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SOME REMARKS ON MINIMAL MODELS I

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Introduction

Let X be a complete non singular algebraic variety, over an algebraically closed field of any characteristic.

We say that X is a relatively minimal model if any birational morphism $f: X \to Y$, over a non singular algebraic variety Y, is actually an isomorphism.

We say also that X is an (absolutely) minimal model if any birational map $g: Y \to X$, where Y is a non singular variety, is actually a morphism. After the Hironaka's main theorem it is known that any birationality class of algebraic varieties has at least a relatively minimal model.

When is such a model an (absolutely) minimal model?

In the theory of surfaces there is a complete answer to this problem: (Castelnuovo–Enriquez–Zariski): Any birationality class of surfaces which is not related to the class of ruled surfaces (i.e. which does not contain a surface of the form $C \times \mathbb{P}^1$, where C is an algebraic curve) has an (absolutely) minimal model.

In this paper we give two examples in order to show that a similar theorem does not hold for 3-dimensional varieties. Our technique is essentially due to Moisezon (see [7]), who has given an example of a not algebraic relatively minimal model using the same method. However we must proceed very cautiously because in this case the models are requested to be algebraic varieties.

Although the following two examples are built using the same method, they show different pathologies.

The first example gives two relatively minimal models X and Y of

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Kodaira dimension >0. Even if they are different, the two models have a big open set U (i.e. codim $X\backslash U$ = codim $Y\backslash U > 1$) where they are isomorphic, so the rank of the Neron–Severi group of X is equal to the rank of the Neron–Severi group of Y.

The second example gives two relatively minimal birationally isomorphic models. The first of them, X, has an ample canonical system. The second one, Y, has a canonical system with fixed components and base curves. So we have rg $NS(X) <$ rg $NS(Y)$.

In this paper we denote by threefold a non singular algebraic three dimensional variety defined over an algebraically closed field of arbitrary characteristic.

1

In this section we recall some definitions and some standard lemmas. Let X be a threefold and let Y be a non singular subvariety of X. We consider the blow up diagram:

$$\begin{array}{ccc} P & \xrightarrow{j} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \sigma} \\ Y & \xrightarrow{i} & X \end{array}$$

where P denotes the exceptional divisor. Then we have:

LEMMA 1.1: *Denoting by $\mathcal{I}_Y$ and $\mathcal{I}_P$ the ideals of Y and P respectively, and by $\mathcal{O}_P(1)$ the tautological sheaf on P, one has $P = \mathbf{P}(i^*\mathcal{I}_Y)$ and $j^*\mathcal{I}_P \simeq \mathcal{O}_P(1)$.*

PROOF: See [2] p. 186.

DEFINITION: Let Z be a subvariety of X, Z not contained in Y. The proper transform $\bar{Z}$ of Z is the Zariski closure in X of $\sigma^{-1}(Z \cap (X\backslash Y))$.

LEMMA 1.2: *Suppose* codim $Z = 1$. *Then we have $\sigma^*(Z) = \bar{Z} + sP$ where $s = \min_{x\in Y} s_x$ and $s_x = \{\max_{s\in\mathbf{N}} s$ such that $z \in \mathfrak{m}_x^s$, z local equation of Z at x.$\}$*

PROOF: See [4] p. 142.

COROLLARY 1.3: *Under the same hypothesis as* 1.2 *we get*:

$$N_{\bar{Z}}^{\bar{X}} \simeq (\sigma \mid \bar{Z})^* N_Z^X \otimes \mathcal{O}_{\bar{Z}}(-p)^{\otimes s}$$

where $N_{\bar{Z}}^{\bar{X}} \simeq Hom(\mathcal{I}_{\bar{Z}}/\mathcal{I}_{\bar{Z}}^2, \mathcal{O}_{\bar{X}})$.

PROOF: It follows from 1.2.

COROLLARY 1.4: *Suppose* $\dim Y = \operatorname{codim} Z = 1$, $Y \subset Z$. *We denote by* F *a fiber of the morphism* $p : P \to Y$. *Then we have*:

$$(\bar{Z} \cdot P) \overset{a}{\sim} (Z \cdot Y)F + c_1(\mathcal{O}_P(s)).$$

PROOF: From 1.2 we obtain: $\sigma^* \mathcal{O}_X(Z) \otimes \mathcal{O}_P \simeq \mathcal{O}_P(\bar{Z}) \otimes \mathcal{O}_P(sP)$. By 1.1 it follows $\mathcal{O}_P(sP) \simeq \mathcal{O}_P(-s)$. Moreover $(\sigma \circ j)^* \mathcal{O}_X(Z) = p^* \mathcal{O}_Y(Z)$ so we have: $\mathcal{O}_P(\bar{Z}) \simeq p^* \mathcal{O}_Y(Z) \otimes \mathcal{O}_P(s)$.

