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SYMMETRIC BASES IN MINKOWSKI SPACES

D. R. Lewis* and P. Wojtaszczyk

An analog of a well-known property of symmetric bases in infinite dimensional Banach spaces is established for finite dimensional spaces. It is shown that if a finite dimensional space E has a basis with the property that each permutation of indices naturally induces an isomorphism of norm at most λ , then E has a (possibly different) 9λ -unconditional basis. Restated in terms of symmetry parameters this answers a question posed by Gordon [2], which is implicit in the paper of Gurarii, Kadec and Macaev [4]. Some examples are given to show the non-isometric nature of the result.

Let $B = (b_i)_{i \in I}$ be a basic sequence (finite or countably infinite) in a normed space E . For π a permutation of I with $\pi(i) \neq i$ only finitely often, g_π is the isomorphism of E defined by $g_\pi(b_i) = b_{\pi(i)}$, $i \in I$; and for $(\varepsilon_i)_{i \in I}$ a sequence of scalars with $|\varepsilon_i| = 1$ for all $i \in I$ and $\varepsilon_i \neq 1$ only finitely often, g_ε is the operator defined by $g_\varepsilon(b_i) = \varepsilon_i b_i$. Three symmetry parameters of B are defined as follows:

the *unconditional basis constant* of B is $x(B) = \sup_\varepsilon \|g_\varepsilon\|$;
the *diagonal symmetry constant* of B is $\delta(B) = \sup_\pi \|g_\pi\|$;
and the *total symmetry constant* of B is $t(B) = \sup_{\varepsilon, \pi} \|g_\pi g_\varepsilon\|$.

Clearly $x(B) \leq t(B)$ and $\delta(B) \leq t(B)$ for every basis B , and it is known that $x(B) \leq 2\beta\delta(B)^2$, where β is the basis constant of B (cf. [7]). But also observe that no inequalities of the form $x(B) \leq f(\delta(B))$ or $\delta(B) \leq f(x(B))$ are valid for all bases, where f indicates a real function independent of the particular basis. A simple sequence of examples showing that the first relation cannot hold may be given as follows.

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For n odd let B be the unit vector basis of \mathbb{R}^n considered with the norm $\|x\| = \max \langle x, \varepsilon \rangle$, where the maximum is taken over all n -tuples $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ of signs with $\sum_{i \leq n} \varepsilon_i = 1$. Then $\delta(B) = 1$ and $x(B) = n$.

For p any one of the parameters x , δ or t define the corresponding symmetry parameter of E by $p(E) = \inf_B p(B)$, with the infimum taken over all bases for E .

The Banach Mazur distance between isomorphic spaces E and F is defined as $d(E, F) = \inf \|u\| \|u^{-1}\|$, the infimum being taken over all isomorphisms u between E and F . It is immediate that each of the three symmetry parameters of E defined above is continuous in the sense that $p(E) \leq d(E, F)p(F)$ holds for all E and F .

Although the diagonal and total symmetry constants of a particular basis in a finite dimensional space may behave quite differently, the diagonal and total symmetry constants of the space itself are equivalent. More precisely,

THEOREM 1: *The relations $\delta(E) \leq t(E) \leq 9\delta(E)$ hold in every finite dimensional space E .*

The first inequality is obvious. To prove the second it is convenient to first consider the case in which E has a basis $B = (b_i)_{i \leq n}$ with $\delta(B) = 1$. The coefficient functionals of the basis are denoted by $(b'_i)_{i \leq n}$, and m is the greatest integer satisfying $2m \leq n$. The group of all permutations of $\{1, 2, \dots, k\}$ is written S_k . The proof of the special case requires two lemmata, the first of which is given without proof.

