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COMPLEMENTS, BASELESS SUBGROUPS AND SYLOW SUBGROUPS OF INFINITE WREATH PRODUCTS

by

B. Hartley

1. Introduction

Throughout this paper $\overline{W}$ will denote the complete standard wreath product $H \,\overline{\wr}\, K$ of two non-trivial groups H and K. We have

$$\overline{W} = \overline{\overline{H}}K,\ \overline{\overline{H}} \cap K = 1,$$

where $\overline{\overline{H}}$ is the base group, that is the set of all functions $f : K \to H$ made into a group by pointwise multiplication, and acted on by K according to the rule

$$f^k(k') = f(k'k^{-1})(f \in \overline{\overline{H}}, k, k' \in K).$$

By the *support* $\operatorname{supp} f$ of an element $f \in H$ we mean as usual the set of all $k \in K$ such that $f(k) \neq 1$. If α is any infinite cardinal then the set $\overline{H}_\alpha$ of all $f \in \overline{\overline{H}}$ such that $|\operatorname{supp} f| < \alpha$ is clearly a normal subgroup of $\overline{W}$. The group

$$W_\alpha = \overline{H}_\alpha K$$

will be denoted by $H \wr_\alpha K$ and called the *α-restricted wreath product* of H by K; $\overline{H}_\alpha$ is its *base group*. When $\alpha = \aleph_0$ of course we obtain the usual standard restricted wreath product $H \wr K$. We shall denote this by W, and its base group by $\overline{H}$.

A subgroup X of W_α will be called *baseless* if $\overline{H}_\alpha \cap X = 1$. Evidently every baseless subgroup of W_α is isomorphic to some subgroup of K. The questions which concern us here are:

Q1. *Which baseless subgroups of W_α are conjugate in W_α to subgroups of K?*

Q2. *What are the isomorphism types of the maximal baseless subgroups of W_α?*

In Section 2 we shall show how some of the answers obtained to Q2 may be used to construct locally finite groups containing certain given sets of locally finite p-groups as Sylow (that is, maximal) p-subgroups, thereby justifying the title of the paper.

The answers to the above questions are particularly simple when $\alpha > |K|$, in which case we are dealing with the complete wreath product. For every baseless subgroup of $\overline{W}$ is conjugate to a subgroup of K (Lemma 3.2 (i), which is essentially [5] Theorem 10.1), and so every maximal baseless subgroup of $\overline{W}$ is isomorphic to K itself. In general, however, the situation is more complicated. Most of the methods of this paper only allow us to deal with baseless subgroups of W_α with the property that every subset of cardinal $< \alpha$ generates a subgroup of cardinal $< \alpha$, and so we call groups with this property *α-bounded*. Of course this is no restriction unless $\alpha = \aleph_0$, in which case it amounts to local finiteness. Our first main result is then as follows; here an *N-group* is one which satisfies the normalizer condition, and a cardinal α is *regular* if α is not the sum of fewer than α cardinals each $< \alpha$.

THEOREM A. *Suppose that $\alpha \leqq |K|$ and let S be any α-bounded N-subgroup of K of cardinal α. Then there exists a maximal baseless α-bounded N-subgroup S^* of W_α such that $\overline{H}_\alpha S^* = \overline{H}_\alpha S$. In particular $S^* \cong S$.*

Suppose further that α is regular, and let T be any baseless α-bounded N-subgroup of W_α. Then either $|T| = \alpha$ or T is conjugate in W_α to a subgroup of K.

We shall see in Section 3 that the assumption that α is regular cannot be omitted above. In the first part of the theorem, notice that we are referring to the maximal members of the set of baseless α-bounded N-subgroups of W_α; similar conventions apply in the sequel. In the case when K itself is an α-bounded N-group and α is regular we obtain from Theorem A a complete answer to Q2. For in that case every subgroup of cardinal α of K is isomorphic to some maximal baseless subgroup of W_α, and the only maximal baseless subgroups of W_α which do not arise in this way, if any, are the conjugates of K. The latter possibility only occurs if $|K| = \beta > \alpha$, and in that case it seems rather surprising that the cardinal of a maximal baseless subgroup of W_α must be either α or β and intermediate cardinals do not occur.

The case $\alpha = \aleph_0$ of Theorem A is of course of special interest and so we state below some of the information obtained from Theorem A for that case. Notice that $\aleph_0$ is regular.

COROLLARY A1. *Suppose that K is a locally finite N-group. Then every countably infinite subgroup of K is isomorphic to some maximal baseless subgroup of W.*

Let T be any baseless subgroup of W. Then either T is countably infinite or T is conjugate in W to a subgroup of K. In particular, if K is uncountable, then the complements to $\overline{H}$ in W are conjugate.

We are not aware to what extent, if any, the normalizer condition may be weakened in Corollary A1. Some information about Q1, and in particular about the conjugacy of the complements, can however be obtained without the hypothesis of local finiteness, and we return to this question later.

In the case of general K, the only restriction which we have been able to obtain on the maximal baseless subgroups of W_α is the elementary fact that they have cardinal at least α, provided that α is regular and $\alpha \geqq |K|$ (Corollary 3.3). Theorem B shows that a wide range of α-bounded subgroups of cardinal α of K may be isomorphic to maximal baseless subgroups of W_α and it seems conceivable that all such subgroups of K may occur in that way. However this is certainly false without the restriction of α-boundedness; for example Theorem D shows that if K is free abelian of rank 3 then the maximal baseless subgroups of $W = H \wr K$ have rank 1 or 3.

