(M) = d'(M)$ *for every bi-noetherian A-bimodule M.*

If this property (d5) is satisfied, we shall say that the notion of dimension under consideration is *symmetric* (for A), or also that A is *symmetric* (for d, d'). So symmetry is always a property of the triple (A, d, d'), and not of A or d alone.

EXAMPLE: GK-dimension is symmetric for any finitely generated k-algebra A. (Apply 2.3.)

PROBLEM: *Is GR-dimension symmetric for every bi-noetherian ring A?.*

The answer is known to be positive for many specific cases, but seems to be a difficult open problem in general, see [19].

2.5. *r-symmetry*

Since the larger part of this paper will deal with one-sided conditions only, let us also introduce a one-sided term generalizing symmetry: By definition, A is called *r-symmetric* (with respect to d, d'), if

$$d(M) \leq d'(M) \text{ for all } r\text{-noetherian bimodules } M,$$

and moreover,

$$d(A) = d'(A).$$

EXAMPLE: We have seen that every finitely generated k-algebra A is r-symmetric for GK-dimension (2.3.).

§3. Homogeneity

3.1. An A-module M is called *homogeneous*, if $d(N) = d(M)$ for all submodules $N \neq 0$ of M. For an A-bimodule M, one has to distinguish the three terms ℓ-homogeneity, r-homogeneity, and bi-homogeneity. Note however, that if we assume A symmetric (2.4), and M bi-noetherian, then these three terms will coincide.

COMMENTS: The following remarks will help to avoid confusion about vocabulary. The present notion was first introduced by R.

Gordon [25] for a very special class of rings in the case of GR-dimension, and was named 'K-homogeneous' or 'α-homogeneous' by Krause–Lenagan–Stafford [15] in their subsequent more general study of this case. In the case of GK-dimension, the notion has been introduced by Joseph–Small [11], using the word 'smooth' instead of homogeneous.

Moreover, there is an extra definition of 'bi-smooth', crucial for the discussion in [11]. Let us explain its relation to our terminology. Considering GK-dimension for a bi-noetherian finitely generated k-algebra A, Joseph–Small define A to be 'bi-smooth' if (i) it is 'left smooth' (ℓ-homogeneous), and (ii) $d(M) = d'(M)$ for all bi-modules M of the form $M = I/J$ with $J \supset I$ are ideals of A. Now observe that the assumptions on A imply A symmetric by our lemma 2.3, and that M is always bi-noetherian, so that (ii) is always satisfied. This proves that the notion 'bi-smooth', where ever it is actually applied in [11], ***agree with each*** of our terms ℓ-homogeneous, r-homogeneous, or bi-homogeneous.

3.2. *Restrictedness*

DEFINITION: An A-bimodule M is called *r-restricted,* if

AmA is r-noetherian for all $m \in M$.

This is the basic definition of the present paper. If M is r-restricted, then *every* finitely generated bi-submodule of M will be even finitely generated as a r-module. If A is r-noetherian, then this is in fact an equivalent way to define r-restrictedness. Trivial *examples* of r-restrictedness are: M itself r-noetherian, or also $M = B \supset A$ a commutative ring-extension. Non-trivial examples occur typically in enveloping algebras, see §8. We note that r-restrictedness passes to sub- and to quotient-bimodules. Since the r-restricted bimodules are inductive unions of r-noetherian bi-submodules, they will inherit some good behaviour. Let us give an example.

LEMMA: *Assume A r-symmetric, and M r-restricted. Then $d(M) \leq d'(M)$.*

The proof, using ($d4$), is straight-forward.

3.3. *Dimension of ideals*

LEMMA: *Assume A r-symmetric and $B \supset A$ an extension-ring r-restricted over A. Then for all $b \in B$*

$$d(BbB) \leq d'(bA).$$

PROOF: Since all monogeneous r-modules xbA with $x \in B$ are homomorphic images of bA, we first obtain

$$d'(bA) \geq d'(BbA)$$

from 1.5(2) and (4). Next r-restrictedness gives

$$d'(BbA) \geq d(BbA)$$

by 3.2. Finally, employ again 1.5(2) and (4) to find

$$d(BbA) = \sup_{a \in A} d(BbA) \geq \sup_{a \in A} \sup_{x \in B} d(Bbax) = d(BbB). \qquad \text{Q.e.d.}$$

3.4. *A homogeneity-theorem*

THEOREM: *Assume A r-symmetric, and $B \supset A$ a prime l- or r-noetherian extension-ring. If B, considered as an A-bimodule, is r-restricted, then it is r-homogeneous.*

PROOF: Let $0 \neq b \in B$. We have to show that $d'(bA) = d'(A)$. Since BbB is a nonzero two-sided ideal of a prime l- or r-noetherian ring B, we know (Goldie) that it contains a non-zero-divisor y. In particular, right multiplication by this element y provides a monomorphism $A \to Ay \subset BbB$ of left A-modules. We conclude that

$$d(A) = d(Ay) \leq d(BbB) \leq d'(bA) \leq d'(A),$$

the inequalities coming form 1.5(1) resp. 3.3 resp. 1.5(2). But $d(A) = d'(A)$ by definition of r-symmetry (2.5). Hence

$$d'(bA) = d'(A). \qquad \text{Q.e.d.}$$

REMARK: Assuming A symmetric and B bi-restricted, the results of this section remain valid without further change.

