COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 31, nº 3 (1975), p. 309-343
<http://www.numdam.org/item?id=CM_1975__31_3_309_0>

COMPOSITIO MATHEMATICA, Vol. 31, Fasc. 3, 1975, pag. 309–343
Noordhoff International Publishing
Printed in the Netherlands

JOINS OF SCHEMES, LINEAR PROJECTIONS

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Introduction

Developed here is the basic theory of the join of two projective schemes. Intuitively, the join is the locus of points on the lines joining two schemes in general position in an ambient projective space. The (projective projecting) cone is the important special case in which one of the two schemes is equal to the base scheme.

The join is an old and fruitful construction in algebraic geometry. Recently, joins have been used in proofs of Chow's moving lemma (cf. Chow [3], Samuel [11], Chevalley [2], Roberts [10]) and by Lascu and Scott [9] in an interesting determination of the change in the Chern classes of a variety undergoing a blowing-up.

The theory of joins is closely related to the theory of maps of the form Proj (a), thought of as linear projections. The theory of linear projections is developed at the end, and it includes the determination of the blow-up of the source along the center of projection. One such determination is a key step in Holme's important work [6] giving a numerical characterization of the smallest projective space into which a given smooth projective variety can be embedded by means of a linear projection. Another is used in the theory of Lefschetz pencils (see Katz [7]).

We ourselves were led to develop a theory of joins because we needed a number of properties of cones to study an example of a divisorial cycle acquiring an embedded component under a flat specialization [1]. There is an alternate treatment of the theory of cones in EGA II, 8, but it does not contain all the results we needed. Moreover, our development of the theory of joins seems more concise, less technical, and easier to comprehend, as well as being more general than EGA's theory of cones.

The first section consists of preliminaries, which could have been in EGA II. It divides naturally into two parts. The first part deals with the Serre map,

$$\alpha_Y^{\#} : \mathscr{S}_Y \to \operatorname{Sym}(\mathscr{O}_Y(1)),$$

where Y is a projective scheme Proj $(\mathscr{S})$. The principal result asserts that the linear embedding, Proj $(\alpha_Y^\sharp)$, is equal to the diagonal map, $\Delta_Y : Y \to Y \times Y$. While the details are not carried out here, this result can easily be made to yield the functorial characterization of T-points of Proj $(\mathscr{S})$ as homogeneous 1-quotients of $\mathscr{S}_T$ (a homogeneous 1-quotient of $\mathscr{S}_T$ is a homogeneous quotient of $\mathscr{S}_T$ that is isomorphic, as a graded $\mathscr{O}_T$-Algebra, to Sym (L) for some invertible $\mathscr{O}_T$-Module L). In fact, the result can easily be made to yield for a map, $a : \mathscr{S} \to \mathscr{T}$ of graded Algebras, the functorial characterization of the T-points of the scheme $G(a)$ as the homogeneous 1-quotients $\mathscr{L}$ of $\mathscr{T}_T$ such that the natural composition,

$$\mathscr{S}_T \overset{a_T}{\to} \mathscr{T}_T \to \mathscr{L},$$

is surjective.

The second part of the first section deals with tensor products of graded Algebras, and the principal result here asserts that the open subscheme $G(c)$ of Proj $(\mathscr{R} \otimes_{\mathscr{S}} \mathscr{T})$ is the product over Proj $(\mathscr{S})$ of the open subschemes $G(a)$ of Proj $(\mathscr{T})$ and $G(b)$ of Proj $(\mathscr{R})$, where $a : \mathscr{S} \to \mathscr{T}$ and $b : \mathscr{S} \to \mathscr{R}$ and $c : \mathscr{S} \to \mathscr{R} \otimes_{\mathscr{S}} \mathscr{T}$ denote the structure maps. (This assertion could also be derived from the above characterization of the T-points of $G(a)$.)

The second section contains the basic definitions and compatibilities of the theory of joins. The join of two projective schemes, Proj $(\mathscr{R})$ and Proj $(\mathscr{S})$, is defined as "Proj" of the tensor product, Proj $(\mathscr{R} \otimes \mathscr{S})$ (thus, it depends upon the embeddings of Proj $(\mathscr{R})$ and Proj $(\mathscr{S})$ in projective space). There are several more basic definitions. The two fundamental embeddings and the two fundamental retractions are defined as linear maps arising naturally from two augmentation maps and two structure maps. The two conjunctive embeddings are defined as linear embeddings arising naturally from Serre maps, and the conjunctive transforms are equal to the images. The compatibilities of these notions with linear embedding and with base change are derived easily from the definitions and are important in the remainder of the development.

The third section contains the principal results of the theory of joins. The key result is the first, which asserts that the conjunctive transform is isomorphic to the join off the exceptional locus and that the exceptional locus is the inverse image of the fundamental subscheme. All the important structural results follow from this result and from the definitions. The main result, which asserts that the conjunctive transform is equal to the blow-up of the join along a fundamental subscheme, is now easy to anticipate, but nevertheless somewhat difficult to prove.

The fourth section deals with linear projections using the ideas developed in the first three sections. The first result asserts that the blow-up of the source of a linear projection along the center of projection is equal to the closure of the graph of the linear projection; in other words, the fundamental points of a linear projection can be eliminated by blowing up along the center of projection. The main results give an explicit description of this blow-up as a certain "Proj" over the target of the linear projection, valid under certain useful hypotheses. Three blow-ups are computed as examples: the blowing-up of a linear subscheme of a projective scheme, the blowing-up of the diagonal of a product of a projective space with itself, and the blowing-up of the axis of an r-fold family of r-codimensional linear space sections.

Blanket notation, see also (*B*1) *and* (*B*7)

Fix S, a base scheme. Let $\mathscr{C}(S)$ denote the category whose objects are the graded, quasi-coherent $\mathcal{O}_S$-Algebras generated as $\mathcal{O}_S$-Algebras by their terms of degree one (and so they are zero in negative degrees and $\mathcal{O}_S$ in degree zero) and whose morphisms are the homogeneous $\mathcal{O}_S$-Algebra homomorphisms. Let $\mathscr{C}(S)_{\text{surj}}$ denote the subcategory with the same objects but with only the surjective maps in $\mathscr{C}(S)$ for morphisms.

Fix an object $\mathscr{S}$ of $\mathscr{C}(S)$; set

$$Y = \text{Proj}(\mathscr{S});$$

and let $f : Y \to S$ denote the structure morphism.

Fix maps $a : \mathscr{S} \to \mathscr{T}$ in $\mathscr{C}(S)$ and $u : \mathscr{S} \to \mathscr{S}'$ in $\mathscr{C}(S)_{\text{surj}}$. Set

$$Z = \text{Proj}(\mathscr{T}),$$

and let g denote the (affine) S-morphism,

$$g = \text{Proj}(a) : G(a) \to Y, \qquad \text{(EGA II, 3.5.1)},$$

where $G(a)$ denotes the open subscheme, $(Z - V_+(a(\mathscr{S}_+)))$, of Z. Set

$$j = \text{Proj}(u) : Y' = \text{Proj}(\mathscr{S}') \to Y,$$

the linear embedding defined by u.

A. General lemmas on Proj

(*A*1) *The canonical surjection* $\alpha_Y^\#$

Let M be a quasi-coherent $\mathcal{O}_S$-Module, and regard it as a graded

$\mathcal{O}_S$-Module concentrated in degree 0. There is a useful functorial isomorphism on Y,

$$f^*M = (M \underset{\mathcal{O}_S}{\otimes} \mathcal{S})^{\sim}, \qquad \text{(cf. last paragraph of EGA II, 3.3.2).} \tag{A1.1}$$

For future reference, note the formula, which holds for any integer n,

$$f^*M \otimes \mathcal{O}_Y(n) = (M[n] \otimes \mathcal{S})^{\sim}, \tag{A1.2}$$

which holds because, clearly, $M[n] \otimes \mathcal{S}$ is equal to $(M \otimes \mathcal{S}) \otimes_{\mathcal{S}} \mathcal{S}[n]$ and because tilde is compatible with tensor product (EGA II, 3.2.6).

For each nonnegative integer n, there is a canonical map,

$$s_n : \mathcal{S}_n \otimes \mathcal{S} \to \mathcal{S}[n]. \tag{A1.3}$$

Applying tilde yields, by (A1.1), a canonical $\mathcal{O}_Y$-homomorphism,

$$\alpha^{\#}_{Y,n} : \mathcal{S}_{Y,n} \to \mathcal{O}_Y(n);$$

(It is equal to the adjoint of the Serre homomorphism, $\alpha_{Y,n} : \mathcal{S}_n \to f_*\mathcal{O}_Y(n)$, by the last paragraph of (EGA II, 3.3.2).) Since $\mathcal{S}$ is generated by $\mathcal{S}_1$, there is a canonical isomorphism,

$$\bigoplus_{n \geqq 0} \mathcal{O}_Y(n) = \mathrm{Sym}\,(\mathcal{O}_Y(1)) \qquad \text{(EGA II, 3.2.7.2).}$$

So, summing the $\alpha^{\#}_{Y,n}$ yields an important map in $\mathcal{C}(Y)$,

$$\alpha^{\#}_Y : \mathcal{S}_Y \to \mathrm{Sym}\,(\mathcal{O}_Y(1)).$$

Since $\mathcal{S}$ is generated by $\mathcal{S}_1$, the maps s_n of (A1.3) are obviously (TN)-surjective; (in fact, their components of degree d are surjective for $d \geqq 0$). So, $\alpha^{\#}_Y$ is surjective by (EGA II, 3.4.2).

(A2) Lemma: *There is a natural commutative diagram on the open subscheme $G(a)$ of Z,*

$$\begin{array}{ccc} \mathcal{S}_{G(a)} & \xrightarrow{a_{G(a)}} & \mathcal{T}_{G(a)} \\ {\scriptstyle g^*(\alpha^{\#}_Y)} \downarrow & & \downarrow {\scriptstyle \alpha^{\#}_Z | G(a)} \\ g^*\,\mathrm{Sym}\,(\mathcal{O}_Y(1)) & \xrightarrow{\eta} & \mathrm{Sym}\,(\mathcal{O}_Z(1))|G(a). \end{array} \tag{A2.1}$$

Moreover η is an isomorphism.

Proof: For each integer n, consider the natural commutative diagram on S,

$$\begin{array}{ccc}
(\mathscr{S}_n \underset{\mathcal{O}_S}{\otimes} \mathscr{S}) \underset{\mathscr{S}}{\otimes} \mathscr{T} & \xrightarrow{a_n \otimes \mathscr{T}} & \mathscr{T}_n \underset{\mathcal{O}_S}{\otimes} \mathscr{T} \\
{\scriptstyle s_n \otimes \mathscr{T}}\downarrow & & \downarrow \\
\mathscr{S}[n] \underset{\mathscr{S}}{\otimes} \mathscr{T} & \longrightarrow & \mathscr{T}[n].
\end{array}$$

The lower horizontal map is obviously an isomorphism. Applying tilde yields the following commutative diagram on Z, in which the bottom map is an isomorphism:

$$\begin{array}{ccc}
((\mathscr{S}_n \underset{\mathcal{O}_S}{\otimes} \mathscr{S}) \underset{\mathscr{S}}{\otimes} \mathscr{T})^{\sim} & \xrightarrow{a_{Z,n}} & \mathscr{T}_{n,Z} \\
{\scriptstyle (s_n \underset{\mathscr{S}}{\otimes} \mathscr{T})^{\sim}}\downarrow & & \downarrow{\scriptstyle \alpha^{\sharp}_{Z,n}} \\
(\mathscr{S}[n] \underset{\mathscr{S}}{\otimes} \mathscr{T})^{\sim} & \xrightarrow{\sim} & \mathcal{O}_Z(n).
\end{array}$$

Restrict this diagram to $G(a)$. Since tilde is compatible with g (EGA II, 3.5.2, (ii)), the diagram becomes a commutative diagram, in which the bottom map is an isomorphism,

$$\begin{array}{ccc}
\mathscr{S}_{G(a),n} & \xrightarrow{a_{G(a),n}} & \mathscr{T}_{G(a),n} \\
{\scriptstyle g^*(\alpha^{\sharp}_{Y,n})}\downarrow & & \downarrow{\scriptstyle \alpha^{\sharp}_{Z,n}|G(a)} \\
g^*(\mathcal{O}_Y(n)) & \xrightarrow{\sim} & \mathcal{O}_Z(n)|G(a).
\end{array}$$

Summing these diagrams yields the desired diagram (A2.1) with isomorphism η.