THEOREM B. *Suppose that* $\alpha \leqq |K|$ *and let* S *be an* α*-bounded subgroup of cardinal* α *of* K. *Suppose further that* S *contains a subgroup* U *such that*

(i) $|U| = \alpha$,

(ii) *for each* α*-bounded subgroup* L *with* $S < L \leqq K$, *there exists an element* $x \in L - S$ *such that* $U^x \leqq S$.

Then there is a maximal baseless α*-bounded subgroup* S^* *of* W_α *such that* $\bar{H}_\alpha S = \bar{H}_\alpha S^*$. *Thus* $S \cong S^*$.

Evidently (ii) holds whenever $U \lhd K$, and in fact it holds whenever U is an ascendant subgroup of K in the sense that there exists an ordinal ρ and subgroups $\{U_\sigma : \sigma \leqq \rho\}$ such that $U_0 = U$, $U_\sigma \lhd U_{\sigma+1}$ if $\sigma < \rho$, $U_\mu = \bigcup_{\sigma<\mu} U_\sigma$ if $\mu \leqq \rho$ is a limit ordinal, and $U_\rho = K$. For suppose $S < L \leqq K$ and let σ be the least ordinal such that $U_\sigma \cap S < U_\sigma \cap L$. Then σ is neither a limit ordinal nor zero and we have $U \leqq U_{\sigma-1} \cap S = U_{\sigma-1} \cap L \lhd U_\sigma \cap L$. Thus $U_\sigma \cap L$ contains elements $x \notin S$, and any such element satisfies $U^x \leqq S$.

We therefore have

COROLLARY B1. *Suppose that* $\alpha \leqq |K|$ *and let* S *be any* α*-bounded subgroup of cardinal* α *of* K *which contains an ascendant subgroup of cardinal* α of K. *Then there is a maximal baseless* α*-bounded subgroup* S^* *of* W_α *such that* $\bar{H}_\alpha S = \bar{H}_\alpha S^*$.

In the case when K is an N-group every subgroup of K is ascendant in K and so Corollary B1 yields an alternative proof of Corollary A1. In fact the obvious similarity between Theorem B and the first half of Theorem A allows us to deduce them both from a simultaneous generalization (Lemma 4.2), the statement of which is unfortunately more complicated than either.

We have not been able to extend the results of Theorem A to cover the baseless locally nilpotent subgroups of W_α for general α, but the following gives some information about the case $\alpha = \aleph_0$.

THEOREM C. *Let π be a set of primes and let S be a countably infinite periodic locally nilpotent π-subgroup of K. Suppose that H is abelian and either*

(i) *H contains a subgroup of order at least 4 which contains no non-trivial π-elements, or*

(ii) *the Sylow 2-subgroup of S is finite and H is not a π-group.*

Then there is a maximal baseless periodic locally nilpotent subgroup S^* of W such that $\bar{H}S = \bar{H}S^*$.

Evidently conditions (i) and (ii) both hold unless H is the direct product of a π-group and a cyclic group of order at most 3. Notice that, in contrast to Theorem A, no information is given here about the uncountable baseless locally finite and locally nilpotent subgroups of W. It does not seem clear whether they necessarily lie in conjugates of K.

In the last section (Section 6) we consider the baseless subgroups of $W = H \wr K$ which contain elements of infinite order. These turn out to be surprisingly well behaved, provided they possess a suitable flavour of generalized solubility.

THEOREM D. *Let L^* be a baseless subgroup of $W = H \wr K$ and suppose that L^* is a radical group whose Hirsch-Plotkin radical is not periodic. Then unless L^* is a polycyclic group of Hirsch number one, L^* is contained in a conjugate of K.*

Conversely, let $L \leqq K$ be polycyclic of Hirsch number one. Then there is a baseless subgroup L^* of W such that $\bar{H}L = \bar{H}L^*$ and L^* is not contained in a conjugate of K.

Here we use the term *radical group* in the sense of Plotkin [6]; a radical group is one possessing an ascending series with locally nilpotent factors. The Hirsch number of a polycyclic group is the number of infinite factors in a cyclic series of the group.

COROLLARY D1. *Suppose that K is a radical group with non-periodic Hirsch-Plotkin radical. Then the complements to $\bar{H}$ in W are conjugate if and only if K is not a polycyclic group of Hirsch number one.*

As a consequence of Corollary D1, Corollary 3.3 and the remarks after Lemma 3.5 we obtain a criterion for the conjugacy of the complements to the base group in the case when K is countable and locally nilpotent.

COROLLARY D2. *Suppose that K is countable and locally nilpotent. Then the complements to $\bar{H}$ in W are conjugate if and only if K is neither infinite periodic nor finite-by-infinite cyclic.*

Some information about the uncountable case can be obtained from Theorem A and Lemma 6.5.

2. Groups with many Sylow subgroups

There are a number of examples in the literature to show that the Sylow p-subgroups of a locally finite group G may fail to be isomorphic in various rather spectacular ways, even when the group G has rather restricted structure (for example [4], [7]). In this section we indicate how a number of other such examples may be constructed on the basis of Theorems B and C. Some rather similar examples have recently been obtained by Heineken [2] in a different way. We make the obvious remark that if a p-subgroup P of a group G happens to be a Sylow π-subgroup of G for some set π of primes containing p, then P is a Sylow p-subgroup of G.

