COMPOSITIO MATHEMATICA

SEBASTIAN J. VAN STRIEN

Unicity of the Lie product

Compositio Mathematica, tome 40, n° 1 (1980), p. 79-85

<http://www.numdam.org/item?id=CM_1980__40_1_79_0>

© Foundation Compositio Mathematica, 1980, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http://http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

COMPOSITIO MATHEMATICA, Vol. 40, Fasc. 1, 1980, pag. 79–85
© 1980 Sijthoff & Noordhoff International Publishers – Alphen aan den Rijn
Printed in the Netherlands

UNICITY OF THE LIE PRODUCT

Sebastian J. van Strien

1. Statement of the result

For a C^∞ manifold M, $\mathfrak{X}(M)$ denotes the linear space of C^∞ vectorfields on M. Let $\chi: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a bilinear operator, defined for every n dimensional manifold M. This operator is called natural if for every smooth open embedding $f: N \to M$ the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X}(M) \times \mathfrak{X}(M) & \xrightarrow{\chi} & \mathfrak{X}(M) \\ \downarrow{\scriptstyle f^* \times f^*} & & \downarrow{\scriptstyle f^*} \\ \mathfrak{X}(N) \times \mathfrak{X}(N) & \xrightarrow{\chi} & \mathfrak{X}(n) \end{array}$$

where M, N are C^∞ manifolds and f^* is the composition $\mathfrak{X}(M) \xrightarrow{r}$ $\mathfrak{X}(f(N)) \xrightarrow{(f^{-1})_*} \mathfrak{X}(N)$, r the restriction operator, i.e. $f^*X(x) = \mathrm{df}(x)^{-1}(X(f(x)))$ for $X \in \mathfrak{X}(M)$. In this note I shall prove that the Lie-product ($[X, Y] = X \cdot Y - Y \cdot X$ for $X, Y \in \mathfrak{X}(M)$) is characterised by this property:

THEOREM: *Let χ be a bilinear natural operator in the above sense, then there exists a constant $\lambda \in \mathbf{R}$ such that $\chi(X, Y) = \lambda \cdot [X, Y]$, for all $X, Y \in \mathfrak{X}(M)$.*

Palais and others [3], [4], [5] prove analogous results for operations on differential forms. Peetre [6] has a similar characterisation of linear (not bilinear) differential operators. The formal techniques are similar to those in [7]. I am indebted to my supervisor Prof. Floris Takens, for suggesting the problem and for his encouragement.

0010-437X/80/01/0079-8 $00.20/0

2. The proof

The naturality of χ implies that it is a local operator, i.e. for U open in M

$$\chi(X, Y) \mid U = \chi(X|U, Y|U).$$

Furthermore if $U, V \subset M$, U, V diffeomorphic and $\chi(X, Y) = \lambda \cdot [X, Y]$ for some constant and all $X, Y \in \mathfrak{X}(U)$, then also $\chi(X, Y) = \lambda \cdot [X, Y]$ for all $X, Y \in \mathfrak{X}(V)$. Therefore I may assume $M = \mathbb{R}^n$. It is sufficient to prove

$$\chi(X, Y)(0) = \lambda \cdot [X, Y](0), \quad \forall X, Y \in \mathfrak{X}(\mathbb{R}^n),$$

because χ commutes with translations. Of course naturality implies

$$f_* \chi(X, Y)(0) = \chi(f_* X, f_* Y)(0) \tag{1}$$

for every diffeomorphism f and every $X, Y \in \mathfrak{X}(\mathbb{R}^n)$.

The main step in the proof is $\chi(X, Y)(0) = \chi(j^1 X(0), j^1 Y(0))(0)$. (Where, for $s \in \mathbb{N}$, $j^s X(p)$ is the polynomial vectorfield of degree s corresponding to the s-jet of X in p, that is, the first s terms of the Taylor expansion of X in p.) In lemma 1 I use naturality to prove this for polynomial vectorfields. In lemmas 2 and 3 this is shown for arbitrary smooth vectorfields, by proving $\chi(X, Y)(0) = 0$ if $X(p)$ or $Y(p)$ has in $p = 0$ a zero of sufficiently high order.