(A3) Lemma: *Use the structure maps to identify* Proj (Sym ($\mathcal{O}_Z(1)$)) *with* Z (EGA II, 3.1.7 *and* 3.1.8, (iii)) *and* Proj ($\mathscr{T}_Z$) with $Z \times_S Z$ (EGA II, 3.5.3). *Then* Proj ($\alpha^{\sharp}_Z$) *is equal to the diagonal map*, $\Delta_Z : Z \to Z \times_S Z$.

Proof: Take $\mathscr{S}$ to be Sym ($\mathscr{T}_1$) and take $a : \mathscr{S} \to \mathscr{T}$ to be the canonical surjection. Then $g = \text{Proj}\,(a) : Z \to Y$ is a (linear) embedding. Consider a diagram,

$$\begin{array}{ccccc}
Y \times_S Y & \xleftarrow{Y \times g} & Y \times_S Z & \xleftarrow{g \times Z} & Z \times_S Z \\
{\scriptstyle \text{Proj}\,(\alpha^{\sharp}_Y)}\uparrow & & \uparrow{\scriptstyle \text{Proj}\,(\alpha^{\sharp}_Y|Z)} & & \uparrow{\scriptstyle \text{Proj}\,(\alpha^{\sharp}_Z)} \\
Y & \xleftarrow{g} & Z & \xleftarrow{id} & Z
\end{array}$$

The left-hand square is commutative because Proj is compatible with base change (EGA II, 3.5.3). The right-hand square is obtained by applying Proj to the commutative diagram (A2.1); so it is commutative because Proj is a 'functor' (EGA II, 2.8.4).

It follows immediately from (EGA II, 4.2.3) (indeed, it is the principal result) that Proj $(\alpha_Y^\sharp)$ is equal to the diagonal map, $\Delta_Y : Y \to Y \times_S Y$ because Y is equal to $\mathbb{P}(\mathscr{T}_1)$. Hence, since the above diagram is commutative, $\Delta_Y \circ g$ is equal to $(g \times g) \circ$ Proj $(\alpha_Z^\sharp)$. Obviously, $\Delta_Y \circ g$ is equal to $(g \times g) \circ \Delta_Z$. Therefore, since $(g \times g)$ is a monomorphism, the maps, Proj $(\alpha_Z^\sharp)$ and Δ_Z, are equal.

(A4) Lemma: *Consider a tensor product diagram (pushout diagram) in the category $\mathscr{C}(S)$, (actually $\mathscr{R}$ and $\mathscr{T}$ need not be generated by their components with degree* 1),

(A4.1)
$$\begin{array}{ccc} \mathscr{R} \otimes_{\mathscr{S}} \mathscr{T} & \xleftarrow{\ t\ } & \mathscr{T} \\ {\scriptstyle r}\uparrow & \nwarrow{\scriptstyle c} & \uparrow{\scriptstyle a} \\ \mathscr{R} & \xleftarrow{\ b\ } & \mathscr{S}, \end{array}$$

where c denotes the composition, $r \circ b = t \circ a$. Then the following diagram of S-schemes is cartesian:

(A4.2)
$$\begin{array}{ccc} G(c) & \xrightarrow{\text{Proj}\,(t)|} & G(a) \\ {\scriptstyle \text{Proj}\,(r)|}\downarrow & \square & \downarrow{\scriptstyle \text{Proj}\,(a)} \\ G(b) & \xrightarrow{\text{Proj}\,(b)} & Y. \end{array}$$

Proof: The assertion is clearly local on S, so we may assume S is affine (cf. EGA I, 3.2). Set $A = \Gamma(S, \mathscr{S})$, set $R = \Gamma(S, \mathscr{R})$, and set $T = \Gamma(S, \mathscr{T})$. Let h be a homogeneous element of A. Then there are canonical isomorphisms (EGA II, 2.4.1 and 2.8.1),

$$Y_h = \text{Spec}\,(A_{(h)}), \qquad \text{Proj}\,(b)^{-1}(Y_h) = \text{Spec}\,(R_{(h)}),$$

$$\text{Proj}\,(a)^{-1}(Y_h) = \text{Spec}\,(T_{(h)}), \qquad \text{Proj}\,(c)^{-1}(Y_h) = \text{Spec}\,((R \otimes_A T)_{(h)}).$$

Assume h has degree 1. Then there is a canonical isomorphism (cf. the proof of EGA II, 2.5.13),

$$(R \otimes_A T)_{(h)} = R_{(h)} \otimes_{A_{(h)}} T_{(h)}.$$

Therefore, by (EGA I, 3.2.1.3), there is a canonical isomorphism,

$$\mathrm{Proj}\,(c)^{-1}(Y_h) = \mathrm{Proj}\,(a)^{-1}(Y_h) \underset{Y_h}{\times} \mathrm{Proj}\,(b)^{-1}(Y_h).$$

Since A is generated by A_1, such Y_h cover Y. Hence $G(c)$ is canonically isomorphic (cf. EGA I, 3.2) to the fibered product $G(a) \times_Y G(b)$.

(A5) Lemma: *Consider the tensor product diagram in* $\mathscr{C}(S)$,

$$\begin{array}{ccc} \mathscr{T} \underset{\mathscr{S}}{\otimes} \mathscr{S}' & \xleftarrow{a'} & \mathscr{S}' \\ {\scriptstyle u'}\uparrow & \otimes & \uparrow{\scriptstyle u} \\ \mathscr{T} & \xleftarrow{a} & \mathscr{S}. \end{array}$$

(i) *The following diagram of S-schemes is cartesian*:

(A5.1)
$$\begin{array}{ccc} G(a') & \xrightarrow{\mathrm{Proj}\,(a')} & Y' \\ {\scriptstyle \mathrm{Proj}\,(u)|}\downarrow & \square & \downarrow{\scriptstyle \mathrm{Proj}\,(u) = j} \\ G(a) & \xrightarrow{\mathrm{Proj}\,(a) = g} & Y. \end{array}$$

(ii) *If a is surjective, then the following diagram of S-schemes is cartesian*:

(A5.2)
$$\begin{array}{ccc} \mathrm{Proj}\,(\mathscr{T} \underset{\mathscr{S}}{\otimes} \mathscr{S}') & \longrightarrow & Y' \\ \downarrow & \square & \downarrow \\ Z & \longrightarrow & Y. \end{array}$$

Proof: (i) Since u is surjective, the images $a'(u(\mathscr{S}_+))$ and $a'(\mathscr{S}'_+)$ are equal; hence, $G(a' \circ u)$ is equal to $G(a')$. Also since u is surjective, $G(u)$ is equal to Y'. So, (A5.1) is cartesian by (A4).

(ii) Since a is surjective, a' is also surjective. So, diagram (A5.1) becomes diagram (A5.2).

B. Definitions and compatibilities in the theory of joins

More blanket notation

For each object $\mathscr{h}$ of $\mathscr{C}(S)$, let ε denote the augmentation map, $\varepsilon : \mathscr{h} \to \mathcal{O}_S$, and let ρ denote the structure map, $\rho : \mathcal{O}_S \to \mathscr{h}$.

Fix an element $\mathscr{R}$ of $\mathscr{C}(S)$, and set

$$X = \mathrm{Proj}\,(\mathscr{R}).$$

(B1) *Joins*

The *join* $J(\mathscr{R}, \mathscr{S})$ of $\mathscr{R}$ and $\mathscr{S}$ is the projective S-scheme defined by the formula,

$$J(\mathscr{R}, \mathscr{S}) = \operatorname{Proj}(\mathscr{R} \underset{\mathcal{O}_S}{\otimes} \mathscr{S}).$$

It will also be called the join of X and Y and denoted $X * Y$ when no confusion is likely.

Join is a (contravariant) functor of two variables on the category $\mathscr{C}(S)_{\mathrm{surj}}$ because of the functoriality of Proj (cf. EGA II, 2.8.4). In geometric terms, the morphism $J(\mathscr{R}, u)$ of $J(\mathscr{R}, \mathscr{S}')$ into $J(\mathscr{R}, \mathscr{S})$ is denoted

$$X * j : X * Y' \to X * Y.$$

Note that it is a (linear) embedding because it is 'Proj' of a surjection. Hence, the join of closed subschemes is in a natural way a closed subscheme of the join of their ambient schemes.

The *fundamental embedding*,

$$v : Y \to X * Y,$$

is defined as the linear S-embedding, $J(\varepsilon, \mathscr{S})$. Its image is denoted V_Y and is called a *fundamental subscheme* of $X * Y$.

The *fundamental retraction*,

$$r : G(\rho \otimes \mathscr{S}) \to Y,$$

is defined as the (affine) S-morphism, $\operatorname{Proj}(\rho \otimes \mathscr{S})$. Since the composition $\varepsilon \circ \rho$ is equal to the identity of $\mathscr{S}$ and since Proj is 'functor' (EGA II, 2.8.4), the fundamental subscheme V_Y is contained in the open subscheme $G(\rho \otimes \mathscr{S})$ of $X * Y$, and there is a formula,

(B1.1) $$r \circ v = id_Y,$$

justifying the use of the term retraction for r.

The formation of the above objects and maps is compatible with base change basically because Proj is so (EGA II, 3.5.3). In particular, for any S-scheme T, these formulas hold:

(B1.2) $$(X \underset{S}{\times} T) * (Y \underset{S}{\times} T) = (X * Y) \underset{S}{\times} T,$$

(B1.3) $$J(\mathscr{R}_T, u_T) = J(\mathscr{R}, u) \underset{S}{\times} T.$$

(B2) EXAMPLES: (i) Take $X = \mathbb{P}(E)$ and $Y = \mathbb{P}(F)$ where E and F are quasi-coherent $\mathcal{O}_S$-Modules. Then, there is a formula,

(B2.1) $$\mathbb{P}(E) * \mathbb{P}(F) = \mathbb{P}(E \oplus F),$$

because of the following formula:

(B2.2) $\mathrm{Sym}\,(E) \otimes \mathrm{Sym}\,(F) = \mathrm{Sym}\,(E \oplus F)$. (EGA I, 9.4.4)

Moreover, the fundamental embedding v of $\mathbb{P}(E)$ in $\mathbb{P}(E) * \mathbb{P}(F)$ is clearly equal to the natural linear embedding of $\mathbb{P}(E)$ in $\mathbb{P}(E \oplus F)$.

(ii) Take $\mathscr{S} = \not{p}/\mathscr{I}$ with $\not{p}$ in $\mathscr{C}(S)$ and $\mathscr{I}$ a homogeneous Ideal of $\not{p}$. Then, $\mathscr{R} \otimes \mathscr{S}$ is clearly isomorphic to the quotient of $\mathscr{R} \otimes \not{p}$ by the homogeneous Ideal generated by $\mathscr{I}$. So, for example, the join of a hypersurface, $X : P(T_0, \ldots, T_m) = 0$, and projective n-space is equal to the hypersurface in projective $(m+n+1)$-space defined by P, thought of as a polynomial in the $(m+n+2)$ variables.

(iii) Take $\mathscr{S} = \mathrm{Sym}\,(L)$ where L is an invertible $\mathcal{O}_S$-Module. Then the join $X * Y$ is called a *twisted S-cone over X*. Geometrically it is the join of X with the base S. The fundamental subschemes V_Y and V_X are called the *vertex subscheme* and the *locus at infinity*.

Since Y is equal to S, and since the fundamental retraction, $r : G(\rho \otimes \mathscr{S}) \to Y$, is an S-morphism, it is equal to the restriction $q|$ of the structure map, $q : X * Y \to S$. Now, by (B1.1), the fundamental embedding of Y in $X * Y$ is a section of r; hence it is a section of q. It is called the *vertex section* of the twisted cone. The fundamental embedding of X in $X * Y$ is called the *embedding at infinity*.

