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TAMAGAWA NUMBER OF REDUCTIVE ALGEBRAIC GROUPS

K.F. Lai

0. Introduction

The purpose of this paper is to give a formula for the Tamagawa number of a reductive quasi-split algebraic group G defined over an algebraic number field in terms of the Tamagawa number of a maximal torus of G (cf. Theorem 7.1).

The Tamagawa numbers of classical groups were determined by Weil [23]. In [15] Langlands determined the Tamagawa number of all split semisimple groups. We extend the result of Langlands to quasi-split groups.

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NOTATIONS:

F = number field
F_v = completion of F at the place v
$\bar{F}$ = algebraic closure of F
$v \mid \infty$ = v is an infinite place
$v < \infty$ = v is a finite place
$0_v = 0_{F_v}$ = ring of integers of F_v ($v < \infty$)
q = order of residue field of F_v
$\tilde{\omega}_v$ = uniformizing element of 0_v ($v < \infty$)
$\mathbb{A}$ = adeles of F, $\mathbb{A}_{\mathcal{S}}$ = adeles trivial outside $\mathcal{S}$
$|\,|_v$ = normalised absolute value at v ($v < \infty$): $|\tilde{\omega}_v|_v = q^{-1}$
$|\,|$ = adelic absolute value.

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For an algebraic group H defined over F, we write

$$H_v = H(F_v)$$
$$H_f = \{(h_v) \in H(\mathbb{A}) \mid h_v = 1 \text{ if } v \mid \infty\}$$
$$H_\infty = \prod_{v \mid \infty} H_v$$
$$H_{\mathcal{S}} = \{(h_v) \in H(\mathbb{A}) \mid h_v = 1 \text{ if } v \notin \mathcal{S}\}$$
$$H^{\mathcal{S}} = \{(h_v) \in H(\mathbb{A}) \mid h_v \in H(0_v) \text{ if } v \notin \mathcal{S}\}.$$

For a complex valued function $f(x)$, write $\bar{f}(x)$ for the complex conjugate of $f(x)$.

1. Quasi-split algebraic groups

1.1. Let G be a connected reductive algebraic group defined over F. We say that G is *quasi-split* if one of the following equivalent conditions is satisfied

(I) G has a Borel subgroup B defined over F,

(II) the centralizer in G of a maximal F-split torus is a maximal torus of G,

(III) G has no anisotropic roots.

In the following G denotes a connected reductive quasi-split group.

1.2. Let A be a maximal torus of G lying in B and defined over F, L the group of characters of A, $\hat{L} = \mathrm{Hom}(L, \mathbb{Z})$, $\Sigma(\hat{\Sigma})$ the set of roots (coroots) of G with respect to A, Δ basis of Σ with respect to B and $\hat{\Delta}$ the elements of $\hat{\Sigma}$ corresponding to Δ. There is a bijection between $\bar{F}$-isomorphism classes of triple (G, B, A) and isomorphism classes of based root system $\psi_0(G) = (L, \Delta, \hat{L}, \hat{\Delta})$. This bijection yields a connected reductive $\mathbb{C}$-group $\hat{G}^0$ with based root system $\psi_0(\hat{G}^0) = (\hat{L}, \hat{\Delta}, L, \Delta)$. Let $\hat{A}^0$ (resp. $\hat{B}^0$) be the maximal torus (resp. Borel subgroup) defined by $\psi_0(\hat{G}^0)$.

Let E be a Galois extension of F such that G splits over E. If $\sigma \in \mathrm{Gal}(E/F)$, $\lambda \in L$, we denote the action of σ on λ by $\sigma\lambda$ where $\sigma\lambda(a) = \sigma(\lambda(\sigma^{-1}a))$ for $a \in A$. As G is quasi-split, $\sigma\Delta = \Delta$. We can define a homomorphism $\mu: \mathrm{Gal}(E/F) \to \mathrm{Aut}\, \psi_0(G)$. Since we have canonical $\mathrm{Aut}\, \psi_0(G) = \mathrm{Aut}\, \psi_0(\hat{G}^0)$, we may view μ as a homomorphism of $\mathrm{Gal}(E/F)$ into $\mathrm{Aut}\, \psi_0(\hat{G}^0)$. Moreover there is a split exact sequence

$$(1) \qquad (1) \to \mathrm{Int}\,\hat{G}^0 \to \mathrm{Aut}\,\hat{G}^0 \to \mathrm{Aut}\,\psi_0(\hat{G}^0) \to (1)$$

and a splitting yields a monomorphism

$$\mathrm{Aut}\,\psi_0(\hat{G}^0) \to \mathrm{Aut}(\hat{G}^0, \hat{B}^0, \hat{A}^0).$$

Together with the μ above we get a homomorphism

$$\mu' : \mathrm{Gal}(E/F) \to \mathrm{Aut}(\hat{G}^0, \hat{B}^0, \hat{A}^0)$$

The associated group to, or L-group of, G is then by definition the semidirect product

$$\hat{G} = \hat{G}^0 \rtimes \mathrm{Gal}(E/F).$$

(See Borel [3]).

1.3. Let Z be the identity component of the centre of G and G' be the derived group of G. Then $G = ZG'$ and $A = ZA'$ where $A' = A \cap G'$. Let ${}^0L^+$ be the group of rational characters of Z and ${}^0L^-$ be the elements of ${}^0L^+$ which are 1 on $Z \cap A'$. Let ${}^1L^-$ be the lattice of roots of A'. (Note that there is a bijection between the roots of (G, A) and (G', A') and the corresponding Weyl groups can be identified. We shall not use a separate notation.) We denote the Weyl group of the root system by W. There exists a non-degenerate W-invariant bilinear form $(.,.)$ on ${}^1L^- \otimes_{\mathbf{Z}} \mathbf{C}$ such that its restriction to ${}^1L^- \otimes_{\mathbf{Z}} \mathbf{R}$ is positive definite. Let ${}^1L$ be the lattice of rational characters of A' and

$${}^1L^+ = \left\{ \lambda \in {}^1L^- \underset{\mathbf{Z}}{\otimes} \mathbf{C} \;\middle|\; \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbf{Z} \text{ for all roots } \alpha \right\}.$$

Set $L^- = {}^0L^- \oplus {}^1L^-$ and $L^+ = {}^0L^+ \oplus {}^1L^+$. We define dual lattices by

$$\hat{L}^+ = \mathrm{Hom}(L^-, \mathbf{Z}) = \mathrm{Hom}({}^0L^-, \mathbf{Z}) \oplus \mathrm{Hom}({}^1L^-, \mathbf{Z}) = {}^0\hat{L}^+ \oplus {}^1\hat{L}^+$$
$$\hat{L} = \mathrm{Hom}(L, \mathbf{Z})$$
$$\hat{L}^- = \mathrm{Hom}(L^+, \mathbf{Z}) = \mathrm{Hom}({}^0L^+, \mathbf{Z}) \oplus \mathrm{Hom}({}^1L^+, \mathbf{Z}) = {}^0\hat{L}^- \oplus {}^1\hat{L}^-.$$

We then have $L^- \subset L \subset L^+ \subset L \otimes_{\mathbf{Z}} \mathbf{C}$ and $\hat{L}^- \subset \hat{L} \subset \hat{L}^+ \subset \hat{L} \otimes_{\mathbf{Z}} \mathbf{C}$.

For the pairing $L \times \hat{L} \to \mathbf{C}$, we use the notation $\langle \lambda, \hat{\lambda} \rangle = \hat{\lambda}(\lambda)$ where $\lambda \in L$, $\hat{\lambda} \in \hat{L}$ and we extend it meaningfully to the other lattices. The form on ${}^1\hat{L}^+ \otimes \mathbf{C}$ adjoint to the one given above on ${}^1L^- \otimes \mathbf{C}$ will also be denoted by $(.,.)$, i.e. if $\mu, \nu \in {}^1L^- \otimes \mathbf{C}$, and if the elements $\hat{\mu}, \hat{\nu}$ of ${}^1\hat{L}^+ \otimes \mathbf{C}$ satisfy the equations

$$\langle \lambda, \hat{\mu} \rangle = (\lambda, \mu) \quad \text{and} \quad \langle \lambda, \hat{\nu} \rangle = (\lambda, \nu)$$

for all $\lambda \in {}^1L^- \otimes \mathbf{C}$, then $(\mu, \nu) = (\hat{\mu}, \hat{\nu})$.

Suppose v is a finite place of F. We define a map $\nu : A(F_v) \to \hat{L} \otimes \mathbf{Q}$ by the condition

$$|\lambda(a)|_v = |\tilde{\omega}_v|_v^{\langle \lambda, \nu(a) \rangle} \tag{2}$$

for all $\lambda \in L$ and $a \in A(F_v)$, where $\tilde{\omega}_v$ is the uniformizing element of F_v and $|\cdot|_v$ is the normalized valuation of F_v. For $\mu \in L \otimes \mathbf{C}$, define $\hat{t}_\mu \in \hat{A}^0 = \mathrm{Hom}(\hat{L}, \mathbf{C}^*)$ by

$$\hat{t}_\mu(\hat{\lambda}) = |\tilde{\omega}_v|_v^{\langle \mu, \hat{\lambda} \rangle} \tag{3}$$

for all $\hat{\lambda} \in \hat{L}$. We sometimes write $\hat{t}$ for $\hat{t}_\mu$.

We write L_F for the lattice of F-rational characters of A. Similar notation will be used for the lattices ${}^0L^+$ etc.

1.4. Next we write down explicitly the Galois action on the derived group $\hat{G}'$ of $\hat{G}^0$. Put $\hat{A}' = \hat{A}^0 \cap \hat{G}'$. Let $\hat{\mathfrak{a}}$ be the Lie algebra of $\hat{A}'$. Choose $H_1, \ldots, H_r \in \hat{\mathfrak{a}}$ so that

$$\lambda(H_i) = \langle \alpha_i, \lambda \rangle$$

where $\lambda \in {}^1\hat{L}^+$ and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ are the simple roots. Choose vectors $X_{\pm\hat{\alpha}_i}$ belong to the $\pm\hat{\alpha}_i$ respectively such that

$$[X_{\hat{\alpha}_i}, X_{-\hat{\alpha}_i}] = H_i.$$

For $\sigma \in \mathrm{Gal}(E/F)$, $\widehat{\sigma\alpha} = \sigma\alpha$ for $\alpha \in \Delta$. If we put $\sigma(\hat{\alpha}_i) = \hat{\alpha}_{\sigma(i)}$, then the Galois action on the Lie algebra $\hat{\mathfrak{g}}'$ of $\hat{G}'$ is the unique isomorphism satisfying

$$\sigma(H_i) = H_{\sigma(i)}, \quad \sigma X_{\pm\hat{\alpha}_i} = X_{\pm\hat{\alpha}_{\sigma(i)}}$$

(see Jacobson [9] Chap. VII).

1.5. Let Σ_F denote the set of F-roots of G with respect to A_d, the maximal F-split torus in A. As G is quasi-split, each element of Σ has a nontrivial restriction to A_d, and Σ_F is equal to the set of restriction to A_d of elements of Σ. In fact, if G splits over a Galois extension E of F, the Galois group Gal(E/F) acts on Σ and each orbit restricts to an element of Σ_F. In each orbit choose a representative α and denote the corresponding orbit by $\mathcal{O}_\alpha$ and the element in Σ_F to which the elements in $\mathcal{O}_\alpha$ restrict, is denoted by α_F, i.e. $\alpha_F = \alpha \mid A_d$.

The Weyl group W of Σ is given by $N(A)/Z(A)$ while the rational Weyl group W_F of Σ_F is $N(A_d)/Z(A_d)$. We can identify W_F as a subgroup of W.