LEMMA 1: *Let w_1, w_2, \dots, w_m and v_1, v_2, \dots, v_m be two finite sequences of non-negative reals with $v_i \geq v_{i+1}$ for $1 \leq i < m$. For $1 \leq k \leq m$ set $u_{2k} = u_{2k-1} = v_k$, and $u_n = 0$ if n is odd. Then*

$$\max_{\substack{\pi \in S_n \\ k \leq m}} \sum [u_{\pi(2k)} + u_{\pi(2k-1)}] w_k = 2 \max_{\substack{\tau \in S_m \\ k \leq m}} \sum v_{\tau(k)} w_k.$$

- LEMMA 2:** *Let $\|\cdot\|$ be any norm on E for which $\delta(B \subset (E, \|\cdot\|)) = 1$. Then*
- (a) $q = n^{-1}(\sum_{i \leq n} b'_i) \otimes (\sum_{k \leq n} b_k)$ is a norm one projection,
 - (b) $p = 2^{-1} \sum_{k \leq m} (b'_{2k} - b'_{2k-1}) \otimes (b_{2k} - b_{2k-1})$ is a norm one projection,
 - (c) $1_E - q = (n-1)m^{-1}(n!)^{-1} \sum_{\pi \in S_n} g_\pi^{-1} p g_\pi$, and
 - (d) $t((b_{2k} - b_{2k-1})_{k \leq m} \subset (E, \|\cdot\|)) = 1$.

PROOF OF LEMMA 2: Part (a) follows from the equality $q = (n!)^{-1} \sum_{\pi} g_\pi$, and (b) is true since $p = 2^{-1}(1 - g_\tau)$, where $\tau \in S_n$ interchanges $2k$ with

$2k-1$ for each $k = 1, 2, \dots, m$.

To verify (c) write $w = (n!)^{-1} \sum_{\pi} g_{\pi}^{-1} p g_{\pi}$. Since w commutes with each g_{π} , $w = s1_E + tq$ for some scalars s and t . Write H for the kernel of q . Then

$$s1_H = (n!)^{-1} \sum_{\pi} (g_{\pi}|H)^{-1} (p|H) (g_{\pi}|H)$$

and $p|H$ is a projection onto an m dimensional subspace of H , so

$$s(n-1) = \text{trace}(s1_H) = \text{trace}(p|H) = m.$$

Also

$$sn + t = \text{trace}(w) = \text{trace}(p) = m,$$

so that $s = -t = m(n-1)^{-1}$.

Finally, given signs $\delta_1, \delta_2, \dots, \delta_m$ and $\tau \in S_m$ let $\pi \in S_n$ be the permutation which maps $\{2k, 2k-1\}$ onto $\{2\tau(k), 2\tau(k)-1\}$, $1 \leq k \leq m$, and which satisfies $\pi(2k) = 2\tau(k)$ if $\delta_k = 1$, $\pi(2k) = 2\tau(k)-1$ if $\delta_k = -1$. For each $k = 1, 2, \dots, m$, $\delta_k(b_{2\tau(k)} - b_{2\tau(k)-1}) = g_{\pi}(b_{2k} - b_{2k-1})$, which proves (d).

PROOF OF THEOREM 1: Assume $\delta(B) = 1$ and let $(())$ be the norm on E defined by

$$((x)) = \max_{\pi, \varepsilon} \|pg_{\varepsilon}g_{\pi}(x)\|,$$

where $\| \cdot \|$ is the given norm on E and the maximum is over all $\pi \in S_n$ and n -tuples of signs ε . Let F denote E under $(())$. Notice that each operator $g_{\varepsilon}g_{\pi}$ is an isometry of F , and hence $t(F) = \delta(B \subset F) = 1$, so the assumptions of Lemma 2 are satisfied by the basis B in both norms, $(())$ and $\| \cdot \|$.

The first claim is that $((x)) = \|x\|$ for all x in $[b_{2k} - b_{2k-1}]_{k \leq m}$, the span of the vectors $b_{2k} - b_{2k-1}$. The inequality $((x)) \geq \|x\|$ is immediate, and for the other direction it is enough, by Lemma 2(d), to consider vectors of the form $x = \sum_{k \leq m} a_k(b_{2k} - b_{2k-1})$ with $|a_k| \geq |a_{k+1}|$ for $1 \leq k < m$. For ε an n -tuple of signs and $\pi \in S_n$, choose $x' \in E'$ so that $\|x'\| = 1$ and

$$\begin{aligned} \|pg_{\varepsilon}g_{\pi}^{-1}(x)\| &= |\langle pg_{\varepsilon}g_{\pi}^{-1}(x), x' \rangle| \\ &\leq 2^{-1} \sum_{k \leq m} [|\langle x, b'_{\pi(2k)} \rangle| + |\langle x, b'_{\pi(2k-1)} \rangle|] |\langle b_{2k} - b_{2k-1}, x' \rangle| \end{aligned}$$

