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# ON A SERIES OF COSECANTS RELATED TO A PROBLEM IN ERGODIC THEORY

by

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In investigating the spectrum of the transformation induced on the space consisting of $[0, 1)$ with a "second floor" above $[0, \beta)$ by translation mod 1 by an irrational $\alpha$, one is led to consider [9] convergence of the series

$$(1) \qquad \sum_{k \neq 0} \frac{1}{k^2} \frac{||k\beta||^2}{||k\alpha||^2},$$

where $|| \ ||$ denotes distance to the nearest integer. Since there are constants $c, c', d, d'$ for which $c|\sin \pi x| \leqq ||x|| \leqq c'|\sin \pi x|$ and $d|1-e^{2\pi i x}| \leqq ||x|| \leqq d'|1-e^{2\pi i x}|$ for all $x \in \boldsymbol{R}$, convergence of this series is equivalent to convergence of

$$(2) \qquad \sum_{k \neq 0} \frac{1}{k^2} \frac{\sin^2 \pi k\beta}{\sin^2 \pi k\alpha}$$

and to convergence of

$$(3) \qquad \sum_{k \neq 0} \frac{1}{k^2} \frac{|1-e^{2\pi i k\beta}|^2}{|1-e^{2\pi i k\alpha}|^2}.$$

Series similar to (2) have been mentioned in an earlier paper of Kac and Salem [6], and problems of convergence of series with small denominators are well known in celestial mechanics.

Let $f(x) = \chi_{[0,\beta)}(x)-\beta$ for $x \in [0, 1)$, and let

$$f_n(x) = \sum_{k=0}^{n-1} f\langle x+k\alpha \rangle$$

for $n = 1, 2, \cdots$, where $\langle y \rangle$ denotes the fractional part of $y \in \boldsymbol{R}$. From the equidistribution mod 1 of $\{\langle x+k\alpha \rangle : k \in \boldsymbol{Z}\}$ it follows that $|f_n(x)| = o(n)$ for each $x \in [0, 1)$. Kesten [7] has proved that $\{|f_n(0)| : n = 1, 2, \cdots\}$ is bounded if and only if $\beta \in \boldsymbol{Z}\alpha$ (mod 1), and recently a simple proof of this and related theorems along with an application to topological

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dynamics have been given by Shapiro [10] and Furstenberg, Keynes, and Shapiro [3]. It will develop that convergence of (1) is equivalent to boundedness of the sequence of $L_2$ norms $\{||f_n||_2 : n = 1, 2, \cdots\}$, and that (1) converges if and only if $\beta \in \mathbf{Z}\alpha \pmod 1$. For earlier literature concerning boundedness of $\{|f_n| : n = 1, 2, \cdots\}$, see [1, p. 226 ff.], [5], and [8].

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THEOREM. *Let* $\alpha, \beta \in [0, 1)$ *with* $\alpha$ *irrational, let* $f(x) = \chi_{[0,\beta)}(x) - \beta$ *for* $x \in [0, 1)$, *and let* $f_n(x) = \sum_{k=0}^{n-1} f\langle x+k\alpha\rangle$ *for* $n = 1, 2, \cdots$. *Then the following statements are equivalent.*

(i) $\sum_{k\neq 0} \frac{1}{k^2} \frac{||k\beta||^2}{||k\alpha||^2} < \infty.$

(ii) $\sup_n \sum_{k\neq 0} \frac{1}{k^2} \frac{||k\beta||^2}{||k\alpha||^2} ||kn\alpha||^2 < \infty.$

(iii) $\sup_n ||f_n||_2 < \infty.$

(iv) *There is* $g \in L^2[0, 1)$ *such that* $f(x) = g(x) - g\langle x+\alpha\rangle$ a.e.

(v) $\beta \in \mathbf{Z}\alpha \pmod 1$.

(vi) *There is an* $x \in [0, 1)$ *for which* $\sup_n |f_n(x)| < \infty$.

