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DAVID D. BLEECKER

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IMMERSED SURFACES WITH CONSTANT CURVATURE NEAR INFINITY

David D. Bleecker

1. Introduction. The Theorem

In Tilla Klotz Milnor's articles [3], [4] and [5], the following conjecture of John Milnor is discussed. Throughout the present work, we assume M is a connected, complete surface, C^4 immersed in Euclidean space (E^3). Consider the following statements:

A. The Gaussian curvature K of M is locally constant outside a compact subset of M.

B. The sum of the squares of the principal curvatures of M is bounded away from zero on M.

C. M has no umbilic points.

D. K changes sign on M or $K \equiv 0$ on M.

E. $\int_M K \, dM = 0$.

CONJECTURE (John Milnor): B and C imply D.

THEOREM: A, B, *and* C *imply* E.

Note that E implies D. In fact, by Cohn-Vossen's Theorem ([1] and [2]), namely that $\int_M K \, dM \leq 2\pi X(M)$, D implies $X(M) \geq 0$ and M is homeomorphic to a plane, cylinder or mobius strip if M is noncompact. We need not concern ourselves with the case where M is compact, since both the Conjecture and Theorem are well known in that case. We also have the following:

COROLLARY: *If* $X(M) < 0$, *then* A *and* B *imply* M *has an umbilic point.*

PROOF: By Cohn-Vossen's Theorem $\int_M K \, dM < 0$. However, A, B and C imply $\int_M K \, dM = 0$. Hence C is false. $\square$

Other corollaries are obtained analogously. We also see that the Theorem is a generalization of Hilbert's Theorem that $K \not\equiv c$ for $c < 0$. Unfortunately, while the conclusion of the Theorem is stronger than that of Milnor's Conjecture, so is the hypothesis.

2. Lemmas

Because of hypothesis C, it is natural to consider the line fields obtained from the principal directions on M. A technical difficulty is encountered if these are not orientable (i.e., they are not generated by a global vector field). Hence we prove

LEMMA 1: *Given a line field L on M, there is a double covering $p: \bar{M} \to M$, such that $\bar{L}$ defined by $p_*(\bar{L}) = L$ is orientable.*

PROOF: Let $\bar{M} = \{X \in TM\colon X \in L \text{ and } \|X\| = 1\}$. If $\pi\colon TM \to M$ is the tangent bundle projection, then $\pi \mid \bar{M}\colon \bar{M} \to M$ is a double covering. The evenly covered neighborhoods are just open sets over which TM is trivial. Now $\bar{L}$ on $\bar{M}$ is generated by the vector field $\bar{X}$ defined by $(\pi \mid \bar{M})_*(\bar{X}_X) = X$. $\square$

First we may assume that M is orientable, for if the Theorem is true for orientable M, then for a nonorientable M with (orientable) Riemannian double cover $\tilde{M}$, $\int_{\tilde{M}} K \, d\tilde{M} = 0$ implies $\int_M K \, dM = 0$. So henceforth, M is orientable, and we have well-defined principal curvature functions K_1 and K_2 relative to a choice of unit normal. Let L_1 and L_2 be the line fields given by the principal directions associated to K_1 and K_2, respectively.

In Lemmas 2 through 9 we assume that M is finitely connected. Then it is well known that M is homeomorphic to $M_0 - \{p_1, \ldots, p_n\}$ where M_0 is a compact surface. Let $\xi\colon PM_0 \to M_0$ be the circle bundle obtained from the unit circle bundle of M_0 (relative to some Riemannian metric on M_0) by identifying each unit vector with its antipode. We can define the index of L_1 about p_i, denoted $L_1(p_i)$ in a way analogous to defining indices of vector fields about singularities (or zeros). Thus we obtain $\Sigma_{i=1}^n L_1(p_i) = 2X(M_0)$, the factor of 2 coming from the antipodal identification. Note that clearly $L_1(p_i) = L_2(p_i)$. We will prove $L_1(p_i) \geq 2$, $1 \leq i \leq n$, using A and B, thus obtaining $2n \leq$

$\Sigma_{i=1}^{n} L_1(p_i) = 2X(M_0)$ or $0 \leq X(M_0) - n = X(M)$ and then we need only consider M which are homeomorphic to the plane if the cylinder.