THEOREM E. *Let q be a given prime, let α be a given infinite cardinal, and let δ be the smallest cardinal satisfying the condition*

$$\gamma < \alpha \Rightarrow \alpha^{\gamma} \leqq \delta.$$

Then there exists a periodic metabelian group G of cardinal δ which contains as a Sylow q'-subgroup a copy of every infinite abelian q'-group of cardinal not exceeding α.

Since, if $\gamma < \alpha$, we have $\alpha^{\gamma} \leqq \alpha^{\alpha} = 2^{\alpha}$, it follows that $\delta \leqq 2^{\alpha}$. Notice, however, that equality may occur. For example, define $\alpha_0 = \aleph_0$ and $\alpha_i = 2^{\alpha_{i-1}}$ for $1 \leqq i < \omega$, the first infinite ordinal. Then if $\alpha = \sum_{i<\omega} \alpha_i$ we have

$$\alpha^{\aleph_0} = \alpha\alpha \cdots \geqq 2^{\alpha_0} 2^{\alpha_1} \cdots = 2^{\Sigma \alpha_i} = 2^{\alpha}.$$

Hence $\delta = 2^{\alpha}$ in this case.

However in the case $\alpha = \aleph_0$ we evidently have $\delta = \aleph_0$, and so we find in particular that there exists a countable periodic metabelian group which contains, as a Sylow p-subgroup, a copy of every countably infinite abelian p-group for which $p \neq q$. This result is in a sense best possible in view of the following:

PROPOSITION. *Let G be a periodic metabelian group which contains, for each prime p, a Sylow p-subgroup S_p of type C_{p^∞}. Then G is locally cyclic and S_p is its unique Sylow p-subgroup.*

PROOF. Let R be the Hirsch-Plotkin radical of G. Then $S_p \cap R$ is the unique Sylow p-subgroup of R and so R is locally cyclic. Therefore R is the union of finite characteristic subgroups and if $C = C_G(R)$ then G/C is residually finite. Hence $S_p \leqq C$ for all primes p. Now clearly C is nilpotent, whence $C = R$ and so $S_p \leqq R$. Therefore S_p is the unique Sylow p-subgroup of R and $S_p \lhd G$. It follows that S_p is the set of p-elements of G, whence $G = \langle S_p;\ \text{all}\ p\rangle = R$, as claimed.

However the proof of Theorem E will show that a *non-periodic* metabelian group may well contain, for every prime p, every abelian p-group of cardinal at most α as a Sylow p-subgroup.

PROOF OF THEOREM E. Let C be the direct product of groups of type C_{p^∞}, one for each prime $p \neq q$, and let K be the direct product of α copies of C. Now every infinite abelian group A can be embedded in a divisible abelian group of cardinal $|A|$ and it follows from this that K contains a copy of every infinite abelian q'-group of cardinal not exceeding α. Let H be a cyclic group of order q and for each infinite cardinal $\beta \leqq \alpha$ let L_β denote the base group of $H \wr_\beta K$. Finally let $L = \mathrm{Dr}_{\beta \leqq \alpha} L_\beta$ be the direct product of the groups L_β (β infinite) and let $G = LK$ be the natural semidirect product of L by K. Then G is periodic and metabelian.

Let S be any infinite subgroup of K. We shall show that G has a Sylow q'-subgroup isomorphic to S. Now $|S| = \beta$ for some $\beta \leqq \alpha$, and since $M_\beta = L_\beta K \cong H \wr_\beta K$, Theorem B shows that there is a subgroup T of M_β which is isomorphic to S and has the property that any larger subgroup of M_β meets L_β non-trivially. Therefore T is a Sylow q'-subgroup of M_β. Let U be a q'-subgroup of G containing T, let $L_\beta^* = \mathrm{Dr}_{\gamma \neq \beta} L_\gamma$ and let $U^* = L_\beta^* U$. Then as $G = L_\beta^* M_\beta$ we have $U^* = L_\beta^*(U^* \cap M_\beta)$ and $U^* \cap M_\beta$ complements L_β^* in U^*. Therefore $U^* \cap M_\beta$ is a q'-subgroup of M_β containing T, whence $U^* \cap M_\beta = T$. Hence $L_\beta^* U = L_\beta^* T$ and $U = (L_\beta^* \cap U)T = T$ since L_β^* is a q-group. Therefore T is a Sylow q'-subgroup of G.

It remains to consider the cardinal of G. Let $\gamma < \alpha$. Now evidently the number of γ-element subsets of K is at most α^γ, and since K can be partitioned into γ subsets each of cardinal α it follows that the number of such subsets is precisely α^γ. Therefore the number of maps of K into H with support of cardinal γ is $\alpha^\gamma 2^\gamma = \alpha^\gamma$. Hence $|L_\beta| = \sum_{\gamma < \beta} \alpha^\gamma$ and so, since L is generated by the sets L_β for all $\beta \leqq \alpha$, we have $|L| \leqq \sum_{\beta \leqq \alpha} (\sum_{\gamma < \beta} \alpha^\gamma) \leqq \alpha\delta = \delta$, since $\delta \geqq \alpha$ and in the double sum there occur at most α summands each of which is at most $\alpha\delta = \delta$. Since $|L_\alpha| \geqq \alpha^\gamma$ for all $\gamma < \alpha$ it follows that in fact $|L| = \delta$, whence $|G| = \delta\alpha = \delta$.