(vii) $\sup_{x,n} |f_n(x)| < \infty.$

PROOF. The series in (ii) converges for each $n$ because always $||kn\alpha|| \leqq n||k\alpha||$. Since $||kn\alpha||^2 \leqq 1$ for all $n$ and $k$, the implication from (i) to (ii) is clear. In order to see that (ii) implies (iii), note that $f$ has the Fourier expansion

$$f(x) = \sum_{k\neq 0} \frac{1}{2\pi ik} (1 - e^{-2\pi ik\beta}) e^{2\pi ikx},$$

so that

$$f_n(x) = \sum_{j=0}^{n-1} f\langle x+j\alpha\rangle = \sum_{k\neq 0} \frac{1}{2\pi ik} (1 - e^{-2\pi ik\beta}) e^{2\pi ikx} \sum_{j=0}^{n-1} e^{2\pi ijk\alpha}$$

$$= \sum_{k\neq 0} \frac{1}{2\pi ik} (1 - e^{-2\pi ik\beta}) \frac{1 - e^{2\pi ikn\alpha}}{1 - e^{2\pi ik\alpha}} e^{2\pi ikx},$$

and thus

$$||f_n||_2^2 = \sum_{k\neq 0} \frac{1}{4\pi^2 k^2} \frac{|1 - e^{-2\pi ik\beta}|^2}{|1 - e^{2\pi ik\alpha}|^2} |1 - e^{2\pi ikn\alpha}|^2,$$

which lies between two constant multiples of

$$\sum_{k\neq 0} \frac{1}{k^2} \frac{||k\beta||^2}{||k\alpha||^2} ||kn\alpha||^2.$$

We prove now that (iii) implies (iv). For $h \in L^2[0, 1)$, let $Uh(x) = h\langle x+\alpha\rangle$, and define $V : L^2[0, 1) \to L^2[0, 1)$ by $Vh = f + Uh$. Let $K$ denote the norm-closed convex cover of $\{f_1, f_2, \cdots\}$, so $K$ is weakly compact. We claim that $VK \subset K$. For if $h \in K$, then there is a sequence of finite convex combinations $\Sigma\, a_\nu f_{n_\nu}$ ($a_\nu \geqq 0$, $\Sigma a_\nu = 1$) converging to $h$. Since $V$ is continuous, $V\Sigma a_\nu f_{n_\nu}$ converges to $Vh$. But, using linearity of $U$, we see that

$$\begin{aligned} V\Sigma a_\nu f_{n_\nu} &= f + U\Sigma a_\nu f_{n_\nu} = \Sigma a_\nu f + \Sigma a_\nu U f_{n_\nu} \\ &= \Sigma a_\nu (f + U f_{n_\nu}) = \Sigma a_\nu f_{n_\nu + 1} \in K, \end{aligned}$$

so $Vh \in K$. Therefore, by the Schauder-Tychonoff Theorem, there is $g \in K$ with $Vg = g$.

Suppose now that (iv) holds and let $\tau(x) = e^{2\pi i g(x)}$ for $x \in [0, 1)$. Then $\tau\langle x+\alpha\rangle = e^{2\pi i [g(x) - f(x)]} = \tau(x) e^{2\pi i [\beta - \chi_{[0,\beta)}(x)]} = e^{2\pi i \beta}\tau(x)$, so $\tau$ is an eigenfunction with eigenvalue $e^{2\pi i \beta}$ of the transformation $x \to x+\alpha$ (mod 1). All eigenvalues of this transformation are known to be of the form $e^{2\pi i n\alpha}$, $n \in \mathbf{Z}$; therefore $\beta \in \mathbf{Z}\alpha$ (mod 1).

Since $||nx|| \leqq n||x||$ for all $x \in \mathbf{R}$, that (v) implies (i) is immediate. For the sake of completeness, we include Hecke's proof [5, p. 70] that (v) implies (vi). Suppose $\beta = \langle j\alpha\rangle$ with $j > 0$; we will show that $|f_n(0)| \leqq j$ for all $n$ (the proof in case $j \leqq 0$ is similar). Note first that

$$\langle (k-j)\alpha\rangle = \langle k\alpha\rangle - \langle j\alpha\rangle + \chi_{[0,\beta)}\langle k\alpha\rangle$$

for $k = 0, 1, 2, \cdots$. Then we have

$$\begin{aligned} f_n(0) &= \sum_{k=0}^{n-1} [\chi_{[0,\beta)}\langle k\alpha\rangle - \langle j\alpha\rangle] = \sum_{k=0}^{n-1} [\langle (k-j)\alpha\rangle - \langle k\alpha\rangle] \\ &= \sum_{k=-j}^{-1} \langle k\alpha\rangle - \sum_{k=n-j}^{n-1} \langle k\alpha\rangle, \end{aligned}$$

so $|f_n(0)| \leqq j$ for all $n$.