Calculating the Chern classes we conclude.

Now we consider the surfaces $\mathbf{F}_n = \operatorname{Proj}(\mathcal{O}_{\mathbf{P}^1}(n) \oplus \mathcal{O}_{\mathbf{P}^1})$ and its canonical projection $p : \mathbf{F}_n \to \mathbf{P}^1$. Then we have:

LEMMA 1.5: *There exist two standard sections* A, B *on* $\mathbf{F}_n$ *such that*:

(i) $A \equiv B + nF$, F *fiber of* $p : \mathbf{F}_n \to \mathbf{P}^1$,

(ii) $A^2 = n$, $A \cdot B = 0$, $B^2 = -n$,

(iii) $c_1(\mathcal{O}_{\mathbf{F}_n}(1)) \equiv A$.

PROOF: For (i), (ii) see [9] p. 153, while (iii) follows from [6] Lemma 6.1.

LEMMA 1.6: *There exists a canonical* $\mathbf{P}^1$*-isomorphism* $f : \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(p) \oplus \mathcal{O}_{\mathbf{P}^1}(q)) \to \mathbf{F}_{p-q}$. *Moreover* $f^* \mathcal{O}_{\mathbf{F}_{p-q}}(1)$ *is canonically isomorphic to* $\mathcal{O}_{\mathbf{P}(\mathcal{O}(P) \oplus \mathcal{O}(q))}(1) \otimes_{\mathcal{O}_{\mathbf{P}^1}} \mathcal{O}_{\mathbf{P}^1}(-q)$.

PROOF: See E.G.A. II, 4.1.4.

We still denote, for the sequel, by A and B the divisors f^*A and f^*B on $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(p) \oplus \mathcal{O}_{\mathbf{P}^1}(q))$.

2

In order to construct the examples, we shall study two special birational maps.

Let X be a threefold, and let C be a curve on X such that:

(i) $C \cong \mathbf{P}^1$,

(ii) $N_C^X \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$.

We take the blow up of X along C:

$$\begin{array}{ccc} Q & \xrightarrow{j} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \sigma} \\ C & \xrightarrow{i} & X \end{array}$$

By 1.1 we have $Q \simeq \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1))$. Since $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(1) \oplus \mathcal{O}_{\mathbf{P}^1}(1))$ is isomorphic to the Segre product $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$, we can consider $p : Q \to C$ as the projection $p_1 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ on the first factor. In this case, by 1.5(ii), sections A, B occur as fibers of the projection $p_2 : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ on the second factor; so, from 1.5(iii), 1.6, the tautological sheaf $\mathcal{O}_Q(1)$ is isomorphic to $\bigotimes_{i=1}^{2} p_i^* \mathcal{O}_{\mathbf{P}^1}(1)$. Therefore, we can use $j^* \mathcal{I}_Q \simeq \mathcal{O}_Q(1)$ and Artin criterion [1] to define a morphism $\tau : Z \to Y$ in the category of algebraic spaces, which is an isomorphism on $Z \backslash Q$ and such that the restriction to Q is the projection on the second factor.

By construction, the map $\tau \circ \sigma^{-1}$ is a birational non regular transformation between two non singular three-dimensional algebraic spaces X and Y, which is defined outside two non singular curves C and $C' = \tau(Q)$ on X and Y respectively.

Now we get a criterion of algebraicity for the algebraic space Y:

PROPOSITION 2.1: *Y is an algebraic (projective) variety if there exists a contraction of C:*

$$\begin{array}{ccc} C & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle h} \\ \{P\} & \longrightarrow & \tilde{X} \end{array}$$

such that $\tilde{X}$ is an algebraic (projective) variety.

PROOF: Take an affine neighborhood U of $\{P\}$. We put $V = (h \circ \sigma)^{-1}(U)$, $\mathcal{F} = \sigma^* \mathcal{L}^{-1} \otimes \mathcal{I}_Q^n$ where $\mathcal{L}$ is an invertible sheaf on X and $\deg \mathcal{L} \otimes \mathcal{O}_C = n > 0$.

We are going to prove that $H^1(V, \mathcal{F}^{\otimes S} \otimes \mathcal{I}_Q) = 0$, where $s > 0$; for it suffices to show that $R^1(h \circ \sigma)_* \mathcal{F}^{\otimes S} \otimes \mathcal{I}_Q = 0$.