§4. Regularity

4.1. *The Small set*

For an ideal I of A, we denote by $C^r(I)$ resp. $C(I)$ the set of all elements of A r-regular resp. bi-regular for A/I. So $C(I) = C^r(I) \cap C^\ell(I)$. Now assume A r-noetherian, denote $\sqrt{0}$ its nilradical, and recall the well-known fact about semiprime noetherian rings telling that $C^r(\sqrt{0}) = C(\sqrt{0}) = C^\ell(\sqrt{0})$. By definition, we call this set $S = C(\sqrt{0})$ the *Small set* of A. By Small's theorem, this set S is bi-regular for A iff A is a r-order in a r-artinian ring; and if such is the case, that r-artinian ring is just AS^{-1}. See [17], theorems 2.10, 2.11, 2.12. We do not assume this theorem here, but eventually reprove it as a corollary of a more general theorem on localizability. See 5.1, 5.7, 5.8. Our present goal is to explain, how homogeneity may imply regularity of the Small set (4.5).

4.2. LEMMA: (A l-noetherian). *Let M be a ℓ-homogeneous A-bimodule of dimension $d(M) = d(A) < \infty$. Let $I \subset A$ be an ideal with $IM = 0$. Then $s \in A$ r-regular for A/I implies s ℓ-regular for M.*

PROOF: Let $m \in M$ such that $sm = 0$. Consider the left A-module-homomorphism $A \to Am$ provided by multiplication. Since its kernel contains As and I, we see that

$$d(Am) \leq d(A/(As + I)) < d(A/I) \leq d(A) = d(M),$$

where the proper inequality follows from (d3), using A ℓ-noetherian, $d(A) < \infty$ and s r-regular for A/I. Now ℓ-homogeneity implies $Am = 0$, hence $m = 0$. Q.e.d.

4.3. LEMMA: (cf. [11], 2.2): *Let M be a ℓ-homogeneous A-bimodule. Let $I \subset A$ be an ideal with $MI \neq 0$. Let $U = \{m \in M \mid mI = 0\}$. Then M/U is a ℓ-homogeneous A-bimodule of dimension $d(M/U) = d(M)$.*

PROOF: Clearly, U and M/U are bimodules. Let $L \subset M$ be any ℓ-submodule containing U properly. Then there exists $a \in I$ such that $La \neq 0$. Now right multiplication by a induces an epimorphism of left A-modules $L/U \to La$. Using ℓ-homogeneity of M, this gives $d(L/U) > d(La) = d(M)$. Hence the lemma. Q.e.d.

4.4. LEMMA: *Assume A ℓ-noetherian ℓ-symmetric and $d(A) < \infty$. Let M be a ℓ-noetherian r-homogeneous r-faithful A-bimodule. Let*

$J \subset A$ be a nilpotent ideal. Then

(a) *M is bi-homogeneous of dimension $d(M) = d(A)$.*

(b) *The set $C'(J)$ is ℓ-regular for M.*

PROOF: (a) The dimension-lemma 2.2 gives $d'(M) = d'(A)$ by r-faithfulness plus ℓ-noetherian. Now for $0 \neq m \in M$, we have $d(Am) = d(AmA) \geq d'(AmA) = d'(M) = d'(A) = d(A)$ by ℓ-symmetry plus r-homogeneity. This proves M ℓ-homogeneous. Note that the assumption 'r-faithful' may be replaced by $d'(M) = d'(A)$.

(b) Let $J^r = 0$ and set $M_i = \{m \in M \mid J^i m = 0\}$. This defines a chain of bimodules $M = M_r \supset M_{r-1} \supset \cdots \supset M_0 = 0$. All its factors $F_i = M_i/M_{i-1}$ satisfy $JF_i = 0$, and are r-homogeneous of dimension $d'(F_i) = d'(A)$ by 4.3. Since they are again ℓ-noetherian, they are even bi-homogeneous by (a). Now $C'(J)$ is ℓ-regular for all F_i by 4.2, and hence also for M. Q.e.d.

4.5. *A regularity-theorem*

THEOREM: *Assume A l-noetherian and ℓ-symmetric. Let M be a ℓ-restricted r-homogeneous A-bimodule of dimension $d'(M) = d'(A) < \infty$. Then the Small set of A is ℓ-regular for M.*

PROOF: Take $J = \sqrt{0}$ in lemma 4.4, and apply it to AmA for $m \in M$. Q.e.d.

COMMENT: It would be more natural, to assume here only ℓ-homogeneity instead of r-homogeneity plus l-symmetry. The rest of this section is dedicated to the proof of an alternative regularity-theorem of such kind (4.8).

4.6. *Torsion*

Let $S \subset A$ be a multiplicative subset. Write $T_S^{\ell}(M)$ for the set of left *S-torsion-elements* of M, i.e. of those $m \in M$ such that $sm = 0$ for some $s \in S$.

LEMMA: *Let S be ℓ-orean* (5.1) *for A. Then for any left A-module M, the left S-torsion-elements form a submodule.*

The proof is straight-forward, and similar to [16], proof of theorem 1.4.

4.7. LEMMA: (*A* ℓ-noetherian): *Let M be a ℓ-homogeneous A-*

bimodule of dimension $d(M) = d(A) < \infty$. Let $J \subset A$ be an ideal such that $J^r M = 0$ for some r. Let $S \subset A$ be a multiplicative set ℓ-orean for A and r-regular for A/J. Then S is ℓ-regular for M.

