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ON DERIVATIONS OF LIE ALGEBRAS

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Let L be a Lie algebra of finite or infinite dimension over a field $\boldsymbol{k}$ of arbitrary characteristic and let DER(L) denote the Lie algebra of all derivations of L. Then we may raise the following question: how far does DER(L) determine L?

In the case of finite-dimensional Lie algebras we have the well-known results obtained by G. F. Leger, N. Jacobson, S. Tôgô, T. Satô and others. But in the case of infinite-dimensional Lie algebras almost nothing is known! Since this type of Lie algebras appears more and more in miscellaneous fields of research, as for instance group theory, differential geometry and operator theory, it becomes important to have the disposal of some general relations between arbitrary Lie algebras and their derivation algebras. In this paper we will prove some initial results.

Section 1 deals with notation and basic definitions. In section 2 we shall give sufficient conditions for a Lie algebra to have outer derivations. The main result of this section is a generalization of a well-known theorem of N. Jacobson, which states that every non-zero nilpotent Lie algebra has outer derivations ([1], Theorem 4). We shall prove: every Lie algebra L with nonzero centre, where $L \urcorner L^2$ contains some nilpotent element, has outer derivations. As a corollary we have: Lie algebras of type (T) have outer derivations d such that $d^2 = 0$.

In section 3 we are concerned with nilpotency and simplicity of derivation algebras. First we shall prove: if the centre of L is not contained in L^2 and DER(L) is nilpotent, then L is one-dimensional. This theorem includes a lemma of S. Tôgô. By means of this theorem we can prove that a non-zero Lie algebra with Abelian derivation algebra is also one-dimensional, a result already proved by S. Tôgô in.the case of finite-dimensional Lie algebras ([5], Lemma 6). Next we will show that non-semisimple Lie algebras may have simple derivation algebras, as distinct from the finite-dimensional case, where only simple Lie algebras have simple derivation algebras. Some properties of such Lie algebras will be derived. Finally we show that Jacobson's theorem concerning finite-dimensional Lie algebras with non-singular (= injective) derivations ([1], Theorem 3) does not remain valid for infinite-dimensional Lie algebras. We shall give an example of a non-nilpotent Lie algebra with non-singular derivations.

1. Notation and terminology

For notation we refer to I. N. Stewart [4]. L will always designate a *Lie algebra* (possibly of infinite dimension) over an arbitrary field $\boldsymbol{k}$. L is *perfect* if $L = L^2$ and *simple* if 0 and L are the only ideals of L. We shall call L *semisimple* if L does not contain solvable ideals $\neq 0$. The *centre* of L will be denoted by $\zeta(L)$. The ideal of DER(L) of all *inner derivations* of L will be referred to as $ad(L)$. If L is a *split extension* of an ideal I by a subalgebra S, we will write $L = I \boxplus S$. In the case of a *direct sum* the sign $\boxplus$ will be replaced by the sign $\oplus$. The radicals of a Lie algebra will be denoted by $\beta(L) =$ the *Baer radical*, $\nu(L) =$ the *Fitting radical*, $\rho(L) =$ the *Hirsch-Plotkin radical* (for their definitions see [4]) and $R_L =$ the *Vasilescu radical* (introduced in [7]).

2. Outer derivations of Lie algebras

We first give some information about the derivation algebra of certain types of split extensions.

LEMMA 1: *Suppose $L = I \boxplus S$, $I^2 = 0$ and $I \neq [I, \zeta(S)]$. Then L has outer derivations.*

PROOF: Every $x \in L$ has a unique representation $x = x_I + x_S$, $x_I \in I$ and $x_S \in S$. Define $dx = x_I$, then $d[x, y] = [x_I, y_S] + [x_S, y_I] = [dx, y] + [x, dy]$, since I is Abelian. Therefore $d \in$ DER(L); d cannot be an inner derivation, for suppose $d = ad(z)$ for some $z \in L$, then $x_I = dx = [z, x]$ for all $x \in L$. Now $z_I = [z, z] = 0$, so $z \in \zeta(S)$. Consequently $I = [I, \zeta(S)]$, a contradiction to our hypothesis.

COROLLARY: *A Lie algebra L, splitting over a non-zero Abelian ideal I by some subideal S, has outer derivations.*

PROOF: Since S si L, $[I, {}_nS] \subset S$ for some n, so $[I, {}_nS] = 0$. Hence $I \neq [I, \zeta(S)]$, for otherwise $I = [I, {}_nS] = 0$.

LEMMA 2: *Let $L = H \oplus K$ and suppose H has outer derivations. Then L has outer derivations.*

PROOF: Let u be some outer derivation of H. Every $x \in L$ can be uniquely written as $x = x_H + x_K$, $x_H \in H$ and $x_K \in K$. We define $dx = ux_H$ for all $x \in L$. Then it is easily verified that $d \in$ DER(L). If d would be an inner derivation, then $d = ad(z)$ for some $z \in L$ and consequently $ux_H = dx = [z, x] = [z_H, x_H] + [z_K, x_K]$, so $ux_H - [z_H, x_H] \in H \cap K = 0$. Hence $u = ad(z_H)$, contrary to its choice.