Now the exact sequence:

$$0 \longrightarrow \mathcal{F}^{\otimes S} \otimes \mathcal{I}_Q^{r+1} \longrightarrow \mathcal{F}^{\otimes S} \otimes \mathcal{I}_Q^{r} \longrightarrow \mathcal{F}^{\otimes S} \otimes \mathcal{I}_Q^{r} \otimes \mathcal{O}_Q \longrightarrow 0$$

gives:

$$R^1(h\circ\sigma)_*\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q^{r+1}\longrightarrow R^1(h\circ\sigma)_*\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q^{r}\longrightarrow$$
$$\longrightarrow H^1(Q,\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q^{r}\otimes\mathcal{O}_Q).$$

It is easily seen that $H^1(Q,\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q^{r}\otimes\mathcal{O}_Q)=H^1(Q,p_1^*\mathcal{O}_Q(r)\otimes p_2^*\mathcal{O}_Q(r+sn))=0$, with $r>0$.

Moreover, since $\mathcal{I}_Q$ is $h\circ\sigma$-ample by E.G.A. III 4.7.1, we have (E.G.A. III 2.2.1):$R^1(h\circ\sigma)\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q^{r+1}=0$ for $r\gg 0$. By induction we get $R^1(h\circ\sigma)\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q=0$, hence we have the exact sequence:

$$0\longrightarrow\Gamma(V,\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q)\longrightarrow\Gamma(V,\mathcal{F}^{\otimes S})\xrightarrow{\rho}\Gamma(Q,p_2^*\mathcal{O}(ns))\longrightarrow 0.$$

Therefore we can lift any neighborhood $W=Q-\mathrm{supp}(s)$, $s\in\Gamma(Q,p_2^*\mathcal{O}(ns))$ of the fibers of p_2 to the open subset $\bar{W}=V-\mathrm{supp}(s^*)$, where $s^*\in\Gamma(V,\mathcal{F}^{\otimes S})$ and $\rho(s^*)=s$.

To prove that Y is an algebraic variety it suffices to show (see [5]) that the morphisms:

$$\Gamma(\bar{W},\mathcal{O}_W)\longrightarrow\Gamma(W,\mathcal{O}_Q)$$
$$\Gamma(\bar{W},\mathcal{I}_Q)\longrightarrow\Gamma(W,\mathcal{O}_Q(1))$$

are surjective for all $\bar{W}$. Taking in account that $\mathcal{F}_{|\bar{W}}\simeq\mathcal{O}_{\bar{W}}$ and $H^1(V,\mathcal{F}^{\otimes S}\otimes\mathcal{I}_Q^{r})=0$ for $r>0$, this follows from the following:

LEMMA 2.2: *Let U be an open subset of an algebraic variety X. Let D be an effective divisor and $V=U\backslash D$. Then for every coherent sheaf $\mathcal{F}$ we have $H^1(V,\mathcal{F})=0$ whenever $H^1(U,\mathcal{F}\otimes\mathcal{O}(nD))=0$, $n\gg 0$.*

PROOF: Take a 1-cocicle $\{a_{ij},U_{ij}\cap V\}\in C^1(V,\mathcal{F})$, where the U_{ij} are affine open sets. We have $a_{ij}\in\Gamma(U_{ij},\mathcal{F}\otimes\mathcal{O}(n_{ij}D))$. Let $N=\max\{n_{ij}\}$. We can suppose $a_{ij}\in\Gamma(U_{ij},\mathcal{F}\otimes\mathcal{O}(ND))$ for any pair (i,j) of the covering. So $\{a_{ij},U_{ij}\}$ is a 1-cocicle for $C^1(U,\mathcal{F}\otimes\mathcal{O}(ND))$. Thus, by hypothesis, it is a trivial cocicle, and so is $\{a_{ij},U_{ij}\cap V\}$.

Let now X be a projective variety. Then we can take on Z the sheaf:

$$(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{I}_Q.$$

We claim: $H^1(Z,(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{I}_Q)=0$ if $n\gg 0$: for the Leray spectral sequence we have:

$$H^1(\tilde{X},\mathcal{O}_{\tilde{X}}(n)\otimes(h\circ\sigma)_*\mathcal{F}\otimes\mathcal{I}_Q)\longrightarrow H^1(Z,(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{I}_Q)$$
$$\longrightarrow H^0(\tilde{X},R^1(h\circ\sigma)_*((h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{I}_Q))$$

The first cohomology group is equal to zero if $n \geq n_0$ for E.G.A. III, 2.2.1; moreover the vanishing of the last cohomology group of the sequence follows by use of the same arguments as previously.

So we get the exact sequence for $n \geq n_0$:

$$0 \longrightarrow H^0(Z, (h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{I}_Q) \longrightarrow H^0(Z, (h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}) \longrightarrow H^0(Q, (h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{O}_Q) \longrightarrow 0$$

and $H^0(Q, (h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}\otimes\mathcal{O}_Q) = H^0(Q, p_2^*\mathcal{O}_{\mathbf{P}^1}(n))$. Then the linear system $|(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}|$ is base points free on Q.