Take $\mathscr{S} = \mathcal{O}_S[t]$, a polynomial $\mathcal{O}_S$-Algebra in one variable t. Then the join (a special case of a twisted S-cone over X) is called the *projective (projecting) cone over X*, and it is denoted $\hat{C}(X)$. The affine S-scheme $\mathrm{Spec}\,(\mathscr{R})$ is called the *affine (projecting) cone*, and it is denoted $C(X)$. We shall see in (B5, (ii)) that it is equal to the complement in $\hat{C}(X)$ of the locus at infinity.

(B3) LEMMA: (i) *The homogeneous Ideal $\mathscr{I} = (\mathscr{R} \otimes \mathscr{S}_+)$ defines the fundamental subscheme V_X in $X * Y$; put symbolically, there is a formula,*

(B3.1) $$V_X = V_+(\mathscr{I}).$$

Moreover, $\tilde{\mathscr{I}}$ is equal to the Ideal of V_X in $X * Y$.

(ii) *The open subscheme $G(\rho \otimes \mathscr{S})$ of $X * Y$ is equal to the complement of the fundamental subscheme V_X; put symbolically, there is a formula,*

(B3.2)
$$G(\rho \otimes \mathscr{S}) = ((X * Y) - V_X).$$

PROOF: (i) The exact sequence of graded $\mathcal{O}_S$-Modules,

$$0 \to \mathscr{S}_+ \to \mathscr{S} \xrightarrow{\varepsilon} \mathcal{O}_S \to 0,$$

yields an exact sequence,

$$0 \to (\mathscr{R} \otimes \mathscr{S}_+) \to (\mathscr{R} \otimes \mathscr{S}) \xrightarrow{\mathscr{R} \otimes \varepsilon} \mathscr{R} \to 0.$$

Hence $\mathscr{I}$ defines V_X because the fundamental embedding, $v : X \to X * Y$, is equal to $\operatorname{Proj}(\mathscr{R} \otimes \varepsilon)$. Moreover, $\tilde{\mathscr{I}}$ is therefore equal to the Ideal of V_X by (EGA II, 3.6.2, (i)).

(ii) The subscheme $G(\rho \otimes \mathscr{S})$ is defined (EGA II, 3.5.1) as the complement of $V_+((\rho \otimes \mathscr{S})(\mathscr{S}_+))$. Since $(\rho \otimes \mathscr{S})(\mathscr{S}_+)$ generates $\mathscr{R} \otimes \mathscr{S}_+$, assertion (ii) follows from (i).

(B4) REMARKS: (i) In view of formula (B3.2), the fundamental retraction, $r : G(\rho \otimes \mathscr{S}) \to Y$, may be written more geometrically

$$r : (X * Y - V_X) \to Y.$$

(ii) The fundamental subscheme V_Y is contained in $(X * Y - V_X)$ by (B3, (ii)) and (B1); so, V_X and V_Y are disjoint.

(B5) PROPOSITION: (i) *Let $\{t_\alpha\}$ be a set of generators of the $(\mathscr{R} \otimes \mathscr{S})$-Ideal $(\mathscr{R} \otimes \mathscr{S}_+)$. Then the complement in $X * Y$ of V_X is equal to the union of the corresponding principal open sets; put symbolically, there is a formula,*

$$(X * Y - V_X) = \bigcup_\alpha (X * Y)_{t_\alpha}.$$

(ii) *There is a canonical isomorphism,*

(B5.1)
$$(\hat{C}(X) - V_X) = C(X).$$

PROOF: (i) The closed set V_X is equal to $V_+(\{t_\alpha\})$ by (B3, (i)), so to $\bigcap_\alpha V_+(t_\alpha)$ by (EGA II, 2.3.2.2). Therefore, (i) holds.

(ii) The subscheme $(\hat{C}(X) - V_X)$ is equal to $\hat{C}(X)_t$ by (i) because t

generates $\mathcal{R} \otimes \mathcal{O}_S[t]_+ = t\mathcal{R}[t]$. The affine Algebra of $\hat{C}(X)_t$ is equal to $\mathcal{R}[t]/(t-1)\mathcal{R}[t]$ by (EGA II, 3.1.4), so to $\mathcal{R}$. Thus, (ii) holds.

(B6) LEMMA: *Set $\mathcal{I} = (\mathcal{R} \otimes \mathcal{S}_+)$. For each positive integer n, there is a canonical surjection,*

$$m_n : \mathcal{S}_n[-n] \otimes (\mathcal{R} \otimes \mathcal{S}) \to \mathcal{I}^n,$$

which is natural in $\mathcal{R}$ and $\mathcal{S}$ and compatible with base change. Moreover, it is an isomorphism if $\mathcal{S}$ has the form Sym (L), *where L is an invertible $\mathcal{O}_S$-Module.*

PROOF: Shifted n places to the right and tensored with $\mathcal{R}$, the map s_n of (A1.3) becomes a map,

$$m'_n : \mathcal{S}_n[-n] \otimes (\mathcal{R} \otimes \mathcal{S}) \to \mathcal{R} \otimes \mathcal{S}.$$

Obviously m'_n is natural in $\mathcal{R}$ and $\mathcal{S}$ and compatible with base change. Since $\mathcal{S}_1$ generates $\mathcal{S}_+$, the image of m'_1 is equal to $\mathcal{I}$. Since $\mathcal{S}_n[-n]$ is, obviously, equal to $(\mathcal{S}_1[-1])^n$, the image of m'_n is equal to $(\mathrm{Im}\,(m'_1))^n$, so to $\mathcal{I}^n$. Hence m'_n induces the required surjection m_n.

Assume $\mathcal{S}$ has the form Sym (L). Then in degree $(s+t+n)$, the map m_n is equal to the natural map,

$$L^{\otimes n} \otimes [\oplus (\mathcal{R}_s \otimes L^{\otimes t})] \to \oplus (\mathcal{R}_s \otimes L^{\otimes (t+n)}),$$

because L is invertible. This map is obviously an isomorphism.

(B7) *The conjunctive transform B_X*

The *conjunctive transform of $X * Y$ with respect to X*, denoted B_X and also $B_X(X * Y)$, is defined as the twisted Y-cone over $X \times_S Y$,

$$B_X = J(\mathcal{R}_Y, \mathrm{Sym}\,(\mathcal{O}_Y(1))).$$

Let p denote its structure map,

$$p : B_X \to Y.$$

The *conjunctive embedding* is defined as the linear Y-embedding.

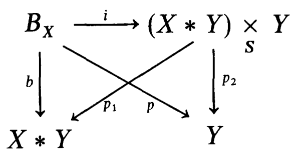
$$i : B_X \to (X * Y) \underset{S}{\times} Y,$$

that is equal to the map,

$$J(\mathscr{R}_Y, \alpha_Y^{\#}) : J(\mathscr{R}_Y, \mathrm{Sym}\,(\mathscr{O}_Y(1))) \to J(\mathscr{R}_Y, \mathscr{S}_Y).$$

Note that i fits into this important commutative diagram:

(B7.1)



where p_1 and p_2 denote the projections and b is defined as the composition $p_1 \circ i$. This map b is called the *conjunctive transformation* of $X * Y$ *with respect to* $v : X \to X * Y$, or *to* X, and it is also denoted b_X. Note for future reference, the formula,

(B7.2) $$i = (b, p).$$

The embedding at infinity is called the *exceptional embedding*, and it is denoted

$$e : X \underset{S}{\times} Y \to B_X.$$

Its image, the locus at infinity, is called the *exceptional locus*, and it is denoted E. The vertex subscheme is denoted V.

(B8) Proposition: *Assume* $X * Y$ *is a twisted* S*-cone over* X, *that is,* $\mathscr{S}$ *has the form* Sym (L) *for a certain invertible* $\mathscr{O}_S$*-Module* L.

(i) *The locus at infinity* V_X *is a divisor, and there is a formula,*

$$\mathscr{O}_{X*Y}(-V_X) = q^*L \otimes \mathscr{O}_{X*Y}(-1),$$

where $q : X * Y \to S$ *denotes the structure morphism.*

(ii) *The conjunctive embedding,* $i : B_X \to (X * Y) \times_S Y$, *and the conjunctive transformation,* $b : B_X \to X * Y$, *are both isomorphisms and are essentially the same map.*

Proof: (i) That the Ideal of V_X is equal to $q^*L \otimes \mathscr{O}_{X*Y}(-1)$ results from applying tilde to the isomorphism m_1 of (B6) and then identifying the terms using formula (A1.2) and assertion (B3, (i)). Hence V_X is a divisor because its Ideal is invertible, and so the formula holds.

(ii) The map, $\alpha_Y^\# : \mathrm{Sym}\,(L)_Y \to \mathrm{Sym}\,(\mathcal{O}_Y(1))$, is an isomorphism (EGA II, 3.4.2) because it is obtained by applying tilde to the maps s_n of (A1) and s_n is obviously (TN)-bijective because L is invertible. So, since i is equal to $\mathrm{Proj}\,(\mathscr{R}_Y \otimes \alpha_Y^\#)$, it too is an isomorphism. Since the structure map, $f : Y \to S$, is an isomorphism by (EGA II, 3.1.7 and 3.1.8, (iii)), the projection $p_1 : (X * Y) \times_S Y \to X * Y$, is also an isomorphism. Hence $b = p_1 \circ i$ is an isomorphism.

(B9) Theorem (Compatibility with linear embedding): *Let* $u' : \mathscr{R} \to \mathscr{R}'$ *be a map in* $\mathscr{C}(S)_{\mathrm{surj}}$, *set* $X' = \mathrm{Proj}\,(\mathscr{R}')$ *and let* $j' : X' \to X$ *denote the linear embedding* $\mathrm{Proj}\,(u')$. *Then the following diagrams are cartesian except for* (iii, a), *which is only commutative*:

(i)
$$\begin{array}{ccc} X' * Y' & \xrightarrow{X' * j} & X' * Y \\ {\scriptstyle j' * Y'}\downarrow & \square & \downarrow{\scriptstyle j' * Y} \\ X * Y' & \xrightarrow{X * j} & X * Y. \end{array}$$

(ii)
$$\begin{array}{ccc} Y' & \xrightarrow{v} & X' * Y' \\ {\scriptstyle j}\downarrow & \square & \downarrow{\scriptstyle j' * j} \\ Y & \xrightarrow{v} & X * Y. \end{array}$$

(iii, a)
$$\begin{array}{ccc} (X * Y' - V_{Y'}) & \xrightarrow{r} & X \\ {\scriptstyle X * j|}\downarrow & \circlearrowleft & \downarrow{\scriptstyle id} \\ (X * Y - V_Y) & \xrightarrow{r} & X. \end{array}$$

(iii, b)
$$\begin{array}{ccc} (X * Y' - V_X) & \xrightarrow{r} & Y' \\ {\scriptstyle X * j|}\downarrow & \square & \downarrow{\scriptstyle j} \\ (X * Y - V_X) & \xrightarrow{r} & Y. \end{array}$$

(iii, c)
$$\begin{array}{ccc} (X' * Y' - (V_{X'} \coprod V_{Y'})) & \xrightarrow{(r|, r|)} & X' \times_S Y' \\ {\scriptstyle j' * j|}\downarrow & \square & \downarrow{\scriptstyle j' \times j} \\ (X * Y - (V_X \coprod V_Y)) & \xrightarrow{(r|, r|)} & X \times_S Y. \end{array}$$

(iv)
$$\begin{array}{ccc} B_{X'}(X' * Y') & \xrightarrow{i} & (X' * Y') \times_S Y' \\ \downarrow & \square & \downarrow{\scriptstyle (j' * j) \times j} \\ B_X(X * Y) & \xrightarrow{i} & (X * Y) \times_S Y. \end{array}$$

PROOF: (i) Form the tensor product diagram of u and u',

$$\begin{array}{ccc} \mathcal{R}' \otimes \mathcal{S}' & \xrightarrow{\mathcal{R}' \otimes u} & \mathcal{R}' \otimes \mathcal{S} \\ {\scriptstyle u' \otimes \mathcal{S}'} \downarrow & \otimes & \downarrow {\scriptstyle u \otimes \mathcal{S}} \\ \mathcal{R} \otimes \mathcal{S}' & \xrightarrow{\mathcal{R} \otimes u} & \mathcal{R} \otimes \mathcal{S}. \end{array}$$

Applying Proj obviously yields diagram (i). So, it is cartesian by (A5, (ii)).