LEMMA 2: A, B *and* C *imply* $L_1(p_i) \geq 2$.

PROOF: Since i is fixed, drop the index i. Let β be a simple, closed noncontractible piecewise smooth curve in the punctured disk T about p (the point at infinity) with non-smooth corners $v_1, \ldots, v_m$ (in cyclic order) so that between consecutive corners β is an integral curve of L_1 or L_2. We assume that m, the number of corners, is minimal subject to these conditions on β. Now β bounds a punctured disk about p, say D. We call v_K positive if the exterior angle of the polygon D at v_K is positive (i.e., $\pi/2$) and otherwise v_K is called negative. Let P be the number of positive corners and N the number of negative corners $(N = m - P)$. One can check that $L_1(p) = 2 + (N - P)/2$. Thus $L_1(p) \geq 2$ if $N \geq P$. Assume $P > N$. Now if $N = 0$ then P is even, since $L_1(p)$ is an integer. Thus $(P, N) = (2, 0)$ is a possibility, but in all other cases $P + N \geq 4$. Let us handle the case $(P, N) = (0, 2)$ later. Now $P + N \geq 4$ and $P > N$ implies that there is a string of corners, say $v_1, \ldots, v_4$ such that v_2 and v_3 are positive. Thus, we have the situation in Figure 1.

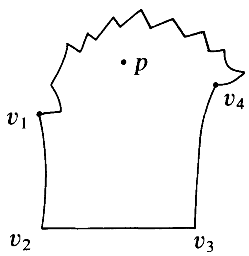

Fig. 1.

Cover the compact segment v_2v_3 with a finite number of coordinate rectangles whose coordinate curves are lines of curvature. For a sufficiently small ϵ, the lines parallel to v_2v_3 and issuing from points on v_1v_2 within ϵ of v_2 run through all the rectangles and meet v_3v_4. We therefore obtain a coordinate rectangle with base v_2v_3, by taking v_2 to correspond to the origin and v_2v_3 to correspond to part of the positive x-axis in such a way that v_2v_3 is parameterized by the variable x with unit speed and the subsegment of v_2v_1 of length ϵ starting at v_2

corresponding to part of the positive y-axis with y parameterizing the subsegment with unit speed. (Here it should be noted that in general not all the coordinate curves are parameterized by arclength since this would imply $K \equiv 0$.) We call such a coordinate rectangle (with base v_2v_3 and left and right sides being subsegments of v_1v_2 and v_3v_4 respectively) a *special coordinate system* (scs). Let R be the maximal scs, (i.e., R is the union of all the scs's). We shall prove that v_1 or v_4 is a top corner of R. Thus $D - \bar{R}$ will be a region bounded by a curve β' of fewer sides than β, contradicting the minimality assumption on β.

Let $\ell(y_0)$ be the length of the coordinate curve of R given by $y = y_0$. If we assume $\ell(y)$ is bounded (this will be proved in Lemma 3), then we can prove that either v_1 or v_4 is a top corner of R: Let $b = \sup\{y : (0, y) \in R\}$. Let the sequence (b_i) be chosen such that $b_i \geq 0$, $b_i \uparrow b$ and $\ell(b_i) \to \delta \in \mathbf{R}$ (recall we are assuming $\ell(y)$ is bounded). Let $(0, b)$ denote the point on v_1v_2 which is a distance b from v_2, and identify the points of R with their coordinates. Define a unit vector field on M equal to $\partial/\partial x$ on v_2v_3 generating the line field which is tangent to v_2v_3 (we may have to lift the whole situation to $\bar{M}$ of Lemma 1). Now since M is complete, this vector field defines a flow ϕ: $M \times \mathbf{R} \to M$. Now ϕ is continuous and since $(b_i, 0) \to (b, 0)$ and $\ell(b_i) \to \delta$, we have $\lim_{i\to\infty}(\ell(0), b_i) = \lim_{i\to\infty}\phi((0, b_i), \ell(b_i)) = \phi(\lim_{i\to\infty}(0, b_i), \lim_{i\to\infty} \ell(b_i)) = \phi((0, b), \delta)$. Now $\lim_{i\to\infty}(\ell(0), b_i)$ is a point on v_3v_4 and $\phi((0, b), \delta)$ is the endpoint of the line of curvature of length δ issuing from $(b, 0)$ into D. By covering this curve by a finite number of small rectangles as before, we get an extension of R (a contradiction) unless $(b, 0) = v_2$ or $\phi((0, b), \delta) = v_4$ (i.e., v_1 or v_4 are corners of R). In either case the curve $\gamma(t) = \phi((0, b), t)$ $0 \leq t \leq \delta$ bypasses v_2 and v_3 and introduces at most one new corner, contradicting the minimality of β. Thus assuming $\ell(y)$ is bounded, our original assumption that $(P, N) \neq (2, 0)$ and $P > N$ is false. Before proving $\ell(y)$ is bounded, let us handle the case $(P, N) = (2, 0)$. Here $L_1(p)$ is 3. It is clear that for L_1 to be orientable on T we must have $L_1(p)$ even. Thus, L_1 is not orientable on T. Let $\bar{T}$ be the two fold cover of T as in Lemma 1. Then taking v_1 to be the beginning (and end) of the loop β, we can lift $\beta + \beta$ to a simple closed noncontractible curve $\bar{\beta}$ (easily verified). Now $\bar{\beta}$ has four positive vertices which by the preceding argument must be vertices of a coordinate rectangle bounded by $\bar{\beta}$ which is impossible since $\bar{\beta}$ is noncontractible. The next lemma completes the proof. □