On the other hand the sheaf $(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}$ is generated by its global sections outside Q if $n \geq n_1$. In fact we have the isomorphism:

$$Z\backslash Q \xrightarrow{\sim} \tilde{X}\backslash\{P\}$$

so outside Q the sheaf $(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(n)\otimes\mathcal{F}$ can be looked as $\mathcal{O}_{\tilde{X}}(n)\otimes(h\circ\sigma)_*\mathcal{F}$ thus the claim follows from E.G.A. III 2.2.1.

Let us assume now $N = \max\{n_0, n_1\}$; then $(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(N)\otimes\mathcal{F}$ is generated by its global sections on Z. Therefore $|(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(N+1)\otimes\mathcal{F}|$ separates points outside Q, and cuts the linear system $|p_2^*\mathcal{O}_{\mathbf{P}^1}(n)|$ on Q.

Thus the morphism induced by the linear system $|(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(N+1)\otimes\mathcal{F}|$ is a contraction of Q on the second factor. Finally, by the unicity of the contraction, it follows that the projective image of $|(h\circ\sigma)^*\mathcal{O}_{\tilde{X}}(N+1)\otimes\mathcal{F}|$ is isomorphic to Y.

By use of the previous construction, we get the following second map: let X be a threefold, and let C be a curve on X with the properties:

(a) $C \simeq \mathbf{P}^1$,

(b) $N_C^X \simeq \mathcal{O}_{\mathbf{P}^1}(-2)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$.

Take the blow up of X along C:

$$\begin{array}{ccc} P & \longrightarrow & \tilde{X} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \rho} \\ C & \longrightarrow & X. \end{array}$$

Let B be the section of negative self-intersection on P. We shall prove that $N_B^{\tilde{X}} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$: by 1.5, 1.6, we have $c_1(N_B^P) = -1$, $c_1(N_P^{\tilde{X}}) = c_1(\mathcal{O}_{\mathbf{P}}(-1)) = -B - 2F$, where F is the fiber of $q: P \to C$. From the exact sequence:

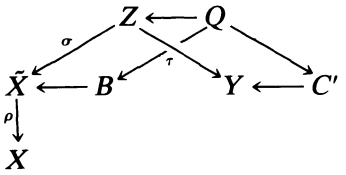
$$0 \longrightarrow N_B^P \longrightarrow N_B^{\tilde{X}} \longrightarrow N_{P|B}^{\tilde{X}} \longrightarrow 0$$

we get:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow N_B^{\tilde{X}} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow 0.$$

It follows $N_B^{\tilde{X}} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, since $\mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$.

Therefore section B satisfies the hypothesis (i), (ii); thus, reasoning on X as previously, we obtain the following diagram:



By construction $f = \tau \circ \sigma^{-1} \circ \rho^{-1}$ is an isomorphism on $X \backslash C$. We shall study the exceptional divisor of f on Y: let $\bar{P}$ be the proper transform of P on Z. By 1.4 we have $(\bar{P} \cdot Q) = (P \cdot B)F + c_1(\mathcal{O}_Q(1))$. On the other hand we recall that $\mathcal{O}_Q(1) \simeq \bigotimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbf{P}^1}(1)$, and $(P \cdot B) = -1$, so $(\bar{P} \cdot Q)$ is a fiber of the projection of Q on the second factor. Therefore τ induces on $\bar{P}$ the contraction of the section B to a point, and it is easy to see that the image of P is a plane E on Y.

Now lemma 1.3 gives the relation:

$$\tau^* N_{E|\bar{P}}^Y \simeq \sigma^* N_{P|\bar{P}}^{\tilde{X}} \otimes \mathcal{O}_{\bar{P}}(-Q),$$

and from 1.5, 1.6 it follows that $\sigma^* N_{P|\bar{P}}^X \simeq \mathcal{O}_{\bar{P}}(-B - 2F)$. So $\tau^* N_E^Y \simeq \mathcal{O}_{\bar{P}}(-2(B+F))$ and $N_E^Y \simeq \mathcal{O}_{\mathbf{P}^2}(-2)$.

Also in this case we get a criterion of algebraicity for Y:

PROPOSITION 2.3: *Y is an algebraic variety (projective) if there exists a contraction of C:*

$$\begin{array}{ccc} C & \longrightarrow & X \\ \downarrow & & \downarrow h \\ \{P\} & \longrightarrow & X' \end{array}$$

such that X' is an algebraic variety (projective).