Let ${}_0\Sigma_F$ be the reduced F-root system consisting of the indivisible F-roots of Σ_F, i.e. ${}_0\Sigma_F = \{\alpha_F \in \Sigma_F \mid \frac{1}{2}\alpha_F \notin \Sigma_F\}$. ${}_0\Sigma_F^+ = {}_0\Sigma_F \cap \Sigma_F^+$.

Next we define the elementary subgroup G_{α_F} of G for $\alpha_F \in {}_0\Sigma_F^+$. Let $A_{\alpha_F} = (\ker \alpha_F)^0$. Then $G_{\alpha_F} = Z_G A_{\alpha_F}$, i.e. we take the centralizer in G of A_{α_F}.

It can be easily proved that G_{α_F} is connected reductive quasi-split group of semi simple F-rank 1.

1.6. There is a non-empty finite set $\mathcal{S}$ of places of F, containing all the infinite places such that the F-group G can be regarded as defined above Spec($0_{\mathcal{S}}$), where $0_{\mathcal{S}}$ is the ring of the elements of F which are integral outside $\mathcal{S}$. Thus $G(0_v)$ is defined for those v not in $\mathcal{S}$.

For $v \mid \infty$, let K_v be a maximal compact subgroup of G_v such that $G_v = B_v \cdot K_v$ is an Iwasawa decomposition. For $v < \infty$, let K_v be a special open maximal compact subgroup of G_v, in the sense of Bruhat–Tits [4]. In particular, for almost all v, K_v can be taken to be $G(0_v)$. Similar considerations can be given to G_{α_F}. Therefore, when we consider the finite set $\{G, G_{\alpha_F}\}_{\alpha_F \in {}_0\Sigma_F}$ of groups taken together, except for a finite number of places, we have simultaneously

$$G_v = B_v G(0_v)$$
$$G_{\alpha_F}(F_v) = B_{\alpha_F}(F_v) G_{\alpha_F}(0_v) \tag{4}$$

where $\alpha_F \in {}_0\Sigma_F$.

Let us now fix $K_f = \Pi_{v<\infty} K_v$, $K_\infty = \Pi_{v\mid\infty} K_v$, $K = K_\infty K_f$. Then $G(\mathbb{A}) = B(\mathbb{A}) \cdot K$.

1.7. Let $X(G)$ be the lattice of rational characters on G. Let $L(s, G)$ be the Artin L-function corresponding to the Gal(E/F)-module $X(G) \otimes \mathbb{Q}$ and let $L_v(s, G)$ be its v-component.

Let χ be a nontrivial character on $\mathbb{A}$ trivial on F. χ defines a nontrivial character χ_v of F_v at each place v of F. Let dx_v be the additive Haar measure on F_v self-dual with respect to χ_v and let $dx = \Pi_v\, dx_v$. For v finite, the Haar measure on $F_v^\times$ is chosen so that the measure of $0_v^\times$ is one.

Let ω be an F-rational left-invariant nowhere vanishing exterior form of highest degree on G. For each v, ω and dx_v defines a measure $|\omega|_v$ on G_v (cf. [23]). We put $dg_v = L_v(1, G)|\omega|_v$, for finite v, and $dg_v = |\omega|_v$ for infinite v. Then the *Tamagawa measure* dg on $G(\mathbb{A})$ is the Haar measure on $G(\mathbb{A})$ defined by

$$dg = \lim_{s\to 1} \frac{1}{(s-1)^r L(s, G)} \prod_v dg_v \tag{5}$$

where r the rank of the lattice of F-rational characters $X(G)_F$ of G (cf. [17]). This measure is independent of choice of χ and ω.

Let $\chi_1, \ldots, \chi_r$ a basis of $X(G)_F$. Then the map $g \to (|\chi_1(g)|, \ldots, |\chi_r(g)|)$ defines a homomorphism $G(\mathbb{A}) \to (\mathbf{R}_+^\times)^r$. Let $G^1(\mathbb{A})$ be the kernel of this homomorphism. Also, the restriction of $\chi_1, \ldots, \chi_r$ to the split component Z_d of the radical of G defines an F-homomorphism δ from Z_d to $\mathrm{GL}(1)^r$. This defines a homomorphism δ_∞ from the identity component of $Z_{d\infty}$ to $\mathrm{GL}(1)_\infty^r$. For each $t \in \mathbf{R}_+^\times$, call $\xi(t)$ the idele $(\xi(t)_v)$ such that $\xi(t)_v = 1$ for every finite place and $\xi(t)_v = t$ for every infinite place. Then $t \to \xi(t)$ is an isomorphism of $\mathbf{R}_+^\times$ onto a subgroup $\mathrm{GL}^+(1)_\infty$ of $\mathrm{GL}(1)_\infty$. Let Z_∞^+ be the identity component of inverse image of $\mathrm{GL}^+(1)_\infty^r$ under δ_∞. Then Z_∞^+ is isomorphic to $(\mathbf{R}_+^\times)^r$ and $G(\mathbb{A}) = G(\mathbb{A})^1 \times Z_\infty^+$. If we put the measure $dt = \wedge_{i=1}^r (dt_i/t_i)$ on $\mathbf{R}_+^\times$, then

$$dg = dg^1 \times dt \tag{6}$$

defines a Haar measure on $G^1(\mathbb{A})$. This measure is independent of choice of $\chi_1, \ldots, \chi_r$. The *Tamagawa number* $\tau(G)$ is the finite number defined by

$$\tau(G) = \int_{G(F)\backslash G^1(\mathbb{A})} dg^1 = \int_{G(F)Z_\infty^+\backslash G(\mathbb{A})} dg. \tag{7}$$

1.8. Let N be the unipotent radical of B. Then we can define Tamagawa measures da (resp. dn) on $A(\mathbb{A})$ (resp. $N(\mathbb{A})$) as in the case of G. We normalize the measure on K_v by the condition

$$\int_{K_v} dk_v = 1.$$

Then we have $\mathrm{d}k = \Pi_v \, \mathrm{d}k_v$ and

$$\int_K \mathrm{d}k = 1.$$

Let ρ be the half sum of the positive roots of G with respect to A. To simplify notation we write ρ for the quasi-character on $A(F)\backslash A(\mathbb{A})$ determined by ρ. Since $G(\mathbb{A}) = B(\mathbb{A}) \cdot K = N(\mathbb{A})A(\mathbb{A})K$, there exists a positive constant κ such that for any $f \in C_c(G(\mathbb{A}))$,

$$\int_{G(\mathbb{A})} f(g)\,\mathrm{d}g = \kappa \int_{N(\mathbb{A})A(\mathbb{A})K} f(nak)\rho^{-2}(a)\,\mathrm{d}n\,\mathrm{d}a\,\mathrm{d}k. \tag{8}$$

According to the Bruhat decomposition of G we have

$$G_v = \bigcup_{w \in W_{F_v}} N_v A_v w N_v. \tag{9}$$

But except for the Weyl group element w_0 that sends all the positive roots to negative roots, the cosets $NAwN$ has lower dimension than that of G, and so $NAwN$ has measure zero. Thus if we write $g_v = n_v a_v w_0 n'_v$, we have

$$\mathrm{d}g_v = \rho^{-2}(a)\,\mathrm{d}n_v\,\overline{\mathrm{d}a_v}\,\mathrm{d}n'_v \tag{10}$$

where $\overline{\mathrm{d}a_v}$ is the local measure on A_v induced by $|\omega|_v$.

2. Eisenstein series and $M(w, \lambda)$

2.1. For our purposes it is sufficient to consider the contribution to the spectral decomposition of $\mathscr{L}^2(Z^+_\infty G(F)\backslash G(\mathbb{A})/K)$ from the Borel subgroup B. We can define the adelic analogue of the function spaces $\mathscr{E}(V, W)$, $\mathscr{D}(V, W)$ and $\mathscr{H}(\mathscr{D}(V, W))$ of §2 and 3 of [13] with respect to the Borel subgroup B, the trivial representation of K and a character λ of $Z^+_\infty A(F)\backslash A(\mathbb{A})$ which is trivial on the image of $B(\mathbb{A}) \cap K$ in $N(\mathbb{A})\backslash B(\mathbb{A})$.

2.2. Define A^+_∞ (resp. $A(\mathbb{A})^1$) in the same way as Z^+_∞ (resp. $G(\mathbb{A})^1$). Let $(Z^+_\infty A(F)\backslash A(\mathbb{A}))^*$ be the set of characters of $Z^+_\infty A(F)\backslash A(\mathbb{A})$. Fix a basis $\{\chi_i\}$ of L_F. Each element $\lambda = \Sigma\, s_i\chi_i$ of $L_F \otimes \mathbf{C}$ can be considered as a character of $Z^+_\infty A(F)\backslash A(\mathbb{A})$ via the formula

$$\lambda(a)=\prod_i |\chi_i(a)|_{\mathbb{A}}^{s_i}.$$

In this way $L_F\otimes\mathbf{C}$ is identified with a subset of $(Z_\infty^+A(F)\backslash A(\mathbb{A}))^*$. From now on we shall consider only those λ in $L_F\otimes\mathbf{C}$.

Let $\mathscr{E}(\lambda)$ be the space of continuous functions on $N(\mathbb{A})B(F)\backslash G(\mathbb{A})/K$ satisfying the condition

$$\Phi(ag)=\lambda(a)\rho(a)\Phi(g) \tag{1}$$

for $a\in A(\mathbb{A})$, $g\in G(\mathbb{A})$.

Let $\mathscr{H}(\lambda)$ be the space of functions $\Phi(\cdot,g)$, with values in $\mathscr{E}(\lambda)$, which is defined and analytic in a tube in $L_F\otimes\mathbf{C}$ over a ball of radius R with $R>(\rho,\rho)^{1/2}$ and which goes to zero at infinity faster than the inverse of any polynomial.

2.3. Let D_0 be the unitary characters of $Z_\infty^+A(F)\backslash A(\mathbb{A})$. Then $(Z_\infty^+A(F)\backslash A(\mathbb{A}))^*$ is also the union of sets of the form

$$D_\sigma=\{\chi\in(Z_\infty^+A(F)\backslash A(\mathbb{A}))^* \mid |\chi|=\sigma\}$$

where σ is a fixed character with values in $\mathbf{R}_+^\times$. We equip D_0 with the dual Haar measure via Pontrjagin duality and give D_σ the measure obtained by transport of structure from D_0.

We write $\mathscr{D}$ for the space spanned by functions of the form

$$\phi(g)=\int_{\operatorname{Re}\lambda=\lambda_0}\Phi(\lambda,g)|d\lambda| \tag{2}$$

where $\Phi\in\mathscr{H}(\lambda)$ and λ_0 is a character with values in $\mathbf{R}_+^\times$. By means of Fourier transform we get

$$\Phi(\lambda,g)=\int_{Z_\infty^+A(F)\backslash A(\mathbb{A})}\phi(ag)\lambda^{-1}(a)\rho^{-1}(a)\,da. \tag{3}$$

According to Langlands [13, 14], for $\phi\in\mathscr{D}$ the theta series

$$\tilde{\phi}(g)=\sum_{\gamma\in P(F)\backslash G(F)}\phi(\gamma g) \tag{4}$$

belongs to $\mathscr{L}^2(Z_\infty^+G(F)\backslash G(\mathbb{A}))$. Combining with (2), we get

$$\tilde{\phi}(g)=\int_{\operatorname{Re}\lambda=\lambda_0}E(g,\Phi,\lambda)\,d\lambda \tag{5}$$

where

$$(6) \qquad E(g, \Phi, \lambda) = \sum_{\gamma \in P(F)\backslash G(F)} \Phi(\lambda, \gamma g)$$

is an Eisenstein series. It converges uniformly for g in compact subsets of $G(\mathbb{A})$ and $\lambda \in L_F \otimes \mathbb{C}$ such that $\operatorname{Re}(\lambda, \alpha) > (\rho, \alpha)$ for every positive root α.