REMARK: Lemma 2 makes clear, that in T. Satô's result ([2], Corollary

1 of Proposition 10) the conditions of finite-dimensionality, semisimplicity and solvability are redundant.

LEMMA 3: *$\zeta(L) \not\subset L^2$ implies that L has outer derivations.*

PROOF: $\zeta(L)$ contains some element e such that $e \notin L^2$. Consequently $L = M \oplus \boldsymbol{k}e$, where $L^2 \leqq M$ for some ideal M of L. We now apply Lemma 2 or the Corollary of Lemma 1.

THEOREM 1: *If $\zeta(L) \neq 0$ and $L \urcorner L^2$ contains some nilpotent element, then L has outer derivations.*

PROOF: By the preceding Lemma we only have to consider the case $\zeta(L) \subset L^2$. Suppose all derivations of L are inner. By the hypothesis L contains some nilpotent element e such that $e \notin L^2$. Now L can be written as $L = M \boxplus \boldsymbol{k}e$, where $L^2 \leqq M$ for some M. We have $0 \neq \zeta(L) \subset \zeta(M)$. Let $0 \neq z \in \zeta(M)$. Every $x \in L$ can be uniquely written as $x = x_M + \alpha(x)e$, $x_M \in M$ and $\alpha(x) \in \boldsymbol{k}$. Define $d: L \to L$ by $dx = \alpha(x)z$ for all $x \in L$, then $d[x, y] = 0$, since $[x, y] \in L^2 \leqq M$ and $[dx, y] + [x, dy] = \alpha(x)\alpha(y)[z, e] + \alpha(y)\alpha(x)[e, z] = 0$, so $d \in \mathrm{DER}(L)$. Consequently $d = ad(m + \lambda e)$ for some $m \in M$ and some $\lambda \in \boldsymbol{k}$. But $z = de = [m, e]$ and $0 = dm = \lambda[e, m]$, so $\lambda = 0$, since $z \neq 0$. Moreover, $0 = dM = [m, M]$, so $0 \neq m \in \zeta(M)$. We can proceed in the same way with m. Hence $\zeta(M)$ contains a sequence $m = m_1, m_2, m_3, \cdots$ such that $z = [m_i, {}_ie]$. But e is nilpotent, so $z = 0$. This is a contradiction! Therefore $ad(L) \neq \mathrm{DER}(L)$.

COROLLARY: *Lie algebras of type (T) have outer derivations d such that $d^2 = 0$.*

PROOF: A Lie algebra L is called *of type* (T), if there exists some subspace $U \neq 0$ of L with the following properties:

$$L = L^2 + U, \quad L^2 \cap U = 0, \quad [L^2, U] = 0, \quad U^2 = \boldsymbol{k}c,$$

where $0 \neq c \in \zeta(L)$, and the mapping $\tau: U \times U \to \boldsymbol{k}$, defined by $[u, v] = \tau(u, v)c$, is non-singular. This notion was first introduced by S. Tôgô for finite-dimensional Lie algebras in [6]. It is clear that every $0 \neq u \in U$ is not contained in L^2 and has the property $[L, {}_2u] = 0$. Moreover we have $\zeta(L) \subset L^2$, since $x \in \zeta(L)$ implies $x_U = 0$, for τ is non-singular. From the proof of Theorem 1 it follows now immediately, that L has outer derivations d such that $d^2 = 0$. In the case of finite-dimensional Lie algebras this result was already obtained by Tôgô.

3. Nilpotency and simplicity of derivation algebras

The following result is due to S. Tôgô ([5], Lemma 5): Suppose L to be non-Abelian nilpotent and $\zeta(L) \not\subset L^2$. Then DER(L) is not nilpotent.

We will give the following sharpening.

THEOREM 2: *If* $\dim L \neq 1$ *and* $\zeta(L) \not\subset L^2$, *then* DER(L) *is not nilpotent.*

PROOF: $\zeta(L) \not\subset L^2$, so $\zeta(L)$ contains some element e such that $e \notin L^2$. Consequently $L = M \oplus \boldsymbol{k}e$, where $L^2 \leqq M$. It is easy to verify that $\zeta(L) = \zeta(M) \oplus \boldsymbol{k}e$. If $\zeta(M) = 0$, then $\zeta(L) = \boldsymbol{k}e$, so L is not nilpotent by a Lemma of Schenkman ([3], Lemma 4). If $\zeta(M) \neq 0$, then we may choose $0 \neq z \in \zeta(M)$. Every $x \in L$ can be represented in the form $x = x_M + \alpha(x)e$, $x_M \in M$ and $\alpha(x) \in \boldsymbol{k}$. We now define $d_1 : L \to L$ by putting $d_1 x = \alpha(x)z$ and $d_2 : L \to L$ by putting $d_2 x = \alpha(x)e$ for all $x \in L$. Then $d_i \in$ DER(L) for $i = 1, 2$ and we have $[d_1, d_2]x = \alpha(x)d_1 e - \alpha(x)d_2 z = \alpha(x)z = d_1 x$. The theorem now follows.