Applying Lemma 1 with $v_k = |a_k|$ and $w_k = |\langle b_{2k} - b_{2k-1}, x' \rangle|$ shows that

$$\begin{aligned} \|pg_\pi g_\pi^{-1}(x)\| &\leq \max_{\tau \in S_m} \sum_{k \leq m} |a_{\tau(k)}^{-1}| |\langle b_{2k} - b_{2k-1}, x' \rangle| \\ &= \max_{\tau, |\delta|=1} |\langle \sum_{k \leq m} \delta_k a_k (b_{2\tau(k)} - b_{2\tau(k)-1}), x' \rangle| \leq \|x\|, \end{aligned}$$

the last by part (d) of Lemma 2.

We next assert that the inequalities

$$\|(1-q)(x)\| \leq 2((x)) \quad \text{and} \quad (((1-q)(x))) \leq 2\|x\|$$

hold for all $x \in E$. For the first, applying Lemma 2 with both $\| \cdot \|$ and $((\cdot))$ yields

$$\begin{aligned} \|(1-q)(x)\| &\leq (n-1)m^{-1}(n!)^{-1} \sum_{\pi} \|g_\pi^{-1} p g_\pi(x)\| \\ &\leq 2(n!)^{-1} \sum_{\pi} \|p g_\pi(x)\| \\ &= 2(n!)^{-1} \sum_{\pi} ((p g_\pi(x))) \\ &\leq 2(n!)^{-1} \sum_{\pi} ((g_\pi(x))) \\ &= 2((x)), \end{aligned}$$

and the other inequality follows by interchanging the rôles of $\| \cdot \|$ and $((\cdot))$.

Now let λ be the constant satisfying

$$\lambda \|\sum_{i \leq n} b_i\| = ((\sum_{i \leq n} b_i))$$

and define $u: F \rightarrow E$ by $u = 1_E + (\lambda - 1)q$. Since $((q(x))) = \|q(x)\|\lambda$ for all $x \in E$, $\|u(x)\| \leq \|(1-q)(x)\| + ((q(x))) \leq 3((x))$ by the preceding paragraph and Lemma 2, and hence $\|u\| \leq 3$. But $u^{-1} = 1_E + (\lambda^{-1} - 1)q$ so the same proof gives $((u^{-1}(x))) \leq 3\|x\|$ and thus $d(E, F) \leq 9$. Then

$$t(E) \leq t(F)d(E, F) \leq 9.$$

More generally for $B \subset E$ any basis let H be E with the norm $|x| = \max_{\pi} \|g_{\pi}(x)\|$. Since $\delta(B \subset H) = 1$ and $d(E, H) \leq \delta(B)$ the special case shows that $t(E) \leq d(E, H)t(H) \leq 9\delta(B)$. Taking the infimum over all possible bases for E completes the proof of the theorem.

REMARK 1: In [4] Gurarii, Kadec and Macaev define the symmetry parameter α of a finite dimensional space E by $\alpha(E) = \inf_B x(B)\delta(B)$. The theorem implies that $\delta(E) \leq \alpha(E) \leq 9\delta(E)^2$, answering a question raised by Gordon [2] and by Lewis [5].

REMARK 2: Let E be a Banach space with a diagonally symmetric basis $B = (b_i)_{i \geq 1}$. Then for any $\varepsilon > 0$ and any finite dimensional $F \subset E$ there is finite dimensional W with $F \subset W \subset E$ and $t(W) \leq (9 + \varepsilon)\delta(B)$. This follows from a routine perturbation argument and the fact that $t([b_1, b_2, \dots, b_n]) \leq 9\delta(B)$ for all n . Thus, although the unconditional basis constant of B depends on the basis constant of B , the local unconditional structure of E depends only on $\delta(B)$.

Following [1] define the *asymmetry constant* of a finite dimensional space E by

$$s(E) = \inf_G \sup_{g \in G} \|g\|,$$

with the infimum taken over all compact groups G of isomorphisms of E which have the property that only scalar multiples of the identity commute with the elements of G .