We define the constant term of the Eisenstein series $E(g, \Phi, \lambda)$ by

$$(7) \qquad E_0(g, \Phi, \lambda) = \int_{N(F)\backslash N(\mathbb{A})} E(ng, \Phi, \lambda)\, \mathrm{d}n.$$

2.4. Proposition: *The constant term is given by the following formula*:

$$E_0(g, \Phi, \lambda) = \sum_{w \in W_F} M(w, \lambda)\Phi(\lambda, g)$$

where W_F is the F-rational Weyl group of G and

$$(8) \qquad M(w, \lambda)\Phi(\lambda, g) = \int_{w^{-1}B(F)w \cap N(F)\backslash N(\mathbb{A})} \Phi(\lambda, wng)\, \mathrm{d}n.$$

Proof: We have

$$E_0(g, \Phi, \lambda) = \int_{N(F)\backslash N(\mathbb{A})} \sum_{B(F)\backslash G(F)} \Phi(\lambda, \gamma ng)\, \mathrm{d}n.$$

The proposition is immediate once we break up the sum over $B(F)\backslash G(F)$ into a sum over $W_F = B(F)\backslash G(F)/N(F)$ (Bruhat decomposition) and a sum over $(w^{-1}B(F)w \cap N(F))\backslash N(F)$.

2.5. We can define local version of $\mathscr{E}(\lambda)$ as the space $\mathscr{E}_v(\lambda)$ of continuous functions Φ_v on $N_v\backslash G_v/K_v$ satisfying

$$\Phi_v(a_v g_v) = \lambda(a_v)\rho(a_v)\Phi(g_v)$$

(here $\rho(a_v)$ is to be interpreted as $|\rho(a_v)|_v$).

For $\Phi \in \mathscr{E}(\lambda)$, we let Φ_v denote its restriction to G_v. Since Φ is right invariant under $K = \Pi K_v$ where $K_v = G(0_v)$ almost all v, and

$G(A)$ is the direct limit of $G^{\mathscr{S}}$, we can write

$$\Phi(g) = \prod \Phi_v(g_v).$$

(Here it is understood that $\Phi(1) = 1$.)

Furthermore, $M(w, \lambda)$ is a linear map from $\mathscr{E}(\lambda)$ to $\mathscr{E}(\lambda^w)$ where $\lambda^w(a) = \lambda(waw^{-1})$. In fact it is just multiplication by a constant to be calculated below. Moreover, $M(1, \lambda) = 1$ because $\mathrm{vol}(N(F)\backslash N(\mathbb{A})) = 1$.

2.6. Proposition: *Let* ${}^wN = w^{-1}Nw \cap N$ *and* $N^w = w^{-1}\bar{N}w \cap N$ *where* $\bar{N}$ *is the unipotent subgroup opposite to* N. *Define a linear transform* $M_v(w, \lambda): \mathscr{E}_v(\lambda) \to \mathscr{E}_v(\lambda^w)$ *by*

$$M_v(w, \lambda)\Phi(g) = \int_{N_v^w} \Phi(wng)\, \mathrm{d}n \tag{9}$$

for $g \in G_v$. *Then we have*

$$M(w, \lambda) = \prod M_v(w, \lambda). \tag{10}$$

(Here one regard the $M_v(w, \lambda)$ *as complex numbers.)*

Proof: First we have $N = {}^wN \cdot N^w$. So

$${}^wN(F)\backslash N(\mathbb{A}) = ({}^wN(F)\backslash {}^wN(\mathbb{A})) \cdot \mathrm{N}^w(\mathbb{A}).$$

It follows that, for $\Phi \in \mathscr{E}(\lambda)$

$$\begin{aligned} M(w, \lambda)\Phi(g) &= \int_{{}^wN(F)\backslash N(\mathbb{A})} \Phi(\mathrm{wng})\, \mathrm{d}n \\ &= \int_{{}^wN(F)\backslash {}^wN(\mathbb{A})} \int_{N^w(\mathbb{A})} \Phi(wn_1w^{-1} \cdot wn_2g)\, \mathrm{d}n_2\, \mathrm{d}n_1. \end{aligned}$$

The formula (10) now follows from the above and the fact that we have normalized our measure such that

$$\int_{{}^wN(F)\backslash {}^wN(\mathbb{A})} \mathrm{d}n_1 = 1.$$

3. $M_v(w, \lambda)$ in the rank one case

3.1. We shall compute $M_v(w, \lambda)$ for those places v of F satisfying the following conditions:

(i) G is a connected reductive quasi-split group over F_v.

(ii) G splits over an unramified extension of F_v.

(iii) $G_v = B_v K_v$ and $K_v = G(0_v)$.

(iv) G is of semisimple F_v-rank one.

Let us write E_v for the unramified extension of F_v over which G splits and write $\tilde{\omega}$ for the uniformizing element of both E_v and F_v. We denote by σ the Frobenius element in $\mathrm{Gal}(E_v/F_v)$.

Under the assumption, the F_v-rational Weyl group $W_{F_v} = \{1, w_0\}$, where w_0 sends all the positive roots to negative roots. We know that

$$M_v(1, \lambda) = 1.$$

It remains to calculate $M_v(w_0, \lambda)$. As $\mathscr{E}_v(\lambda)$ is one dimensional it suffices to calculate

$$M_v(w_0, \lambda) = M_v(w_0, \lambda)\Phi(\lambda, 1) = \int_{N_v^{w_0}} \Phi(\lambda, w_0 n)\, dn \tag{1}$$

where $\Phi(\lambda)$ is $\mathscr{E}(\lambda)$ is chosen to satisfy

$$\Phi(\lambda, 1) = 1.$$

G has F_v-rational rank 1 also implies that $L_{F_v} \otimes \mathbf{C}$ is isomorphic to $\mathbf{C}$ and hence can be replaced by the set $\{\rho^s \mid s \in \mathbf{C}\}$. Thus it suffices to consider $M(w_0, \rho^s)$. We define $\Phi(\rho^s)$ by:

$$\Phi(\rho^s, a) = |\rho(a)|_v^{s+1} \quad \text{if } a \in A_v,$$

$$\Phi(\rho^s, ngk) = \Phi(\rho^s, g) \quad \text{if } n \in N_v, k \in K_v.$$

Let us write $M(s)$ for $M(w_0, \rho^s)$. Then (1) becomes

$$M(s) = \int_{N_v^{w_0}} \rho^{s+1}(w_0 n)\, dn.$$

We can further assume that $w_0 \in K_v$, then changing variable by the map $n \to w_0 n w_0^{-1}$, we have

$$M(s) = \int_{\bar{N}_v} \rho^{s+1}(\bar{n})\, d\bar{n}, \tag{2}$$

and

$$\rho^s(a) = (|\tilde{\omega}|_{F_v}^{\langle \rho, \nu(a) \rangle})^s.$$

3.2. PROPOSITION: *Let* $\hat{\mathfrak{n}}$ *be the subspace of the Lie algebra of* $\hat{G}$ *spanned by the positive root vectors. Then*

$$M(s) = \frac{\det(I - |\tilde{\omega}|_{F_v} \sigma \text{ Ad } \hat{t}|_{\hat{\mathfrak{n}}})}{\det(I - \sigma \text{ Ad } \hat{t}|_{\hat{\mathfrak{n}}})} \tag{3}$$

where $\hat{t} = \hat{t}_{s\rho}$.

Let G' be the derived subgroup of G. Then the unipotent radical of the Borel subgroup of G' is the same as that of the corresponding Borel subgroup B of G. Thus we only need to compute the integral $M(s)$ for connected semisimple quasi-split groups of F_v-rank one. Henceforth, in this subsection we shall assume G to be of such type.

According to Steinberg's variation of Chevalley's theme, the quasi-split form of G is determined up to F_v-isomorphism by its Dynkin diagram and the twisted action of galois group (modulo inner twisting). As a result, up to central isogeny, G can only be one of the following types:

(I) G splits over G_v and has a connected Dynkin diagram, i.e. $G = \mathrm{SL}_2$.

(II) G is a twisted form of a F_v-split group whose Dynkin diagram is type A_2, i.e. $G(F_v) = \mathrm{SU}_3(E_v/F_v) = (g \in \mathrm{SL}_3(E_v) \,|\, {}^t\bar{g}Jg = J\}$ where E_v/F_v is a quadratic extension; the conjugation by the nontrivial element of the Galois group $\mathrm{Gal}(E_v/F_v)$ is denoted by $x \to \bar{x}$; ${}^t\bar{g}$ is the conjugate-transpose of the matrix $g : J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$ is the matrix of the Hermitian form with respect to the nontrivial element of $\mathrm{Gal}(E_v/F_v)$.

(III) G is a twisted form of a F_v-split group whose Dynkin diagram consists of n copies of A_1, i.e. there exists an extension E_v/F_v of degree n and $G(F_v) = \mathrm{SL}_2(E_v)$.

(IV) G is a twisted form of F_v-split group whose Dynkin diagram consists of n copies of A_2; there exists field extensions E_v, E'_v of F such that $[E_v : E'_v] = 2$, $[E_v : F_v] = 2n$. If $x \to \bar{x}$ is the nontrivial action of the Galois group $\mathrm{Gal}(E_v/E'_v)$ then $G(F_v) = \mathrm{SU}_3(E_v/E'_v) =$

$$\{g \in \mathrm{SL}_3(E_v) \mid {}^t\bar{g}Jg = J\} \text{ where } J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

It is obvious that it suffices to calculate (2) up to isogeny (see for example [18] §4.3). Moreover Rapoport [18] pointed out that it is possible to avoid the calculation of (2) for the cases (III) and (IV) by proving a general lemma on the behaviour of (2) under restriction of ground field.

3.3. When G is SL_2, it is well known that

$$M(s) = \frac{1 - q^{-(s+1)}}{1 - q^{-s}}.$$

The Lie algebra $\hat{\mathfrak{n}}$ in this case is one dimensional and it is trivial to check the formula (3). We shall omit the details.

3.4. PROPOSITION: *Let E_v/F_v be an unramified quadratic extension of local fields such that 2 is a unit in E_v. Then for the quasi-split group $\mathrm{SU}_3(E_v/F_v)$ we have*

$$M(s) = \frac{(1 - q^{-2(s+1)})(1 + q^{-2s-1})}{(1 - q^{-2s})(1 + q^{-2s})} = \frac{\det(I - |\tilde{\omega}|_{F_v}\sigma \text{ Ad } \hat{t}|_{\hat{\mathfrak{n}}})}{\det(I - \sigma \text{ Ad } \hat{t}|_{\hat{\mathfrak{n}}})}$$

PROOF: First we have

$$A(F_v) = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & \bar{a}^{-1} \end{pmatrix} \;\middle|\; \begin{matrix} a, b \in E_v^\times \\ b\bar{b} = 1, ab\bar{a}^{-1} = 1 \end{matrix} \right\},$$

$$N(F_v) = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \;\middle|\; \begin{matrix} y + \bar{y} + x\bar{x} = 0 \\ x, y \in E_v \end{matrix} \right\},$$

$$K = \mathrm{SU}_3(0_{E_v}).$$

E is an unramified quadratic extension of F, so there exists an element $c \in 0_{F_v} - \tilde{\omega}0_{F_v}$ such that its image in $0_{F_v}/\tilde{\omega}0_{F_v}$ is not a square and $E_v = F_v(\sqrt{c})$. Let the map $\mathrm{ord}_{F_v}: F_v^\times \to Z$ be defined by the condition

$$|x|_{F_v} = |\tilde{\omega}|_{F_v}^{\mathrm{ord}_F x} \quad \text{for } x \in F_v^\times.$$

Similar condition defines ord_{E_v}. Note if $x \in F_v$, then $|x|_{E_v} = |x|^2_{F_v}$ implies $\mathrm{ord}_{F_v} x = \mathrm{ord}_{E_v} x$.