(ii) Consider the diagram,

$$\text{(B9.1)} \qquad \begin{array}{ccccc} Y' & \xrightarrow{j} & Y & \xrightarrow{id} & Y \\ {\scriptstyle v} \downarrow & & {\scriptstyle v} \downarrow & & {\scriptstyle v} \downarrow \\ X' * Y' & \xrightarrow{X' * j} & X' * Y & \xrightarrow{j' * Y} & X * Y. \end{array}$$

Since Y' and Y are equal to the joins, $J(\mathcal{O}_S, \mathcal{S}')$ and $J(\mathcal{O}_S, \mathcal{S})$, both squares of (B9.1) are cartesian by (i). Hence, (ii) is cartesian.

(iii, a) Applying the 'functor' Proj $(\mathcal{R} \otimes \_)$ to the commutative diagram,

$$\begin{array}{ccc} \mathcal{S}' & \xrightarrow{\rho} & \mathcal{O}_S \\ {\scriptstyle u} \downarrow & & \downarrow {\scriptstyle id} \\ \mathcal{S} & \xrightarrow{\rho} & \mathcal{O}_S. \end{array}$$

yields diagram (iii, a) by virtue of (B3, (ii)); hence, (iii, a) is commutative.

(iii, b) Form the tensor product diagram of u and ρ,

$$\begin{array}{ccc} \mathcal{R} \otimes \mathcal{S}' & \xrightarrow{\rho \otimes \mathcal{S}'} & \mathcal{S}' \\ {\scriptstyle \mathcal{R} \otimes u} \downarrow & \otimes & \downarrow {\scriptstyle u} \\ \mathcal{R} \otimes \mathcal{S} & \xrightarrow{\rho \otimes \mathcal{S}} & \mathcal{S}. \end{array}$$

Applying the 'functor' Proj yields, by (A5, (i)) a cartesian diagram,

$$\begin{array}{ccc} G(\rho \otimes \mathcal{S}') & \xrightarrow{r} & Y' \\ {\scriptstyle X * j|} \downarrow & \square & \downarrow {\scriptstyle j} \\ G(\rho \otimes \mathcal{S}) & \xrightarrow{r} & Y, \end{array}$$

which, by (B3, (ii)), is equal to (iii, b); hence (iii, b) is cartesian.

(iii, c) Consider the diagram,

(B9.2)
$$\begin{array}{ccccc}
(X * Y' - (V_X \coprod V_Y)) & \longrightarrow & (X * Y' - V_X) & \xrightarrow{r} & Y' \\
\downarrow{\scriptstyle X*j|} & & \downarrow{\scriptstyle X*j|} & & \downarrow{\scriptstyle j} \\
(X * Y - (V_X \coprod V_Y)) & \longrightarrow & (X * Y - V_X) & \xrightarrow{r} & Y.
\end{array}$$

The right-hand square is cartesian by (iii, b), and, clearly, the left-hand square is cartesian by (ii). Hence, the outer rectangle is cartesian.

Next, consider the diagram,

$$\begin{array}{ccccc}
(X' * Y' - (V_{X'} \coprod V_{Y'})) & \longrightarrow & (X * Y' - (V_X \coprod V_{Y'})) & \longrightarrow & Y' \\
\downarrow & & \downarrow & & \downarrow \\
(X * Y - (V_{X'} \coprod V_Y)) & \longrightarrow & (X * Y - (V_X \coprod V_Y)) & \longrightarrow & Y \\
\downarrow & & \downarrow & & \\
X' & \longrightarrow & X. & &
\end{array}$$

The upper left-hand square is clearly cartesian by (i) and (ii); the other two are cartesian because (B9.2) is cartesian. It now follows formally, by considering T-points, that (iii, c) is cartesian.

(iv) Consider the diagram,

(B9.3)
$$\begin{array}{ccccc}
J(\mathscr{R}_{Y'}, \mathrm{Sym}\,(\mathcal{O}_{Y'}(1))) & \longrightarrow & J(\mathscr{R}_{Y'}, [\mathrm{Sym}\,(\mathcal{O}_Y(1))]_{Y'}) & \longrightarrow & J(\mathscr{R}_Y, \mathrm{Sym}\,(\mathcal{O}_Y(1))) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
J(\mathscr{R}_{Y'}, \mathscr{S}'_{Y'}) & \longrightarrow & J(\mathscr{R}_{Y'}, \mathscr{S}_{Y'}) & \longrightarrow & J(\mathscr{R}_Y, \mathscr{S}_Y).
\end{array}$$

The right-hand square is cartesian because join is compatible with base change (B1.3). The left-hand square is commutative, so cartesian, because it is obtained by applying $J(\mathscr{R}_{Y'}, \_)$ to the commutative diagram (A2.1) with $u : \mathscr{S} \to \mathscr{S}'$ for $a : \mathscr{S} \to \mathscr{T}$. Hence the outer rectangle is cartesian.

Next consider the diagram,

$$\begin{array}{ccccc}
B_{X'}(X' * Y') & \longrightarrow & B_X(X * Y') & \longrightarrow & B_X(X * Y) \\
\downarrow & & \downarrow & & \downarrow \\
(X' * Y') \times_S Y' & \longrightarrow & (X * Y') \times_S Y' & \longrightarrow & (X * Y) \times_S Y.
\end{array}$$

The right-hand square is equal to the outer rectangle of (B9.3); hence it is cartesian. The left-hand square is cartesian by (i) because a conjunctive transform is a join. Hence, (iv) is cartesian.

C. Properties of conjunctive transforms

Still more blanket notation

Let p_1 and p_2 denote the first and second projections from any fibered product of schemes.

(C1) Theorem: (i) *The following diagram is cartesian*:

(C1.1)
$$\begin{array}{ccc} X \underset{S}{\times} Y & \xrightarrow{p_1} & X \\ {\scriptstyle e}\downarrow & \square & \downarrow{\scriptstyle v} \\ B_X & \xrightarrow{b_X} & X * Y. \end{array}$$

In particular, there is a formula,

$$b_X^{-1}(V_X) = E.$$

(i′) *The following diagram is cartesian*:

(C1.2)
$$\begin{array}{ccc} Y & \xrightarrow{id} & Y \\ {\scriptstyle v}\downarrow & \square & \downarrow{\scriptstyle v} \\ B_X & \xrightarrow{b_X} & X * Y. \end{array}$$

In particular, there is a formula,

$$b_X^{-1}(V_Y) = V.$$

(ii) *The conjunctive transformation, $b_X : B_X \to X * Y$, induces an isomorphism $b_X|$, which fits into a commutative diagram,*

$$\begin{array}{ccc} (B_X - E) & \xrightarrow[\simeq]{b_X|} & (X * Y - V_X) \\ {\scriptstyle p|}\downarrow & & \downarrow{\scriptstyle r} \\ Y & \xrightarrow{id} & Y. \end{array}$$

(ii′) *The conjunctive embedding, $i : B_X \to (X * Y) \times_S Y$, carries $(B_X - E)$ isomorphically onto the image of the graph morphism, $\Gamma_r =$ (id, r), of the fundamental retraction, $r : (X * Y - V_X) \to Y$.*

Proof: (i) Consider this diagram:

$$\begin{array}{ccccc}
X \times_S Y & \xrightarrow{id} & X \times_S Y & \xrightarrow{p_1} & X \\
\downarrow e & & \downarrow v & & \downarrow v \\
B_X & \xrightarrow{i} & (X * Y) \times_S Y & \xrightarrow{p_1} & X * Y.
\end{array}$$

Its two squares are cartesian because fundamental embeddings are compatible with base change (B1), and with linear embedding (B9, (ii)). Hence diagram (C1.1) is cartesian.

(i′) Consider this diagram:

$$\begin{array}{ccccc}
Y & \xrightarrow{\mathrm{Proj}(\alpha_Y^\sharp)} & Y \times_S Y & \xrightarrow{p_1} & Y \\
\downarrow v & & \downarrow v & & \downarrow v \\
B_X & \xrightarrow{i} & (X * Y) \times_S Y & \xrightarrow{p_1} & X * Y.
\end{array}$$

Its two squares are cartesian because fundamental embeddings are compatible with base change (B1) and with linear embedding (B9, (ii)). Since Proj $(\alpha_Y^\sharp)$ is equal to Δ_Y by (A3), diagram (C1.2) is therefore cartesian.

(ii) and (ii′) The appropriate form of diagram (B9,(iii, b)) expressing the compatibility of the fundamental retraction with linear embedding is equal to the cartesian diagram,

$$\begin{array}{ccc}
(B_X - E) & \xrightarrow{i|} & ((X * Y) - V_X) \times_S Y \\
\downarrow p| & \square & \downarrow r \times Y \\
Y & \xrightarrow{\Delta_Y} & Y \times_S Y,
\end{array}$$

because Δ_Y is equal to Proj $(\alpha_Y^\sharp)$ by (A3), because $p|$ is equal to the fundamental retraction, $r : (B_X - E) \to Y$, by (B2, (iii)), and because $r \times_S Y$ is equal to the fundamental retraction, $r : ((X * Y) - V_X) \times_S Y \to Y \times_S Y$, by compatibility of r with base change (B1). Since the diagram,

$$\begin{array}{ccc}
(X * Y - V_X) & \xrightarrow{\Gamma_r} & (X * Y - V_X) \times_S Y \\
\downarrow r & & \downarrow r \times Y \\
Y & \xrightarrow{\Delta} & Y \times_S Y,
\end{array}$$

is also cartesian, assertions (ii) and (ii′) follow.

(C2) Corollary: *If $\mathscr{R}$ is isomorphic to* Sym (F) *where F is a locally free $\mathcal{O}_S$-Module with a finite rank, then the fundamental retraction, $r:(X * Y - V_X) \to Y$, is smooth.*

Proof: The map r is isomorphic to the restriction $p|$ of the structure map, $p: B_X \to Y$, by (C1, (ii)), while p is smooth because it is isomorphic to the structure morphism of $\mathbb{P}(F_Y \oplus \mathcal{O}_Y(1))$ by (B2.1).

(C3) Theorem: (i) *The exceptional locus E is a divisor on B_X.*
(ii) *There are formulas,*

$$\mathcal{O}_{B_X}(-E) = p^*\mathcal{O}_Y(1) \otimes b^*\mathcal{O}_{X*Y}(-1), \tag{C3.1}$$

$$e^*\mathcal{O}_{B_X}(E) = p_2^*\mathcal{O}_Y(-1) \otimes p_1^*\mathcal{O}_X(1). \tag{C3.2}$$

(iii) *The invertible sheaf $\mathcal{O}_{B_X}(-E)$ is relatively very ample for the conjunctive transformation, $b: B_X \to X * Y$.*

Proof: Since E is equal to the locus at infinity of the twisted Y-cone B_X over $X \times_S Y$, it is a divisor by (B8, (i)). Moreover, (B8, (i)) yields the following formula:

$$\mathcal{O}_{B_X}(-E) = p^*\mathcal{O}_Y(1) \otimes \mathcal{O}_{B_X}(-1). \tag{C3.3}$$

Now, since b is $p_1 \circ i$, there is a formula,

$$b * \mathcal{O}_{X*Y}(1) = \mathcal{O}_{B_X}(1),$$

because the formation of $\mathcal{O}(1)$ is compatible with p_1, a base change (EGA II, 3.5.3), and with i, a linear embedding (EGA II, 3.5.2). Therefore, formula (C3.3) yields formula (C3.1).