It is clear that $s(E) \leq t(E)$, so the following theorem strongly indicates the non-isometric nature of the relationship between $\delta(E)$ and $t(E)$.

THEOREM 2: *There is a sequence $(E_n)_{n \geq 5}$ of Minkowski spaces with $\dim E_n = n$, $\delta(E_n) = 1$ and $\liminf_n s(E_n) \geq (2^{-1} + 2^{-\frac{1}{2}})^{\frac{1}{2}}$.*

PROOF: Let $e_i \in l_\infty^{n+1}$ and $e'_i \in l_1^{n+1}$ be the unit vectors, $1 \leq i \leq n$ and $E_n \subset l_\infty^{n+1}$ be the kernel of $\sum_{i \leq n} e'_i + \lambda e'_{n+1}$ (a sequence of values for λ will be specified later). The basis $b_i = e_i - \lambda^{-1}e_{n+1}$, $1 \leq i \leq n$, has diagonal symmetry constant one so $\delta(E_n) = 1$. To estimate $s(E_n)$ from below we use the inequality [1]

$$s(E_n)^2 \geq n^{-1} \gamma_\infty(E_n) \pi_1(E_n),$$

with $\gamma_\infty(E_n)$ and $\pi_1(E_n)$ denoting, respectively, the projection constant of E_n and the 1-absolutely summing norm [6] of the identity on E_n . Write G for the group of isometries of l_∞^{n+1} of form $g(e_i) = e_{\pi(i)}$ for some $\pi \in S_{n+1}$ with $\pi(n+1) = n+1$.

If w is a projection of l_∞^{n+1} onto E_n with $\|w\| = \gamma_\infty(E_n)$, then $u = |G|^{-1} \sum_{g \in G} g^{-1}wg$ is also a projection onto E_n with norm $\gamma_\infty(E_n)$. Since u commutes with each element of G

$$u = 1 - \left(\sum_{i \leq n} e'_i + \lambda e'_{n+1} \right) \otimes \left(t \sum_{i \leq n} e_i + s e_{n+1} \right)$$

for some scalars s and t with $tn + \lambda s = \text{trace}(1 - u) = 1$. Thus

$$\begin{aligned} \gamma_\infty(E_n) &= \|u\| = \max \{ |1 - t| + (n + \lambda - 1)|t|, n|s| + |1 - s\lambda| \} \\ &\geq \inf_t \max \{ |1 - t| + (n + \lambda - 1)|t|, n\lambda^{-1}|1 - nt| + n|t| \} \\ &= 2n(n - 1)(n^2 - 2\lambda + \lambda^2)^{-1}. \end{aligned}$$

To estimate $\pi_1(E_n)$ from below, there is by Pietsch's Theorem [6] a measure μ on $\Omega = \{e'_i | E_n : 1 \leq i \leq n + 1\}$ such that $\|\mu\| = \pi_1(E_n)$ and $\|x\| \leq \mu(|\langle x, \cdot \rangle|)$ for all $x \in E_n$. Let ν be a measure on Ω given by $\nu(f) = |G|^{-1} \sum_{g \in G} \mu(fog)$, so that $\|\nu\| = \pi_1(E_n)$, $\|x\| \leq \nu(|\langle x, \cdot \rangle|)$ for $x \in E_n$ and $\nu(f) = \nu(fog)$ for all $f \in C(\Omega)$ and $g \in G$. The last implies that $s = \nu(\{b'_i\})$ is independent of i , $1 \leq i \leq n$. Setting $t = \nu(\{b'_{n+1}\})$ gives scalars s and t satisfying $\pi_1(E_n) = sn + t$ and

$$\|x\| \leq s \sum_{i \leq n} |\langle x, e'_i \rangle| + t|\langle x, e_{n+1} \rangle|, x \in E_n.$$

Substituting $e_1 - \lambda^{-1}e_{n+1}$ and $e_1 - e_2$ in the last inequality shows that $s + t\lambda^{-1} \geq 1$ and $2s \geq 1$, so that

$$\pi_1(E_n) = sn + t \geq 2^{-1}(n + \lambda).$$

Now vary λ with n by taking $\lambda_n = [2n(n - 1)]^{\frac{1}{2}} - n$ for $n \geq 5$. Combining inequalities yields the desired lower estimate.