Next, let us determine the measure dn on the nilpotent group $N(F_v)$. Let x, $y \in E_v$ such that $y + \bar{y} + x\bar{x} = 0$. Then we can write $y = y_1\sqrt{c} - \frac{x\bar{x}}{2}$ where $y_1 \in F_v$. Note that $x\bar{x} = N_{E_v/F_v}(x)$ also belongs to F_v.

A typical element of $N(F_v)$ can now be written as

$$\begin{pmatrix} 1 & x & y \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & -\frac{x\bar{x}}{2} \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y_1\sqrt{c} \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Thus we can write $N(F_v) = N_1 N_2$ (as sets) and take dn to be the image of the product of the measure on E_v and F_v respectively under the maps;

$$x \mapsto n_1 = \begin{pmatrix} 1 & x & -\frac{x\bar{x}}{2} \\ & 1 & -\bar{x} \\ & & 1 \end{pmatrix}, \quad x \in E_v,$$

$$y_1 \mapsto n_2 = \begin{pmatrix} 1 & 0 & y_1\sqrt{c} \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad y_1 \in F_v.$$

We normalize the measures on E_v and F_v by the condition that the volume of the respective maximal compact subrings is one.

The nontrivial element of the Weyl group corresponds to the matrix

$$w_0 = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

We have

$$\bar{N}_v = \left\{ \begin{pmatrix} 1 & & \\ -\bar{x} & 1 & \\ y & x & 1 \end{pmatrix} \middle| \begin{matrix} y + y + x\bar{x} = 0 \\ x, y \in E_v \end{matrix} \right\}.$$

If$\bar{n} \in \bar{N}_v$, then by Iwasawa decomposition of $\mathrm{SU}_3(E_v/F_v)$, we get

$$\bar{n} = \begin{pmatrix} 1 & & \\ -\bar{x} & 1 & \\ y & x & 1 \end{pmatrix} = n \begin{pmatrix} \bar{a}^{-1} & & \\ & b & \\ & & a \end{pmatrix} k$$

for some $n \in N_v$, $k \in K_v$.

As noted we can write $y = y_1\sqrt{c} - \dfrac{x\bar{x}}{2}$ for some $y_1 \in F$.

Then $\operatorname{ord}_{E_v} y = \inf(\operatorname{ord}_{E_v} y_1, 2\operatorname{ord}_{E_v} x)$ and

$$|a|_{E_v} = |\tilde{\omega}|_{E_v}^{\inf(0,\, \operatorname{ord}_{E_v} x,\, \operatorname{ord}_{E_v} y)}.$$

The zero in the "inf" is put into account for the case when both x and y are integral, and $\bar{n} \in K_v$.

Direct calculation using the definition of ρ^s gives

$$\rho^s\left(\begin{pmatrix} a & & \\ & b & \\ & & \bar{a}^{-1} \end{pmatrix}\right) = |a|_{E_v}^s, \quad s \in \mathbf{C}.$$

To calculate the value of $\rho^{s+1}(\bar{n})$, we have to consider four cases:

1. $\operatorname{ord}_{E_v} x \geq 0$ and $\operatorname{ord}_E y_1 \geq 0$
   $\Rightarrow \operatorname{ord}_{E_v} y \geq 0$
   $\Rightarrow \inf(0, \operatorname{ord}_{E_v} x, \operatorname{ord}_{E_v} y) = 0$
   $\Rightarrow \rho^{s+1}(\bar{n}) = 1.$

2. $2\operatorname{ord}_{E_v} x \geq \operatorname{ord}_{E_v} y_1$, $\operatorname{ord}_{E_v} y_1 < 0$, $\operatorname{ord}_{E_v} y_1$ is even.
   if $\operatorname{ord}_{E_v} x \geq 0$ then $\operatorname{ord}_{E_v} y_1 < \operatorname{ord}_{E_v} x$.
   If $\operatorname{ord}_{E_v} x < 0$ then $\operatorname{ord}_{E_v} y_1 \leq 2\operatorname{ord}_{E_v} x < \operatorname{ord}_{E_v} x$.
   Thus $\inf(0, \operatorname{ord}_{E_v} x, \operatorname{ord}_{E_v} y) = \operatorname{ord}_{E_v} y_1$ and
   $\rho^{s+1}(\bar{n}) = |\bar{a}^{-1}|_{E_v}^{s+1} = q^{2(s+1)\operatorname{ord}_{E_v} y_1}.$

Note: if $\operatorname{ord}_{E_v} y_1 = -2m$ then

$$\operatorname{ord}_{E_v} x \geq \frac{\operatorname{ord} y_1}{2} = -m.$$

3. $2\operatorname{ord}_{E_v} x \geq \operatorname{ord}_{E_v} y_1 < 0$, $\operatorname{ord}_{E_v} y_1$ is odd
   $\Rightarrow \inf(0, \operatorname{ord}_{E_v} x, \operatorname{ord}_E y) = \operatorname{ord}_{E_v} y_2$
   $\Rightarrow \rho^{s+1}(\bar{n}) = q^{2(s+1)\operatorname{ord}_{E_v} y_1}.$

Note: if $\mathrm{ord}_{E_v} y_1 = -(2m-1)$, $m \geq 1$ then

$$\mathrm{ord}_{E_v} x \geq -m + \tfrac{1}{2} \text{ or } \mathrm{ord}_{E_v} x \geq -(m-1).$$

4. $2\,\mathrm{ord}_{E_v} x < \mathrm{ord}_{E_v} y_1$, $\mathrm{ord}_{E_v} x < 0$
   $\Rightarrow \mathrm{ord}_{E_v} y = 2\,\mathrm{ord}_{E_v} x$
   $\Rightarrow \rho^{s+1}(\bar{n}) = q^{2(s+1)2\,\mathrm{ord}_{E_v} x}$.

Note: if $\mathrm{ord}_{E_v} x = -m$ then $\mathrm{ord}_{E_v} y_1 > -2m \geq -(2m-1)$.

Now we are ready to calculate the integral $M(s)$. We break the integral up into four pieces corresponding to the four cases above and transfer the integral over $\bar{N}(F_v)$ to those over $E_v \times F_v$, viz.,

$$M(s) = \int_{\bar{N}(F_v)} \rho^{s+1}(\bar{n})\,\mathrm{d}\bar{n} = \int_{\bar{N}_1}\int_{\bar{N}_2} \rho^{s+1}(\bar{n}_1\bar{n}_2)\,\mathrm{d}\bar{n}_1\,\mathrm{d}\bar{n}_2$$

$$= \int_{0_{E_v}}\int_{0_{F_v}} \mathrm{d}x\,\mathrm{d}y_1 + \sum_{m=1}^{\infty} \int_{P_{E_v}^{-2m} - P_{F_v}^{-2m}} \int_{P_{f_v}^{-(2m-1)}} q^{s(s+1)(-2m)}\,\mathrm{d}x$$

$$+ \sum_{m=1}^{\infty} \int_{P_{E_v}^{-(m-1)}} \int_{P_{F_v}^{-(2m-2)} - P_{F_v}^{-(2m-2)}} q^{-2(s+1)(2m-1)}\,\mathrm{d}x\,\mathrm{d}y_1$$

$$+ \sum_{m=1}^{\infty} \int_{P_{E_v}^{-m} - P_{E_v}^{-(m-1)}} \int_{P_{E_v}^{-(2m-1)}} q^{2(s+1)(-2m)}\,\mathrm{d}x\,\mathrm{d}y_1$$

where P_{E_v} (resp. P_{E_v}) is the maximal prime ideal of E_v (resp. F_v). We normalized measure on E_v, F_v by $\int_{0_{E_v}} \mathrm{d}x = 1$ and $\int_{0_{F_v}} \mathrm{d}y_1 = 1$.

Further calculation gives

$$\int_{0_{E_v}}\int_{0_{F_v}} \mathrm{d}x\,\mathrm{d}y_1 = 1.$$

$$\sum_{m=1}^{\infty} \int_{P_{F_v}^{-m} - P_{F_v}^{-(2m-1)}} q^{2(s+1)(-2m)}\,\mathrm{d}x\,\mathrm{d}y_1$$

$$= \sum_{m=1}^{\infty} q^{2m}(q^{2m} - q^{2m-1})q^{-4(s+1)m},$$

$$= (1-q^{-1}) \sum_{m=1}^{\infty} (q^{-4s})^m = \frac{(1-q^{-1})q^{-4s}}{1-q^{-4s}}.$$

$$\sum_{m=1}^{\infty} \int_{P_{E_v}^{-(m-1)}} \int_{P_{F_v}^{-(2m-1)} - P_{F_v}^{-(2m-2)}} q^{-2(s+1)(2m-1)} \, dx \, dy_1$$

$$= \sum_{m=1}^{\infty} q^{2m-2}(q^{2m-1} - q^{2m-2}) q^{-2(s+1)(2m-1)},$$

$$= (q^{-1} - q^{-2}) q^{2s} \sum_{m=1}^{\infty} (q^{-4s})^m,$$

$$= \frac{(q^{-1} - q^{-2}) q^{-2s}}{1 - q^{-4s}}.$$

$$\sum_{m=1}^{\infty} \int_{P_{E_v}^{-m} - P_{E_v}^{-(m-1)}} \int_{P_{F_v}^{-(2m-1)}} q^{2(s+1)(-2m)} \, dx \, dy_1$$

$$= \sum_{m=1}^{\infty} (q^{2m} - q^{2m-2}) q^{2m-1} q^{-4m(s+1)},$$

$$= (1 - q^{-2}) q^{-1} \sum_{m=1}^{\infty} (q^{-4s})^m,$$

$$= \frac{(q^{-1} - q^{-3}) q^{-4s}}{1 - q^{-4s}}.$$

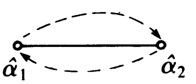
Adding all the terms, we have

$$M(s) = \frac{(1 - q^{-2s-2})(1 + q^{-2s-1})}{(1 - q^{-2s})(1 + q^{-2s})}.$$

To complete the proof of the proposition, let us look at the Lie algebra $\hat{\mathfrak{g}}$ of the analytic group $\hat{G}$ associated with G. We can take $\hat{\mathfrak{g}}$ to be $\mathfrak{sl}_2(\mathbb{C})$ and let $\hat{\Sigma}^+ = \{\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3\}$, $\hat{\alpha}_3 = \hat{\alpha}_1 + \hat{\alpha}_2$. There exists root vectors $X_{\hat{\alpha}_1}, X_{\hat{\alpha}_2}, X_{\hat{\alpha}_3}$ such that

$$[X_{\hat{\alpha}_1}, X_{\hat{\alpha}_2}] = X_{\hat{\alpha}_2}.$$

$\hat{\mathfrak{g}}$ has a Dynkin diagram of type A_2



the arrows indicate the action of $\sigma \in \mathrm{Gal}(E/F)$, i.e. $\sigma(X_{\hat{\alpha}_1}) = X_{\hat{\alpha}_2}$. Since this action is to be extended to a Lie algebra isomorphism, i.e. $\sigma[X_{\hat{\alpha}_1}, X_{\hat{\alpha}_2}] = [\sigma X_{\hat{\alpha}_1}, \sigma X_{\hat{\alpha}_2}]$, so $\sigma X_{\hat{\alpha}_3} = [X_{\hat{\alpha}_2}, X_{\hat{\alpha}_1}] = -X_{\hat{\alpha}_3}$.