PROOF: Consider the left S-torsion submodule (4.6) $T := T^{\ell}_S(M)$, and assume $T \neq 0$. Since $J^r T = 0$, there is $0 \neq t \in T$ such that $Jt = 0$. Let $s \in S$ with $st = 0$. Now we conclude, as in the proof of 4.2, that

$$d(At) \leq d(A/(At + J)) < d(A/J) \leq d(A) = d(M),$$

and hence $At = 0$ by ℓ-homogeneity, contradicting the choice of t. Thus $T = 0$. Q.e.d.

4.8. *An alternative*

THEOREM: *Assume A ℓ-noetherian, and the Small-set S of A ℓ-orean for A. Let M be a ℓ-homogeneous A-bimodule of dimension $d(M) = d(A) < \infty$. Then S is ℓ-regular for M.*

PROOF: Take $J = \sqrt{0}$ in 4.7. Q.e.d.

§5. Localizability

5.1. *A localization-theorem*

Let $S \subset A$ be a multiplicative subset, and M an A-bimodule. We call S *r-orean* for M, if

$$mS \cap sM \neq \emptyset \text{ for all } m \in M,\ s \in S.$$

THEOREM: *Let M be a r-restricted bimodule over some r-noetherian ring A. If the Small set of A is ℓ-regular for M, then it is r-orean for M.*

The proof will proceed in a series of lemmas (5.2–5.6), where we keep the assumptions of the theorem, and denote by S the Small set of A. Replacing M by a r-noetherian bi-submodule containing the proposed element m, say by AmA, we shall always assume without loss of generality that M is even r-noetherian.

5.2. LEMMA: *$s \in A$ ℓ-regular for M implies sM r-essential for M.*

PROOF: Let $R \subset M$ be a nonzero r-submodule. We have to show $R \cap s\ M \neq 0$. Note first that $s^\nu R \neq 0$ for all $\nu = 0, 1, 2, \ldots$, since s is

ℓ-regular. Since M is assumed r-noetherian, the infinite sum $R + sR + s^2R + \cdots$ of r-submodules $\neq 0$ is not direct. So there exist $r_0, \ldots, r_n \in R$, not all zero, (for some n) such that $r_0 + sr_1 + s^2r_2 + \cdots + s^nr_n = 0$. Since cancellation by s on the left is possible (by ℓ-regularity), we may assume $r_0 \neq 0$. But then $r_0 \in R \cap sM$, on the other hand, implies $R \cap sM \neq 0$. Q.e.d.

5.3. LEMMA: *Theorem* 5.1. *holds if* $M\sqrt{0} = 0$.

PROOF: Let $s \in S$ and $m \in M$ be given. Let $^-$ denote the canonical homomorphism of A onto $A/\sqrt{0} = \bar{A}$. Consider the right ideal $\bar{R} := \{\bar{a} \mid a \in A,\ ma \in sM\}$ of $\bar{A}$. (What we have to prove is that $\bar{R}$ meets $\bar{S}$).

It will be sufficient to prove that $\bar{R}$ is r-essential for $\bar{A}$, for the following reason: Since A is semiprime r-noetherian, $\bar{R}$ would then (after Goldie, cf. [2], 2.7c)) have to contain a regular element. By definition of the Small set S, this would mean $\bar{S} \cap \bar{R} \neq \emptyset$, that is $mt \in sM$ for some $t \in S$ by definition of $\bar{R}$, hence $mS \cap sM \neq \emptyset$, as desired.

Now only r-essentiality of $\bar{R}$ is left to be proved. Let $c \in A$ be such that $\bar{c} \neq 0$; we have to show $\bar{R} \cap \bar{c}A \neq 0$. If $m\bar{c} = 0$, then $\bar{c} \in \bar{R}$, and we are done. So let us assume $m\bar{c} \neq 0$. By lemma 5.2, we conclude that $m\bar{c}A \cap sM \neq 0$, say $0 \neq m\bar{c}a \in sM$, where $a \in A$. By definition of $\bar{R}$, this means $0 \neq \bar{c}a \in R \cap \bar{c}A$, and thus finishes the proof. Q.e.d.

5.4. LEMMA: *Let* $T \subset A$ *be a multiplicative subset. Let* $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ *be a chain of bi-submodules. If* T *is r-orean for all factors* M_i/M_{i-1} *then it is r-orean for* M.

PROOF: By a straight-forward induction on r, which is left to the reader.

5.5. LEMMA: *Let* I *be a subset of* A *such that the* ℓ*-submodule* $U = \{m \in M \mid mI = 0\}$ *is* $\neq M$. *Then* $s \in A$ ℓ*-regular for* M *implies* s ℓ*-regular for* M/U.

PROOF: If $m \in M$ is such that $sm \in U$, them $smI = 0$. Hence ℓ-regularity of s implies $mI = 0$, which means just $m \in U$. Q.e.d.

5.6. PROOF OF THEOREM 5.1: Let r be the smallest positive integer such that $\sqrt{0}^r = 0$. Set $M_i := \{m \in M \mid m\sqrt{0}^i = 0\}$ for $i = 1, \ldots, r$. This gives a chain of bimodules

$$0 := M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_r = M$$

with factors $F_i := M_i/M_{i-1}$ which are all r-noetherian, $\neq 0$, and such that $F_i\sqrt{0} = 0 (1 \leq i \leq r)$. By the assumptions of 5.1, the Small set S is ℓ-regular for M. By lemma 5.5, it is also ℓ-regular for M/M_i, and hence for the factors $F_i (1 \leq i \leq r)$. By lemma 5.3, S is r-orean for all F_i. Finally, by lemma 5.4, this implies that S is also r-orean for M. Q.e.d.