By similar arguments we can establish the following result which is somewhat more general than Theorem 3 of Wehrfritz [7], although the ideas behind the proof are essentially the same.

Theorem F. *Let q be a given prime and for each prime $p \neq q$ let $n_p \geqq 0$ be an integer. Then there exists a countable periodic metabelian group satisfying Min-p for all $p \neq q$ and containing, for any $p \neq q$, a Sylow p-subgroup isomorphic to any given countably infinite abelian p-group of rank not exceeding n_p.*

Proof. We may assume without loss of generality that $n_p > 0$ for some $p \neq q$. Let K be the direct product of a collection of groups consisting of n_p copies of C_{p^∞} for each $p \neq q$, let H be cyclic of order q, and let $W = H \wr K$. Then it is immediate from any of Theorems A–C that W has the properties required by Theorem E.

Theorem G. *Let $\{P_\lambda : \lambda \in \Lambda\}$ be a given set of infinite locally finite p-groups. Then there exists a locally finite and locally soluble group G containing a copy of each P_λ as a Sylow p-subgroup.*

It will be seen from the proof that G may often be chosen to inherit special properties from the P_λ; for example if the P_λ are all soluble and of bounded derived length then G may be chosen soluble, and so on. The case $|\Lambda| = 2$ of the Theorem G has been known for some time and is due to Heineken [3]. His construction is rather different from ours in that it starts from the free product of the two groups in question. It has since been substantially generalized by Heineken [2].

Proof of Theorem G. Let K be the direct product of the P_λ and suppose that $|K| = \alpha$. Let H be a cyclic group of order $q \neq p$ and, for each infinite $\beta \leqq \alpha$, let L_β be the base group of $H \wr_\beta K$. Let $L = \mathrm{Dr}_{\beta \leqq \alpha} L_\beta$ and let G be the semidirect product LK, which is clearly both locally finite and locally soluble.

Let $\lambda \in \Lambda$. Then $|P_\lambda| = \beta$ for some infinite $\beta \leqq \alpha$. Now $M_\beta = L_\beta K \cong H \wr_\beta K$ and Theorem B shows, since $P_\lambda \lhd K$, that M_β has a Sylow p-subgroup $T \cong P_\lambda$. It then follows as in the proof of Theorem E that T is a Sylow p-subgroup of G.

Theorem G allows us, for example, to obtain a locally soluble group containing every countably infinite locally finite p-group as a Sylow p-subgroup; however the group so obtained has cardinal $2^{\aleph_0}$. We can improve on this by using Theorem C, but since it does not seem to be known whether there exists a countable locally finite and locally soluble group in which every countable locally finite p-group can be embedded, we have to sacrifice local solubility.

Theorem H. *There exists a countable locally finite group which contains, for each prime p, a copy of every countably infinite locally finite p-group as a Sylow p-subgroup.*

PROOF. It was shown by P. Hall [1] that there exists a countable locally finite group U which contains every countable locally finite group as a subgroup. Let $p_1, p_2, \cdots$ be the primes in natural order, let H_2 be a cyclic group of order 9 and let H_i be a cyclic group of order p_i for $i > 2$. We now define $G_1 = U$ and inductively $G_{i+1} = H_{i+1} \wr G_i$ for $i \geqq 1$. Let $G = \bigcup_{i=1}^{\infty} G_i$, G_i being embedded in G_{i+1} in the obvious way. Then for $i > 0$ we have $G = N_{i+1} G_{i+1}$, $N_{i+1} \cap G_{i+1} = 1$, where N_{i+1} is the product of the base groups of $G_{i+2}, G_{i+3}, \cdots$. Let P be any countably infinite locally finite p_i-group. Then P is isomorphic to a subgroup of G_i and Theorem C shows that there is a subgroup $P^* \cong P$ of G_{i+1} which is such that any larger locally nilpotent subgroup of G_{i+1} meets the base group of G_{i+1} non-trivially. Hence P^* is a Sylow p_i-subgroup of G_{i+1}. Since N_{i+1} is a p_i'-group it follows easily that P^* is a Sylow p_i-subgroup of G, as claimed.

3. Preliminary results

In the notation already established, let $\boldsymbol{X}$ be a set of baseless subgroups of W_α. We shall say that *$\boldsymbol{X}$ is W_α-contained in K*, and write $\boldsymbol{X} \leqq_{W_\alpha} K$, if every member of $\boldsymbol{X}$ is conjugate in W_α to a subgroup of K. Let $\boldsymbol{B}$ denote the set of all baseless α-bounded subgroups of W_α and let $\boldsymbol{B}_\beta$ be the set of all members of $\boldsymbol{B}$ of cardinal $< \beta$, where β is some infinite cardinal. We shall be interested in investigating the $\boldsymbol{B}_\beta$ with respect to the property of being W_α-contained in K, and the following elementary remark will often be used without mention.

LEMMA 3.1. *Let $G = AB$, $A \lhd G$, $A \cap B = 1$ be the semidirect product of two groups A and B.*

(i) *Suppose $C^* \leqq G$ satisfies $C^* \cap A = 1$, and let $C = AC^* \cap B$. Then C^* is conjugate in G to a subgroup of B if and only if $C^{*a} = C$ for some $a \in A$.*

(ii) *Let $X \leqq B$ and $a, a' \in A$. Then $X^a \leqq B^{a'}$ if and only if $a'a^{-1} \in C_A(X)$.*

PROOF. (i) Suppose $C^{*g} \leqq B$ for some $g \in G$. Then writing $g = ab$ ($a \in A$, $b \in B$), we obviously have $C^{*a} \leqq B \cap AC^* = C$. Since also $C \leqq AC^{*a}$, we obtain $C = (A \cap C)\ C^{*a} = C^{*a}$, as required. The converse is clear.

