

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 33, n° 3 (1976), p. 289-295

<[http://www.numdam.org/item?id=CM\\_1976\\_\\_33\\_3\\_289\\_0](http://www.numdam.org/item?id=CM_1976__33_3_289_0)>

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## NEW UNIQUE MODELS OF REPRESENTATIONS OF UNITARY GROUPS

Mark E. Novodvorsky

### Introduction

Let  $\pi$  be an irreducible admissible representation of a unitary group  $G$  over a  $p$ -adic field, with sufficiently small anisotropic kernel. This paper introduces a family of  $Z$ -eigen-functionals on the space of  $\pi$ ,  $Z$  a subgroup of  $G$ , which are unique up to a scalar multiplier. This is equivalent to the uniqueness of corresponding Whittaker models of the representation  $\pi$ , and allows one to obtain Euler products for some meromorphic functions attached to automorphic forms on orthogonal groups – the details of the latter will appear in a forthcoming paper.

When  $G$  is a split orthogonal group of type  $B_2$  realized as the group of  $4 \times 4$  symplectic matrices, the functionals of this paper are essentially the same as those considered by I. I. Pyatetsky-Shapiro and the author in [5]. The proof follows the modification of arguments of I. M. Gelfand and D. A. Kajdan [3] introduced by the author in [6].

### 1. The result and its reduction

Let  $k$  be a locally compact non-discrete disconnected field,  $\ell$  an extension of it,

$$(1) \quad \lambda \rightarrow \bar{\lambda}, \quad \lambda \in \ell,$$

an involution of  $\ell$  trivial on  $k$  only (particularly,  $\ell : k \leq 2$ ). Suppose the characteristic of  $k$  different from 2.

Algebraic groups defined over  $k$  will be identified throughout this paper with the groups of their  $k$ -points in the natural locally compact topology. The linear span of a set of vectors  $\{e_i : 1 \leq i \leq n\}$  in a linear space over  $\ell$  will be denoted  $\{e_i | 1 \leq i \leq n\}$ .

Let  $G$  denote the group of all automorphisms of a Hermitian form  $(\dots, \dots)$  on a linear space over  $\ell$  with base  $\{e_i : 1 \leq i \leq n\}$ . Let  $e$  be a linear combination of vectors  $e_i$ ,  $r \leq i \leq n - r$ , with coefficients from  $k$ .

Suppose

$$(2) \quad \begin{cases} (e_i, e_j) = (e_j, e_i) \forall_{i,j} \leq n; \\ (e_i, e_j) = \begin{cases} 0 & \text{if } i+j \neq n+1, \\ 1 & \text{if } i+j = n+1, \end{cases} & 1 \leq i \leq r, 1 \leq j \leq n; \\ (e, e) \neq 0. \end{cases}$$

Define

$$\begin{aligned} T &= \{g \in G : ge = e; ge_i = e_i \text{ if } i \leq r \text{ or if } i > n-r\}, \\ (3) \quad U &= \{g = (a_{ij}) \in G : a_{ii} = 1 \forall_i; a_{ij} = 0 \text{ if } j < i \text{ or} \\ Z &= T \times U. & \text{if } r < i < j \leq n-r\}, \end{aligned}$$

$T$  and  $U$  are subgroups of  $G$ ;  $T$  normalizes  $U$ ; hence,  $Z$  is a subgroup too.

Take a non-trivial character  $\chi$  of the additive group of the field  $k$  and a character  $\tau$  of the group  $T$  trivial on the subgroup of unimodular matrices from  $T$  if  $\ell \neq k$  and of order 2 if  $\ell = k$ .

Define

$$(4) \quad \begin{cases} \theta(u) = \chi \left( \text{tr} \left( \sum_{i=1}^{r-1} a_{i,i+1} + (e_{n-r+1}, ue) \right) \right), & u = (a_{ij}) \in U; \\ \alpha(z) = \tau(t) \cdot \theta(u), & z = t \cdot u \in Z. \end{cases}$$

Then  $\theta$  and  $\alpha$  are characters of the subgroups  $U$  and  $Z$ , respectively.

Assume that the dimension  $r'$  of a maximal isotropic subspace of the form  $(\dots, \dots)$  does not exceed  $r+1$ .