5.7. *One-sided Small theorem*

COROLLARY: *Assume A r-noetherian, and its Small-set S ℓ-regular for A. Then* (a) *S is r-orean for A, and* (b) *AS^{-1} exists and is r-artinian.*

PROOF: (a) is the special case $M = A$ of theorem 5.1. (b) Using (a), the construction of the ring of quotients AS^{-1} is performed as usual. The proof that it is artinian is as in Small's paper [17], 2.11. Q.e.d.

COMMENT: The canonical map $A \to AS^{-1}$ will have as kernel exactly the right S-torsion $T_S^r(A) = T$. The construction of AS^{-1} can also be reduced to the bi-regular case by the following argument: By (a) and 4.6, T is an ideal; and S is bi-regular for A/T (use 5.5). Denote $\bar{S}$ the image of S in $\bar{A} = A/T$. Then AS^{-1} identifies with $\overline{AS}^{-1}$, and the formation of $\overline{AS}^{-1}$ is the case of [17]. Note that A *embeds* into AS^{-1} iff $T = 0$. This gives:

5.8. COROLLARY (Small [17]): *Assume A r-noetherian. If the Small-set of A is bi-regular, then A is a r-order in a r-artinian ring.*

5.9. COMMENTS

(a) Let us give an example of a ring A (first discussed by Small [24], see also [9], p. 239) not an order in an artinian ring, for which 5.7 still applies. Let $\mathsf{F}_p = \mathsf{Z}/p\mathsf{Z}$ denote a finite field with a prime number p of elements. Then the ring $A = \begin{bmatrix} \mathsf{Z} & \mathsf{F}_p \\ 0 & \mathsf{F}_p \end{bmatrix}$ is bi-noetherian, and has Small set $S = \begin{bmatrix} \mathsf{Z}^* & 0 \\ 0 & \mathsf{F}_p^* \end{bmatrix}$ with left S-torsion

$$T_S^\ell(A) = \begin{bmatrix} 0 & \mathsf{F}_p \\ 0 & 0 \end{bmatrix} = \sqrt{0}$$

and no right S-Torion. Then by 5.7, we obtain a (commutative) artinian ℓ-quotient ring $S^{-1}A \simeq \mathbb{Q} \times \mathbb{F}_p$, with kernel$(A \to S^{-1}A) = \sqrt{0}$.

(b) For the localization theorem (5.1.), we have assumed A and M both r-noetherian. However, much weaker finiteness assumptions on A and M are sufficient, as may be noted by inspection of the proofs: It suffices to know that mod. some nilpotent ideal J (5.5.), the ring A/J is semiprime r-Goldie (5.3), and that M has 'reduced finite rank' (5.2), meaning in the notation of 5.6, that all F_i have finite right Goldie rank. (Cf. [17], 2.14–2.27 for a detailed discussion of the non-noetherian case.)

5.10. COROLLARY: *Assume A ℓ-noetherian, and its Small set S r-regular for A. Let M be a ℓ-homogeneous A-bimodule of dimension $d(M) = d(A) < \infty$. Then S is ℓ-regular for M.*

PROOF: Since S is ℓ-orean for A by 5.7, this follows from 4.8. Q.e.d.

§6. Equidimensionality

6.1. *Uniprime modules*

An A-module $M \neq 0$ is called *uniprime*, if all the annihilators of non-zero submodules coincide, i.e. if $\mathrm{Ann}\, U = \mathrm{Ann}\, M$ for all submodules $0 \neq U \subset M$. As it is easily verified, the annihilator $\mathrm{Ann}\, M$ of a uniprime module M is necessarily a prime P of A. Let us say then that M is P-uniprime.

LEMMA: *Assume A r-noetherian. Let M be a r-noetherian left P-uniprime A-bimodule. Then M is ℓ-homogeneous of dimension $d(M) = d(A/P)$.*

PROOF: The bimodule UA generated by an arbitrary ℓ-submodule $U \neq 0$ is r-noetherian, and has ℓ-annihilator P, by uniprimality. By 1.5(2), (4) and the dimension-lemma (2.2), we conclude $d(U) \geq \sup_{a \in A} d(Ua) = d(UA) = d(A/P) = d(M)$. This gives the lemma.

Q.e.d.

6.2. *Associated primes*

A prime P of A is called *associated* to a module M, if M contains a P-uniprime submodule. The set of all associated primes of M is denoted by $\mathrm{Ass}\, M$. For instance, all maximal annihilator ideals are associated primes. Now assume A r-noetherian. Then every right A-module M will contain a uniprime submodule (consider a maximal annihilator ideal), or to put it in the style of [6]: $M \neq 0$ iff Ass $M \neq \emptyset$. A considerable part of the theory of associated primes in commutative algebra may be transferred to noncommutative algebra along the lines of Bourbaki [6], chap. 4. For some purposes however, it turns out to be necessary to consider the class of 'affiliated primes' defined below, in order to obtain a useful analogue to commutative theory. The former notion seems to be due to Gabriel [2], 1.1, the latter to Stafford [19], [20].