(ii) Suppose that $X^a \leqq B^{a'}$, and let $c = a'a^{-1}$. Then $X \leqq B^c$ and so, for $x \in X$, we have $x = b^c = [c, b^{-1}]b$ for some $b \in B$. Since the product AB is semidirect it follows that $x = b$ and $[c, b^{-1}] = 1$; therefore $[c, x^{-1}] = 1$ for all $x \in X$ and we have $c \in C_A(X)$. The converse is again clear.

LEMMA 3.2. (i) *If* $\alpha > |K|$ *then* $\boldsymbol{B} \leqq_{W_\alpha} K$.

(ii) *Suppose that* $\alpha \leqq |K|$ *and let* β *be the least cardinal such that* α *is the sum of* β *cardinals each* $< \alpha$. *Then* $\boldsymbol{B}_\beta \leqq_{W_\alpha} K$.

COROLLARY 3.3. *If* α *is regular then* $\boldsymbol{B}_\alpha \leqq_{W_\alpha} K$.

PROOF OF LEMMA 3.2. (i) As previously explained, this is essentially well known (cf. [5] Theorem 10.1) since the condition $\alpha > |K|$ simply states that we are considering the complete wreath product $H \mathbin{\overline{\wr}} K$. However it will be useful to give a proof.

Let S be a subgroup of $\overline{W} = H \mathbin{\overline{\wr}} K$ such that $\overline{\overline{H}} \cap S = 1$. Then

$$\overline{\overline{H}}S = \overline{\overline{H}}T$$

where $T = \overline{\overline{H}}S \cap K$, and each element of S is uniquely of the form $h_t t$ with $t \in T$. We have, if $t_1, t_2 \in T$,

$$h_{t_1} t_1 h_{t_2} t_2 = h_{t_1} h_{t_2}^{t_1^{-1}} t_1 t_2,$$

whence

$$h_{t_1 t_2} = h_{t_1} h_{t_2}^{t_1^{-1}}$$

or, evaluating at $k \in K$ and rearranging,

(1) $$h_{t_1 t_2}(k)^{-1} h_{t_1}(k) h_{t_2}(kt_1) = 1.$$

Let $u \in \overline{\overline{H}}$. Then $(h_t t)^u = u^{-1} h_t u^{t^{-1}} t$. We wish to choose u so that this element lies in K for all $t \in T$; this is equivalent to the condition $u^{-1} h_t u^{t^{-1}} = 1$ for all $t \in T$, or

(2) $$u(k)^{-1} h_t(k) u(kt) = 1$$

for all $k \in K$, $t \in T$.

Let $\{s_\lambda\}$ be a right transversal to T in K. Thus $K = \bigcup_\lambda s_\lambda T$ and $s_\lambda T \cap s_\mu T = \emptyset$ if $\lambda \neq \mu$. Define $u \in \overline{\overline{H}}$ by

$$u(s_\lambda t) = h_{t^{-1}}(s_\lambda t) \quad (t \in T).$$

Then, if $t_1, t_2 \in T$ and we substitute $k = s_\lambda t_2$, $t = t_1$ in (2), we obtain

$$u(s_\lambda t_2)^{-1} h_{t_1}(s_\lambda t_2) u(s_\lambda t_2 t_1) = h_{t_2^{-1}}(k) h_{t_1}(k) h_{(t_2 t_1)^{-1}}(kt_1),$$

which is 1 as can be seen by replacing t_2 by $(t_2 t_1)^{-1}$ in (1).

(ii) This follows by the argument of (i). For let S be a subgroup of cardinal $< \beta$ of W_α. Then we can view W_α in a natural way as a subgroup of $\overline{W}$, and so we have $S^u \leqq K$, where u is the element of $\overline{\overline{H}}$ constructed in (i). It will clearly suffice to show that $u \in \overline{H}_\alpha$. Now u clearly takes the value 1 at all points outside the union of the supports of the $h_t (t \in T)$. Since $h_t \in \overline{H}_\alpha$ each such support has cardinal $< \alpha$, and since $|T| < \beta$ it follows that the support of u has cardinal $< \alpha$. Thus $u \in \overline{H}_\alpha$, as required.

The corollary is immediate since if α is regular then $\beta = \alpha$ in Lemma 3.2. Notice that the regularity of α is essential in Corollary 3.3. For let α be any irregular cardinal and write $\alpha = \sum_{\lambda \in \Lambda} \gamma_\lambda$, where Λ is a set of cardinal $\beta < \alpha$ and $\gamma_\lambda < \alpha$ for all $\lambda \in \Lambda$. Let K be any group of cardinal α which contains a free abelian subgroup L of rank β. Then $|K : L| = \alpha$ and so the set of right cosets kL of L in K may be partitioned as the union $\bigcup_{\lambda \in \Lambda} C_\lambda$ of pairwise disjoint sets C_λ such that C_λ consists of γ_λ cosets. Let $W_\alpha = H \wr_\alpha K$, where H is some non-trivial group. Further let $\{x_\lambda : \lambda \in \Lambda\}$ be a basis of L, let $1 \neq t \in H$, and let $y_\lambda = h_\lambda x_\lambda$, where h_λ is the element of $\overline{H}_\alpha$ taking the value t at each point which belongs to some coset in C_λ, and 1 elsewhere. Then the support of h_λ has cardinal $\beta\gamma_\lambda = \max\{\beta, \gamma_\lambda\} < \alpha$ and so h_λ does in fact belong to $\overline{H}_\alpha$.