In lemma 4 I show that there exist constants $\gamma_1, \ldots, \gamma_4$ such that:

$$\chi(X, Y)(0) = \gamma_1 \cdot \nabla_X Y(0) + \gamma_2 \cdot \nabla_Y X(0) + \gamma_3 \cdot ((\operatorname{div} Y)(0)) \cdot X(0) + \gamma_4 \cdot ((\operatorname{div} X)(0)) \cdot Y(0).$$

$$\left(\text{Where } \nabla_X Y = \sum X_j \frac{\partial Y_i}{\partial x_j} \frac{\partial}{\partial x_i}, \quad \text{if } X = \sum X_i \frac{\partial}{\partial x_i}, \; Y = \sum Y_i \frac{\partial}{\partial x_i}.\right)$$

In these lemmas I use the naturality property, but only with affine diffeomorphisms f in equation (1).

Finally in the proof of the theorem one needs non-linear diffeomorphisms f in (1) to show that the constants $\gamma_1, \ldots, \gamma_4$ satisfy $\gamma_1 = -\gamma_2$, $\gamma_3 = \gamma_4 = 0$; i.e.: $\chi(X, Y) = \gamma_1 [X, Y]$.

LEMMA 1: *For monomial vectorfields*

$$X(x_1, \ldots, x_n) = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i},$$

$$Y(x_1, \ldots, x_n) = x_1^{\beta_1} \ldots x_n^{\beta_n} \frac{\partial}{\partial x_j},$$

$\chi(X, Y)(0) = 0$ if $\Sigma \alpha_i + \Sigma \beta_i \neq 1$.

PROOF: Let

$$\chi(X, Y)(0) = \chi\left(x_1^{\alpha_1} \ldots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, x_1^{\beta_1} \ldots x_n^{\beta_n} \frac{\partial}{\partial x_j}\right)(0)$$

$$= c_1 \frac{\partial}{\partial x_1}\bigg|_0 + \cdots + c_n \frac{\partial}{\partial x_n}\bigg|_0.$$

Define a diffeomorphism by $\Phi(x) = \lambda \cdot x$, $\lambda \neq 0$. Then

$$\Phi_* X = \lambda^{-\Sigma\alpha_i+1} \cdot X, \quad \Phi_* Y = \lambda^{-\Sigma\beta_i+1} \cdot Y,$$

hence, using (1),

$$\Phi_*(\chi(X, Y)) = \chi(\Phi_* X, \Phi_* Y) = \lambda^{-\Sigma\alpha_i-\Sigma\beta_i+2} \cdot \chi(X, Y).$$

However, the left side at 0 is equal to $\lambda \cdot \chi(X, Y)(0)$. This proves the lemma.

LEMMA 2: *For X a C^∞ vectorfield, there exists a C^∞ vectorfield $\bar{X}$ and sequences $p_s \to 0$, $q_s \to 0$ such that*: (1) $\bar{X} \mid U_s = (X - j^s X(p_s)) \mid U_s$, *$U_s$ a neighbourhood of p_s.* (2) $\bar{X} \mid V_s = 0$, *V_s a neighbourhood of q_s.*

PROOF: In fact lemmas 2 and 3 use a classical theorem of E. Borel and a technique of J. Peetre [6]. Take for example $p_s = (1/s, 0, \ldots, 0)$, $q_s = -p_s$. Define a smooth function $\alpha: \mathbf{R}^n \to \mathbf{R}$ such that

$$\alpha(x) = \begin{cases} 1 \text{ for } \|x\| \leq \frac{1}{2} \\ 0 \text{ for } \|x\| \geq 1. \end{cases}$$

and $\tilde{X}_s = X - j^s X(p_s)$, for every $s \in \mathbf{N}$. Choose $\epsilon_s > 0$ so small that

$$\frac{1}{s+1} + \epsilon_{s+1} < \frac{1}{s} - \epsilon_s$$

so that

$$\left\| \alpha\left(\frac{x-p_s}{\epsilon_s}\right)\tilde{X}_s \right\|_i < 2^{-i} \quad \text{for } i = 1, \ldots, s-1;$$

$$\|f\|_i = \sup_{\substack{|\nu|=i \\ x\in\mathbf{R}^n}} |D^\nu f(x)|.$$

Then

$$\bar{X} = \sum_s \alpha\left(\frac{x-p_s}{\epsilon_s}\right)\tilde{X}_s$$

converges and has the desired properties.