6.3. *Affiliated primes*

The set Aff M of primes affiliated to a noetherian r-module over a r-noetherian ring A is defined as follows. Pick some maximal associated prime $P_1 \in \text{Ass } M$, and set $M_1 = \text{Ann}_M P_1 = \{m \in M \mid mP_1 = 0\}$. Next pick a maximal prime $P_2 \in \text{Ass } M/M_1$ and put $M_2 = \{m \in M \mid mP_2 \subset M_1\}$, the preimage of $\text{Ann}_{M/M_1} P_2$. Continue this procedure to find recursively a chain of submodules $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_r = M$ (finite, since M is noetherian). By construction, each factor M_i/M_{i-1} is P_i-uniprime for some maximal associated prime P_i of M/M_{i-1}. Following Stafford [19], we call such a chain an *affiliated chain* of submodules of M, and $\{P_1, \ldots, P_r\}$ an *affiliated set* of primes for M. Each prime P of A occuring in some affiliated set for M is called *affiliated* to M. The set of all primes affiliated to M is denoted Aff M. Clearly, this contains all maximal associated primes.

6.4. *Minimal primes*

In commutative algebra, all minimal primes of A are associated to A. In non-commutative algebra, this is no longer true. However, the minimal primes are still r-affiliated to A by the following lemma. Denote by Spec A resp. Minspec A the set of all resp. of minimal primes of A.

LEMMA: *Assume A r-noetherian, and M a faithful noetherian r-module for A. Then each affiliated set of primes for M contains all minimal primes of A. In particular,* Minspec $A \subset$ Aff M.

PROOF: Using the notation of 6.3, set $I = P_1 \cap \cdots \cap P_r$. Then $MI^r = 0$ by construction, hence $I^r = 0$ by faithfulness. Thus I is contained in the nilradical, which is also the intersection of the finitely many minimal primes of A. This implies that each minimal prime must equal one of the P_i's. Q.e.d.

6.5. *An equidimensionality theorem*

THEOREM: *Assume A r-noetherian r-symmetric. Let a r-noetherian A-bimodule M be ℓ-homogeneous of dimension $d(M) = d(A)$. Then*

$$d'(A/P) = d'(A) \text{ for all } P \in r\text{-Aff } M.$$

PROOF: Considering M as a r-module, let us construct an affiliated chain of submodules as in 6.3. Observe that all submodules M_i and hence all factors $F_i = M_i/M_{i-1}$ of the chain are actually bimodules, by construction. By lemma 4.3, each F_i is again ℓ-homogeneous of dimension $d(F_i) = d(M) = d(A)$. On the other hand, F_i is right P_i-uniprime, so $d'(F_i) \le d'(A/P_i) \le d'(A)$. Finally, from $d(A) = d(F_i) \le d'(F_i) \le d'(A/P_i) \le d'(A) = d(A)$ by r-symmetry, we conclude that equality must hold everywhere. Q.e.d.

6.6. COROLLARY: *Assume in addition M r-faithful and $d(A) < \infty$. Then* Minspec $A = r$-Aff M.

PROOF: To prove '$\supset$', let P be any r-affiliated prime for M, and $Q \subset P$ a minimal prime of A. By the theorem, $d'(A) \ge d'(A/Q) \ge d'(A/P) = d'(A)$, which implies $P = Q$ by 1.6. and $d'(A) = d(A) < \infty$. Hence P is minimal. The converse inclusion was proved in 6.4. Q.e.d.

6.7. Applying 6.5. and 6.6. to the case $M = A$, we find that the minimal primes P of A are exactly the primes r-affiliated to A, and satisfy $d'(A/P) = d'(A)$ in that situation. In particular, we have proved the following:

COROLLARY: *If A is r-noetherian r-symmetric and ℓ-homogeneous of dimension $d(A) < \infty$, then all minimal primes of A have the same (r-)dimension.*

COMMENT: This generalizes [11], Proposition 2.6 (i), which was proved there by a different kind of argument (loc. cit., lemma 2.4).

§7. Additivity

7.1. *The artinian case*

Consider an extension-ring B of a (ℓ- or r-) noetherian ring A. For a prime P of B, we want to prove some special properties of the ideal $P \cap A$, which is in contrast to commutative algebra generally not prime. Let $P_1, \ldots, P_r$ denote the primes of A minimal over $P \cap A$. Thus $P_1 \cap \cdots \cap P_r = \sqrt{P \cap A}$. Let us call P *additive over* A, if there are positive integers $z_1, \ldots, z_r$ such that

$$\mathrm{rk}(B/P) = \sum_{i=1}^{r} z_i \,\mathrm{rk}(A/P_i)$$

holds for the Goldie-ranks. Similarly, call P *(ℓ-)equidimensional* over A, if $d(A/P_i) = d(A)$ for $i = 1, \ldots, r$.

LEMMA (Joseph–Small): *If A is artinian, then every prime of B is additive over A.*

For the proof, which is basically some manipulation with idempotents, we refer to [11], 3.8.

7.2. MAIN THEOREM: We are now ready to state and prove our main result on 'good behaviour' of primes in a restricted ring-extension. This extends the theory of Joseph–Small [11], in particular their 'additivity-principle', to a more general level.

THEOREM: *Assume A bi-noetherian bi-symmetric with $d(A) < \infty$. Let B be a r-noetherian ℓ-restricted extension-ring. Let P be any prime of B. Set $\bar{B} = B/P$, $\bar{A} = A/P \cap A$, and S = the Small-set of $\bar{A}$. Then:*

(1) *$\bar{A}$ and $\bar{B}$ are ℓ-homogeneous.*
(2) *S is bi-regular for $\bar{A}$ and $\bar{B}$.*
(2′) *S is bi-orean for $\bar{A}$, and $\bar{A}S^{-1}$ is artinian.*
(3) *S is ℓ-orean for $\bar{B}$.*
(4) *P is equidimensional over A.*
(5) *P is additive over A.*

REMARK: The given dimension for A-modules defines also a dimension of $\bar{A}$-modules, and (1), (4) refer to this dimension. We shall use the phrase 'P is well-behaved over A' to express that (1)–(5) hold.