Now the following is well known and easy to verify:

LEMMA 3.4. *Let* $\overline{W} = H \,\overline{\wr}\, K$, *where* H *and* K *are any groups, let* $S \leqq K$ *and let* $f \in \overline{\overline{H}}$. *Then* f *centralizes* S *if and only if* f *is constant on each right coset of* S *in* K.

Thus the elements h_λ all centralize L and so the y_λ generate an abelian group M. Any non-trivial element $y \in M$ has the form $y = y_{\lambda_1}^{n_1} \cdots y_{\lambda_k}^{n_k}$ where $k > 0$, the λ_i are distinct elements of Λ, and the n_i are non-zero integers. Since $y \equiv x_{\lambda_1}^{n_1} \cdots x_{\lambda_k}^{n_k} \bmod \overline{H}_\alpha$ no such element lies in $\overline{H}_\alpha$, and so $\overline{H}_\alpha \cap M = 1$ and $M \in \boldsymbol{B}_\alpha$.

On the other hand let $h \in \overline{H}_\alpha$. Then supp h has cardinal $\delta < \alpha$ and so we have $\delta = |\mathrm{supp}\, h| < \gamma_\lambda$ for some $\lambda \in \Lambda$. Then

$$y_\lambda^h = (h_\lambda x_\lambda)^h = h_\lambda^h[h, x_\lambda^{-1}]x_\lambda .$$

Now the support of $[h, x_\lambda^{-1}]$ has cardinal at most δ while that of h_λ^h has cardinal $\geqq \gamma_\lambda > \delta$, and so $[h, x_\lambda^{-1}]$ takes the value 1 at some point of the support of h_λ^h. At such a point the value of $h_\lambda^h[h_\lambda, x_\lambda^{-1}]$ is different from 1. Hence $y_\lambda^h \notin K$ and it follows that $M^h \nleqq K$. Therefore M is not conjugate to a subgroup of K.

The following result provides a partial converse to Lemma 3.2 (i).

LEMMA 3.5. *Suppose that* $\boldsymbol{B} \leqq_{W_\alpha} K$. *Then* $|S| < \alpha$ *for every* α*-bounded subgroup* S *of* K.

The proof requires a technical lemma which will find further application later.

LEMMA 3.6. *Let* α *be an infinite cardinal, let* S *be an* α*-bounded group of cardinal* α *containing a subset* U *also of cardinal* α, *and let* $n > 0$ *be an integer. Then there exists a tower* $\{S_\sigma : \sigma < \alpha\}$ *of subgroups of* S *such that* $\bigcup_{\sigma < \alpha} S_\sigma = S$, $|S_\sigma| < \alpha$ if $\sigma < \alpha$, *and* $|(U \cap S_{\sigma+1})S_\sigma : S_\sigma| \geqq n$ *for all* $\sigma < \alpha$.

Here we are thinking of α as an ordinal which is not equivalent to any of its predecessors. By the statement that $\{S_\sigma : \sigma < \alpha\}$ is a tower we mean that $S_0 = 1$, $S_\sigma \leqq S_{\sigma+1}$ for $\sigma+1 < \alpha$, and $S_\mu = \bigcup_{\sigma<\mu} S_\sigma$ for limit ordinals $\mu < \alpha$. The notation $|(U \cap S_{\sigma+1})S_\sigma : S_\sigma|$ denotes the number of right cosets tS_σ of S_σ contained in the set $(U \cap S_{\sigma+1})S_\sigma$.

Proof of Lemma 3.6. In the case $\alpha = \aleph_0$, S is a countably infinite locally finite group and we require simply a tower $1 = S_0 < S_1 < \cdots$ of finite subgroups of S such that $S = \bigcup_{i=0}^{\infty} S_i$ and

$$|(U \cap S_{i+1})S_i : S_i| \geqq n$$

for all i. The construction of such a tower is completely straightforward.

In the case $\alpha > \aleph_0$ the restriction of α-boundedness is of course vacuous. In this case, let $\{s_\tau : \tau < \alpha\}$ and $\{u_\tau : \tau < \alpha\}$ be the elements of S and U respectively. It will clearly suffice to construct, for $\sigma < \alpha$, subgroups S_σ of S satisfying

(i) $\tau < \sigma \Rightarrow s_\tau, u_\tau \in S_\sigma$,
(ii) $|S_\sigma| \leqq \max(\aleph_0, |\sigma|)$,
(iii) $|(U \cap S_{\sigma+1})S_\sigma : S_\sigma| \geqq n$.

For (i) gives $S = \bigcup_{\sigma<\alpha} S_\sigma$ and (ii) gives $|S_\sigma| < \alpha$ if $\sigma < \alpha$, since $\alpha > \aleph_0$ and α is not equivalent to any of its predecessors.