LEMMA 3: *For all* $X, Y \in \mathcal{X}(\mathbf{R}^n)$

$$\chi(X, Y)(0) = \chi(j^1X(0), j^1Y(0))(0).$$

PROOF: This lemma can also be immediately deduced from Peetre [6], because $X \to \chi(X, Y)$ is a local linear operator. But for the sake of completeness an elementary proof will be given here. Take $\bar{X}$, $\bar{Y}$ as in lemma 2.

First I shall prove that for all $p \in \mathbf{R}^n$:

$$(2) \qquad \chi\left(x_1^{\alpha_1}\ldots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, Y\right)(p) = \chi\left(x_1^{\alpha_1}\ldots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(p)\right)(p).$$

$\chi(Z, \bar{Y})(0) = 0$ for every $Z \in \chi(\mathbf{R}^n)$, because $\chi(Z, \bar{Y})(q_s) = 0$ and $q_s \to 0$. Furthermore for $a = (a_1, \ldots, a_n)$ $x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ can be considered as a polynomial in $x_1 - a_1, \ldots, x_n - a_n$ and using lemma 1:

$$\chi\left(x_1^{\alpha_1}\ldots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^tY(p_s)\right)(p_s) = \chi\left(x_1^{\alpha_1}\ldots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(p_s)\right)(p_s)$$

for every $t \in \mathbf{N}$.

But, for every s, $j^1Y(p_s)$ is a linear combination of

$$\frac{\partial}{\partial x_j}, x_k\frac{\partial}{\partial x_l}, j, k, l = 1, 2, \ldots, n.$$

Since any linear operator on a finite dimensional vectorspace is

continuous:

$$\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(p_s)\right)(p_s)\to\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(0)\right)(0),$$

for $s\to\infty$. This together implies that the limit of:

$$\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, Y\right)(p_s)=\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, \bar{Y}\right)(p_s)$$
$$+\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(p_s)\right)(p_s)$$

is

$$\chi\left(x_1^{\alpha_1}\dots x_n^{\alpha_n}\frac{\partial}{\partial x_i}, j^1Y(0)\right)(0).$$

Translation gives (2) for any $p\in\mathbf{R}^n$. Therefore:

$$\chi(X,Y)(p_s)=\chi(\bar{X},\bar{Y})(p_s)+\chi(j^1X(p_s),j^1Y(p_s))(p_s)$$

which goes to $\chi(j^1X(0),j^1Y(0))(0)$ for $s\to\infty$.

Compare this with the proof of the continuity of local operators on vectorfields in [2], XVIII. 13. problem 1.

LEMMA 4: *There are constants* $\gamma_1,\dots,\gamma_4$ *such that*

$$\chi(X,Y)(0)=\gamma_1\cdot\nabla_XY(0)+\gamma_2\cdot\nabla_YX(0)+\gamma_3\cdot((\operatorname{div}Y)(0))\cdot X(0)$$
$$+\gamma_4\cdot((\operatorname{div}X)(0))\cdot Y(0).$$

PROOF: Lemmas 1 and 3 imply that χ can be written as:

$$\chi(X,Y)(0)=(M_1(\mathrm{d}Y(0)))\cdot X(0)+(M_2(\mathrm{d}X(0)))\cdot Y(0) \tag{3}$$

with M_i linear maps from the $(n\times n)$-matrices to the $(n\times n)$-matrices. The lemma is proved when I show that for certain constants γ_1,γ_3:

$$M_1(A)=\gamma_1\cdot A+\gamma_3\cdot\operatorname{Tr}(A)\cdot I. \tag{4}$$

for all matrices A.