Also, we have

$$
\begin{aligned}
(\operatorname{Ad}\hat{t})X_{\hat{\alpha}} &= \hat{\alpha}(\hat{t})X_{\hat{\alpha}} = |\tilde{\omega}|_{F_v}^{s\langle\rho,\hat{\alpha}\rangle}X_{\hat{\alpha}} \\
&= |\tilde{\omega}|_{F_v}^{s}X_{\hat{\alpha}} \quad \text{if } \hat{\alpha} = \hat{\alpha}_1 \text{ or } \hat{\alpha}_2, \\
\text{or} \qquad &= |\tilde{\omega}|_{F_v}^{2s}X_{\hat{\alpha}} \quad \text{if } \hat{\alpha} = \hat{\alpha}_3,
\end{aligned}
$$

because $\langle\rho,\hat{\alpha}\rangle = \dfrac{2(\rho,\alpha)}{(\alpha,\alpha)} = 1$ if α simple and

$$\langle\rho,\hat{\alpha}_3\rangle = \langle\rho,\alpha_1\rangle + \langle\rho,\alpha_2\rangle = 2.$$

We take $\hat{\mathfrak{n}} = \mathbb{C}X_{\hat{\alpha}_1} + \mathbb{C}X_{\hat{\alpha}_2} + \mathbb{C}X_{\hat{\alpha}_3}$. Then

$$
\begin{aligned}
&\det(I - \sigma \operatorname{Ad}\hat{t}|_{\hat{\mathfrak{n}}}) \\
&= \det\left(I - \begin{pmatrix} 0 & |\tilde{\omega}|_{F_v}^{s} & 0 \\ |\tilde{\omega}|_{F_v}^{s} & 0 & 0 \\ 0 & 0 & -|\tilde{\omega}|_{F_v}^{2s} \end{pmatrix}\right), \\
&= (1 - q^{-2s})(1 + q^{-2s}),
\end{aligned}
$$

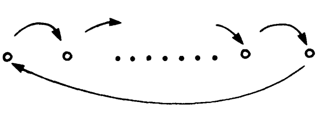
and

$$
\begin{aligned}
&\det(I - |\tilde{\omega}|_{F_v}\sigma \operatorname{Ad}\hat{t}|_{\hat{\mathfrak{n}}}) \\
&= (1 - q^{-2s-s})(1 + q^{-2s-1}).
\end{aligned}
$$

This completes the proof of the proposition.

3.5. Let us now consider the case (III). G is a connected semi-simple quasi-split algebraic group defined over F_V splits over an unramified extension E_v/F_v of degree n.

The absolute Dynkin diagram of G consists of n copies of A_1, and the action of the Frobenius σ in $\operatorname{Gal}(E_v/F_v)$ is the cyclic permutation as indicated



The action has only one orbit; G is of F-rank 1 and $G(F_v) = \mathrm{SL}_2(E_v)$. The integral that we are interested in becomes $M(s) = \int_{\bar{N}_v} \rho^{s+1}(\bar{n})\,d\bar{n}$

where

$$\bar{N}_v = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| x \in E_v \right\},$$

$$A_v = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in E_v^x \right\},$$

and

$$|a|_{E_v}^{s+1} = \rho^{s+1} \left(\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right).$$

So by §3.3

$$\begin{aligned} M(s) &= \frac{1 - q_{E_v}^{-(s+1)}}{1 - q_{E_v}^{-s}} \quad \text{where } q_{E_v} = \text{modulus of } E_v = q^n, \\ &= \frac{1 - q^{-n(s+1)}}{1 - q^{-ns}}. \end{aligned}$$

But on the other hand $\hat{\mathfrak{n}} = \overbrace{\mathfrak{sl}_2 x \ldots x \mathfrak{sl}_2}^{n \text{ copies}}$. Let $X_{\hat{\alpha}_i}$ be the root vector corresponding to the positive root $\hat{\alpha}_i$ of the i^{th} copy of $\mathfrak{sl}_2$ in the product. Then

$$(\mathrm{Ad}\, \hat{t}) X_{\hat{\alpha}_i} = \hat{\alpha}_i(\hat{t}) X_{\hat{\alpha}_i} = |\tilde{\omega}|_{F_v}^{s\langle \rho, \hat{\alpha}_i \rangle} X_{\hat{\alpha}_i} = q^{-s} X_{\hat{\alpha}_i}$$

because $\rho = \frac{1}{2} \Sigma \alpha_i$ and as the diagram is disconnected $\langle \alpha_j, \hat{\alpha}_i \rangle = 0$ if $i \neq j$, and $\left\langle \frac{\alpha_i}{2}, \hat{\alpha}_i \right\rangle = 1$. So,

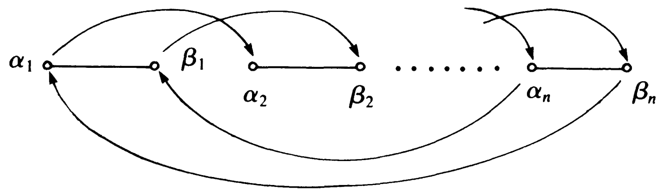
$$\begin{aligned} \det(I - \mathrm{Ad}\, \hat{t}|_{\hat{\mathfrak{n}}}) &= \begin{bmatrix} 1 & & & & \\ -q^{-s} & 1 & & & \\ & -q^{-s} & \ddots & & \\ & & \ddots & 1 & \\ & & & -q^{-s} & 1 \end{bmatrix}, \\ &= 1 - q^{-ns}. \end{aligned}$$

Similarly,

$$\det(I - |\tilde{\omega}|_{F_v} \sigma \,\mathrm{Ad}\, \hat{t}|_{\hat{\mathfrak{n}}}) = 1 - q^{-n(s+1)},$$

and we are done.

3.6. Finally, let us look at the last case IV. Here G is a F_v-form of a split group with a Dynkin diagram consisting of n copies of A_2. G is defined over F_v splits over an unramified extension E_v of degree $2n$; there exists a field E'_v in E_v/F_v such that $[E'_v:F_v]=n$; the non-trivial element of $\mathrm{Gal}(E_v/E'_v)(\subset \mathrm{Gal}(E_v/F_v))$ give rise to the twisting; the action of this element is shown in the diagram



This determines a special unitary group $\mathrm{SU}_3(E_v/E'_v)$ with respect to the form

$$J=\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}$$

such that

$$G(F)\approx \mathrm{SU}_3(E_v/E'_v)=\{g\in \mathrm{SL}_3(E_v)\mid {}^t\bar{g}g=J\}.$$

Thus, using the result in §3.4, we get

$$M(s)=\frac{(1-q^{-2n(s+1)})(1+q^{-n(2s+1)})}{(1-q^{-2ns})(1+q^{-2ns})}$$

(Note: modulus of $E_v=q^{2n}$.)

To establish the formula

$$M(s)=\frac{\det(I-|\tilde{\omega}|_{F_v}\sigma\ \mathrm{Ad}\,\hat{t}|_{\hat{\mathfrak{n}}})}{\det(I-\sigma\ \mathrm{Ad}\,\hat{t}|_{\hat{\mathfrak{n}}})}$$

we shall evaluate the determinants directly.

Let us denote the simple root system Δ by $\{\alpha_1,\beta_1;\ldots;\alpha_n,\beta_n\}$. We calculate

$$\begin{aligned}(\mathrm{Ad}\,\hat{t})X_{\hat{\alpha}_i}&=\hat{\alpha}_i(\hat{t})X_{\hat{\alpha}_i}=|\tilde{\omega}|_{F_v}^{s\langle\rho,\hat{\alpha}_i\rangle}X_{\hat{\alpha}_i}\\&=q^{-s}X_{\hat{\alpha}_i}.\end{aligned}$$

Here $\rho = \frac{1}{2}\sum_{i=1}^{n}(\alpha_i + \beta_i + (\alpha_i + \beta_i))$,

$$\langle \rho, \hat{\alpha}_i \rangle = \sum_{j=1}^{n} \langle \rho_j, \hat{\alpha}_i \rangle \quad \text{where } \rho_j = \alpha_j + \beta_j,$$

because $i \neq j$

$$\langle \rho_j, \hat{\alpha}_i \rangle = 0,$$

and

$$\langle \rho_i, \hat{\alpha}_i \rangle = 1.$$

Similarly

$$(\mathrm{Ad}\,\hat{t})X_{\hat{\beta}_i} = q^{-s}X_{\hat{\beta}_i},$$

and

$$(\mathrm{Ad}\,\hat{t})X_{\hat{\alpha}_i+\hat{\beta}_i} = q^{-2s}X_{\hat{\alpha}_i+\hat{\beta}_i}.$$

Next we write down the effect of the Galois action as indicated by the arrows in the above diagram. For $1 \le i \le n-1$,

$$\sigma X_{\hat{\alpha}_i} = X_{\hat{\alpha}_{i+1}},$$

$$\sigma X_{\hat{\beta}_i} = X_{\hat{\beta}_{i+1}},$$

$$\begin{aligned}\sigma X_{\hat{\alpha}_i+\hat{\beta}_i} &= \sigma[X_{\hat{\alpha}_i}, X_{\hat{\beta}_i}] = [\sigma X_{\hat{\alpha}_i}, \sigma X_{\hat{\beta}_i}] \\ &= [X_{\hat{\alpha}_{i+1}}, X_{\hat{\beta}_{i+1}}] = X_{\hat{\alpha}_{i+1}+\hat{\beta}_{i+1}},\end{aligned}$$

and

$$\sigma X_{\hat{\alpha}_n} = X_{\hat{\beta}_1},$$

$$\sigma X_{\hat{\beta}_n} = X_{\hat{\alpha}_1},$$

$$\sigma X_{\hat{\alpha}_n+\hat{\beta}_n} = [\sigma X_{\hat{\alpha}_n}, \sigma X_{\hat{\beta}_n}] = [X_{\hat{\beta}_1}, X_{\hat{\alpha}_1}] = -X_{\hat{\alpha}_1+\hat{\beta}_1}.$$

If we take the basis of $\mathfrak{n}$ to be $X_{\hat{\alpha}_1}, X_{\hat{\beta}_1}, X_{\hat{\alpha}_1+\hat{\beta}_1}, \ldots, X_{\hat{\alpha}_n}, X_{\hat{\beta}_n}, X_{\hat{\alpha}_n+\hat{\beta}_n}$ (in that order), then it is trivial to show that

$$\begin{aligned}&\det(I - \sigma\,\mathrm{Ad}\,\hat{t}|_{\hat{\mathfrak{n}}}) \\ &= (1 - q^{-2ns})(1 + q^{-2ns}),\end{aligned}$$

and

$$\det(I - \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}})$$
$$= (1 - q^{-2n(s+1)})(1 + q^{-n(2s+1)}).$$

Thus the required formula is proved. With this we complete the proof of Proposition 3.2.

4. Reduction to rank one

To determine the local factor $M_v(w, \lambda)$ for almost all v for G of arbitrary F-rank, we use the method of reduction to F-rank one which was first studied by Bhanu-Murti [1] and was extended by Gindikin and Karpelevich [6]. This method has also been used in Langlands' Euler Product (Yale, 1971) and in the thesis of Jacquet (Paris) and Lai (Yale). Here we shall follow Shiffmann [19].

4.1. We want to calculate the integral (9) of §2. For $\lambda \in L_F \otimes \mathbb{C}$, $\mathscr{E}(\lambda) \neq 0$ and so $\mathscr{E}_v(\lambda) \neq 0$ for all v. We have $W_F \subset W_{F}$. We can consider w as an element of W_{F_v} and do the rest of the calculation over F_V. Moreover for almost all v, $\mathscr{E}_v(\lambda)$ is one dimensional. It is sufficient to evaluate the integral for the following function in $\mathscr{E}_v(\lambda)$:

(1) $$\Phi(g_v) = |\lambda(a_v)\rho(a_v)|_v$$

where $g_v = n_v a_v k_v \in G_v$. The linear transformation $M_v(w, \lambda)$ is just multiplication by the following constant which we also denoted by $M_v(w, \lambda)$:

$$M_v(w, \lambda) = \int_{N_v^w} \Phi(wn)\, dn.$$

Changing the variable by $n \to w^{-1}nw$ and writing $\bar{N}^w = wN^w w^{-1} = wNw^{-1} \cap N$, we have

(2) $$M_v(w, \lambda) = \int_{\bar{N}_v^w} \Phi(nw)\, dn.$$

Recall that the length $\ell(w)$ of w is the smallest integer g of such that there exists g simple F_v-roots $\beta_1, \ldots, \beta_g$ with

(3) $$w = s_{\beta_1}, \ldots, s_{\beta_g}$$

(s_{α_j} is the symmetry with respect to α_j). Moreover the F_v-roots $\alpha_j = s_{\beta_{\ell(w)}} \ldots s_{\beta_{j+1}}(\beta_j)$ $j = 1, \ldots, \ell(w)$ are positive and if we write

$${}_0\Sigma^+_{F_v}(w) = \{\alpha \in {}_0\Sigma^+_{F_v} \mid {}^w\alpha < 0\}$$

then

$${}_0\Sigma^+_{F_v}(w) = \{\alpha_1, \ldots, \alpha_{\ell(w)}\}.$$

We quote the following lemma from Schiffmann ([19], Prop. 1.3).