Clearly formula (C3.3) yields a formula,

$$e * \mathcal{O}_{B_X}(E) = e * p * \mathcal{O}_Y(-1) \otimes e * \mathcal{O}_{B_X}(1). \tag{C3.4}$$

Now, there are canonical isomorphisms,

$$e * \mathcal{O}_B(1) = \mathcal{O}_{\mathrm{Proj}\,(\mathscr{R}_Y)}(1) = p_1 * \mathcal{O}_X(1),$$

because the formation of $\mathcal{O}(1)$ is compatible with e, a linear embedding (EGA II, 3.5.2), and with p_1, a base change (EGA II, 3.5.3). Now, e is a Y-morphism; that is, $p \circ e$ is equal to p_2. Hence formula (C3.4) yields formula (C3.2).

Finally, recall the situation in diagram (B7.1). The sheaf $p_2^*\mathcal{O}_Y(1)$ is relatively very ample for p_1 by (EGA II, 4.4.10, (iii)), and so $p^*\mathcal{O}_Y(1)$ is relatively very ample for b by (EGA II, 4.4.10, (i bis)). So, since $\mathcal{O}_{B_X}(-E)$ is isomorphic to $p^*\mathcal{O}_Y(1) \otimes b^*\mathcal{O}_{X*Y}(-1)$ by (C3.1), it is also relatively ample for b by (EGA II, 4.4.9, (i)).

(C4) Lemma: *Let T be a scheme, D an effective divisor on T, and T' a closed subscheme of T.*

(i) *If $T' \cap (T-D)$ is a divisor and if the Ideal of T' is everywhere locally principal (but possibly generated by a zero-divisor), then T' is a divisor.*

(ii) *The scheme T is equal to the scheme-theoretic closure of $(T-D)$; that is, if a closed subscheme T'' of T contains $(T-D)$, then T'' is equal to T.*

Proof: Both assertions are clearly local on T. So, we may assume that T is affine and that D and T' are principal. Set $A = \Gamma(T, \mathcal{O}_T)$, and let $a = 0$ be an equation of D and $t = 0$ an equation of T'. Clearly $(T-D)$ is equal to T_a, so it is affine with ring A_a. Since D is a divisor, a is a nonzero-divisor; so the restriction map, $\ell : A \to A_a$, is injective.

(i) Clearly $\ell(t) = 0$ is an equation of the divisor $T' \cap (T-D)$, so $\ell(t)$ is a nonzero-divisor. Since ℓ is injective, t is clearly a nonzero-divisor. Thus, T' is a divisor.

(ii) Let I denote the ideal of T''. Then $\ell(I)$ generates the ideal of $T'' \cap (T-D)$. Since this ideal is equal to zero by assumption, $\ell(I)$ is equal to zero. So, since ℓ is injective, I is equal to zero. Thus, T'' is equal to T.

(C5) Proposition: (i) *The open subscheme $(B_X - E)$ is scheme-theoretically dense in B_X.*

(ii) *The conjunctive embedding, $i : B_X \to (X * Y) \times_S Y$, carries B_X isomorphically onto the scheme-theoretic closure Γ of the graph subscheme of the fundamental retraction, $r : (X * Y - V_X) \to Y$.*

(iii) *The inverse image under the conjunctive transformation, $b : B_X \to X * Y$, of each effective divisor on $X * Y$ is a well-defined divisor on B_X.*

(iv) *If the structure map, $f : Y \to S$, is surjective, then the conjunctive transformation, $b : B_X \to X * Y$, is surjective and birational.*

Proof: (i) Since E is a divisor on B_X by (C3, (i)), the open subscheme $(B_X - E)$ is scheme-theoretically dense in B_X by (C4, (ii)).

(ii) By (C1, (ii')), the map i carries $(B_X - E)$ isomorphically onto the graph subscheme of r. So, since i is a closed embedding, it carries the closure of $(B_X - E)$ isomorphically onto Γ. Hence assertion (ii) follows from assertion (i).

(iii) Let D be an effective divisor on $X * Y$. The Ideal of $b^{-1}(D)$ is clearly everywhere locally principal, and $b^{-1}(D) \cap (B_X - E)$ is a divisor because b is an isomorphism on $(B_X - E)$ by (C1, (ii)). So, $b^{-1}(D)$ is a divisor by (C4, (i)) because E is.

(iv) Assume f is surjective. Then, b carries E onto V_X because diagram (C1, (i)) is commutative. It carries $(B_X - E)$, which is dense in B_X by (i), isomorphically onto $(X * Y - V_X)$ by (C1, (ii)). Hence b is surjective, and $(X * Y - V_X)$ is dense in $X * Y$. Thus b is birational.

(C6) THEOREM: *Considered as an $(X * Y)$-scheme via the conjunctive transformation, B_X is canonically isomorphic over $X * Y$ to the blow-up $B\ell_X$ of $X * Y$ with center V_X, and the isomorphism carries E onto the exceptional divisor D of $B\ell_X$.*

PROOF: Set $\mathscr{I} = (\mathscr{R} \otimes \mathscr{S}_+)$, set $\mathscr{B} = (\mathscr{R}_Y \otimes \mathrm{Sym}\,(\mathscr{O}_Y(1)))$, and set $\mathscr{J} = (\mathscr{R}_Y \otimes \mathrm{Sym}\,(\mathscr{O}_Y(1))_+)$. The map m_n of (B6) is natural and compatible with base change, so there exists a commutative diagram on Y,

(C6.1)
$$\begin{array}{ccc}
\mathscr{S}_{n,Y}[-n] \otimes (\mathscr{R}_Y \otimes \mathscr{S}_Y) \underset{(\mathscr{R}_Y \otimes \mathscr{S}_Y)}{\otimes} \mathscr{B} & \xrightarrow{(m_n)_Y \underset{(\mathscr{R}_Y \otimes \mathscr{S}_Y)}{\otimes} \mathscr{B}} & \mathscr{I}^n_Y \underset{(\mathscr{R}_Y \otimes \mathscr{S}_Y)}{\otimes} \mathscr{B} \\
\| & & \\
\mathscr{S}_{n,Y}[-n] \otimes \mathscr{B} & & \downarrow \\
\downarrow{\scriptstyle \alpha^{\#}_{Y,n}[-n] \otimes \mathscr{B}} & & \\
\mathscr{O}_Y(n)[-n] \otimes \mathscr{B} & \xrightarrow{\quad m_n \quad} & \mathscr{J}^n.
\end{array}$$

Moreover, the bottom map is an isomorphism by (B6).

Applying tilde to (C6.1) yields this commutative diagram on B_X, in which the bottom map is an isomorphism,

(C6.2)
$$\begin{array}{ccc}
b^*(\mathscr{S}_{X*Y,n}) \otimes \mathscr{O}_{B_X}(-n) & \xrightarrow{b*(\tilde{m}_n)} & b^*[(\mathscr{J}^n)^{\sim}] \\
\| & & \\
p^*(\mathscr{S}_{Y,n}) \otimes \mathscr{O}_{B_X}(-n) & & \downarrow \\
\downarrow{\scriptstyle p*(\alpha^{\#}_{Y,n}) \otimes \mathscr{O}_{B_X}(-n)} & & \\
p^*(\mathscr{O}_Y(n)) \otimes \mathscr{O}_{B_X}(-n) & \xrightarrow{\ \sim\ } & (\mathscr{J}^n)^{\sim},
\end{array}$$

in view of (A1.2) and of the compatibility of tilde with i, a linear embedding, (EGA II, 3.5.3), and with p_1, a base change (EGA II, 3.5.2, (ii)) (recall the definition, $b = p_1 \circ i$).

Summing (C6.2) over n and applying Proj yields a commutative diagram of B_X-schemes, in which the bottom map is an isomorphism,

(C6.3)
$$\begin{array}{ccc}
B_X \underset{(X*Y)}{\times} [(X*Y) \underset{S}{\times} Y] & \xleftarrow{B_X \times \operatorname{Proj}(\oplus \tilde{m}_n)} & B_X \underset{(X*Y)}{\times} B\ell_X \\
\Vert & & \uparrow \\
B_X \underset{Y}{\times} (Y \underset{S}{\times} Y) & & \\
\uparrow {\scriptstyle B_X \underset{Y}{\times} \operatorname{Proj}(\alpha_Y^{\#})} & & \\
B_X \underset{Y}{\times} Y & \xleftarrow{\sim} & \operatorname{Proj}(\oplus (\mathscr{J}^n)^{\sim}),
\end{array}$$

in view of (EGA II, 3.1.8, (iii)) (it asserts that $\operatorname{Proj}(\oplus(\mathscr{P}_n \otimes L^{\otimes n}))$ is canonically isomorphic to $\operatorname{Proj}(\mathscr{P})$ for an arbitrary graded quasi-coherent Algebra $\mathscr{P}$ and invertible sheaf L), of (EGA II, 3.5.3) (it asserts that Proj is compatible with base change), and of (B3, (i)) (it asserts $\tilde{\mathscr{J}}$ is the Ideal of V_X).

Diagram (C6.3) yields a commutative diagram of $(X * Y)$-schemes, in which the bottom maps are isomorphisms,

(C6.4)
$$\begin{array}{ccccc}
B_X \underset{S}{\times} Y & \xrightarrow{b \times Y} & (X*Y) \underset{S}{\times} Y & \xleftarrow{\operatorname{Proj}(\oplus \tilde{m}_n)} & B\ell_X \\
\Vert & & \uparrow i & & \uparrow \\
B_X \underset{Y}{\times} (Y \underset{S}{\times} Y) & & & & \\
\uparrow {\scriptstyle B_X \underset{S}{\times} \Delta_Y} & & & & \\
B_X \underset{Y}{\times} Y & \xleftarrow{(id,\, p)} & B_X & \xleftarrow{\sim} & \operatorname{Proj}(\oplus (\mathscr{J}^n)^{\sim}),
\end{array}$$

because $\operatorname{Proj}(\alpha_Y^{\#})$ is equal to Δ_Y by (A3) and because i is equal to (b, p) by (B7.2).

There is an $(X * Y)$-morphism,

$$i' : B_X \to B\ell_X,$$

satisfying the relation, $i' \circ \operatorname{Proj}(\oplus \tilde{m}_n) = i$, because the bottom right-hand map in diagram (C6.4) is an isomorphism. The map i' is a closed embedding because i is so, (EGA I, 4.3.6, (iv)). The map i' carries the open subscheme $(B_X - E)$ isomorphically onto the open subscheme $(B\ell_X - D)$

because, by (C1, (ii)) and (EGA II, 8.1.3), both open subschemes are isomorphic to the same open subscheme of the base, namely to $(X * Y - V_X)$, and because, by (EGA II, 8.1.8), the second open subscheme is the full inverse image. Hence i' is an isomorphism because $B\ell_X$ is the scheme-theoretic closure of $(B\ell_X - D)$ by (C4, (ii)).

Finally, since i' is an $(X * Y)$-morphism, since E is the inverse image of V_X under b_x by (C1, (i)), and since D is the inverse image of V_X under the blowing-up, clearly $(i')^{-1}(D)$ is equal to E.

(C7) *The conjunctive transform B*

The *conjunctive transform of* $X * Y$ *with respect to* $X \coprod Y$, denoted B and also $B(X * Y)$, is defined as the $(X \times_S Y)$-scheme,

$$B = \mathbb{P}(p_1^* \mathcal{O}_X(1) \oplus p_2^* \mathcal{O}_Y(1)).$$

It is equal to the join,

$$J(\operatorname{Sym}(p_1^* \mathcal{O}_X(1)), \operatorname{Sym}(p_2^* \mathcal{O}_Y(1))),$$

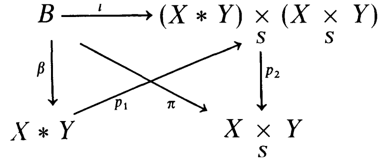
by (B2, (i)), and is canonically embedded in $(X * Y) \times_S (X \times_S Y)$ via the linear embedding,

$$\iota : B \to (X * Y) \underset{S}{\times} (X \underset{S}{\times} Y),$$

that is equal to the map,

$$J(p_1^* \alpha_X^\#, p_2^* \alpha_Y^\#) : J(\operatorname{Sym}(p_1^* \mathcal{O}_X(1)), \operatorname{Sym}(p_2^* \mathcal{O}_Y(1))) \to J(p_1^* \mathcal{R}_X, p_2^* \mathcal{S}_Y).$$

Note that ι fits into this commutative diagram:



where π denotes the structure map, and β, the composition $p_1 \circ \iota$. The map β is called the *conjunctive transformation of* $X * Y$ *with respect to* $X \coprod Y$. Let

$$e_X : \mathbb{P}(p_1^* \mathcal{O}_X(1)) \to B$$

denote the canonical linear embedding, and let E_X denote its scheme-theoretic image.