**THEOREM:** For any irreducible admissible representation  $\pi$  of the group  $G$  the dimension  $\dim(\pi, \alpha)$  of all linear  $Z$ -eigenfunctionals with character  $\alpha$  is, at most, 1.

Define an involution of the group  $G$ :

$$\begin{aligned} \tilde{g} &= I^{-1}(\bar{a}_{ij})^{-1}I, \quad g = (a_{ij}) \in G, \\ (5) \quad Ie_i &= \begin{cases} (-1)^i e_i & \text{if } i \leq r, \\ (-1)^{r+1} e_i & \text{if } r < i \leq n-r, \\ (-1)^{n+1-i} e_i & \text{if } n-r < i. \end{cases} \end{aligned}$$

(Note that this involution preserves the subgroups  $U$  and  $T$  and the characters  $\theta$  and  $\tau$ .)

The main technical step of the proof is the following:

**LEMMA:** For every  $g \in G$  there exist  $z_1, z_2 \in Z$  such that either both  $z_1, z_2$  are unipotent and

$$(6) \quad z_1 g \tilde{z}_2 = g, \quad \alpha(z_1 z_2) \neq 1,$$

or

$$(7) \quad z_1 g \tilde{z}_2 = \tilde{g}, \quad \alpha(z_1 z_2) = 1.$$

This lemma, in view of Theorem 1 from [6], provides the inequality:

$$(8) \quad \dim(\pi, \alpha) \times \dim(\pi^*, \hat{\alpha}) \leq 1, \quad \hat{\alpha}(z) = \alpha(\tilde{z}^{-1}),$$

where  $\pi^*$  denotes the contragradient representation to  $\pi$ .

Now the theorem follows directly from the inequality (8) and the isomorphism  $\hat{\pi} \simeq \pi^*$ ,  $\hat{\pi}(g) = \pi(\tilde{g}^{-1})$ , proved by I. M. Gelfand and A. D. Kajdan [2] providing that the elements  $g$  and  $\tilde{g}$  are conjugate in  $G$  for all  $g \in G$ ; the last fact is an immediate consequence of Milnor's classification of conjugate classes of  $G$  in [4].

The theorem is true for finite field  $k$  too; in that case it follows directly from the lemma, which, together with its proof, is true for any field  $k$  of characteristic different from 2.

## 2. The proof of the lemma

1. Note that if (1.6) is true for  $g \in G$ ,  $z_1(g), z_2(g) \in U$ , then for  $g' = z' g z''$ ,  $z', z'' \in Z$  the same formulas are true when  $z_1(g') = z' z_1(g)(z')^{-1}$  and  $z_2(g') = \tilde{z}'' z_2(g)(\tilde{z}'')^{-1}$ ; similarly, if (1.7) is true for  $g \in G$ ,  $z_1(g), z_2(g) \in Z$ , then it is true for  $g' = z' g z''$ ,  $z', z'' \in Z$ , when  $z_1(g') = \tilde{z}'' z_1(g)(z')^{-1}$ ,  $z_2(g') = z' z_2(g)(\tilde{z}'')^{-1}$ . Therefore, it is enough to consider any complete set of representatives of the double cosets  $Z \backslash G / Z$ .

The definition of the groups  $U$ ,  $T$ , and  $Z$  and the characters  $\theta$  and  $\alpha$  does not depend on the base  $\{e_i : r < i \leq n - r\}$  of the subspace  $\{e_i | r < i \leq n - r\}$ . Changing it, if necessary, we may assume that

$$(1) \quad (e_i, e_j) = \begin{cases} 0 & \text{if } i + j \neq n + 1, \\ 1 & \text{if } i + j = n + 1, \\ 0 & \text{if } i \neq j, r' < i \leq n - r', 1 \leq j \leq n; \end{cases} \quad 1 \leq i \leq r', 1 \leq j \leq n;$$

and that the restriction of the form  $(\dots, \dots)$  to the subspace  $\{e_i | r' < i \leq n - r'\}$  is anisotropic; if  $r' = r + 1$  we may assume that vectors  $e$  and  $e_{r+1}$  are not orthogonal. Then the representatives can be chosen in the form

$$(2) \quad g = m_1 w m_2, \quad w e_i = \omega_i e_i, \quad \omega_i \in \ell, \quad w e_i = e_i \quad \text{if both } e_i, \\ w e_i \in \{e_j | r < j \leq n - r\}$$

$$m_1, m_2 \in M = \{g \in G : g e_i = e_i, \quad \text{if } i \leq r \quad \text{or} \quad i > n - r\}.$$