LEMMA 3: *The function $\ell(y)$ in the proof of Lemma 2 is bounded.*

PROOF: First note that α (the constant Gaussian curvature of M in T) is not positive, since if $q \in D$ is a point of distance greater than $\pi\alpha^{-1/2}$ from β, then by a familiar result there is no length minimizing geodesic from q to β, contradicting the completeness of M. Now in the coordinate rectangle R the metric of M is given by $E\,\mathrm{d}x^2 + G\,\mathrm{d}y^2$. Let K_1 be the principal curvature for the $y = \text{constant}$ curves, etc. Then, the Codazzi-Mainardi equations of embedding and Gauss equation are:

$$\frac{\partial K_1}{\partial y} = \frac{1}{2E}\frac{\partial E}{\partial y}(K_2 - K_1)$$

$$\frac{\partial K_2}{\partial x} = \frac{1}{2G}\frac{\partial G}{\partial x}(K_1 - K_2)$$

$$K_1 K_2 = K = \alpha \leq 0.$$

If $\alpha \neq 0$, then these equations yield the explicit expressions for E and G:

$$E = C(x)(K_1^2 - \alpha)^{-1} \quad G = D(y)(K_2^2 - \alpha)^{-1}$$

where $C(x) = (K_1^2(x, 0) - \alpha)$ and $D(y) = (K_2^2(0, y) - \alpha)$ are chosen in order that $E(x, 0) = G(0, y) = 1$.

Let $C_0 = \max\{C(x) : 0 \leq x \leq \ell(0)\}$. We compute a bound for $\ell(y)$ as follows:

$$\ell(y) = \int_0^{\ell(0)} \sqrt{E(x, y)}\,\mathrm{d}x \leq \int_0^{\ell(0)} \sqrt{C_0}[K_1^2 - \alpha]^{-1/2}\,\mathrm{d}x \leq \ell(0)[C_0/-\alpha]^{1/2}.$$

If $\alpha = 0$ and $K_1 \neq 0$, then we have $E = C(x)K_1^{-2}$ and $\ell(y) \leq \ell(0)[C_0/b]^{1/2}$ since $K_1^2 + K_2^2 = K_1^2 > b > 0$. If $\alpha = 0$ and $K_1 = 0$, then $E \equiv 1$, and $\ell(y) \equiv \ell(0)$. In any case $\ell(y)$ is bounded. This concludes the proofs of Lemmas 2 and 3. □

By the remarks preceding the statement of Lemma 2, we now have that A, B, and C imply $X(M) \geq 0$ and hence M is homeomorphic to $\mathbb{R}^2$, or a cylinder.

Going back to the proofs of Lemmas 2 and 3, one might wonder whether β can have any positive corners at all. In fact we have

LEMMA 4: *The curve β defined in the proof of Lemma 2 has no positive corners unless $N = P = 1$.*

PROOF: We know from the proof of Lemma 2, that $N \geq P$ and no two positive corners occur consecutively. Thus if there is a positive corner, it is flanked by two negative corners or $N = P = 1$. In the first case we have the following situation illustrated in Figure 2.