(C8) **Proposition**: (i) *There is a cartesian diagram,*

(C8.1)
$$\begin{array}{ccc} B & \xrightarrow{\beta_X} & B_Y \\ {\scriptstyle \beta_Y}\downarrow & \square & \downarrow{\scriptstyle b_Y} \\ B_X & \xrightarrow{b_X} & X * Y. \end{array}$$

Moreover, the compositions $b_X \circ \beta_Y$ and $b_Y \circ \beta_X$ are both equal to β.

(ii) *The map β_Y is isomorphic to the conjunctive transformation of B_X with respect to the (vertex) map, $v : Y \to B_X$.*

(iii) *In the diagram,*

(C8.2)
$$\begin{array}{ccccc} \mathbb{P}(p_1^*\mathcal{O}_X(1)) = & X \underset{S}{\times} Y & \xrightarrow{p_1} & X \\ {\scriptstyle e_X}\downarrow & {\scriptstyle v}\downarrow & \square & \downarrow{\scriptstyle \iota} \\ \mathbb{P}(p_1^*\mathcal{O}_X(1) \oplus p_2^*\mathcal{O}_Y(1)) = & B & \xrightarrow{\beta} & X * Y, \end{array}$$

the left-hand square is commutative and the right-hand square is cartesian. In particular, there is a formula,

$$\beta^{-1}(V_X) = E_X.$$

(iv) *The conjunctive transformation, $\beta : B \to X * Y$, induces an isomorphism $\beta|$, which fits into a commutative diagram,*

(C8.3)
$$\begin{array}{ccc} (B - (E_X \coprod E_Y)) & \underset{\simeq}{\xrightarrow{\beta|}} & (X * Y - (V_X \coprod V_Y)) \\ {\scriptstyle \pi|}\downarrow & & \downarrow{\scriptstyle (r|,\, r|)} \\ X \underset{S}{\times} Y & \xrightarrow{id} & X \underset{S}{\times} Y. \end{array}$$

Proof: (i) Form the tensor product diagram of $p_1^*\alpha_X^\#$ and $p_2^*\alpha_Y^\#$,

$$\begin{array}{ccc} \mathrm{Sym}\,(p_1^*\mathcal{O}_X(1)) \otimes \mathrm{Sym}\,(p_2^*\mathcal{O}_Y(1)) & \longleftarrow & \mathrm{Sym}\,(p_1^*\mathcal{O}_X(1)) \otimes p_2^*\mathcal{S}_Y \\ \uparrow & \otimes & \uparrow \\ p_1^*\mathcal{R}_X\, \mathrm{Sym}\,(p_2^*\mathcal{O}_Y(1)) & \longleftarrow & p_1^*\mathcal{R}_X \otimes p_2^*\mathcal{S}_Y. \end{array}$$

It is obviously equal to the diagram,

$$\begin{array}{ccc}
\operatorname{Sym}(p_1^*\mathcal{O}_X(1)\oplus p_2^*\mathcal{O}_Y(1)) & \longleftarrow & ((\operatorname{Sym}(\mathcal{O}_X(1)\otimes\mathcal{S}_X))_{X\underset{S}{\times}Y} \\
\uparrow & \otimes & \uparrow \\
(\mathcal{R}_Y\otimes\operatorname{Sym}(\mathcal{O}_Y(1)))_{X\underset{S}{\times}Y} & \longrightarrow & (\mathcal{R}\otimes\mathcal{S})_{X\underset{S}{\times}Y}.
\end{array}$$

Applying Proj yields the commutative diagram of $(X\times_S Y)$-schemes,

(C8.4)
$$\begin{array}{ccc}
B & \xrightarrow{(\beta_Y,\pi)} & B_Y\underset{X}{\times}(X\underset{S}{\times}Y) \\
{\scriptstyle(\beta_X,\pi)}\downarrow & \searrow^{\iota} & \downarrow{\scriptstyle i\times Y} \\
B_X\underset{Y}{\times}(X\underset{S}{\times}Y) & \xrightarrow{i\times X} & (X*Y)\underset{S}{\times}(X\underset{S}{\times}Y),
\end{array}$$

defining β_X and β_Y. The outer square is cartesian by (A5, (ii)). It now follows formally that diagram (C8.1) is cartesian and that β is equal to $b_X\circ\beta_Y$ and to $b_Y\circ\beta_X$.

(ii) It is obvious from the definitions that β_Y is isomorphic to the conjunctive transformation of B_X with respect to the vertex map v.

(iii) The left-hand square of diagram (C8.2) is commutative by (B2, (i)). Since β is equal to $b_X\circ\beta_Y$ and since b_X and β_Y are both conjunctive transformations, the right-hand square is cartesian by (C1, (i) and (i′)).

(iv) Consider the following diagram:

$$\begin{array}{ccccc}
(B-(E_X\coprod E_Y)) & \xrightarrow{\beta_Y|} & (B_X-(E\coprod V)) & \xrightarrow{b_X|} & (X*Y-(V_X\coprod V_Y) \\
{\scriptstyle\pi|}\downarrow & & \downarrow{\scriptstyle r} & & \\
X\underset{S}{\times}Y & \xrightarrow{id} & X\underset{S}{\times}Y & & \downarrow{\scriptstyle r|} \\
 & & \downarrow{\scriptstyle p_2} & & \\
 & & Y & \xrightarrow{id} & Y.
\end{array}$$

Both top maps are isomorphisms and both squares are commutative by (C1, (i), (i′), and (ii)). Moreover by (i), the top composition is equal to $\beta|$. Hence $\beta|$ is an isomorphism. Moreover, there is a similar diagram for the composition $(b_Y|)\circ(\beta_X|)$. Combining these two diagrams yields the commutative diagram (C8.3).

(C9) PROPOSITION: *There is a canonical isomorphism,*

(C9.1) $$(B-E_X) = V(p_1^*\mathcal{O}_X(1)\otimes p_2^*\mathcal{O}_Y(-1)).$$

PROOF: Set $L = p_1^*\mathcal{O}_X(1)$ and $M = p_2^*\mathcal{O}_Y(1)$. It follows from (EGA II, 4.1.4) that there is a commutative diagram with isomorphisms,

$$\text{(C9.2)}\qquad \begin{array}{ccc} \mathbb{P}(L \oplus M) & \xrightarrow{\sim} & \mathbb{P}((L \otimes M^{-1}) \oplus \mathcal{O}_{X \times Y}) \\ \uparrow & & \uparrow \\ \mathbb{P}(L) & \xrightarrow{\sim} & \mathbb{P}(L \otimes M^{-1}). \end{array}$$

Since, by (B2, (i)), the projective $(X \times_S Y)$-cone over $\mathbb{P}(L \otimes M^{-1})$ is equal to $\mathbb{P}((L \otimes M^{-1}) \oplus \mathcal{O}_{X \times_S Y})$ and since the affine $(X \times_S Y)$-cone over $\mathbb{P}(L \otimes M^{-1})$ is equal to $V(L \otimes M^{-1})$, the canonical isomorphism (B5.1) becomes a canonical isomorphism,

$$(\mathbb{P}((L \otimes M^{-1}) \oplus \mathcal{O}_X) - \mathbb{P}(L \otimes M^{-1})) = V(L \otimes M^{-1}).$$

Since $(B - E_X)$ is equal to $(\mathbb{P}(L \oplus M) - \mathbb{P}(L))$, the isomorphisms of diagram (C9.2) yield the isomorphism (C9.1).

(C10) THEOREM: *The conjunctive transformation β is isomorphic to the blowing-up of $X * Y$ with center $V_X \coprod V_Y$.*

PROOF: By (C8, (ii)), the map β_Y is isomorphic to the conjunctive transformation of B_X with respect to $v : Y \to B_X$. So, by (C6), it is isomorphic to the blowing-up with center $b_X^{-1}(V_Y)$, and b_X is isomorphic to the blowing-up with center V_X. Hence, β is isomorphic to the blowing-up of $X * Y$ with center $V_X \coprod V_Y$ as can be seen readily by means of a local analysis.

(C11) *A geometric description of the join*

Let E and F be quasi-coherent $\mathcal{O}_S$-Modules, and suppose X and Y are closed subschemes of $\mathbb{P}(E)$ and $\mathbb{P}(F)$ (that is, suppose $\mathcal{R}$ and $\mathcal{S}$ are isomorphic to quotients of $\mathrm{Sym}(E)$ and $\mathrm{Sym}(F)$). Then the join $X * Y$ is a closed subscheme of the projective space $\mathbb{P}(E \oplus F)$, the join of $\mathbb{P}(E)$ and $\mathbb{P}(F)$ (B2, (i)). Identify X and Y with their images, the fundamental subschemes, in the join $X * Y$. We shall now see that the join $X * Y$ is the locus of points lying on lines of $\mathbb{P}(E \oplus F)$ determined by pairs of points of X and Y, that the conjunctive transform $B = B(X * Y)$ is the 'disjoint' union of these lines, and that the conjunctive transformation, $\beta : B \to X * Y$, is the natural map preserving these lines. Thus β is the natural map identifying the end points of these lines. (This is in keeping with (C10), which asserts that β is the blowing-up whose center is the union of X and Y).

Let (x, y) be a pair of T-points of X/S and Y/S. The line of $\mathbb{P}(E \oplus F)$ determined by the pair (x, y) is just the relatively one-dimensional linear subscheme $x * y$ of $\mathbb{P}(E_T \oplus F_T)$ where x and y are viewed in the natural way as relatively zero-dimensional linear subschemes of $\mathbb{P}(E_T \oplus F_T)$. It lies in $(X * Y)_T$ by the compatibility of join with linear embedding (B1). In fact, the cartesian diagram expressing the compatibility of β with linear embedding is easily seen to be

$$\begin{array}{ccc} B(x*y) & \xrightarrow{\simeq} & x*y \\ {\scriptstyle\simeq}\downarrow & \square & \downarrow \\ B_T & \longrightarrow & (X*Y)_T \end{array}$$

with the two indicated isomorphisms (cf. B9, (iv) and B8, (ii)). Hence β_T carries B_T isomorphically onto the line $x * y$ determined by x and y; in other words, β preserves lines, and $\beta_T^{-1}(x * y)$ is equal to the fiber over (x, y) of the structure map, $\pi : B \to X \times_S Y$.

On the other hand, given a T-point z of the complement in the join $X * Y$ of the two fundamental subschemes, let x and y be the pair of T-points of X and Y obtained by applying the two fundamental retractions to z. Then z lies on the line $x * y$ determined by x and y because the following diagram expressing the compatibility of fundamental retractions with linear embedding is cartesian (cf. B9, (iii, c)):

$$\begin{array}{ccc} ((x*y)-(x\coprod y)) & \xrightarrow{(r|, r|)} & x \times y \\ \downarrow & \square & \downarrow \\ ((X*Y)-(X\coprod Y))_T & \xrightarrow{(r|, r|)} & (X \times Y)_T. \end{array}$$

Thus $X * Y$ is the locus of points lying on the lines of $\mathbb{P}(E \oplus F)$ determined by pairs of points of X and Y.

D. Linear projections

(D1) *Basics*

Linear projection is a name for a map of the form $g = \operatorname{Proj}(a)$ from the open subscheme $G(a)$ of $Z = \operatorname{Proj}(\mathcal{T})$ to $Y = \operatorname{Proj}(\mathcal{S})$. The closed subscheme $X = \operatorname{Proj}(\mathcal{T}/\mathcal{J})$ of Z, where $\mathcal{J}$ is the Ideal generated by $a(\mathcal{S}_1)$, is called the *center* (*of projection*). Obviously X is a linear subscheme because $a(\mathcal{S}_1)$ concentrated in degree 1. Obviously the complement $(Z-X)$ is equal to the open subscheme $G(a)$.