Take an element  $g$  of the form (2) for which (1.6) is wrong for all unipotent  $z_1, z_2 \in Z$ . Define a function  $\varphi(i)$ ,  $1 \leq i \leq n$ , by the equality

$$(3) \quad we_i = \omega_i e_{\varphi(i)}.$$

2. If  $i < r$  then  $\varphi(i) \neq r$ .

PROOF: The vectors  $\{m_1 e_j : r < j \leq n - r\}$  form a base in the space  $\{e_j | r < j \leq n - r\}$ . Let

$$(4) \quad e = \sum_j \lambda_j m_1 e_j, \quad \lambda_j \in \ell, \quad r < j \leq n - r.$$

Since  $(e, e) \neq 0$  there exists an index  $j = s$  such that either  $\lambda_s \neq 0$  and  $(e_s, e_s) \neq 0$  or  $\lambda_s \neq 0$  and  $\lambda_{n+1-s} \neq 0$ . Therefore, putting either  $\bar{e} = m_1 e_s$  or  $\bar{e} = m_1 e_{n+1-s}$ , we obtain

$$(5) \quad g^{-1} \bar{e} \in \{e_j | 1 \leq j \leq n - r\}, \quad (\bar{e}, e) \neq 0.$$

Define

$$(6) \quad z_1 e_j = \begin{cases} e_j & \text{if } 1 \leq j \leq r \text{ or } n + 2 - r \leq j \leq n; \\ e_j - \lambda \bar{e} - \mu e_r, \quad \mu = \lambda \cdot \bar{\lambda} \cdot (\bar{e}, \bar{e})/2, & \text{if } j = n + 1 - r; \\ e_j + (e_j, \lambda \bar{e}) e_r & \text{if } r < j \leq n - r; \end{cases}$$

$$\tilde{z}_2 = g^{-1} z_1^{-1} g, \quad \lambda \in \ell.$$

Explicit formulas for  $\bar{e}$ ,  $z_1$ , and  $z_2$  allow one to check that if  $we_i = e_r$ ,  $i < r$ , then  $z_2 \in U$  and  $\theta(z_2) = 1$ ; since  $\theta(z_1) = \chi(\text{tr}(\lambda \bar{e}, e))$ , the elements  $g, z_1, z_2$  satisfy (1.6) for a suitable  $\lambda$ , which contradicts our choice of  $g$ .

3. If  $\varphi(i) > r$ ,  $i \leq r$ , and  $i' < i$ , then  $\varphi(i') > r$ .

PROOF: Otherwise there exist  $i \leq r$ ,  $i \geq 2$ , such that  $\varphi(i) > r$  and  $\varphi(i-1) \leq r$ ; hence,  $\varphi(i-1) < r$ . Then  $g$  would have satisfied (1.6) when

$$(7) \quad z_2 e_j = \begin{cases} e_j & \text{if } j \neq i, j \neq n + 2 - i, \\ e_j + \lambda e_{j-1}, & \text{if } j = i, \\ e_j - \bar{\lambda} e_{j-1} & \text{if } j = n + 2 - i; \end{cases}$$

$$z_1 = g(\tilde{z}_2)^{-1} g^{-1}, \quad \chi(\text{tr } \lambda) \neq 1, \quad \lambda \in \ell.$$

4. If the set  $\{i : i \leq r, r < \varphi(i) \leq n - r\}$  is empty, then the set  $\{i : i > n - r, r < \varphi(i) \leq n - r\}$  is empty too, and, consequently, the set  $\{i : r < i \leq n - r\}$  is invariant, hence, stable under  $\varphi$  (cf. no. 1). Therefore,

$$(8) \quad g = mw = wm, \quad m = m_1 \cdot m_2; \quad \tilde{g} = \tilde{w} \tilde{m} = \tilde{m} \tilde{w}.$$