PROOF: Since all assumptions pass to homomorphic images, we may reduce to the case $P = 0$ by replacing B by $\bar{B}$, and A by $\bar{A}$. Property (1): Our homogeneity theorem (3.4) implies B ℓ-homogeneous of dimension $d(B) = d(A)$ by ℓ-symmetry. In particular, A is ℓ-homogeneous. Using r-symmetry and 3.3, we see that A is even bi-homogeneous. Now our regularity-theorem (4.5) implies S bi-regular for A, and therefore (2′) follows by Small's theorem (5.7). Now (1) implies S ℓ-regular for B by 5.10, and in fact even bi-regularity, that is property (2), since B is prime r-noetherian [2], 2.7b). Property (4) follows from (1) by our equidimensionality-theorem (6.7). Finally, to derive additivity, we repeat the argument in [11], 3.9: Left localization by S of A and B is possible by (3), and injective by (2), and preserves Goldie ranks by [2], 2.10. Consequently, it suffices to prove 0 in $S^{-1}B$ to be additive over $S^{-1}A$. Since $S^{-1}A$ is artinian by (2′), this follows from 7.1. Q.e.d.

7.3. *An alternative proof for* (3)

Instead of refering to the localization theorem 5.1, we may establish localizability (3) in the proof of the main-theorem also by the following simpler argument. We show that (2), (2′) imply (3) by ℓ-restrictedness of B over A.

Let $b \in B$, $s \in S$ be given. First note that AbA is ℓ-noetherian over A, and therefore $N := S^{-1}AbA$ is ℓ-artinian over $S^{-1}A$, where we use ℓ-restrictedness and (2′), and also the concept of Ore localization of a module. Next consider the endomorphism φ of N as a left $S^{-1}A$-module, which extends right multiplication by s(in AbA), and use (2) to conclude that φ is injective. Since N is artinian, φ must also be surjective. Hence $b \in N = \varphi(N)$, say $b = \varphi(t^{-1}c)$ for some $t \in S$, $c \in AbA$. Now we conclude that $tb = t\varphi(t^{-1}c) = \varphi(c) = cs$. This proves the left Ore condition. Q.e.d.

§8. Applications

8.1. *The Joseph–Small case*

In this last section, we consider the case where A is a homomorphic image of an enveloping algebra $U(\mathfrak{a})$ of some finite-dimensional k-Lie algebra $\mathfrak{a}$, say $A = U(\mathfrak{a})/I$. Furthermore, we shall only *consider GK-dimension* now. So by 2.3, A is symmetric. Since A is also bi-noetherian, and $d(A) \leq \dim \mathfrak{a} \leq \infty$, A satisfies all assumptions of theorem 7.2. If we take an extension-ring $B \supset A$, which is ℓ-noetherian as an A-module, then B is also ℓ-noetherian as a ring, and trivially ℓ-restricted over A. So theorem 7.2. applies to prove:

COROLLARY: *Assume A of the form $U(\mathfrak{a})/I$. Let B be an extension-ring, ℓ-noetherian as A-module. Then all primes of B are well-behaved, in particular equidimensional and additive, over A.*

COMMENTS: To see that this covers the results of [11], observe that the additional assumptions ad $\mathfrak{a}$ locally finite on B, and B finitely generated as A-bimodule, imply B bi-noetherian as A-bimodule (see [11], 3.1). The Joseph–Small situation occurs in a natural way in the study of primitive ideals for the case $\mathfrak{a}$ semisimple, and has extremely interesting applications there. For these, we refer to subsequent work of Joseph [12], [13], [14].

8.2. *Restrictedness in enveloping algebras*

The next lemma indicates, why 'restrictedness' is a quite natural notion for the study of enveloping algebras. It provides also an example, where restrictedness is satisfied in a not completely trivial way.

LEMMA: *Let $U(\mathfrak{b})$ be the enveloping algebra of a finite dimensional k-Lie-algebra $\mathfrak{b}$ with subalgebra $\mathfrak{a}$. Then $U(\mathfrak{b})$ is bi-restricted over $U(\mathfrak{a})$.*

PROOF: The natural filtration of $U(\mathfrak{b})$ by finite-dimensional k-subspaces $U_n(\mathfrak{b})$ satisfies $\mathfrak{a}U_n(\mathfrak{b}) = U_n(\mathfrak{b})\mathfrak{a}$; hence $U(\mathfrak{a})U_n(\mathfrak{b}) = U_n(\mathfrak{b})U(\mathfrak{a})$ is a $U(\mathfrak{a})$-bimodule finitely generated on both sides. Any proposed $u \in U(\mathfrak{b})$ is in some $U_n(\mathfrak{b})$. Hence $M = U(\mathfrak{a})uU(\mathfrak{a})$ is contained in a bi-noetherian bi-module over the bi-noetherian ring $U(\mathfrak{a})$. This shows that M is bi-noetherian. Hence the lemma. Q.e.d.

REMARK: To elucidate the meaning of restrictedness, let us mention the following more general fact, which is easily proved in a similar way: For any k-subalgebra A of a finitely generated k-algebra B, the following are equivalent:

(i) B is ℓ-restricted over A.

(ii) For some finite-dimensional W generating B as a k-algebra, we have $WA \subset AW$.