There is a natural minimal target for the linear projection $g = \text{Proj}\,(a)$, namely, the closed subscheme Proj (Im (a)) of Y. Moreover, g with this target is dominating because then g is equal to Proj of an injection (EGA II, 3.7.5), although it need not be surjective (for example, consider the projection of the plane quadric, $Z : T_1 T_2 = T_0^2$, to the T_1 (or T_2) axis).

Any linear subscheme Proj $(\mathscr{T}/\mathscr{I})$ of Z is the center of a linear projection, namely, the projection defined by the inclusion map of $O_S[\mathscr{I}_1]$ into $\mathscr{T}$. Any closed subscheme of Z whose homogeneous Ideal $\mathscr{I}$ is finitely generated can be made the center of a linear projection by reembedding Z in projective space using a suitable d-fold Veronese embedding so that the subscheme becomes linear (that is, replace $\mathscr{T}$ by $\mathscr{T}^{(d)} = \oplus_n \mathscr{T}_{nd}$ and $\mathscr{I}$ by $\mathscr{I}^{(d)} = \oplus_n \mathscr{I}_{nd}$ (cf. EGA II, 3.1.8) for a suitable d where $\mathscr{I}_d$ generates $\mathscr{I}^{(d)}$).

Linear projection is compatible with base change because Proj is so (EGA II, 3.5.3). In other words, for any S-scheme T, the linear projection,

$$\text{Proj}\,(a_T) : G(a_T) \to \text{Proj}\,(\mathscr{S}_T),$$

with center, Proj $((\mathscr{T}/\mathscr{I})_T) = X \times_S T$, is equal to the map,

$$g_T : G(a) \underset{S}{\times} T \to Y \underset{S}{\times} T.$$

Linear projection is compatible with linear embedding in the following sense. Let $Z' = \text{Proj}\,(\mathscr{T}')$ be a closed subscheme of $Z = \text{Proj}\,(\mathscr{T})$, and let $Y' = \text{Proj}\,(\mathscr{S}')$ be a closed subscheme of $Y = \text{Proj}\,(\mathscr{S})$ such that the map $a : \mathscr{S} \to \mathscr{T}$ induces a map $a' : \mathscr{S}' \to \mathscr{T}'$. Then since Proj is a 'functor' (cf. EGA II, 2.8.4), the projection $g = \text{Proj}\,(a)$ with center X restricts to the projection $g' = \text{Proj}\,(a')$ with center $X' = X \cap Z'$. In other words, there is a commutative diagram,

(D1.1)
$$\begin{array}{ccc} (Z' - X') & \xrightarrow{g'} & Y' \\ \downarrow & & \downarrow \\ (Z - X) & \xrightarrow{g} & Y. \end{array}$$

If $\mathscr{T}'$ is equal to $\mathscr{T} \otimes_{\mathscr{S}} \mathscr{S}'$, then diagram (D1.1) is cartesian by (A5, (i)) and then X is contained in Z' – in other words, X is equal to X' – because $\mathscr{T}/\mathscr{S}_1\mathscr{T}$ is obviously isomorphic to $\mathscr{T}'/\mathscr{S}'_1\mathscr{T}'$.

By way of example, suppose g is the natural linear projection from a projective space $\mathbb{P}(F)$ with center a linear subspace $\mathbb{P}(F')$; that is, suppose g is the map,

$$g = \mathrm{Proj}\,(a) : (\mathbb{P}(F) - \mathbb{P}(F')) \to \mathbb{P}(G),$$

where a denotes the inclusion of Sym (G) in Sym (F) and G denotes the kernel of the canonical surjection $u : F \to F'$. Suppose F, F' and G are free. Then for a suitable choice of coordinates, g is clearly given by the formula,

$$\text{(D1.2)} \qquad g(t_0, t_1, \ldots; u_0, u_1, \ldots) = (t_0, t_1, \ldots).$$

In view of the compatibility of projection with embedding (D1.1), any restriction of g is also given by such a formula. Moreover, it is evident that any linear projection is such a restriction, so given by such a formula, at least locally over the base.

(D2) *The Y-scheme $B(g)$*

The scheme and maps introduced in (B7) for a join $X * Y$ will now be generalized for the linear projection $g = \mathrm{Proj}\,(a)$. The Y-scheme,

$$B(g) = \mathrm{Proj}\,(\mathscr{T}_Y \underset{\mathscr{S}_Y}{\otimes} \mathrm{Sym}\,(\mathscr{O}_Y(1))),$$

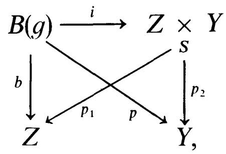
generalizes the conjunctive transform $B_X(X * Y)$. The natural linear Y-embedding,

$$i : B(g) \to Z \underset{S}{\times} Y,$$

equal to the map,

$$\mathrm{Proj}\,(\mathscr{T}_Y \underset{\mathscr{S}_Y}{\otimes} \alpha_Y^{\sharp}) : \mathrm{Proj}\,(\mathscr{T}_Y \underset{\mathscr{S}_Y}{\otimes} \mathrm{Sym}\,(\mathscr{O}_Y(1))) \to \mathrm{Proj}\,(\mathscr{T}_Y),$$

generalizes the conjunctive embedding. It clearly fits into an important commutative diagram (similar to (B7.1)),



where p is the structure map of $B(g)$, where p_1 and p_2 are the projections and where b is defined as $p_1 \circ i$.

The linear projection Proj (a_Y) of $Z \times_S Y$ to $Y \times_S Y$ is equal to g_Y and its center is $X \times_S Y$ by (D1). It restricts, by (D1), to a projection,

$$h = \operatorname{Proj}\,(a_Y \underset{\mathscr{S}_Y}{\otimes} \operatorname{Sym}\,(\mathscr{O}_Y(1))),$$

of the closed subscheme $B(g)$ of $Z \times_S Y$ to the closed subscheme $Y = \operatorname{Proj}\,(\operatorname{Sym}\,(\mathscr{O}_Y(1)))$ of $Y \times_S Y$. The center of h will be denoted by E. It is equal to the center $X \times_S Y$ of g_Y by (D1) because the homogeneous coordinate ring of $B(g)$ is equal to the tensor product $\mathscr{T}_Y \otimes_{\mathscr{S}_Y} \operatorname{Sym}\,(\mathscr{O}_Y(1))$. Moreover, h is equal to the restriction $p|$ to $(B(g)-E)$ of the structure map p of $B(g)$ because h is a Y-morphism.

It is easy, in view of the preceding paragraph, to verify, following the proof of (C1, (i), (ii), and (ii')), that E is equal to the inverse image $b^{-1}(X)$ of X in $B(g)$, that b induces an isomorphism $b|$, which fits into a commutative diagram,

(D2.1)
$$\begin{array}{ccc} (B(g)-E) & \xrightarrow{p|} & Y \\ {\scriptstyle b|}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle id} \\ (Z-X) & \xrightarrow{g} & Y, \end{array}$$

and that i carries $(B(g)-E)$ isomorphically onto the graph of g. In particular, E is the exceptional locus of b.

We shall now verify that the formation of $B(g)$ is compatible with linear embedding in the following sense: For each closed subscheme Z' of Z and restriction $g':(Z'-X') \to Y'$, the following diagram is cartesian:

(D2.2)
$$\begin{array}{ccc} B(g') & \longrightarrow & B(g) \\ \downarrow & \square & \downarrow \\ Z' \underset{S}{\times} Y' & \longrightarrow & Z \underset{S}{\times} Y. \end{array}$$

To verify this, consider the diagram,

$$\begin{array}{ccccc} B(g|) \underset{Y}{\times} Y' & \longrightarrow & B(g|) & \longrightarrow & B(g) \\ \downarrow & & \downarrow & & \downarrow \\ Z' \underset{S}{\times} Y' & \longrightarrow & Z' \underset{S}{\times} Y & \longrightarrow & Z \underset{S}{\times} Y. \end{array}$$

where $g|$ denotes the restriction, $g|:(Z'-X') \to Y$, of g. The left-hand square is obviously cartesian; the right-hand square is cartesian by (A5, (ii)) because it can be obtained by applying Proj to the diagram,

$$\begin{array}{ccc} \mathcal{T}'_Y \underset{\mathcal{S}_Y}{\otimes} \operatorname{Sym}(\mathcal{O}_Y(1)) & \longleftarrow & \mathcal{T}_Y \underset{\mathcal{S}_Y}{\otimes} \operatorname{Sym}(\mathcal{O}_Y(1)) \\ \uparrow & \otimes & \uparrow \\ \mathcal{T}'_Y \underset{\mathcal{S}_Y}{\otimes} \mathcal{S}_Y & \longleftarrow & \mathcal{T}_Y \underset{\mathcal{S}_Y}{\otimes} \mathcal{S}_Y, \end{array}$$

which is obviously a tensor product diagram in $\mathscr{C}(Y)$. Moreover, there are relations,

$$\begin{aligned} B(g|) \underset{Y}{\times} Y' &= \operatorname{Proj}((\mathcal{T}'_Y \underset{\mathcal{S}_Y}{\otimes} \operatorname{Sym}(\mathcal{O}_Y(1)))_{Y'}) && \text{(EGA II, 3.5.3)} \\ &= \operatorname{Proj}(\mathcal{T}'_{Y'} \underset{\mathcal{S}'_{Y'}}{\otimes} \operatorname{Sym}(\mathcal{O}_{Y'}(1))) && \text{(A2)} \\ &= B(g'). \end{aligned}$$

Hence diagram (D2.2) is cartesian.

(D3) THEOREM: (i) *The blow-up $B\ell(g)$ of Z along X is equal to the closure in $Z \times_S Y$ of the graph of g, and the complement in $B\ell(g)$ of the graph is the support of the exceptional divisor.*

(ii) *$B\ell(g)$ is a closed subscheme of $B(g)$. If $B\ell(g)$ is equal to $B(g)$, then E is equal to the exceptional divisor. If E is a divisor, then $B\ell(g)$ is equal to $B(g)$ and E is equal to the exceptional divisor. (In general, $B\ell(g)$ is not equal to $B(g)$, for E is equal to $X \times_S Y$ while the exceptional divisor often has a different form.)*

PROOF: (All the assertions are proved by reasoning similar to that in (C6).) For each positive integer n, there is a canonical surjection,

(D3.1) $$m_n(a) : \mathcal{S}_n[-n] \otimes \mathcal{T} \to \mathcal{I}^n,$$

similar to the one defined in (B6), which is natural in a and compatible with base change, where $\mathcal{I}$ is, as always, the Ideal of $\mathcal{T}$ generated by $a(\mathcal{S}_1)$. Then, the map,

$$\operatorname{Proj}(\oplus (m_n(a))^{\sim}) : B\ell(g) = \operatorname{Proj}(\oplus(\tilde{\mathcal{I}})^n) \to \operatorname{Proj}(\oplus \mathcal{S}_{n,Z} \otimes \mathcal{O}_Z(-n)) = Z \underset{S}{\times} Y,$$

is a closed Z-embedding of $B\ell(g)$ in $Z \times_S Y$. The reasoning of (C6) shows that $B\ell(g)$ contains the graph of g as an open subscheme whose complement is equal to the support of the exceptional divisor. By (C4, (i)), the blowup $B\ell(g)$ is therefore equal to the closure of the graph of g in

$Z \times_S Y$. Moreover, the reasoning of (C6) shows (1) $B\ell(g)$ is a closed subscheme of $B(g)$ (for the right-hand vertical map of the diagram analogous to (C6.4) is always an isomorphism) and (2) if $B(g)$ is equal to $B\ell(g)$, then E is equal to the exceptional divisor of $B\ell(g)$. Finally, if E is a divisor in $B(g)$, then since E is equal to the complement of the graph of g in $B(g)$ by (D2), clearly $B(g)$ is equal to $B\ell(g)$ by (C4, (i)).