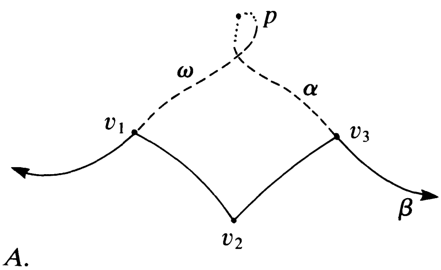

Fig. 2.

The integral curve ω (dashed) is the extension of the side of β containing v_1 as shown. We prove that ω stays in D. If ω leaves D, $\bar{\omega}$ (the part of ω between v_1 and the next time it meets β) will divide D into two subregions, one of which "contains" p say D'. Now D' has no more sides than D. In fact D' will have fewer sides than D if the end of $\bar{\omega}$ does not lie on v_1v_2 (the end of $\bar{\omega}$ cannot lie on the remaining side of β containing v_1, since then ω would have a self intersection whence ω would be a closed orbit and would have to circle p and give a contradiction to β minimality). Now the end of $\bar{\omega}$ cannot lie on v_1v_2, say at q, since $qv_1 + \bar{\omega}$ would circle p because integral curves of L_1 and L_2 cannot intersect twice in a simply connected region (readily verified). Now $qv_1 + \bar{\omega}$ has only two corners and β has at least two corners by assumption. Thus $qv_1 + \bar{\omega}$ is minimal and has two positive corners at v_1 and q which is impossible. Thus D' has fewer sides than D which is a contradiction and so ω stays in D. The same reasoning applies to α shown in Figure 2. Consider special coordinate systems with base v_1v_2 and sides being subsegments of α and ω. Using the same reasoning as in Lemmas 2 and 3 and the fact that ω has infinite length, we get a maximal scs, R, with v_3 as an upper corner. However the top edge of R is α, so α intersects ω and $D - R$ is bounded by a curve with fewer sides than β. $\square$

LEMMA 5: *Let the punctured disk T (in M) about p (a point at infinity) have constant negative curvature α. Then T has finite area.*

PROOF: It is convenient to introduce coordinate systems with the lines $x = \text{constant}$ and $y = \text{constant}$ being asymptotic lines. These systems are the classical Tchebycheff nets. If the curvature is constant the metric in a suitable coordinate system of this type is $dx^2 + 2\cos\rho(x, y)\, dx\, dy + dy^2$ where $\rho(x, y)$ is a function ranging between 0 and π, and is in fact the angle between the asymptotic directions. The Gaussian curvature α is given by $-\partial^2\rho/\partial x\,\partial y = \alpha \sin\rho$. Hence the total curvature of a rectangle satisfies:

$$0 \geq \int_a^b \int_c^d \alpha \sin\rho\, dx\, dy = \int_c^d \int_a^b -\partial^2\rho/\partial x\,\partial y\, dx\, dy$$
$$= \rho(b, c) + \rho(a, d) - \rho(b, d) - \rho(a, c) > -2\pi.$$

This says any rectangle in asymptotic coordinates has total curvature less than 2π in magnitude. Taking limits, an asymptotic half-plane or quarter-plane has finite total curvature, and since $K = \alpha$ is constant, finite area. Now, we can find a simple closed noncontractible piecewise smooth curve β in T such that each smooth piece of β is an asymptotic segment. Let us assume β has a minimal number of corners. Again it is possible to prove that there do not exist consecutive positive corners as in the proofs of Lemmas 2 and 3. In fact here $\ell(y)$ is constant. The only difficulty in making the proofs go through in this case is that the unit vector field generating the flow φ on M does not exist in general since the asymptotic lines are not global if M has points of non-negative curvature. However, inside T (or $\bar{T}$ of Lemma 1) there is no difficulty and we simply extend the vector field to the rest of M and note that the integral curves of the vector field are asymptotic lines while they are in T, which is really all we need. Now we cover the open region D, bounded by β, by a finite number of closed asymptotic half or quarter planes in the following fashion: Each side of β is flanked by two negative corners or a negative and a positive. In the former case we take as an element of our cover, the half-plane with the side contained in the boundary of the half-plane and extending into D. In the latter case, we take the quarter-plane with corner at the positive vertex and extending into D. Thus we have a finite set U of plane sections—one for each side of β. We wish to show U is a cover of D. Now $(\cup U) \cap D$ is closed since U consists of a finite number of closed sets, and $(\cup U) \cap D$ is open since $(\cup U) \cap D = (\cup U') \cap D$, where U' consists of the plane sections in U without their boundaries, since we note that each bounding ray or line of a plane section once it enters D is contained in the interior of a flanking section. Thus, $(\cup U) \cap D$ is open and closed in D, and hence