PROOF: We have only to prove that the couple $(\tilde{X}, B)$ satisfies the hypothesis of the proposition 2.1.

Take an affine neighborhood U of the point $\{P\}$. We put $V = (h\circ\rho)^{-1}(U)$, $\mathcal{F} = \rho^*\mathcal{L}^{-1}\otimes\mathcal{I}_P^n$, where $\mathcal{L}$ is an invertible sheaf on X and $\deg\mathcal{L}\otimes\mathcal{O}_C = n > 0$. We have $\mathcal{F}\otimes\mathcal{I}_P^r\otimes\mathcal{O}_P \simeq \mathcal{O}_P((n+r)B + (n+2r)F)$, hence $H^1(P, \mathcal{F}\otimes\mathcal{I}_P^r\otimes\mathcal{O}_P) = 0$ for $r > 0$. Moreover $\mathcal{I}_P$ is $h\circ\rho$-ample. Hence, as in proposition 2.1, we have the exact sequence:

$$(*) \qquad \Gamma(V, \mathcal{F}) \longrightarrow \Gamma(P, \mathcal{O}_P(nB + nF)) \longrightarrow 0.$$

On the other hand the rational map associated to the linear system $|\mathcal{O}_P(nB + nF)|$ contracts B to a point. Therefore we can take a section $s \in \Gamma(P, \mathcal{O}_P(nB + nF))$ such that $W^* = P - \operatorname{supp}(s)$ is a complete neighborhood of the section B. By (*) it is possible to lift W^* to an open subset W of X such that $\mathcal{F}_{|W} \simeq \mathcal{O}_W$ on W. Then we consider the morphism $W \to \operatorname{Spec}\Gamma(W, \mathcal{O}_W)$, induced from the isomorphism of the ring of regular section. We want to show that this is the required contraction.

For it suffices to see that the sections of $\Gamma(W, \mathcal{O}_W)$ separate points outside B, or equivalently $H^1(W, \mathcal{I}_x\otimes\mathcal{I}_y) = 0$ for all not necessary distinct points $x, y \in W\backslash B$.

From the exact sequence:

$$0 \longrightarrow \mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y\otimes\mathcal{I}_P^{r+1} \longrightarrow \mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y\otimes\mathcal{I}_P^{r} \longrightarrow$$
$$\longrightarrow \mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y\otimes\mathcal{I}_P^{r}\otimes\mathcal{O}_P \longrightarrow 0$$

we have:

$$R^1(h\circ\rho)_*\mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y\otimes\mathcal{I}_P^{r+1} \longrightarrow$$
$$R^1(h\circ\rho)_*\mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y\otimes\mathcal{I}_P^{r} \longrightarrow$$
$$\longrightarrow H^1(P, \mathcal{O}_P((sn+r)B + (sn+2r)F)).$$

It is easy to see that $H^1(P, \mathcal{O}_P((sn+r)B + (sn+2r)F) = 0$, hence $R^1(h\circ\rho)_*\mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y = 0$ by descending induction on r; therefore $H^1(V, \mathcal{F}^{\otimes S}\otimes\mathcal{I}_x\otimes\mathcal{I}_y) = 0$.

We can conclude by using lemma 2.2.

Suppose now that X is projective. To prove the proposition in this case we have to show that the map associated to the linear system $|(h\circ\rho)^*\mathcal{O}_{X'}(n)\otimes\mathcal{F}|$ or the map associated to a sufficiently large multiple of this system is the required contraction. This follows by using the same argument of the proposition 2.1.

3

In order to construct the counterexamples as we mentioned in the introduction we take a threefold X such that:

(1) X is a relatively minimal model,

(2) $P_n(X) \neq 0$ for some $n > 0$,

(3) X contains a curve C satisfying the properties:

(i) $C \simeq \mathbf{P}^1$,

(ii) $N_C^X \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-n)$, $n = 1, 2$,

(iii) there exist a projective variety $\tilde{X}$ and a morphism $h: X \to \tilde{X}$, which is the contraction of C.

Let now Y be the projective model obtained from X as in the previous section. We have to check that Y is a relatively minimal model.

We shall prove the existence of threefolds satisfying the properties (1), (2), (3).

Henceforth we assume $k = \mathbf{C}$.

$n = 1$ Take the hypersurface F in $\mathbf{A}^4$ defined by the equation:

$$XG_1(XYZT) + TG_2(X..T) + XH_1(X..T) + TH_2(X..T),$$

where G_i are homogeneous forms of degree two, and $H_i \in \mathfrak{m}_{(0)}^3$, so that F is non singular outside the origin O. Thus the hypersurface F has an isolated triple point in the origin O, and it contains the plane $X = T = 0$.