8.3. (*Notation* 8.2.)

THEOREM: *All primes of $U(\mathfrak{b})$ are well-behaved, in particular equidimensional and additive, over $U(\mathfrak{a})$.*

This follows from 8.2 and 7.2.

COMMENTS: Let us mention some special cases.

(a) If $\mathfrak{a}$ is even an ideal of $\mathfrak{b}$, then it is well-known that $P \cap U(\mathfrak{a})$ is even a prime for every prime P of $U(\mathfrak{b})$. So the theorem is of no new interest in this case. However, if $\mathfrak{a}$ is not an ideal in $\mathfrak{b}$, then even the statement on GK-dimensions seems to be new.

(b) For the prime $P = 0$, all statements of the theorem are trivial, except for the Ore condition, i.e. 7.2(3), saying here that $S = U(\mathfrak{a})\backslash\{0\}$ is (bi-) orean in $U(\mathfrak{b})$. This is [5], Satz 3.3. Note that the arguments of the present paper, if directly applied to this very special case, provide a new, even simpler proof of this old result (use 7.3).

8.4. *On tensoring with finite-dimensional representations*

Let us assume the notations of [3], §2: For a semisimple complex Lie algebra $\mathfrak{g}$, and for each weight λ, the finite set $\underline{X}_\lambda$ of primitive ideals of $U(\mathfrak{g})$ corresponding to the central character of the Verma-module of $M(\lambda)$ is considered. For each pair λ, μ of weights with $\lambda - \mu$ integral, a 'translation operator' T_λ^μ is defined. If λ, μ are in the same Weyl chamber, T_λ^μ defines a bijection $\underline{X}_\lambda \to \underline{X}_\mu$. If only λ is in some chamber, and μ on one of its walls, then each $J \in \underline{X}_\lambda$ either 'translates' onto some $T_\lambda^\mu J \in \underline{X}_\mu$, or $T_\lambda^\mu J = U(\mathfrak{g})$ 'degenerates'. But if we start with λ on a wall, and "translate" to μ inside a chambre, then $T_\lambda^\mu J$ will usually decompose into *several* primitive ideals $J_1, \ldots, J_r \in \underline{X}_\mu$, i.e. $J_1 \cap \cdots \cap J_r = \sqrt{T_\lambda^\mu J}$, where the J_i are the primes minimal over $T_\lambda^\mu J$. This phenomenon played already some role in [3], §4, and was employed by D. Vogan [22] for the definition of his 'generalized τ-invariant'.

COROLLARY: *All J_i above have the same GK-dimension.*

PROOF: Apply theorem 8.3. with $\mathfrak{a} = \mathfrak{g}$ diagonally embedded into $\mathfrak{b} = \mathfrak{g} \times \mathfrak{g}$. Q.e.d.

8.5. Finally, let us only mention another type of application: Our general results on regularity and localizability for bimodules may be directly applied to certain Harish–Chandra modules occuring in Joseph's study of Goldie ranks in the enveloping algebra of a semisimple Lie algebra (see [14], e.g. Proposition 4.3).

8.6. *Outlook on further applications*

In the *enveloping algebra situation* 8.3, or – more generally – whenever A is finitely generated by a k-Lie algebra $\mathfrak{a}$ and B is ad $\mathfrak{a}$-finite, (char $k = 0$), a prime P of B satisfies not only conditions (1)–(5) of good behaviour, but in fact much more is true about the relation between any two of the minimal primes $P_1, \ldots, P_n$ of A over $P \cap A$, say P_i and P_j:

(1) Their *associated zero-varieties* in $\mathfrak{a}^*$ are equal: $V(\operatorname{gr} P_i) = V(\operatorname{gr} P_j)$; which is a more precise statement than just equality of their dimensions, i.e. equidimensionality (4) for GK-dimension.

(2) *If* ad $\mathfrak{a}$ *is locally nilpotent on B, then* $n = 1$. Hence in this case, we have a map Spec $B \to$ Spec A, $P \mapsto \sqrt{P \cap A}$. It follows that $P \cap Z(A)$ is a primary ideal without embedded components in the center $Z(A)$ of A; this gives an explanation for a result in [5], 2.4c).

(3) *If* ad $\mathfrak{a}$ *is locally trigonalizable on B, then P_i is conjugate to P_j* under some automorphism of $U(\mathfrak{a})$. In particular $A/P_i \cong A/P_j$. The isomorphism is given by tensoring with some 1-dimensional ad $\mathfrak{a}$-subquotient of B.

(3′) For example let $B = U(\mathfrak{b})$ with $\mathfrak{b}$ semi-simple and $\mathfrak{a}$ a split Cartan-subalgebra for $\mathfrak{b}$, and let P be the annihilator of a finite-dimensional B-module E. Then $A/P_i = k$, rk $B/P = \dim E$. The various P_i correspond to the weights of E, and the numbers z_i (notat. 7.1) are the corresponding multiplicities. Thus the additivity-formula rk $B/P = \Sigma_i z_i$ rk A/P_i for this very special case is nothing else but *Weyl's character formula.*

(4) If $\mathfrak{a}$ is *semi-simple*, then P_i and P_j are *related by tensoring with some finite-dimensional representation.* More precisely, P_i occurs as annihilator of a subquotient of $E_{ij} \otimes A/P_j$ for some simple ad $\mathfrak{a}$-subquotient of B. In particular, their central characters differ by an integral translation. To make this more precise, let φ denote the map from $\mathfrak{h}^*$ (dual of a Cartan subalgebra for $\mathfrak{a}$) onto the set of maximal ideals of the center $Z(\mathfrak{a})$ of $U(\mathfrak{a})$, mapping λ onto the central character of a module with highest weight λ. Then there are irreducible closed subsets Λ_i of $\mathfrak{h}^*$ such that $\varphi(\Lambda_i) = V(P_i \cap Z(\mathfrak{a}))$ and $\Lambda_i = \Lambda_j + \mu_{ij}$ for some integral weight μ_{ij}. Furthermore, certain (non-trivial) quotient-categories of the categories of left A/P_i- resp. A/P_j-modules are Morita equivalent.