D connected implies $\cup U = D$. Since each element of U has a finite area, D has finite area and T has finite area. $\square$

COROLLARY: *If M is finitely connected and has locally constant negative curvature outside a compact set, then $\int_M K\, \mathrm{d}M = 2\pi X(M)$.*

PROOF: We have shown that M must have finite area, and Huber [2] has shown that the Gauss-Bonnet formula holds in this case. $\square$

Thus we have the Theorem in the case where M is homeomorphic to a cylinder with locally constant *negative* curvature outside a compact. We must prove the following Lemma 5, to handle the case of M homeomorphic to $\mathbb{R}^2$.

LEMMA 6: *If M is homeomorphic to the plane and has constant negative curvature outside a compact set, then M has an umbilic point.*

PROOF: There is only one point p at infinity, and so the index of L_1 about p is four. Thus $N - P = 4$ by the formula before Lemma 2. Now the index of an asymptotic line field A defined in T about p is also four, since A is homotopic to L_1. Now Lemma 4 is also valid if the line fields on T are the asymptotic line fields (the proof goes through easily). Hence $N = 4$, $P = 0$. Consider the compact region $R \subset \mathbb{R}^2$ bounded by a minimal, four-sided curve β with sides being asymptotic line segments. By the previous corollary, we know $\int_M K\, \mathrm{d}M = 2\pi$. Since K is negative outside R, we then have $\int_R K\, \mathrm{d}M > 2\pi$. Let us assume that R is chosen large enough so that there is a neighborhood U of β in which the curvature is a negative constant. Let A_1 and A_2 be the asymptotic line fields in U and let ρ, $(0 < p < \pi)$, be the positive oriented angle from A_1 to A_2. Let $\beta_1, \ldots, \beta_4$ be the oriented sides of β (i.e., $\beta = \beta_1 + \cdots + \beta_4$ where we parameterize β so that R lies to the left as we traverse β). Suppose that A_1 is tangent to β_1. If we select a Tchebycheff net in U about some point of β_1 with β_1 being the $y = 0$ curve parameterized with unit speed by x, the metric takes the form $\mathrm{d}x^2 + 2\cos\rho\, \mathrm{d}x\, \mathrm{d}y + \mathrm{d}y^2$, and after some computation we find that the geodesic curvature of β_1 is $-\partial\rho(x, 0)/\partial x$ inside the net. Since this can be done at each point of β_1 we get that the geodesic curvature function along β_1 is $k_g(s) = -\mathrm{d}\rho(\beta_1(s))/\mathrm{d}s$ where "s" denotes arclength. Thus the total geodesic curvature of β_1 is $\rho(x_1) - \rho(x_2)$ where x_1 is the initial point of β_1 and x_2 is the end point of β_1. The geodesic curvature along β_2 is given by

$d\rho(\beta_2(s))/ds$. Here, there is no minus sign since the positive oriented angle from L_2 to L_1 is $\pi - \rho$. Thus the total geodesic curvature of β_2 is $\rho(x_3) - \rho(x_2)$, and we find that:

$$\sum_{i=1}^{4} \int_{\beta_i} k_g \, ds = 2(\rho(x_1) + \rho(x_3) - \rho(x_2) - \rho(x_4))$$

where $x_1, \ldots, x_4$ are corners of β as shown in Figure 3.

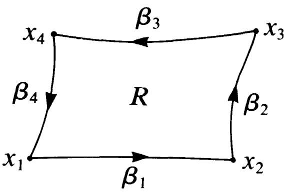

Fig. 3.