(D4) PROPOSITION: (i) *The Ideal I of X in Z is equal to the image of the canonical map, $\mathcal{S}_{Z,1} \otimes \mathcal{O}_Z(-1) \to \mathcal{O}_Z$, induced by a.*

(ii) *Assume E is a divisor in $B(g)$. Then there are formulas,*

$$\mathcal{O}_{B(g)}(-E) = p^*\mathcal{O}_Y(1) \otimes b^*\mathcal{O}_Z(-1)$$
$$\mathcal{O}_{B(g)}(E)|E = p_2^*\mathcal{O}_Y(-1) \otimes p_1^*\mathcal{O}_X(1).$$

PROOF: (i) Since Im $(m_1(a))$ (see (D3.1)) generates the homogeneous Ideal $\mathcal{I}$ of X in Z, clearly Im $((m_1(a))^{\sim})$ is equal to $I = \tilde{\mathcal{I}}$. So, the assertion follows from (A1.2).

(ii) The surjective map (see (D3.1)),

$$m_1(a_Y \underset{\mathcal{S}_Y}{\otimes} \mathrm{Sym}\,(\mathcal{O}_Y(1)))^{\sim} : p^*\mathcal{O}_Y(1) \otimes \mathcal{O}_{B(g)}(-1) \to \mathcal{O}_{B(g)}(-E),$$

is bijective because both its source and target are locally free with rank 1. The formulas now result from reasoning similar to that in (C3, (ii)).

(D5) THEOREM: *Assume the map $a : \mathcal{S} \to \mathcal{T}$ is locally isomorphic to a map of the form $\rho \otimes 1 : \mathcal{S} \to \mathcal{R} \otimes \mathcal{S}$. Then $B(g)$ is equal to $B\ell(g)$, and E is equal to the exceptional divisor.*

PROOF: Locally Z is equal to the join $X * Y$ of X and Y. Hence, E is locally a divisor in $B(g)$ by (B8, (i)). So, E is a divisor in $B(g)$, and the assertions hold by (D3, (ii)).

(D6) LEMMA: *Consider a diagram with exact rows,*

(D6.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & V & \xrightarrow{\alpha} & L & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u''} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \\ 0 & \longrightarrow & H & \longrightarrow & \mathcal{O}_S & \longrightarrow & \mathcal{O}_T & \longrightarrow & 0, \end{array}$$

in which u'' is surjective, V is locally generated by r sections, and L is locally *free with rank s. Assume that S is locally noetherian and that the Ideal u(V) defines a regularly embedded subscheme with codimension r in S. Then the Ideal u′(L) defines a regularly embedded subscheme with codimension s in T.*

PROOF: Let t be a point of T. Set $A = \mathcal{O}_{S,t}$, set $I = u(V)_t$ and set $J = u'(L)_t$. Consider the commutative diagram of $k(t)$-vector spaces,

$$\text{(D6.2)}\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & K_t \otimes k(t) & \longrightarrow & V_t \otimes k(t) & \longrightarrow & L_t \otimes k(t) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle u_t \otimes k(t)} & & \downarrow & & \\ 0 & \longrightarrow & H_t \otimes k(t) & \longrightarrow & I \otimes k(t) & \longrightarrow & J \otimes k(t) & \longrightarrow & 0. \end{array}$$

The vertical maps are obviously surjective and will now be proved bijective; the top row is exact because L is locally free and the bottom row will now be proved exact.

Since $u(V)$ defines a regularly embedded subscheme with codimension r, the (A/I)-module (I/I^2) is free with rank r (EGA IV$_4$, 16.9.2, 16.9.3); hence $I \otimes k(t)$ is r-dimensional. Therefore, since V_t has r generators, the surjection $u_t \otimes k(t)$ is bijective. It follows easily that the vertical maps of (D6.2) are bijective. Thus, the bottom row is exact, and $J \otimes k(t)$ has dimension s.

By Nakayama's lemma, there is therefore a minimal set of generators $(f_1, \ldots, f_r)$ of I such that $(f_1, \ldots, f_{r-s}$ generates H_t and such that the image of $(f_{r-s+1}, \ldots, f_r)$ in J generates J. Since A is noetherian and I is regular, the sequence $(f_1, \ldots, f_r)$ is A-regular (EGA IV$_4$, 16.9.5 and 19.5.2). So since $\mathcal{O}_{T,t}$ is equal to $A/(f_1 A + \ldots + f_{r-s}A)$, clearly $J = f_{r-s+1}\mathcal{O}_{T,t} + \ldots + f_r \mathcal{O}_{T,t}$ defines a regularly embedded subscheme with codimension s in T.

(D7) THEOREM: *Assume that Z is locally noetherian, that Y is flat and of finite type over S, that X is regularly embedded in Z with codimension r, and that* $\mathcal{S}_1$ *is locally generated by r sections. Then B(g) is equal to Bℓ(g) and E is equal to the exceptional divisor.*

PROOF: Since Z is locally noetherian and Y is of finite type over S, clearly $Z \times_S Y$ is locally noetherian. Moreover, since Y is flat over S, clearly $Z \times_S Y$ is flat over Z; hence $(\mathcal{I}_Y)^{\sim}$, which is equal to $(\mathcal{I}^{\sim})_Y$ by (EGA II, 3.5.3), is a regular Ideal of $\mathcal{O}_Z \otimes_{SY}$ (EGA IV$_4$, 19.1.5, (ii)). The Ideal of $B(g) = \operatorname{Proj}(\mathcal{T}_Y \otimes_{\mathcal{S}_Y} \operatorname{Sym}(\mathcal{O}_Y(1)))$ in $Z \times_S Y = \operatorname{Proj}(\mathcal{T}_Y)$ is obviously equal to the image H of the tilde of $\operatorname{Ker}(\mathcal{T}_Y \otimes_{S_Y} \alpha_Y^{\#})$. By (A1.2), this tilde is equal to $\operatorname{Ker}(p^*(\alpha_{Y,1}^{\#}))(-1)$. By (D4, (i)), the Ideal of E in

$B(g)$ is equal to the image of $p^*\mathcal{O}_Y(1) \otimes \mathcal{O}_{B(g)}(-1)$ *in* $\mathcal{O}_{B(g)}$. So, by (D6) with the diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}(p_2^*(\alpha_{Y,1}^\#))(-1) & \longrightarrow & p_2^*\mathcal{S}_{Y,1}(-1) & \longrightarrow & p_2^*\mathcal{O}_Y(1) \otimes \mathcal{O}_{B(g)}(-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H & \longrightarrow & \mathcal{O}_{Z \times_S Y} & \longrightarrow & \mathcal{O}_{B(g)} & \longrightarrow & 0,
\end{array}$$

for (D6.1), the Ideal of E in $B(g)$ is invertible. Thus E is a divisor in $B(g)$. Therefore the assertions hold by (D3, (ii)).

(D8) *The blow-up of a projective space along a linear subscheme Compare* [8], *Corollary* (4.5).

Let $u : F \to F'$ be a surjective $\mathcal{O}_S$-homomorphism of quasi-coherent $\mathcal{O}_S$-Modules, and let G denote the kernel of u. Assume u is locally split; this holds, for example, when F' is locally free. Then the blow-up of $P = \mathbb{P}(F)$ along the linear subscheme $L = \mathbb{P}(F')$ is equal to the scheme $W = \mathbb{P}(H)$ over $Y = \mathbb{P}(G)$, where H is equal to the $\mathcal{O}_Y$-Module defined by the commutative diagram with exact rows and columns,

(D8.1)
$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Ker}(\alpha) & \xrightarrow{id} & \operatorname{Ker}(\alpha) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_Y & \longrightarrow & F_Y & \xrightarrow{u_Y} & F'_Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & \mathcal{O}_Y(1) & \longrightarrow & H & \longrightarrow & F'_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where all the maps are the canonical ones.

Indeed, W is obviously equal to $B(\gamma)$, with $\gamma = \operatorname{Proj}(c)$ where c is the inclusion of $\operatorname{Sym}(G)$ in $\operatorname{Sym}(F)$. Since u is locally split, c is locally isomorphic to the natural map of $\operatorname{Sym}(G)$ into $\operatorname{Sym}(G \oplus F') = \operatorname{Sym}(G) \otimes \operatorname{Sym}(F')$. Hence $W = B(\gamma)$ is equal to the blow-up of P along L by (D5).

(D9) *The blow-up of the diagonal in* $\mathbb{P}(V) \times_S \mathbb{P}(V)$

Let R denote the projective space $\mathbb{P}(V)$ associated to a quasi-coherent $\mathcal{O}_S$-Module V. In (D8), take R for the base, and take the Serre map, $\alpha^{\#}_{R,1} : V_R \to \mathcal{O}_R(1)$, for the map, $u : F \to F'$ (it is locally split because $\mathcal{O}_R(1)$ is locally free). Then, the linear embedding $\mathbb{P}(\alpha^{\#}_{R,1})$ of R in $R \times_S R$ is equal to the diagonal map by (A3). Set $\Omega(1) = \operatorname{Ker}(\alpha^{\#}_{P,1})$. If V is locally free with a finite rank, then Ω is equal to the Module of 1-differentials, $\Omega^1_{P/S}$, by (GD I, 3.1), and $\mathbb{P}(\Omega(1))$ is isomorphic to the projectivized tangent bundle of R.

The blow-up of $\mathbb{P}(V) \times_S \mathbb{P}(V)$ along the diagonal subscheme is therefore, by (D8), equal to the bundle of projective lines $\mathbb{P}(H)$ over $Y = \mathbb{P}(\Omega(1))$ where H is a suitable extension of $(\mathcal{O}_{\mathbb{P}(V)}(1))_Y$ by $\mathcal{O}_Y(1)$.

(D10) *Families of linear space sections*

Use the notation of (D8). The Y-projective space $\mathbb{P}(H)$ is obviously linearly embedded in $P \times_S Y$. Moreover, the canonical surjection u_Y from F_Y to F'_Y factors through H, so $\mathbb{P}(H)$ contains $L \times_S Y$. Thus, $\mathbb{P}(H)$ is the total space of a family, parametrized by Y, of linear subspaces of P containing L. It is easy to check that $\mathbb{P}(H)/Y$ is the universal family of linear spaces containing L as a subspace with codimension 1 in the sense that, for a family $\mathbb{P}(H')$ parametrized by T, the natural surjection $H' \to F'_T$ has an invertible sheaf for its kernel. Thus it is natural to call L the *axis* of the family, for the members of the family 'rotate' about L. If G is locally free with rank 2, then $\mathbb{P}(H)/Y$ is a 1-parameter family, or pencil, of hyperplanes.

Let $Z = \operatorname{Proj}(\mathscr{T})$ be a closed subscheme of P, and set $X = Z \cap L$. The intersection $\mathbb{P}(H) \cap (Z \times_S Y)$ is clearly the total space of a family, parametrized by Y, of linear space sections of Z containing X. Again X is called the *axis*.

Let γ be the natural projection from P to Y and g its restriction to Z. Then $\mathbb{P}(H)$ is equal to $B(\gamma)$ by (D8). So, $\mathbb{P}(H) \cap Z \times_S Y$ is equal to $B(g)$ by (D2).

Assume that Z is locally noetherian, that G is locally free with rank $(r+1)$, and that X is regularly embedded in Z with codimension $(r+1)$. The latter occurs, for example, when Z is flat and locally of finite presentation over S, when the fibers of Z/S satisfy condition S_{r+1}, (e.g., the fibers are Cohen-Macaulay, or the fibers are normal and $r = 1$ holds), and when the relative codimension of X in Z is $(r+1)$. Then, by (D4) and (D7), the Y-scheme $B(g)$ is equal to the blow-up of Z along X and there are formulas,

$$\mathcal{O}_{B(g)}(-E) = p^*\mathcal{O}_Y(1) \otimes b^*\mathcal{O}_Z(-1)$$
$$\mathcal{O}_{B(g)}(E)|E = p_2^*\mathcal{O}_Y(-1) \otimes p_1^*\mathcal{O}_X(1).$$

In sum, under mild assumptions, the blow-up of the ambient space along the axis of a family of linear space sections is equal to the total space of the family, and there are formulas for the Ideal of the exceptional divisor and for its normal sheaf.

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(Oblatum 10–II–1975)

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