4.2. Lemma: *Let w, w', w'' be three elements of w_F such that $w = w'w''$ with $\ell(w) = \ell(w') + \ell(w'')$. Then the map* (4) $(n', n'') \to n'(w'n''w'^{-1})$ *defines a variety isomorphism $\bar{N}^{w'} \times \bar{N}^{w''} \to \bar{N}^w$.*

4.3. Using the above lemma, and *assuming* the integrals involve converges, we have

$$\begin{aligned} M_v(w, \lambda) &= \int_{\bar{N}_v^{w'} \times \bar{N}_v^{w''}} \Phi(n'w'n''w'^{-1}w)\, dn'\, dn'', \\ &= \int_{\bar{N}_v^{w''}} M_v(w', \lambda)\Phi(n''w'')\, dn'', \end{aligned}$$

and so

(5) $$M_v(w, \lambda) = M_v(w', \lambda^{w''})M_v(w'', \lambda).$$

If we write w as a product of symmetries (as in (3)) then formula (5) allows us to reduce the calculation to the case $\ell(w) = 1$, i.e. the F-rank one case, and in this case the convergence follows from the explicit formula given in §3. To summarize we have

4.4. Proposition: *Let $N_\alpha = G_\alpha \cap N$ for $\alpha \in {}_0\Sigma^+_F$ and $\bar{N}_\alpha$ the unipotent subgroup of G_α opposite to N_α. Then the integral* (2) *converges for $\lambda \in L_F \otimes \mathbb{C}$* with $\mathrm{Re}(\langle\lambda, \hat{\alpha}\rangle) > 0$ *for all $\alpha \in {}_0\Sigma^+_F(w)$,*

(6) $$M_v(w, \lambda) = \prod_{\alpha \in {}_0\Sigma^+_F(w)} \int_{\bar{N}_\alpha(F_v)} \Phi_\alpha(\bar{n})\, d\bar{n}$$

where Φ_α is the restriction of Φ to G_α.

4.5. As each G_α has F_v-rank one we can apply Proposition 3.2 to get

(7)
$$\int_{\bar{N}_\alpha(F_v)} \Phi_\alpha(\bar{n})\,\mathrm{d}\bar{n} = \frac{\det(I - |\tilde{\omega}|_v \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}_\alpha})}{\det(I - \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}_\alpha})}.$$

Let $\hat{\mathfrak{n}}$ be the nilpotent subalgebra of $\hat{\mathfrak{g}}$ spanned by $\hat{\mathfrak{g}}_\alpha$ for $\alpha \in {}_0\Sigma^+_{Fv}(w)$. The action of $\sigma \operatorname{Ad} \hat{t}$ on $\hat{\mathfrak{n}}^w$ preserves the subspaces $\hat{\mathfrak{n}}_\alpha$. Hence

(8)
$$\frac{\det(I - |\tilde{\omega}|_v \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}^w})}{\det(I - \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}^w})} = \prod_{\alpha \in {}_0\Sigma^+_{F_v}(w)} \frac{\det(I - |\tilde{\omega}|_v \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}_\alpha})}{\det(I - \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}_\alpha})}.$$

The following proposition follows immediately from (6), (7) and (8).

4.6. PROPOSITION: *For almost all v, we have*

(9)
$$M_v(w, \lambda) = \frac{\det(I - |\tilde{\omega}|_v \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}^w})}{\det(I - \sigma \operatorname{Ad} \hat{t}|_{\hat{\mathfrak{n}}^w})}$$

where σ is the Frobenius and $\hat{t} = \hat{t}_\lambda$.

5. Value of the local factor at one

5.1. Let $\mathcal{S}$ be a finite set of places of F containing all the infinite place of F, all the ramified places of F and all the places at which the conditions (i) to (iii) of §3.1 are not satisfied. Let us write

$$M_{\mathcal{S}}(s) = \prod_{v \in \mathcal{S}} M_v(w_0, \rho^s)$$

where $s \in \mathbb{C}$ and $w_0 \in W_F$ sends all positive roots to negative roots. Then $M_{\mathcal{S}}(1)$ can be considered as a linear map $E_{\mathcal{S}}(\rho) \to E_{\mathcal{S}}(\rho^{-1})$ and

(1)
$$M_{\mathcal{S}}(1)\Phi(g) = \int_{N_{\mathcal{S}}} \Phi(w_0 n g)\,\mathrm{d}n$$

for $\Phi \in \mathscr{E}_{\mathcal{S}}(\rho)$, $g \in G_{\mathcal{S}}$. Now $G_{\mathcal{S}} = B_{\mathcal{S}} K_{\mathcal{S}}$ implies that $\mathscr{E}_{\mathcal{S}}(\rho)$ is one dimensional and $M_{\mathcal{S}}(1)$ is just multiplication by a constant which we also denoted by $M_{\mathscr{S}}(1)$. We have

$$M_{\mathscr{S}}(1) = \int_{N_{\mathscr{S}}} \rho^2(w_0 n)\, \mathrm{d}n. \tag{2}$$

5.2. Let $L(s, G)$ be the Artin L-function of the Galois action on the rational characters of G, $L_v(s, G)$ be the local factor at v of $L(s, G)$ and

$$\mu_G = \lim_{s \to 1} (s-1)^{r_G} L(s, G)$$

where r_G is the rank of $X(G)_F$. Similar definitions are made with A replacing G.

PROPOSITION: *For $\mathscr{S}$ sufficiently large we have*

$$M_{\mathscr{S}}(1) = \kappa \frac{\mu_G}{\mu_A} \prod_{v \in \mathscr{S}} \frac{L_v(1, A)}{L_v(1, G)} \prod_{v \notin \mathscr{S}} \mathrm{vol}\, K_v \tag{3}$$

where the vol K_v *is calculated by the local measure* $\mathrm{d}g_v$.

PROOF: Let h be an integrable function on $N_{\mathscr{S}} + A_{\mathscr{S}}$. Let f be a function on $G(\mathbb{A})$ which vanishes at g except if $g_v \in K_v$ for all $v \notin \mathscr{S}$ and if the latter condition is satisfied, we have

$$f(g) = f(g_{\mathscr{S}}) = h(n, a)$$

for $g = nak$. First of all we have

$$\int_{G(\mathbb{A})} f(g)\, \mathrm{d}g = \kappa \int_{N_{\mathscr{S}} \times A_{\mathscr{S}}} h(n_2, a_2) \rho^{-2}(a_2)\, \mathrm{d}n_2\, \mathrm{d}a_2. \tag{4}$$

On the other hand, suppose that $g_{\mathscr{S}}$ lies in the large cell $N_{\mathscr{S}} S_{\mathscr{S}} w_0 N_{\mathscr{S}}$ of the Bruhat decomposition: $g_{\mathscr{S}} = n_2 a_2 w_0 n_1$ where $a_2 \in A_{\mathscr{S}}$ and $n_1, n_2 \in N_{\mathscr{S}}$ and if we write $w_0 n_1 = n(n_1) a(n_1) k$ with $n(n_1) \in N_{\mathscr{S}}$ and $a(n_1) \in A_{\mathscr{S}}$, then $g_{\mathscr{S}} = n_2 a_2 n(n_1) a_2^{-1} a_2 a(n_1) k$ and

$$\int_{G(\mathbb{A})} f(g)\, \mathrm{d}g \tag{5}$$
$$= \prod_{v \notin \mathscr{S}} \mathrm{vol}(K_v) \int_{N_{\mathscr{S}} A_{\mathscr{S}} N_{\mathscr{S}}} h(n_2 a_2 n(n_1) a_2^{-1}, a_2 a(n_1)) \rho^{-2}(a_2)\, \mathrm{d}n_2\, \overline{\mathrm{d}a_2}\, \mathrm{d}n_1.$$

After changing the measures, the integral in the above formula becomes

$$\int_{N_{\mathscr{S}}A_{\mathscr{S}}N_{\mathscr{S}}} \rho^2(a(n_1))h(n_2, a_2)\rho^{-2}(a_2)\,\mathrm{d}n_2\,\overline{\mathrm{d}a_1}\,\mathrm{d}n_1.$$

Substitute this and

$$\mathrm{d}a_2 = \Big(\prod_{v\in\mathscr{S}} L_v(1, A)\Big)\,\overline{\mathrm{d}a_2}$$

into (5). Comparing the result with (4), we obtain (3) by noting that the choice of h is arbitrary.

5.3. Corollary: *For $v \notin \mathscr{S}$, if we write*

$$M_v(1) = M_v(w_0, \rho) = \int_{N_v} \rho^2(w_0 n)\,\mathrm{d}n$$

then

$$M_v(1) = \mathrm{vol}(K_v)\cdot L_v(1, A)/L_v(1, G). \tag{6}$$

Proof: Apply the proposition to $\mathscr{S}' = \mathscr{S} \cup \{v\}$. The corollary then follows immediate form

$$M_{\mathscr{S}'}(1) = M_v(1)M_{\mathscr{S}}(1).$$

5.4. Remark: We have followed Rapoport [18] in the proof of corollary 5.3. An alternative approach is given in my thesis (Yale 1974) in which (6) is deduced from (9) of §4 by calculating directly $\mathrm{vol}(K_v)$ via reduction mod v.

6. The constant functions

We calculate in this section the projection of $\mathscr{E}$ into the subspace of constant functions in $\mathscr{L}^2(Z_\infty^+G(F))\backslash G(\mathbb{A}))$.

6.1. Let $\mathscr{L}$ be the closed subspace of $\mathscr{L}^2(Z_\infty^+G(F))\backslash G(\mathbb{A})$ generated by $\tilde{\phi}$ for $\phi \in \mathscr{D}$. Write $\mathscr{H}$ for the union of $\mathscr{H}(\lambda)$ for all λ in $L_F\otimes\mathbf{C}$.

Suppose that f is a complex valued function defined, bounded and analytic in a tube in $L_F \otimes \mathbf{C}$ over a ball of radius R with centre at zero and $R > (\rho, \rho)^{1/2}$. Assume also that $f({}^w\lambda) = f(\lambda)$ for all $w \in W_F$. Then

$$\Phi \to \Psi = f\Phi$$

where $\Psi(\lambda, g) = f(\lambda)\Phi(\lambda, g)$, defines a linear map on $\mathscr{H}$ and induces a bounded linear operator

$$\Lambda(f): \tilde{\phi} \to \tilde{\psi}$$

on $\mathscr{L}$. If $a > (\rho, \rho)$ and $f(\lambda) = (a - (\lambda, \lambda))^{-1}$, then $\Lambda(f)$ is self-adjoint. We define

$$\mathscr{A} = a - \Lambda(f)^{-1}.$$

It is an unbounded self-adjoint operator on $\mathscr{L}$ ($\mathscr{A}$ is introduced in Langlands [14] §6 and [15]). It is obvious that if $\Psi(\lambda, g) = (\lambda, \lambda)\Phi(\lambda, g)$ then $\mathscr{A}\tilde{\phi} = \tilde{\psi}$. The following two lemmas and the corollary are easy to prove.