The sum of the exterior angles of β is

$$\sum_{i=1}^{4} \epsilon_i = \rho(x_2) + \rho(x_4) - \rho(x_3) - \rho(x_1) + 2\pi.$$

Let us define $\delta = \rho(x_2) + \rho(x_4) - \rho(x_3) - \rho(x_1)$. Then:

$$\sum_{i=1}^{4} \int_{\beta_i} k_g \, ds = -2\delta \quad \text{and} \quad \sum_{i=1}^{4} \epsilon_i = \delta + 2\pi.$$

Hence by the Gauss-Bonnet formula $\int_R K \, dM - 2\delta + \delta + 2\pi = 2\pi$ and so $\delta = \int_R K \, dM > 2\pi$ and $\Sigma_{i=1}^{4} \epsilon_i > 4\pi$, but $\epsilon_i < \pi$ so $\Sigma_{i=1}^{4} \epsilon_i < 4\pi$, a contradiction. □

We now consider the case where T about p (at infinity) has curvature 0.

LEMMA 7: *If T about p has curvature* 0, *then* B *and* C *imply that the sides of β are geodesics.*

PROOF: Suppose K_1 ($K_1 > 0$) is the non-zero principle curvature, and L_1 is the line field associated to K_1. Consider a coordinate system with L_1 tangent to y = constant curves and L_2 tangent to x = constant curves. In general, the metric will be given by $E\,dx^2 + G\,dy^2$ where $E = C(x)K_1^{-2}$ and $G \equiv D(y)$ (see proof of Lemma 3). Let us choose $C(x) \equiv D(x) \equiv 1$. The Gauss equation says (where $H = \sqrt{EG}$)

$$0 = K_1K_2 = K = \frac{-1}{2H}\left(\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial x} \div H\right) + \frac{\partial}{\partial y}\left(\frac{\partial E}{\partial y} \div H\right)\right)$$

or

$$0 = \frac{\partial}{\partial y}\left(\frac{\partial (K_1^{-2})}{\partial y} \div K_1^{-1}\right)$$

or

$$2f(x) = \frac{\partial (K_1^{-2})}{\partial y}\cdot K_1 = \frac{\partial (K_1^{-1})^2}{\partial y}\cdot K_1 = 2K_1^{-1}\frac{\partial (K_1^{-1})}{\partial y}K_1 = 2\frac{\partial (K_1^{-1})}{\partial y}.$$

Thus $K_1^{-1} = f(x)y + g(x)$ or $K_1 = (f(x)y + g(x))^{-1}$ for some functions f and g. Now consider a minimal curve β in T, and an integral curve γ of L_2 entering D (the punctured disk bounded by β) from a point on a side of β which is an integral curve of L_1. We prove γ cannot leave D. If β is a closed orbit of L_1, then if γ leaves D, γ divides D into two components, one of which is simply connected, but integral curves of L_1 and L_2 cannot intersect twice in a simply connected set. Thus γ cannot leave D in this case. Since $L_1(p)$ is an integer, if β is not a closed orbit, β must have at least two sides. Now if γ leaves D through the side where it (γ) started, then either γ plus the asymptotic segment between the endpoints of γ forms a minimal β with two positive corners (a contradiction) or we have two integral curves intersecting twice in a simply connected set (a contradiction). Now γ cannot leave through any side of β adjacent to the side it enters since these are also integral curves of L_2. Thus γ bypasses at least two corners of β and hence γ divides D into D' and D'', both having boundaries with no more sides than D. One of these, say D', "contains" p, and has a positive corner at the initial point of γ. Thus D' has only two corners by Lemma 4, and so β has two sides, but we have shown that γ cannot leave through either of these. So we have in general that γ cannot leave D.

From the relation $K_1 = (f(x)y + g(x))^{-1}$ we get that the limit of K_1 along γ (parameterized by y) is 0 if $f(x) \neq 0$. However, $K_1 > \sqrt{b} > 0$ by B, thus $f(x) = 0$ on any side of β which is an integral curve of L_1. Thus K_1 is constant on γ, and $\partial K_1/\partial y = 0$ along a side with L_1 tangents. A simple calculation now reveals that the sides of β are geodesics. □

LEMMA 8: *If M has zero curvature outside a compact set, then* B *and* C *imply* $\int_M K \,\mathrm{d}M = 0$.