COROLLARY: *If* $L \neq 0$ *and* DER(L) *is Abelian, then* L *is one-dimensional.*

PROOF: $ad(L)$ is Abelian, so $L^3 = 0$. Let U be some subspace of L complementary to L^2. Then every $x \in L$ can be uniquely written as $x = x_2 + x_U$, $x_2 \in L^2$ and $x_U \in U$. Define $d : L \to L$ by setting $dx = 2x_2 + x_U$ for all $fx \in L$, then $d \in$ DER(L), as an easy computation shows. But $ad(dx) = [d, ad(x)] = 0$, since DER(L) is Abelian. Thus $[dx, y] = 0$ for all x and for all y; choose $x, y \in U$, then $[x, y] = 0$. Hence U is Abelian and consequently $L^2 = U^2 = 0$. Now $L = \zeta(L) \not\subset L^2 = 0$, so $\dim L = 1$ by Theorem 2.

THEOREM 3: *Suppose* DER(L) *is simple. Then either* L *is one-dimensional or* L *satisfies each of the following conditions*:

1. *All derivatives of* L *are inner.*
2. DER(L) *is centreless.*
3. L *is perfect.*
4. $L \neq \zeta(L) = \sum_{\substack{H \text{ si } L \\ \neq}} H$.
5. *All radicals of* L *equal the centre of* L.

PROOF: $ad(L) \lhd$ DER(L), so we have to consider two cases.

Case I: $ad(L) = 0$.

$L^2 = 0$, so DER(L) consists of all linear mappings of L into itself. Therefore scalar multiplications form an ideal of DER(L) and since the identity mapping of L is not trivial, we must have that all linear mappings are scalar multiplications. This implies that L is one-dimensional.

Case II: $ad(L) =$ DER(L).

Then $L/\zeta(L)$ is simple. If Abelian, it must be one-dimensional, so that L is Abelian and Case I applies. So we may assume $L/\zeta(L)$ simple non-Abelian. In particular its centre is 0. Now $L = \zeta(L) + L^2$, so $L = L^2$, since by Lemma 3 $\zeta(L) \subset L^2$. If $H \lhd L$, then either $H \subset \zeta(L)$ or $L = H + \zeta(L)$.

But L is perfect, so the latter case implies $H = L$. We can now immediately conclude that the centre of L is the sum of all proper subideals of L. Since L is not solvable, $\zeta(L) = \beta(L) = \nu(L)$.

L is not locally nilpotent, for otherwise $L/\zeta(L)$ would be locally nilpotent and also satisfy the minimum condition for ideals, since DER(L) is simple; consequently $L/\zeta(L)$ would be solvable by a result of I. N. Stewart ([4], Lemma 14.3), contrary to the fact that L is not solvable. It now follows that $\zeta(L) = \rho(L)$.

The center of L is evidently a primitive ideal of L and therefore contains R_L. But $\zeta(L) \leqq R_L$, so $\zeta(L) = R_L$. This finishes the proof.

REMARK: The centre of L does not necessarily vanish, if L is infinite-dimensional in Case II. M. Favre has drawn my attention to the following counterexample, due to P. de la Harpe:

Let H be an arbitrary infinite-dimensional complex Hilbert space, $B(H)$ the algebra of all its bounded linear operators and $C(H)$ the ideal of compact operators. Then $B(H)/C(H)$ is a C^*-algebra. By considering this associative algebra as a Lie algebra L, it turns out that $ad(L) = \text{DER}(L)$ is simple and $\zeta(L) \neq 0$. The proof is rather technical and will be omitted.

A finite-dimensional Lie algebra over char. 0 with simple derivation algebra is itself simple. This well-known result also follows immediately from Theorem 3 by using the *Levi decomposition.*

We will finish by showing that infinite-dimensional Lie algebras with non-singular derivations need not to be nilpotent. For example, take $L = C(\mathbb{R})$, the vector space of all continuous real functions, where the Lie product is given by $[f, g] = f(0)g - g(0)f$. This Lie algebra L is infinite-dimensional, but not nilpotent, since $[x, {}_n x+1] = (-1)^n x$ for all n. However, $d: L \to L$, defined by $df = xf$, is a non-singular derivation of L.

Finally, I wish to thank my referee for his useful comments.

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