REMARK 3: As is observed above every Minkowski space satisfies $x(E) \leq t(E)$ and $s(E) \leq t(E)$. Some other possible relations between the three parameters x , t and s are known to be false. The space $A_n = l_1^n \oplus l_2^n$ has unconditional basis constant one but $s(A_n)$ and $t(A_n)$ behave asymptotically like $n^{\frac{1}{2}}$ [1], and the tensor product $B_n = l_2^n \hat{\otimes} l_2^n$ has asymmetry constant one but $x(B_n)$ and $t(B_n)$ both act asymptotically like $n^{\frac{1}{2}}$ [3]. Such examples suggest the following problem. Is there a real function f of two variables such that $t(E) \leq f(x(E), s(E))$ for all finite dimensional E ?

The answer to this problem is negative as is shown by the following example due to J. Lindenstrauss.

EXAMPLE: Let

$$E_n = \left(\sum_{k=1}^n E_k^n \right)_{l_4}$$

where each E_k^n is isometric to an n^2 dimensional Hilbert space. Then $s(E_n) = x(E_n) = 1$ but $t(E_n) \rightarrow_{n \rightarrow \infty} \infty$. The first two statements are clear so we will prove only the last one. Let us start with the observation that E_n is isometric to a subspace of $L_4[0, 1]$.

LEMMA 1: Let $B = (b_i)_{i \leq n}$ be a normalized basic sequence in $L_4[0, 1]$ with $t(B) = \alpha$. Then $\text{span} \{b_i\}_{i \leq n}$ is α -isomorphic to a subspace of $l_2^n \oplus_4 l_4^n$.

PROOF: The expression

$$\int_0^1 \left| \sum_{i=1}^n \lambda_i b_i \right|^4$$

is α -equivalent to

$$\frac{1}{2^n n!} \sum_{\varepsilon} \sum_{\sigma} \int_0^1 \left| \sum_{i=1}^n \varepsilon_i \lambda_i b_{\sigma(i)} \right|^4$$

where $\varepsilon = (\varepsilon_i)_{i=1}^n$ ranges over all 2^n choices of signs and σ over all $n!$ permutations of $\{1, 2, \dots, n\}$. This latter sum is of the form

$$a \sum_{i=1}^n \lambda_i^4 + c \left(\sum_{i=1}^n \lambda_i^2 \right)^2$$

for suitable positive a and c .

LEMMA 2: Let $E \subset l_4^n$ $\dim E \geq n^{\frac{3}{2}}$ then $d(E, l_2^{\dim E}) \rightarrow_{n \rightarrow \infty} \infty$.

This Lemma follows immediately from Corollary 3.1 of [8]. Using those two Lemmas we will estimate $t(E_n)$. Suppose $t(E_n) \leq C$ for $n = 1, 2, 3, \dots$ then E_n embeds uniformly into $l_2^{n^3} + l_4^{n^3}$. Denote

$$\beta_k = \|P \circ \varphi|E_k^n\|$$

where φ is an isomorphic embedding from E_n into $l_2^{n^3} \oplus l_4^{n^3}$ and P is a projection from $l_2^{n^3} \oplus l_4^{n^3}$ onto $l_2^{n^3}$ annihilating $l_4^{n^3}$. Let

$$z_k \in E_k^n, \quad \|z_k\| = 1, \quad \|P \circ \varphi(z_k)\| = \beta_k.$$

Then for some ε_k , $|\varepsilon_k| = 1$

$$n^{\frac{1}{2}} = \left\| \sum_{k=1}^n \varepsilon_k z_k \right\| \geq \frac{1}{C} \|P \circ \varphi(\sum_{k=1}^n \varepsilon_k z_k)\| \geq \frac{1}{C} \left(\sum_{k=1}^n |\beta_k|^2 \right)^{\frac{1}{2}}.$$

But this implies that for n big enough at least one β_k must be very small. Then an easy perturbation argument implies that $l_4^{n^3}$ contains uniformly $l_2^{n^2}$ which contradicts Lemma 2.

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