(5) Let $B = U(\mathfrak{b})$ and $A = U(\mathfrak{a})$ where $\mathfrak{b} \supset \mathfrak{a}$ are both semi-simple. Let $\sim$ denote the relation by tensoring with finite-dimensional representations as defined above. Then $P \mapsto P \cap A$ induces a map on the sets of classes of related primes:

Spec $B/\sim \to$ Spec $A/\sim$; (which is some substitute for the "morphisms" in commutative algebra).

(6) All of the above results result from the following fact: There exists a bi-subquotient U_{ij} of the A-bisubmodule B which is P_i-uniprime on the left and P_j-uniprime on the right.

Details, proofs, and further applications shall be given in a second, complenentary part to this paper.

REFERENCES

[1] I.N. BERNSTEIN: Modules over a ring of differential operators. Study of the fundamental solutions of equations with constant coefficients. *Funct. Anal. Appl. 5* (1971) 89–101.

[2] W. BORHO, P. GABRIEL, and R. RENTSCHLER: Primideale in Einhüllenden auflösbarer Lie-Algebren. *Springer LN 357* (1973).

[3] W. BORHO and J.C. JANTZEN: Über primitive Ideale in der Einhüllenden einer halbeinfachen Lie-Algebra. *Invent. Math. 39* (1977) 1–53.

[4] W. BORHO and H. KRAFT: Über die Gelfand–Kirillov-Dimension. *Math. Ann. 220* (1976) 1–24.

[5] W. BORHO and R. RENTSCHLER: Oresche Teilmengen in Einhüllenden Algebren. *Math. Ann. 217* (1975) 201–210.

[6] N. BOURBAKI: *Algèbre commutative*, Chap. 3 et 4. Paris, Hermann 1967.

[7] P. GABRIEL and R. RENTSCHLER: Sur la dimension des anneaux et ensembles ordonnés. *C.R. Acad. Sci. Paris 265* (1967) 712–715.

[8] I.M. GELFAND and A.A. KIRILLOV: Sur les corps liés aux algèbres enveloppantes des algèbres de Lie. *Publ. IHES 31* (1966) 5–19.

[9] A.W. GOLDIE: The structure of Noetherian Rings, *Springer LN 245* (1972) 213–321.

[10] R. GORDON and J.C. ROBSON: Krull Dimension. *Mem. AMS 133* (1973).

[11] A. JOSEPH and L.W. SMALL: An additivity principle for Goldie rank. *Israel J. Math. 31* (1978) 105–114.

[12] A. JOSEPH: Towards the Jantzen conjecture I, II. *Compos. Math. 40* (1980) 35–67.

[13] A. JOSEPH: Kostant's problem, Goldie rank and the Gelfand–Kirillov conjecture. *Invent. Math. 56* (1980) 191–213.

[14] A. JOSEPH: Goldie rank in the enveloping algebra of a semisimple Lie algebra, I, II. *J. Algebra 66* (1980) 269–306.

[15] G. KRAUSE, T.H. LENAGAN and J.T. STAFFORD: Ideal invariance and Artinian quotient rings. *J. Algebra 55* (1978) 145–154.

[16] L. LEVY: Torsion-free and divisible modules over non-integral domains. *Canad. J. Math. 15* (1963) 132–151.

[17] L.W. SMALL: Orders in Artinian rings. *J. Algebra 4* (1966) 13–41.

[18] P.F. SMITH: On two-sided Artinian quotient rings. *Glasgow Math. J. 13* (1972) 159–163.

[19] J.T. STAFFORD: *Regular elements of Noetherian rings.* Proc. 1978 Antwerp Conference, Marcel Dekker Co., New York.

[20] J.T. STAFFORD: *Noetherian full quotient rings.* preprint, Cambridge–La Jolla 1980.

[21] J.T. STAFFORD: Stable structure of non-commutative Noetherian rings. *J. Algebra 47* (1977) 244–267.

[22] D. VOGAN: A generalized τ-invariant for the primitive spectrum of a semisimple Lie-algebra. *Math. Ann. 242* (1979) 209–224.

[23] A. JOSEPH: *Dimension en algèbre non-commutative,* Cours de troisième cycle, Paris 1980 (unpublished notes).

[24] G. BERGMAN: Gel'fand–Kirillov-dimension can go up in extension modules. *Communications in Algebra* (to appear), preprint, Berkeley 1981.

[25] R. GORDON: Artinian quotient rings of FBN-rings. *J. Algebra 35* (1975) 304–307.

[26] W. BORHO: Invariant dimension and restricted extension of Noetherian rings; preprint, Wuppertal, June 1981 (to appear in Séminaire d'Algèbre (Dubreil–Malliavin), Springer Lecture Notes in Mathematics).

(Oblatum I–VI–1981)

FB Mathematik
Universitat-GHS Wuppertal
Gaußstr. 20
56 Wuppertal 1
BRD