Let $S_0 = 1$ and suppose that, for some $0 < \rho < \alpha$, we have the subgroups S_σ for $\sigma < \rho$. If ρ is a limit ordinal we put $S_\rho = \bigcup_{\sigma<\rho} S_\sigma$. Then (i) holds, and since $|S_\sigma| \leqq \max(\aleph_0, |\rho|)$ for $\sigma < \rho$ we have $|S_\rho| \leqq \max(\aleph_0|\rho|, |\rho|^2) = \max(\aleph_0, |\rho|)$. If ρ has the form $\sigma+1$ then $|S_\sigma| < \alpha$ by (ii) and so $|US_\sigma : S_\sigma| = \alpha$. Therefore there exists a least ordinal λ such that $|\{u_\tau : \tau < \lambda\}S_\sigma : S_\sigma| \geqq n$. Now $\lambda > \sigma$ by (i) and the fact that $n > 0$, and so if we put $S_{\sigma+1} = \langle S_\sigma, s_\tau, u_\tau : \tau < \lambda\rangle$, then (i) holds with σ replaced by $\sigma+1$. Further, λ necessarily has the form $\mu+1$ and we have $\{u_\tau : \tau < \mu\} \leqq \{u_\tau : \tau < \mu\}S_\sigma$, which is the union of at most $n-1$ right cosets of S_σ and so has cardinal at most $\max(\aleph_0, |\sigma|)$. Therefore $|\mu| \leqq \max(\aleph_0, |\sigma|)$ and so

$$|\lambda| = |\mu+1| = |\mu| \leqq \max(\aleph_0, |\sigma|) = \max(\aleph_0, |\rho|),$$

recalling that $\rho = \sigma+1$. It follows that $|S_\rho| \leqq \max(\aleph_0, |\rho|)$, and so (ii) holds. Since (iii) holds by the choice of λ, the proof is complete.

Proof of Lemma 3.5. Suppose that K contains an α-bounded subgroup of cardinal exceeding α. Then K contains an α-bounded subgroup S of cardinal α precisely. Taking $U = S$ and $n = 3$ in Lemma 3.5, we obtain a tower $\{S_\gamma : \gamma < \alpha\}$ of subgroups of S satisfying

(3) $$|S_{\gamma+1} : S_\gamma| \geqq 3 \text{ if } \gamma < \alpha$$

and

(4) $$|S_\gamma| < \alpha \text{ if } \gamma < \alpha.$$

For each $\gamma < \alpha$ choose a right coset $D_\gamma \neq S_\gamma$ of S_γ in $S_{\gamma+1}$. Then for each $\beta < \alpha$ the set $\bigcup_{\gamma < \beta} D_\gamma$ is a subset of S_β and so has cardinal $< \alpha$ by (4). Let $t \neq 1$ be a selected element of H and let y_β be the element of $\overline{H}_\alpha$ which takes the value t at each point of $\bigcup_{\gamma < \beta} D_\gamma$ and 1 elsewhere. Since $D_{\beta_1} \cap D_{\beta_2} = \emptyset$ if $\beta_1 \neq \beta_2$ it is clear that if $\beta < \gamma$ then $y_\gamma y_\beta^{-1}$ takes the value t on $\bigcup_{\beta \leqq \sigma < \gamma} D_\sigma$ and 1 elsewhere. Thus $y_\gamma y_\beta^{-1}$ is constant on each right coset of S_β and so by Lemma 3.4 we have

(5) $$y_\gamma y_\beta^{-1} \in C_{\overline{H}_\alpha}(S_\beta) \text{ if } \beta < \gamma < \alpha.$$

Let $S_\beta^* = S_\beta^{y_\beta}$ $(\beta < \alpha)$. Then if $\beta < \gamma < \alpha$ we have from (5) that $S_\beta^* = S_\beta^{y_\gamma} \leqq S_\gamma^*$, and so $\bigcup_{\beta < \alpha} S_\beta^*$ is a subgroup S^* of W_α. Clearly $S^* \cap \overline{H}_\alpha = 1$ so that $S^* \in \boldsymbol{B}$.

To complete the proof it remains to show that $S^* \nleqq K^y$ for all $y \in \overline{H}_\alpha$. Suppose then that $S^* \leqq K^y$. Then for $\gamma < \alpha$ we have

$$S_{\gamma+1}^{y_{\gamma+1}} \leqq K^y$$

whence by Lemma 3.1 we have $y y_{\gamma+1}^{-1} \in C_{\overline{H}_\alpha}(S_{\gamma+1})$. Hence we have $y = c_{\gamma+1} y_{\gamma+1}$, where $c_{\gamma+1}$ is constant on each right coset of $S_{\gamma+1}$ in K. In particular $c_{\gamma+1}$ takes the same value at each point of $S_{\gamma+1}$. Unless that value is t^{-1} it follows that the support of y contains D_γ. If it is t^{-1} then supp y contains $S_{\gamma+1} - (S_\gamma \cup D_\gamma)$. Therefore, by (3), we obtain in either case a coset $E_\gamma \neq S_\gamma$ of S_γ in $S_{\gamma+1}$, which is contained in supp y. Therefore supp y contains $\bigcup_{\gamma < \alpha} E_\gamma$. Since this set can evidently be mapped onto S its cardinal must be α, which contradicts the assumption that $y \in \overline{H}_\alpha$ and completes the proof.

Notice that if $\alpha = |K|$ and K is α-bounded then we can choose S to be K itself, thereby showing that the complements to $\overline{H}_\alpha$ in W_α are not conjugate in that case.

4. Proofs of Theorems A and B

The following elementary result will frequently be required.