PROOF: Let the punctured disks, with curvature zero, about the points at infinity by $T_1, \dots, T_n$. In each T_i we choose a minimal curve β_i. Now the β_i $(i = 1, \dots, n)$ bound a compact region R of M, outside of which the curvature is zero, thus $\int_M K \,\mathrm{d}M = \int_R K \,\mathrm{d}M$. The Gauss-Bonnet formula yields $\int_R K \,\mathrm{d}M + \Sigma_R = 2\pi X(M)$ where Σ_R is the sum of all the exterior angles of R. Note that the geodesic curvature term is zero by Lemma 7. However, $\Sigma_R = \frac{1}{2}\pi(N - P)$ where N is the total number of negative corners of the punctured D_i bounded by the β_i, etc. Thus, as in the proof of Lemma 2,

$$\Sigma_R = \tfrac{1}{2}\pi\,(N - P) = \left(\sum_{i=1}^{n} L_1(p_i) - 2n\right)\pi = 2(X(M_0) - n)\pi = 2\pi X(M).$$

Hence, $0 = \int_R K \,\mathrm{d}M = \int_M K \,\mathrm{d}M$. □

LEMMA 9: *If M is finitely connected, then* A, B, *and* C *imply* $\int_M K \,\mathrm{d}M = 0$.

PROOF: We know from the remark after the proof of Lemma 3, that M is homeomorphic to $\mathbb{R}^2$ or a cylinder. Now $\mathbb{R}^2$ has only one point at infinity, and so M has constant non-positive outside a compact set in this case, and the Corollary to Lemma 5, Lemma 6 and Lemma 8 imply Lemma 9 for $M \approx \mathbb{R}^2$, or M homeomorphic to a cylinder with negative or zero curvature outside a compact set, but not both.

Remaining is the case where M is homeomorphic to a cylinder where the curvature is zero in T_1 and negative in T_2. We know $L(p_1) \geq 2$ and $L(p_2) \geq 2$, by Lemma 2, and $L(p_1) + L(p_2) = 4$. Hence $L(p_1) = L(p_2) = 2$. Since T_2 has finite area by Lemma 5, we can use the strong form of Huber's Theorem ([2]), that $\int_M K \,\mathrm{d}M = 2\pi X(M)$ for M of finite area, which says that we can find a sequence of simple non-contractible closed curves in T_2 tending to p_2 such that the total geodesic curvature and exterior angle contributions go to 0. Now in T_1 of zero curvature, a minimal curve β has geodesic sides by Lemma 7, and the sum of the exterior angle contribution is $\frac{1}{2}\pi(N - P)$ as in Lemma 8. However, $L(p_1) = 2$ implies $N - P = 0$. Thus combining this result for T_1 and Huber's result for T_2, we get $\int_M K \,\mathrm{d}A = 0$ for M homeomorphic to a cylinder. □

To conclude, we have the following to handle the case where M is infinitely connected.

LEMMA 10: A *and* C *imply that* M *is not infinitely connected.*

PROOF: We can exhaust M by an increasing sequence of compact subsurfaces S_n with a finite number, depending on n, of smooth closed bounding curves each bounding one component of $M - S_n$. Now n may be chosen large enough so that $M - S_n$ has constant curvature in each component T_i. In each T_i we define a minimal curve β_i as before, only we require that β_i be homotopic to the boundary of T_i. In this case we can still prove that β_i cannot have consecutive positive corners (see proofs of Lemmas 2 and 3). Hence as in the proof of Lemma 5 we can show that if T_i has constant negative curvature, then the total curvature of T_i is finite. Now since each of the T_i (finite in number) has either negative or zero constant curvature, we get that M has finite total curvature. However, Huber [2] proves that for M infinitely connected $\int_M K \, \mathrm{d}M = -\infty$. $\square$

This completes the proof of the Theorem.

3. Conclusion

Now, although Milnor's conjecture in its full generality seems difficult enough to deal with, I would like to introduce the following stronger conjecture:

CONJECTURE: C and D imply that M can be exhausted by an increasing sequence of compact subsurfaces with boundaries such that the total curvatures of the subsurfaces tend to 0.

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Department of Mathematics
University of Hawaii
Honolulu, Hawaii 96822