Let $V \subset \mathbf{P}^4$ be the projective closure of F, and let $\tilde{\sigma}: \tilde{V} \to V$ be the blow up of the point O. We have the diagram:

$$\begin{array}{ccccc} S & \longrightarrow & \tilde{V} & \longrightarrow & P \\ \downarrow & & \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle \sigma} \\ \{O\} & \longrightarrow & V & \longrightarrow & \mathbf{P}^4 \end{array}$$

We shall prove that $\tilde{V}$ satisfies the properties (1), (2), whenever the degree m of F is big enough.

Let $E = \mathbf{P}^3$ be the exceptional divisor of the blow up $\sigma: P \to \mathbf{P}^4$. We have $\Omega_P \simeq \sigma^* \mathcal{O}_{\mathbf{P}^4}(-5) \otimes \mathcal{O}_P(3E)$, hence, by adjunction formula:

$$\Omega_{\tilde{V}} \simeq \sigma^* \mathcal{O}_{\mathbf{P}^4}(-5) \otimes \mathcal{O}_P(3E) \otimes \mathcal{O}_P(\tilde{V}) \otimes \mathcal{O}_{\tilde{V}}.$$

Since $\tilde{V}$ is the proper transform of V in P, we get $\mathcal{O}_P(\tilde{V}) \simeq \mathcal{O}_P(-3E) \otimes \sigma^* \mathcal{O}_{\mathbf{P}^4}(m)$ using 1.2 and the fact that $F \in \mathfrak{m}_{(0)}^3$. In con-

clusion we have $\Omega_{\tilde{V}} \simeq \sigma^* \mathcal{O}_{\mathbf{P}^4}(m-5) \otimes \mathcal{O}_{\tilde{V}}$, therefore the canonical sheaf Ω_V is effective and generated by its global sections for $m > 5$. So $\tilde{V}$ is a relatively minimal model[1] of positive genus. Now let ℓ be the intersection line of the cubic S with the proper transform L on the plane $X = T = 0$. We have $N_\ell^S \simeq \mathcal{O}_{\mathbf{P}^1}(-1)$ since ℓ is an exceptional line on the cubic S, and $N_S^{\tilde{V}} \simeq \mathcal{O}_S(-1)$ since S is the exceptional divisor of the blow up $\tilde{\sigma}$, therefore $N_{S|\ell}^{\tilde{V}} \simeq \mathcal{O}_{\mathbf{P}^1}(-1)$. So we get the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow N_\ell^{\tilde{V}} \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow 0.$$

Whence $N_\ell^{\tilde{V}} \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, since $\mathrm{Ext}^1(\ell, \mathcal{O}_{\mathbf{P}^1}(-1), \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$. If $\mathcal{L}(n)$ denotes the sheaf $\sigma^* \mathcal{O}_V(n) \otimes \mathcal{O}_{\tilde{V}}(L) \otimes \mathcal{O}_{\tilde{V}}(1)$ one has $\mathcal{L}(n) \cdot C > 0$ for $n \gg 0$ and for all curves $C \neq \ell$ on V. Taking in account that ℓ is the exceptional divisor of the blow up $\sigma|_L$, we get $(L \cdot \ell) = (\ell \cdot \ell)_L = -1$. Thus $\mathcal{L}(n) \cdot \ell = 0$, since $\tilde{\sigma}^* \mathcal{O}_V(n) \cdot \ell = 0$ and $\mathcal{O}_{\tilde{V}}(1) \cdot \ell = 1$.

Hence in order to find a morphism which contracts ℓ, it suffices to prove that $|\mathcal{L}(n)|$ is base points free for $n \gg 0$.

It is easily seen that $\mathcal{L}(n)$ is generated by global sections on $\tilde{V} \backslash S$ for $n \gg 0$, in fact, by EGA III 2.2.1, the sheaf $\tilde{\sigma}_* \mathcal{L}(n)$ is generated by its global sections for large n.

Moreover one has $\mathcal{L}(n) \otimes \mathcal{O}_S \simeq \mathcal{O}_S(1) \otimes \mathcal{O}_S(\ell)$. Since the sheaf $\mathcal{O}_S(1) \otimes \mathcal{O}_S(\ell)$ is generated by its global sections, from the exact sequence:

$$0 \longrightarrow \mathcal{L}(n) \otimes \mathcal{I}_S \longrightarrow \mathcal{L}(n) \longrightarrow \mathcal{L}(n) \otimes \mathcal{O}_S \longrightarrow 0$$