Arguments of R. Steinberg [2], proof of Theorem 49, show that  $w = \bar{w}$ . Since

$$(9) \quad (me, e) = (e, m^{-1}e) = \overline{(m^{-1}e, e)} = (\tilde{m}e, e),$$

Witt's theorem (cf. [1], chap. 9, no. 3) provides an element

$$(10) \quad t \in T, \quad tme = \tilde{m}e = \tilde{m}\tilde{t}e.$$

Changing  $g$  into  $tg$  (cf. no. 1) we obtain

$$(11) \quad me = \tilde{m}e.$$

Now  $g$  satisfies (1.7) with

$$(12) \quad z_1 = \tilde{m}m^{-1}, \quad z_2 = id.$$

(When  $k = \ell$ ,  $\tilde{m} = m^{-1}$ , so  $z$  preserves both vectors  $e$  and  $me$  and coincides with  $m^{-2}$  on their orthogonal complement; if  $k \neq \ell$ ,  $\det z_1 = 1$ ; in both cases our restrictions on the character  $\tau$  guarantee the equality  $\alpha(z_1 z_2) = 1$ .)

Now consider the case when the set  $\{i : i \leq r, r < \varphi(i) \leq n - r\}$  is non-empty; then it consists of only one element. Denote it  $i_0$ . Necessarily  $r' = r + 1$ , and either  $\varphi(i_0) = r + 1$  or  $\varphi(i_0) = n - r$ .

5. If  $\varphi(i) > r$  and  $i \geq i_0$ , then  $i = i_0$ . Otherwise, in view of no. 3,  $\varphi(i_0 + 1) > n - r$ . Then  $g$  satisfies (1.6) with

$$(13) \quad z_2 e_j = \begin{cases} e_j & \text{if } j \neq i_0 + 1, j \neq n + 1 - i_0; \\ e_j + \lambda e_{j-1} & \text{if } j = i_0 + 1; \\ e_j - \bar{\lambda} e_{j-1} & \text{if } j = n + 1 - i_0; \end{cases}$$

$$z_1 = g(\tilde{z}_2)^{-1} g^{-1}, \quad \lambda \in \ell, \quad \chi(\text{tr } \lambda) \neq 1,$$

which contradicts our choice of  $g$ .

6. If one of the formulas (1.6), (1.7) is valid for an element  $g \in G$ , it is valid for  $\tilde{g}$  too.

PROOF: Applying the antiautomorphism (1.5) to the equations (1.6) and (1.7) we obtain the statement immediately with

$$(14) \quad z_1(\tilde{g}) = z_2(g), \quad \widetilde{z_2(\tilde{g})} = \widetilde{z_1(g)}.$$

Applying the statements of nos. 2–5 to  $\tilde{g}$  we obtain:

$$\varphi(r) \geq r \quad (\text{cf. no. 2});$$

$\varphi^{-1}(i) > n - r$  if and only if  $i < i_0$  (cf. no. 3 and no. 5);

$\varphi^{-1}(i_0)$  equals either  $r' = r + 1$  or  $n - r$  (cf. no. 4).

7. Matrices  $m_1$  and  $m_2$  can be chosen so that  $w = \bar{w}$ .

PROOF: Changing, if necessary,  $m_1$  and  $m_2$  obtain:

$$(15) \quad w(e_{r+1}) = -w^{-1}(e_{r+1}) = e_{i_0}.$$

Now the subspaces  $\{e_i | i < i_0 \text{ or } i > n+1-i_0\}$  and  $\{e_i | i_0 < i \leq r \text{ or } n-r < i \leq n-i_0\}$  are invariant under  $w$ . Applying Steinberg's arguments (loc. cit.) to the restrictions of  $w$  on these subspaces, we complete the proof.

8. Suppose  $w$ ,  $m_1$ , and  $m_2$  are chosen as in no. 7. If  $i_0 < r$ , then in view of no. 2  $\varphi(r) = r$ , and  $\varphi(n-r+1) = \varphi(n-r+1)$ . Consequently,  $g$  satisfies (1.6) with

$$(16) \quad z_1 e_i = \begin{cases} e_i & \text{if } i \neq n-r, i \neq n+1-r; \\ e_i + \lambda e_{r+1} & \text{if } i = n+1-r; \\ e - \bar{\lambda} e_r & \text{if } i = n-r; \end{cases}$$

$$z_2 = g^{-1} z_1 g$$

for a suitable  $\lambda \in \ell$ . Hence,  $i_0 = r$ .