Suppose now that (vi) holds, so that there are $x$ and $M$ with $|f_n(x)| \leqq M$ for all $n = 1, 2, \cdots$. Then

$$|f_n\langle x+j\alpha\rangle| = |f_{n+j}(x) - f_j(x)| \leqq 2M,$$

and $|f_n|$ is bounded by $2M$ on $\{\langle x+j\alpha\rangle : j = 0, 1, 2, \cdots\}$, a dense subset of $[0, 1)$. But each $f_n$ is a step function with only finitely many jumps; therefore we must have $|f_n(y)| \leqq 2M$ for all $y \in [0, 1)$, and we have proved (vii).

Since the implication from (vii) to (iii) is obvious, the proof is complete.

The convergence of (1) only in case $\beta \in \boldsymbol{Z}\alpha \pmod 1$ has some obvious applications to the theory of Diophantine approximation; for example, if $\alpha$ is irrational and $\beta \notin \boldsymbol{Z}\alpha \pmod 1$, then for each $\varepsilon > 0$ and $c > 0$ there are infinitely many $k \in \boldsymbol{Z}$ for which $||k\beta|| > ck^{\frac{1}{2}-\varepsilon}||k\alpha||$. Also, it is apparent that the foregoing theorem contains another easy proof of the theorem of Kesten mentioned above.

The equivalence of conditions (iii) and (iv), which is analogous to a theorem of Gottschalk and Hedlund [4, Theorem 14.11], is easily generalized to the case of any linear operator $U$ acting continuously on a reflexive Banach space $B$: an element $f \in B$ has $||f+Uf+ \cdots +U^{n-1}f||$ bounded in $n$ only if there is $g \in \overline{co}\{f+Uf+ \cdots +U^{n-1}f : n = 1, 2, \cdots\}$ such that $f = g - Ug$. Essentially the same result has been obtained earlier by Butzer and Westphal [2]. Further generalizations to cases such as $B$ locally convex and $\{f+Uf+ \cdots +U^{n-1}f : n = 1, 2, \cdots\}^-$ compact are also possible.

The proof that (iii) ⇒ (iv) ⇒ ($v$) applies also to the case of a general measure-preserving transformation $T : X \to X$ of a probability space $(X, \mathscr{B}, \mu)$. Let $A \subset X$ be a measurable set with $\mu(A) = \beta$, and let

$$f_n(x) = \sum_{k=0}^{n-1} [\chi_A(T^k x) - \beta]$$

for $n = 1, 2, \cdots$. If $\{||f_n||_2 : n = 1, 2, \cdots\}$ is bounded, then $e^{2\pi i\beta}$ must be in the spectrum of $T$. The same observation has been made independently by Furstenberg, Keynes, and Shapiro [3, Theorem 2.4].

## REFERENCES

P. Bohl

[1] Über ein in der Theorie der säkulären Störungen vorkommendes Problem, Jour. f. d. reine und angew. Math. 135 (1909) 189–283.

P. L. Butzer and U. Westphal

[2] The mean ergodic theorem and saturation, Indiana Univ. Math. Jour. 20 (1970/71) 1163–1174.

H. Furstenberg, H. Keynes and L. Shapiro

[3] Prime flows in topological dynamics, in preparation.

W. H. Gottschalk and G. A. Hedlund

[4] *Topological Dynamics*, A.M.S. Coll. Pubs. Vol. XXXVI, Providence, R. I., 1955.

E. Hecke

[5] Analytische Funktionen und die Verteilung von Zahlen mod. eins, Abh. Math. Semin. Hamburg Univ. 1 (1922) 54–76.

M. Kac and R. Salem

[6] On a series of cosecants, K. Akad. v. Wet. Amsterdam Proc. (Series A) 60 (1957) 265–267.

H. Kesten
[7] On a conjecture of Erdös and Szüsz related to uniform distribution mod 1, Acta Arith. 12 (1966) 193–212.

A. Ostrowski
[8] Notiz zur Theorie der Diophantischen Approximationen und zur Theorie der linearen Diophantischen Approximationen, Jahresber. d. Deutschen Math. Ver. 36 (1927) 178–180 and 39 (1930) 34–46.

K. Petersen
[9] Spectra of induced transformations, *Recent Advances in Topological Dynamics*, Springer-Verlag, New York, 1973, 226-230.

L. Shapiro
[10] Irregularities of distribution in dynamical systems, *Recent Advances in Topological Dynamics*, Springer-Verlag, New York, 1973, 249-252

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