we can deduce that $\mathcal{L}(n)$ is generated by its global sections on $\tilde{V}$ if we prove that $H^1(\tilde{V}, \mathcal{L}(n) \otimes \mathcal{I}_S) = 0$. By E.G.A. III 2.2.1 we have $H^1(V, \tilde{\sigma}_*(\mathcal{L}(n) \otimes \mathcal{I}_S)) = H^1(V, \mathcal{O}_V(n) \otimes \tilde{\sigma}_*(\mathcal{O}(L) \otimes \mathcal{I}_S(1))) = 0$ for $n \gg 0$. We recall that $\mathcal{I}_S$ is $\tilde{\sigma}$-ample, thus, from the exact sequence:

$$R^1 \tilde{\sigma}_* \mathcal{L}(n) \otimes \mathcal{I}_S^{r+1} \longrightarrow R^1 \tilde{\sigma}_* \mathcal{L}(n) \otimes \mathcal{I}_S^r \longrightarrow H^1(S, \mathcal{L}(n) \otimes \mathcal{O}_S(r))$$

we get by induction $R^1 \tilde{\sigma}_* \mathcal{L}(n) \otimes \mathcal{I}_S = 0$. So we can prove the vanishing of $H^1(\tilde{V}, \mathcal{L}(n) \otimes \mathcal{I}_S)$ by use of Leray spectral sequence. Hence $\tilde{V}$ also satisfies the property (3).

Then, by using the construction of the section 2, we can find a threefold Y and a birational map $f: \tilde{V} \to Y$.

[1] In fact the exceptional divisor of a birational morphism $F: X \to Y$ of smooth varieties would be a fixed component of the canonical system $|K_X|$.

Finally we shall prove that Y is not isomorphic to $\tilde{V}$. To do this, we remark that there exist no surfaces $\tilde{S}$ on $\tilde{V}$ which are isomorphic to the "Del Pezzo" surface $S^4 \subset \mathbf{P}^4$, having the conormal sheaf isomorphic to $\mathcal{O}_{S^4}(1)$. In fact for such a surface it follows by adjunction formula:

$$\Omega_{\tilde{V}} \otimes \mathcal{O}_{\tilde{S}} \simeq \mathcal{O}_{\tilde{S}}.$$

But on $\tilde{V}$ this is true only for the cubic surface S, for $\Omega_{\tilde{V}}$ is the pull-back of an ample sheaf.

Conversely it is easy to see that the image of the cubic S on Y is the "Del Pezzo" surface $S^4 \subset \mathbf{P}^4$ with conormal sheaf isomorphic to $\mathcal{O}_{S^4}(1)$.

$n = 2$ Take the product $\mathbf{P}^1 \times \mathbf{P}^2$ and its projections $p_1 : \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^1$ and $p_2 : \mathbf{P}^1 \times \mathbf{P}^2 \to \mathbf{P}^2$. We consider in a plane $\{a\} \times \mathbf{P}^2$ a line ℓ and two distinct points P_1, P_2 on ℓ. Consider the following blow up diagram:

$$\begin{array}{ccc} E_1 \cup E_2 & \longrightarrow & V \\ \downarrow & & \downarrow \sigma \\ P_1 \cup P_2 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^2 \end{array}$$

of P_1, P_2. Let $\bar{\ell}$ be the proper transform of ℓ on V. We want to show that $N_{\bar{\ell}}^V \simeq \mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-2)$. Let L be the proper transform of $\{a\} \times \mathbf{P}^2$. From 1.3 we have $N_{\bar{\ell}}^L \simeq \mathcal{O}_{\mathbf{P}^1}(-1)$ and $N_L^V \simeq \mathcal{O}_L(-E_1 - E_2)$. Since P_1, P_2 belong to ℓ, the restriction of N_L^V to $\bar{\ell}$ coincides with $\mathcal{O}_{\mathbf{P}^1}(-2)$. Then by the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow N_{\bar{\ell}}^V \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-2) \longrightarrow 0$$

and by the vanishing of $\mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}^1}(-2), \mathcal{O}_{\mathbf{P}^1}(-1))$, we get $N_{\bar{\ell}}^V \simeq \mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$. Let us denote by $\mathcal{L}$ the sheaf $(p_1 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes (p_2 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^2}(2) \otimes \mathcal{O}_V(-E_1 - E_2)$ on V. We claim: there exists a section $\alpha \in \Gamma(V, \mathcal{L}^{\otimes 6})$ such that $\mathrm{div}(\alpha) \cap \bar{\ell} = \emptyset$, with $D = \mathrm{div}(\alpha)$ irreducible and non singular.

It suffices to show that $\mathcal{L}$ is generated by its global sections. For this we choose on V the linear system $|(p_2 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^2}(1) \otimes \mathcal{I}_{E_1 + E_2}|$ of the proper transform of the quadrics trough ℓ. We only have to prove that:

α) $\mathcal{L}$ cuts on $\bar{Q} \in |(p_2 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^2}(1) \otimes \mathcal{I}_{E_1 + E_2}|$ a complete linear system.