6.2. Lemma: *Let $(\ ,\ )$ be the inner product on $\mathscr{L}^2(Z_\infty^+ G(F))\backslash G(\mathbb{A})$ and 1 be the constant function. For $\tilde{\phi} \in \mathscr{L}$, we have*

$$(\tilde{\phi}, 1) = \kappa\Phi(\rho, 1). \tag{1}$$

6.3. Lemma: *For $\tilde{\phi} \in \mathscr{L}$ and $\mathscr{A}$ as defined above we have*

$$(\mathscr{A}\tilde{\phi}, 1) = (\rho, \rho)(\tilde{\phi}, 1). \tag{2}$$

6.4. Corollary: $\mathscr{A}1 = (\rho, \rho)1$.

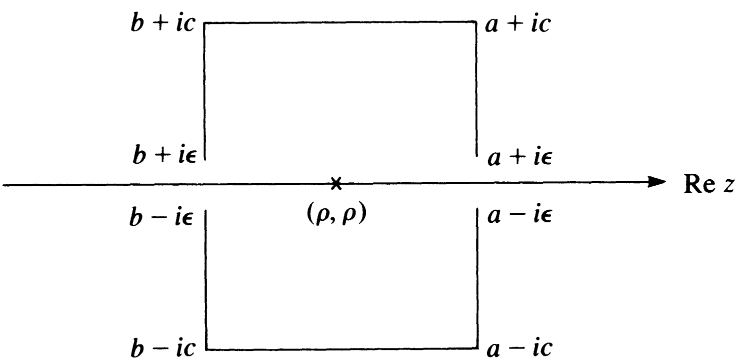
6.5. For $z \in \mathbf{C}$, let $R(z, \mathscr{A}) = (z - \mathscr{A})^{-1}$ be the resolvent of $\mathscr{A}$. For $\lambda_0 \in L_F \otimes \mathbf{R}$ if $\operatorname{Re} z > (\lambda_0, \lambda_0)$, then it is easy to show that

$$(R(z, \mathscr{A})\tilde{\phi}, \tilde{\psi}) = \kappa \sum_{w \in W_F} \int_{|\lambda| = \lambda_0} \frac{M(w, \lambda)\Phi(\lambda)\bar{\Psi}(-{}^w\bar{\lambda})}{z - (\lambda, \lambda)} \, d\lambda. \tag{3}$$

Let $E(x)$, $-\infty < x < \infty$ be a right continuous spectral resolution of the self-adjoint operator $\mathscr{A}$. It is obvious that (ρ, ρ) belongs to the point spectrum of $\mathscr{A}$ and corollary 6.4 implies that the constant functions are in the range of the projection $E((\rho, \rho)) - E((\rho, \rho) - 0) = E$(say). Suppose $a > (\rho, \rho) > b$, and $a - b$ is small, then $(E\tilde{\phi}, \tilde{\psi})$ is given by Stieljes inversion,

$$(4)\quad \tfrac{1}{2}\{(E(a)\tilde{\phi},\tilde{\psi})+(E(a-0)\tilde{\phi},\tilde{\psi})\}-\tfrac{1}{2}\{(E(b)\tilde{\phi},\tilde{\psi})+(E(b-0)\tilde{\phi},\tilde{\psi})\} = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{C(a,b,c,\epsilon)}(R(z,\mathscr{A})\tilde{\phi},\tilde{\psi})\,\mathrm{d}z$$

where $C(a,b,c,\epsilon)$ is the following contour:



6.6. Next we want to determine the dual measure for the Fourier transform on A.

We have put on $A(\mathbb{A})$ the Tamagawa measure da which can be written as $\mathrm{d}a=\mathrm{d}a^1\,\mathrm{d}t$ *corresponding to the decomposition* $A(\mathbb{A})=A^1(\mathbb{A})A^+_\infty$. In §2.3 we put a measure on $(Z^+_\infty A(F)\backslash A(\mathbb{A}))^*$ via Pontryagin duality. But

$$(Z^+_\infty A(F)\backslash A(\mathbb{A}))^*=(A(F)\backslash A(\mathbb{A})^1)^*\times \mathrm{Hom}(Z^+_\infty\backslash A^+_\infty,\mathbf{C}^*)$$

and $(A(F)\backslash A(\mathbb{A})^1)^*$ is discrete, $\mathrm{Hom}(Z^+_\infty\backslash A^+_\infty,\mathbf{C}^*)$ is a vector space over $\mathbf{C}$. Thus we can give $(Z^+_\infty A(F)\backslash A(\mathbb{A}))^*$ the structure of a complex manifold; as such, it has a natural measure which gives the measure 1 to the identity element of the Pontryagin dual of the compact abelian group $A(F)\backslash A^1(\mathbb{A})$; while the dual measure to da^1 gives the measure $1/\mathrm{vol}(A(F)\backslash A^1(\mathbb{A}))$ to the identity element.

The measure on A^+_∞ (resp. A'^+_∞, Z^+_∞) is fixed by identifying it with a power of $\mathbf{R}^\times$ by means of a basis of the lattice L_F (resp. ${}^1L_F$, ${}^0L^+_F$). Since $A^+_\infty=Z^+_\infty A'^+_\infty$, we see that the dual measure to da gives the measure $1/f$ to the identity element of ${}^1L_F$, where $f=[{}^1L_F\oplus{}^0L^+_F:L_F]/[{}^0L^+_F:{}^0L^-_F]$.

Now A'_∞ is identified with ${}^1\hat{L}_F\otimes\mathbf{R}$. Let $\{\mu_j\}$ be a basis of ${}^1L_F$ and let

$\{\hat{\mu}_k\}$ be a dual basis in ${}^1\hat{L}_F \otimes \mathbf{R}$ defined by $\langle \mu_j, \hat{\mu}_k \rangle = \delta_{jk}$. Take the Euclidean measure $d\lambda$ on ${}^1L_F \otimes \mathbf{R}$ to be the one induced by identification of ${}^1L_F \otimes \mathbf{R}$ with $\mathbf{R}^r$ via the basis $\{\mu_j\}$, where r is the rank of ${}^1L_F$. Suppose we change the basis of ${}^1L_F \otimes \mathbf{R}$, namely, we use the Euclidean measure $d\lambda^+$ with respect to ${}^1L_F^+ \otimes \mathbf{R}$. Then $d\lambda^+ = e\, d\lambda$ where $e = [{}^1L_F^+ : {}^1L_F]$. Choose a basis $\{\mu_j^+\}$ of ${}^1L_F^+$ such that $\langle \mu_j^+, \hat{\alpha}_k \rangle = \delta_{jk}$, where $\{\alpha_k\}$ is the set of simple F-roots. Let

$$\lambda : \mathbf{C}^r \to {}^1L_F \otimes \mathbf{C}$$

be the isomorphism defined by

$$\langle \lambda(s_1, \ldots, s_r), \hat{\alpha}_k \rangle = s_k, \quad 1 \le k \le r. \tag{5}$$

That is we identify ${}^1L_F \otimes \mathbf{C}$ with $\mathbf{C}^r$ via the basis $\{\mu_j^+\}$. Then $e\, d\lambda = ds_1, \ldots, ds_r$. Finally we remark that for Fourier inversion in Euclidean space, the dual measure to ${}^1\hat{L}_F \otimes \mathbf{R} \approx \mathbf{R}^r$ is $(2\pi i)^{-r}$ times the measure on ${}^1L_F \otimes \mathbf{R}$.

To summarize we have the following lemma.

6.7. Lemma: *The measure induced on* ${}^1L_F \otimes \mathbf{C}$ *by that of* $(Z_\infty^+ A(F) \backslash A(\mathbb{A}))^*$ *is*

$$ds_1 \ldots ds_r / c \operatorname{vol}(A(F) \backslash A^1(\mathbb{A}))(2\pi i)^r \tag{6}$$

where

$$c = ef = [L_F^+ : L_F]/[{}^0L_F^+ : {}^0L_F^-].$$

6.8. Remark: In the remainder of this section we essentially reproduce Langlands [15] in adelic form. We follow Rapoport [18] in the proofs of lemma 6.9 and 6.10.

6.9. Lemma: *All the local factors $M_v(w, \lambda(s))$ are holomorphic in s in an open half space of $\mathbf{C}^r$ containing the point* $(1, \ldots, 1)$.

Proof: Rewriting the formula (6) of §4 as

$$M_v(w, \lambda(s)) = \prod_{\alpha \in {}_0\Sigma_F^+(w)} M_v^{G_\alpha}(\langle \lambda(s), \hat{\alpha} \rangle) \tag{7}$$

we see that it is sufficient to consider the F-rank 1 case. And in this case, if ϕ is a locally constant function with compact support on F_v, then the integral of $\phi(\rho(a(\bar{n})))$ over $\bar{N}_V$ exists.

Thus there exists a non-negative measure $\mathrm{d}\mu$ on F_v such that

$$\int_{\bar{N}_v} \phi(\rho(a(\bar{n}))\,\mathrm{d}\bar{n} = \int_{F_v} \phi(t)\,\mathrm{d}\mu$$

for all reasonable functions ϕ on F_v. In particular, for $\phi: t \to |t|^{s+1}$ ($\mathrm{Re}\, s > t$), we get

$$M_v(s) = \int_{F_v} |t|^{s+1}\,\mathrm{d}\mu.$$

That is $M_v(s)$ is the Mellin transform of a non-negative measure and is continuous at 1 (§5). 6.9 now results from a variant of Landau's lemma.

6.10. Lemma: *$M(w, \lambda(s))$ is meromorphic in s. There exists a positive number ϵ such that the only singularities of $M(w, \lambda)$ in the region $1-\epsilon < \mathrm{Re}\, s_i < 1+\epsilon$ $(i = 1, \ldots, r)$ are simple poles in the hyperplane $s_i = 1$ for i corresponding to a simple positive root in ${}_0\Sigma_F^+(w)$.*

Proof: By the preceding lemma, we can leave out a finite number of factors $M_v(w, \lambda)$ from $M(w, \lambda)$. In the relative rank 1 case, up to a finite number of factors, there are four cases:

(I) $M(s) = \dfrac{\zeta_F(s)}{\zeta_F(s+1)}$

(II) $M(s) = \zeta_F(2s) \displaystyle\prod_{v<\infty} \frac{(1-|\tilde{\omega}_v|_{F_v}^{2(s+1)})(1+|\tilde{\omega}_v|_{F_v}^{2s+1})}{(1+|\tilde{\omega}_v|_{F_v}^{2s})}$

(III) $M(s) = \dfrac{\zeta_E(s)}{\zeta_E(s+1)}$

(IV) $M(s) = \zeta_E(2s) \displaystyle\prod_{v<\infty} \frac{(1-|\tilde{\omega}_v|_{E_v}^{2(s+1)})(1+|\tilde{\omega}_v|_{E_v}^{2s+1})}{(1+|\tilde{\omega}_v|_{E_v}^{2s})}$

where ζ_F (resp. ζ_E) is the Dedekind zeta function of F (resp. E). It is clear that in the cases (I) and (III) $M(s)$ has a simple pole at $s = 1$ and in cases (II) and (IV) $M(s)$ is holomorphic in an open half-space of $\mathbb{C}$ containing 1. The higher rank case now follows immediately from (7).

6.11. Proposition: *For Φ, $\Psi \in \mathcal{H}$, we have*

(8)
$$(E\tilde{\phi}, \tilde{\psi}) = \frac{\kappa\mu_A}{\mu_G c\tau(A)} \lim_{s\to 1} \frac{L(s,G)}{L(s,A)} M(w_0, s\rho)\Phi(s\rho)\bar{\Psi}(\bar{s}\rho)$$

where $w_0 \in W_F$ is the unique element which sends all the positive roots to negative roots.