Now take $Y(0)=0$, $A=\mathrm{d}Y(0)$, $f(x)=L\cdot x$ (L a linear invertible map) and use naturality (1) in equation (3). This implies:

$$M_1(L^{-1}\cdot A\cdot L)=L^{-1}\cdot M_1(A)\cdot L. \tag{5}$$

Let L run over all diagonal and permutation matrices and deduce from (5) that there exist constants γ_1, γ_3 such that (4) is true for all diagonal matrices A. Therefore (4) is true for all diagonalisable matrices A. But every matrix is a sum of diagonalisable matrices. This proves (4).

Proof of the theorem

a) The constants in lemma 4 satisfy $\gamma_1 = -\gamma_2$, $\gamma_3 = -\gamma_4$. That is:

$$\chi(X, Y) = \gamma_1 \cdot [X, Y] + \gamma_3 \cdot ((\operatorname{div} Y)X - (\operatorname{div} X)Y). \tag{6}$$

To show this, it is now sufficient to prove χ is antisymmetric, i.e. that $\chi(X, X) = 0\ \forall X \in \mathcal{X}(\mathbb{R}^n)$.

If $X(0) = 0$, then lemmas 1 and 3 give $\chi(X, X)(0) = 0$.

If $X(0) \neq 0$ the flow-box theorem [1] gives a local diffeomorphism $\varphi: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

$$\varphi_* X(0) = \frac{\partial}{\partial x_1}: \quad \varphi_* \chi(X, X)(0) = \chi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right)(0) = 0$$

and again $\chi(X, X)(0) = 0$.

b) For $n = 1$: $(\operatorname{div} Y)X - (\operatorname{div} X)Y = [X, Y]$ and we are done.

c) If $n \geq 2$: The operator $(X, Y) \to (\operatorname{div} Y)X - (\operatorname{div} X)Y$ does not commute with every diffeomorphism φ and therefore $\gamma_3 = 0$ in equation (6). To see this, take

$$X = \sum_{i=1}^{\alpha} X_i(x_1, \ldots, x_\alpha) \frac{\partial}{\partial x_i}, \quad Y = \sum_{i=\alpha+1}^{n} Y_i(x_{\alpha+1}, \ldots, x_n) \frac{\partial}{\partial x_i}$$

and a (non-measure preserving) local diffeomorphism $\tilde{\varphi}_1: (\mathbb{R}^\alpha, 0) \to (\mathbb{R}^\alpha, 0)$ such that $\operatorname{div} X(0) = \operatorname{div} Y(0) = 0$, $Y(0) \neq 0$ and $\operatorname{div}((\varphi_1)_* X) \neq 0$.

Define $\varphi(x_1, \ldots, x_n) = (\tilde{\varphi}_1(x_1, \ldots, x_\alpha), x_{\alpha+1}, \ldots, x_n)$, then $\operatorname{div}(\varphi_* X)(0) \neq 0$, $\operatorname{div}(\varphi_* Y) = 0$, that is $(\operatorname{div} Y)X - (\operatorname{div} X)Y$ does not commute with this φ.

REFERENCES

[1] V.I. ARNOLD: *Ordinary Differential Equations MIT press.* Cambridge (1973).

[2] J. DIEUDONNÉ: *Eléments d'Analyse, tome III.* Gauthier-Villars, Paris, 1970.

[3] L. JONKER: A note on a Paper of Palais. *Proc. of the Amer. Math. Soc., 27* (1971) 337–340.

[4] H. LEICHER: Natural Operations on Covariant Tensor Fields. *J. Diff. Geom., 8* (1973) 117–123 (MR 51 #14130).

[5] R. PALAIS: Natural Operations on Differential Forms., *Trans. Amer. Soc.*, *92* (1959) 125–141.
[6] J. PEETRE: Une Characterisation Abstraite des Operateurs Différentiels. *Math. Scand.*, *7* (1959) 211–218 (also *Math. Scand.*, *8* (1960) 116–120).
[7] F. TAKENS: Derivations of Vector Fields. *Compositio Mathematica*, *20* (1973) 151–158.

(Oblatum 26-I-1978 & 4-X-1978)

Mathematisch Instituut
Rijksuniversiteit Groningen
Postbus 800
9700 AV Groningen
The Netherlands

Current address:
Mathematisch Instituut
Rijksuniversiteit Utrecht
Budapestlaan
3508 TA Utrecht
The Netherlands