β) $\mathcal{L}_{\bar{Q}}$ is base points free.

By the exact sequence:

$$0 \longrightarrow (p_1 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes (p_2 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^2}(1) \longrightarrow (p_1 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes$$
$$\otimes (p_2 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^2}(2) \otimes \mathcal{I}_{E_1+E_2} \longrightarrow \mathcal{L}_{\tilde{Q}} \longrightarrow 0.$$

(α) is equivalent to: $H^1(V, (p_1 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes (p_2 \circ \sigma)^* \mathcal{O}_{\mathbf{P}^2}(1)) = 0$, and this follows by Leray spectral sequence and Künneth formula. (β) is almost straightforward. Then we can define a double covering $\pi : \tilde{V} \to V$ ramified exactly on supp D such that $(\pi^* D)_{\text{red}}$ defines a section of $\Gamma(\tilde{V}, \pi^* \mathcal{L}^{\otimes 3})$. (See [8]). We prove that $\tilde{V}$ verifies properties (1), (2), (3). Since $D \cap \bar{\ell} = \emptyset$, the line $\bar{\ell}$ is contained in the unramified set of π. Therefore $\pi^* \bar{\ell}$ splits in two disjoint components ℓ_1, ℓ_2, isomorphic to $\bar{\ell}$ and such that $N_{\ell_i}^{\tilde{V}} \simeq N_{\bar{\ell}}^{V}$, $i = 1, 2$. We shall prove that there exists a morphism which contracts ℓ_i to a point. For the claim the sheaf $\mathcal{L}$ is generated by its global sections and so it is $\pi^* \mathcal{L}$. Moreover it is easy to see that $\mathcal{L} \cdot C > 0$ for all curves $C \neq \bar{\ell}$ and $\mathcal{L} \cdot \bar{\ell} = 0$. So by projection formula $\pi^* \mathcal{L} \cdot \tilde{C} > 0$ for all curves $\tilde{C} > 0$, $\tilde{C} \neq \ell_i$ and $\pi^* \mathcal{L} \cdot \ell_i = 0$.

Then the map associated to the linear system $|\pi^* \mathcal{L}^n|$ contracts the two curves ℓ_1, ℓ_2 to a same point; now we can factorize this map by use of the Stein factorization, and the resulting morphism with connected fibers is the required contraction.

We conclude by showing that $\Omega_{\tilde{V}}$ is ample. We have $\Omega_{\tilde{V}} = \pi^*(\Omega_V \otimes \mathcal{L}^{\otimes 3})$. Since $\Omega_V \otimes \mathcal{L}^{\otimes 3} \simeq p_1^* \mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^* \mathcal{O}_{\mathbf{P}^2}(3) \otimes \mathcal{I}_{E_1+E_2}$, we can write: $\Omega_V \otimes \mathcal{L}^{\otimes 3} = \mathcal{L} \otimes p_2^* \mathcal{O}_{\mathbf{P}^2}(1)$. This sheaf is generated by its global sections as tensor product of sheaves generated by its global sections. Moreover, we have: $p_2^* \mathcal{O}_{\mathbf{P}^2}(1) \cdot \bar{\ell} = 1$, $p_2^* \mathcal{O}(1) \cdot C \geq 0$ for all curves $C > 0$, so we get $\Omega_V \otimes \mathcal{L}^{\otimes 3} \cdot C > 0$, $\forall C > 0$. Hence the ampleness of $\Omega_{\tilde{V}}$ follows from [3] Prop. 4.6, 4.4. Now we can use the map of section 2 to get a birational model Y of $\tilde{V}$. For the ampleness of $\Omega_{\tilde{V}}$, we have that the unique fixed component of the pluri-canonical system of Y is the plane E, i.e. the exceptional divisor of the map $f : \tilde{V} \to Y$. But this plane has normal bundle isomorphic to $\mathcal{O}_{\mathbf{P}^2}(-2)$, therefore E is not contractible to a simple point. Consequently Y is a relatively minimal model.

Added in Proof

The construction of the counterexample in the case $n = 1$ is incorrect since the hypersurface V takes some ordinary double points outside the origin. However, if W is the blow up of the double points,

we can work on W exactly as we work on V in the example, being the desingularization of W a relatively minimal model in the category of algebraic varieties. (In fact the quadrics blow up of the double points are not contractible to regular varieties (see for example M. Cornalba, Two theorems on modification of analytic spaces, Inventiones Math. 20, p. 244) and they are the only fixed components of the canonical system). So the conclusions of the example are true for the model W.

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