First we introduce some functions:

$$f_r(w; s) = M(w, \lambda(s))\Phi(\lambda(s))\bar{\Psi}(-{}^w\overline{\lambda(s)})$$

$$f_q(w; s_1, \ldots, s_q) = \operatorname*{Res}_{s_{q+1}=1} f_{q+1}(w; s_1, \ldots, s_{q+1}) \text{ for } 0 \le q \le \tau - 1$$

$$Q_r(s) = (\lambda(s), \lambda(s))$$

$$Q_q(s_1, \ldots, s_q) = Q_r(s_1, \ldots, s_q, 1, \ldots, 1).$$

We also write s^q for $(s_1, \ldots, s_q)$.

6.12. Lemma: (i) *For $0 \le q \le r$, the functions $f_q(w, s^q)$ are meromorphic in all the s^q-spaces. In the region*

$$\{s^q \mid \operatorname{Re} s_i > 1, 1 \le i \le q\}$$

$f_q(w, s^q)$ is holomorphic, goes to zero faster than the inverse of all polynomials as the imaginary past of s^q goes to infinity and the real part stays in a compact subset of this region.

(ii) *There exists a positive number ϵ* such that the only singularities of $f_q(w; s^q)$ in the region

$$\{s^q \mid 1-\epsilon < \operatorname{Re} s_i < 1+\epsilon; i = 1, \ldots, q\}$$

are simple poles lying the hyperplane $s_i = 1$.

Proof: (i) is just a restatement of the corresponding property of property of $M(w, \lambda)$ which is a consequence of the global theory of Eisenstein series (cf. [14]). (ii) follows from lemma 6.10.

6.13. It follows from §6.4 and 6.5 that

(9)
$$c\ \mathrm{vol}(A(F)\backslash A^1(\mathbb{A}))(E\tilde{\phi}, \tilde{\psi})$$
$$= \lambda \sum_{w\in W_F} \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{C(a,b,c,\epsilon)} \left\{ \frac{1}{(2\pi i)^r} \int_{\mathrm{Re}\, s = s_0} \frac{f_r(w; s)}{z - Q_r(s)} \mathrm{d}s_1 \ldots \mathrm{d}s_r \right\} \mathrm{d}z$$

provided each of these limits exists. We shall show by induction that there exists the limit

(10)
$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{C(a,b,c,\epsilon)} \mathrm{d}z \left\{ \frac{1}{(2\pi i)^q} \times \int_{\mathrm{Re}\, s^q = s_0^q} \frac{f_q(w; s^q)}{z - Q_q(s^q)} \mathrm{d}s_1 \ldots \mathrm{d}s_q \right\}$$

if $s_0^q = s(s_{0,1}, \ldots, s_{0,q})$ with $s_{0,i} > 1$, $1 \le i \le q$. Note that analyticity implies that expression is independent of the actual value of s_0^q, provided its coordinates are strictly greater than one.

Take two small positive real numbers u, and v such that u is much smaller than v. Set $s_0^q = (1+u, \ldots, 1+u, 1+v)$ and $s_0^{q-1} = (1+u, \ldots, 1+u)$. Then $Q_q(1+u, \ldots, 1+u, 1-v) < (\rho, \rho)$. Pick b such that $Q(1+u, \ldots, 1+u, 1-v) < b < (\rho, \rho)$. Then, we can find a constant τ such that if either

$$\begin{cases} \mathrm{Re}\, s_i = 1+u, & 1 \le i \le q-1 \\ \mathrm{Re}\, s_q = 1-v \end{cases}$$

or

$$\begin{cases} \mathrm{Re}\, s_i = 1+u, \quad 1 \le i \le q-1 \\ 1-v \le \mathrm{Re}\, s_q \le 1+v \\ |\mathrm{Im}\, s_q| \ge \tau \end{cases}$$

then

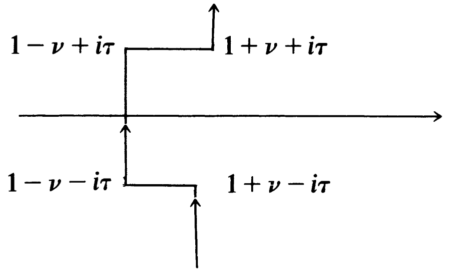
$$\mathrm{Re}\, Q_q(s^q) < b - \frac{1}{\tau}.$$

We integrate

$$\frac{1}{(2\pi i)^q} \int_{\mathrm{Re}\, s^q = s_0^q} \frac{f_q(w; s^q)}{z - Q_q(s^q)} \mathrm{d}s_1 \ldots \mathrm{d}s_q$$

first with respect to s_q; we change the contour $\mathrm{Re}\, s_q = s_{0,q}$ to

$C^q(\nu, \tau)$:



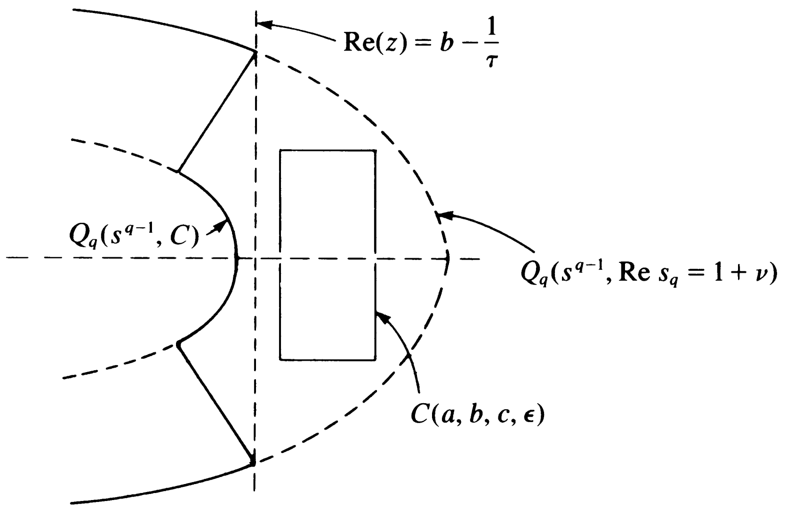
The result is

$$\frac{1}{(2\pi i)^{q-1}} \int_{\mathrm{Re}\, s^{q-1} = s_0^{q-1}} \frac{f_{q-1}(w; s^{q-1})}{z - Q_{q-1}(s^{q-1})} \mathrm{d}s_1 \ldots \mathrm{d}s_{q-1}$$

plus

$$\frac{1}{(2\pi i)^q} \int_{\mathrm{Re}\, s^{q-1} = s_0^{q-1}} \left\{ \int_{C^q(\nu,\tau)} \frac{f_q(w; s^{q-1})}{z - Q_q(s^q)} \mathrm{d}s_q \right\} \mathrm{d}s_1 \ldots \mathrm{d}s_{q-1}.$$

For s^{q-1} fixed and s_q in $C^q(\nu, \tau)$, the image in the Z-plane of $C = C^q(\nu, \tau)$ under Q_q is given in the following diagram



It follows that for $\mathrm{Re}\, s^{q-1} = s_0^{q-1}$ and $s_q \in C$ the function $1/(z - Q_q(s^q))$ is holomorphic in a region containing $C(a, b, c, \epsilon)$ such that

$$\lim_{\epsilon \to 0^+} \int_{C(a,b,c,\epsilon)} \frac{\mathrm{d}z}{z - Q_q(s^q)} = 0$$

and (10) becomes

$$\lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{C(a,b,c,\epsilon)} dz \frac{1}{(2\pi i)^{q-1}} \times \int_{\operatorname{Re} s^{q-1}=s_0^{q-1}} \frac{f_{q-1}(w; s^{q-1})}{z - Q_{q-1}(s^{q-1})} ds_1 \ldots ds_{q-1}.$$

Finally, we get, for $q = 0$

$$\lim_{\epsilon\to 0^+} \frac{f_0(w)}{2\pi i} \int_{C(a,b,c,\epsilon)} \frac{dz}{z - (\rho, \rho)} = f_0(w).$$

But it follows from lemma 6.10 that $f_0(w)$ is zero unless $w = w_0$ and w_0 takes ρ to $-\rho$. We have

$$f_0(w_0) = \lim_{s\to 1} (s-1)^r M(w_0, s\rho)\Phi(s\rho)\bar{\Psi}(w_0(-\bar{s}\rho)).$$

Hence

$$(E\tilde{\phi}, \tilde{\psi}) = \frac{\kappa \lim_{s\to 1} (s-1)^r M(w_0, s\rho)\Phi(s\rho)\bar{\Psi}(\bar{s}\rho)}{c \operatorname{vol}(A(F)\backslash A^1(\mathbb{A}))}$$

and (8) now follows from Ono's formula for Tamagawa number of the torus A (cf. [17]).

Using the formula

$$M(w_0, \lambda) = M_{\mathcal{S}}(w_0, \lambda) \prod_{v \notin \mathcal{S}} M_v(w_0, \lambda)$$

and the result in §5 for the values of M, we see immediately that

$$(E\tilde{\phi}, \tilde{\psi}) = \kappa^2 (c\tau(A))^{-1} \Phi(\rho)\bar{\Psi}(\rho). \tag{9}$$

7. Computation of Tamagawa number

7.1. THEOREM: *Let G be a connected reductive quasi-split group defined over an algebraic number field F. Let A be a maximal torus of G defined over F lying inside the Borel subgroup of G defined over F. Then*

$$\tau(G) = c\tau(A)$$

where $\tau(G)$ (resp. $\tau(A)$) denotes the Tamagawa number of G (resp. A), and $c = [L_F^+ : L_F]/[{}^0L_F^+ : {}^0L_F^-]$.

PROOF: In the Hilbert space $\mathscr{L}^2(Z_\infty^+ G(F)\backslash G(\mathbb{A}))$ we have

$$(\tilde{\phi}, 1)(1, \tilde{\psi}) = (1, 1)(\mathscr{P}\tilde{\phi}, \mathscr{P}\tilde{\psi}). \tag{1}$$

According the last formula of §6, the dimension of the image of E is at most one. As we have already pointed out that the constant functions are in the image of E, we get $E = \mathscr{P}$ and so

$$(\mathscr{P}\tilde{\phi}, \mathscr{P}\tilde{\psi}) = \kappa^2(c\tau(A))^{-1}\Phi(\rho)\bar{\Psi}(\rho).$$

Since $(\tilde{\phi}, 1) = \kappa\Phi(\rho), (1, \tilde{\psi}) = \kappa\bar{\Psi}(\rho)$ and $\tau(G) = (1, 1)$ the theorem is proved.

7.2. Weil conjectured that the Tamagawa number of a semi-simple simply-connected connected algebraic group is one [17]. This conjecture holds for all classical groups ($\neq {}^3D_4, {}^6D_4$) (Tamagawa, Weil, Mars), for some exceptional groups (Mars, Demazure) and for Chevalley groups (Langlands), but it is not yet completely solved. We shall show that the Weil conjecture is true for simply-connected connected semi-simple quasi-split group G. This in fact follows immediately from our formula

$$\tau(G) = c\tau(A)$$

where A is a maximal torus of G.

First, we observe that G is simply-connected implies $L_F^+ = L_F$, i.e. $c = 1$; and the representation of the Galois group in the lattice of weights in a direct sum of permutation representation. Thus by duality theory of algebraic tori, we have

$$A \approx \prod_{i=1}^{n} R_{E_i/F}(G_m)$$

where E_i are finite separable extension of F which is the field of definition of G, and G_m is the 1-dimensional multiplicative group. Now we have (by Ono [17])

$$\tau_F(A) = \prod_{i=1}^{n} \tau_F(R_{E_i/F}(G_m)) = \prod_{i=1}^{n} \tau_{E_i}(G_m) = 1,$$

because $\tau(G_m) = 1$ (which follows from the value of the residue of zeta function ζ_E at 1).

Thus by the formula of the preceeding subsection $\tau(G) = c\tau(A) = 1$ for a simply-connected semi-simple quasi-split connected algebraic group.

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