If

$$(17) \quad (m_1 e_{r+1}, e) \neq 0.$$

Witt's theorem (Bourbaki, loc. cit.) provides an element  $t \in T$  such that

$$(18) \quad t m_1 e_{r+1} = \lambda e_{r+1}, \lambda \in \ell.$$

Replacing  $g$  by  $tg$  we may suppose  $m_1$  to be identity. If, in addition,

$$(19) \quad (m_2^{-1} e_{r+1}, e) \neq 0,$$

similar arguments lead us to equalities

$$(20) \quad g = wm, m e_{r+1} = \mu e_{r+1}, m e_{n-r} = \mu^{-1} e_{n-r}, \mu \in \ell.$$

Moreover,  $\mu = 1$ . Otherwise,  $g$  would have satisfied (1.6) with  $z_1, z_2$  given by (16) for a suitable  $\lambda \in \ell$  (remember that we assumed  $(e_{r+1}, e) \neq 0$ , cf. no. 1)).

Denote  $e'$  the orthogonal projection of the vector  $e$  on the subspace  $\{e_i | r+1 < i < n-r\}$ . According to our assumption (cf. no. 1) this subspace is anisotropic. Therefore,  $(e', e') \neq 0$ , and the arguments similar to those of no. 4 prove (1.7) for the element  $g$ .

If

$$(21) \quad (m_2^{-1} e_{r+1}, e) = 0$$

(while (17) is valid, i.e.  $m_1 = id$ ),  $g$  satisfies (1.6) with  $z_1, z_2$  given by (16) again.

Similarly, if

$$(22) \quad (m_1 e_{r+1}, e) = 0, (m_2^{-1} e_{r+1}, e) \neq 0.$$

At last, let both

$$(23) \quad (m_j e_{r+1}, e) = 0, j = 1, 2.$$

Then the subspace  $\{e_i | r+1 < i < n-r\}$  contains a vector  $\bar{e}$  for which

$$(24) \quad (m_1 \bar{e}, e) \neq 0$$

(otherwise  $e$  would have been proportional to  $m_1 e_{r+1}$ , which is impossible, for  $(e, e) \neq 0$ ). Therefore,  $g$  satisfies (1.6) with

$$(25) \quad m_1 z_1 (m_1)^{-1} e_i = \begin{cases} e_i & \text{if } i \leq r \text{ or } i > n+1-r \\ e_i + \lambda \bar{e} - \mu e_r, & \mu = \lambda \cdot \bar{\lambda} \cdot (\bar{e}, \bar{e})/2, \text{ if } i = n+1-r \\ e_i - (e_i, \lambda \bar{e}) e_r & \text{if } r < i \leq n-r, \end{cases}$$

$$\tilde{z}_2 = g^{-1} z_1^{-1} g$$

for a suitable  $\lambda \in \ell$ .

This completes the proof of the lemma and, therefore, of the theorem.

#### REFERENCES

- [1] N. BOURBAKI: *Éléments*. . . XXIV, Algèbre, Hermann, Paris, 1959.
- [2] R. STEINBERG: *Lectures on Chevalley Groups*. Yale University, 1967.
- [3] I. M. GELFAND, D. A. KAJDAN: *Representations of the Group  $GL(n, K)$ , where  $K$  is a Local Field*. The Institute for Applied Mathematics, No. 242, 1971, Moscow.
- [4] J. MILNOR: On isometries of inner product spaces. *Inventiones math.*, v. 8 (1969) 83–97.
- [5] M. E. NOVODVORSKY, I. I. PYATETSKY-SHAPIRO: Generalized Bessel models for symplectic group of rank 2. *Math. Sborn.*, v. 90 (132) 246–256.
- [6] M. E. NOVODVORSKY: On theorems of uniqueness of generalized Bessel models. *Math. Sborn.*, v. 90, 275–287.

(Oblatum 14–XI–1975)

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