LEMMA 4.1. *Let c, d, y be functions from a group K to another group H and let $f = cdy$. Let U be a proper subgroup of K and $x \in K - U$. Suppose that c is constant on each right coset of $\langle x \rangle$ in K and d is constant on each right coset of U in K. Suppose further that u and v are elements of U such that $y(u) \neq y(v)$ and ux, vx lie in a single right coset C of U which satisfies $C \cap$ supp $y = \emptyset$. Then $f(t) \neq 1$ for some $t \in \{u, v, ux, vx\}$.*

Proof. We have $c(u) = c(ux) = \lambda$, $c(v) = c(vx) = \mu$ say; also $d(u) = d(v) = \sigma$, $d(ux) = d(vx) = \rho$ since by assumption ux, vx lie in a common right coset of U. Further $y(u) = \alpha$, $y(v) = \beta$, $y(ux) = y(vx) = 1$. Therefore f takes the values $\lambda\sigma\alpha$, $\mu\sigma\beta$, $\lambda\rho$, $\mu\rho$ at u, v, ux, vx respectively. The assumption that all these values are equal evidently gives $\lambda = \mu$ and hence $\alpha = \beta$, contrary to our hypotheses.

We shall deduce Theorem **B** and the first half of Theorem A from our next lemma, which generalizes both.

Lemma 4.2. *Let* $W_\alpha = H \wr_\alpha K$ *and suppose that* $\alpha \leqq |K|$. *Let* $U \leqq S$ *be α-bounded subgroups of cardinal α of K and let* $\mathbf{T}$ *be the set of all subgroups T of K containing S and satisfying the condition*

(*) *if* $S < L \leqq T$ *and L is α-bounded, then* $U^x \leqq S$ *for some* $x \in L - S$.

Then there is a baseless subgroup S^* *of* W_α *satisfying*

(i) $\bar{H}_\alpha S = \bar{H}_\alpha S^*$.

(ii) *For all* $T \in \mathbf{T}$, S^* *is maximal among the baseless α-bounded subgroups of* W_α *contained in* $\bar{H}_\alpha T$.

Now the hypotheses of Theorem B imply that $K \in \mathbf{T}$; thus Theorem B is an immediate consequence of Lemma 4.2. To deduce the first half of Theorem A we take S to be any α-bounded N-subgroup of cardinal α of K and $U = S$. If S^* is as in Lemma 4.2 and T^* is a baseless α-bounded N-subgroup of W_α containing it, then $\bar{H}_\alpha T^* = \bar{H}_\alpha T$, where $T = \bar{H}_\alpha T^* \cap K$ is an N-subgroup of K. Clearly $T \in \mathbf{T}$, whence Lemma 4.2 gives $S^* = T^*$, as required.

Proof of Lemma 4.2. As in the proof of Lemma 3.5 we find it useful to think of α as an ordinal which is equivalent to none of its predecessors. Then by Lemma 3.6 with $n = 3$, there exists a tower $\{S_\sigma : \sigma < \alpha\}$ of subgroups of S such that

(i) $\bigcup_{\sigma<\alpha} S_\sigma = S$,

(ii) $|S_\sigma| < \alpha$ if $\sigma < \alpha$,

(iii) if $\sigma < \alpha$ then there exist elements u_σ, $u'_\sigma \in U \cap S_{\sigma+1}$ such that the three cosets S_σ, $u_\sigma S_\sigma$, $u'_\sigma S_\sigma$ are all distinct.

Now let $D_\sigma = u_\sigma S_\sigma$ for $\sigma < \alpha$ and let $1 \neq b \in H$. The sets D_σ are pairwise disjoint, and there exists for each $\sigma < \alpha$ a uniquely defined element y_σ in $\bar{H}_\alpha$ which takes the value b at each point of $\bigcup_{\lambda<\sigma} D_\lambda$ and 1 elsewhere. Notice that $|\bigcup_{\lambda<\sigma} D_\lambda| \leqq |S_\sigma| < \alpha$ by (ii). Clearly if $\tau < \sigma$ then $y_\sigma y_\tau^{-1}$ is constant on each right coset of S_τ and so belongs to $C_{\bar{H}_\alpha}(S_\tau)$. Let $S^*_\sigma = S_\sigma^{y_\sigma}$ $(\sigma < \alpha)$. Then $S^*_\tau \leqq S^*_\sigma$ if $\tau < \sigma$ and so $S^* = \bigcup_{\sigma<\alpha} S^*_\sigma$ is a subgroup of W_α which clearly satisfies

(1) $$\bar{H}_\alpha S^* = \bar{H}_\alpha S, \bar{H}_\alpha \cap S^* = 1.$$

Thus $S^* \in \boldsymbol{B}$.

Suppose now that $T \in \boldsymbol{T}$ and suppose that $S^* \leqq L^* \leqq \bar{H}_\alpha T$ for some baseless α-bounded subgroup L^*. Then $\bar{H}_\alpha L^* = \bar{H}_\alpha L$ for some subgroup L of T containing S. We wish to show that $L^* = S^*$, or equivalently, that $L = S$. Suppose then that this is not the case. Then by condition (*) of the lemma and the fact that $L \cong L^*$, we have

(2) $$U^x \leqq S \textit{ for some } x \in L - S.$$

For each $\sigma < \alpha$ let $T_\sigma = \langle S_\sigma, x\rangle$. Now for $u \in L$ let u^* denote the unique element of L^* congruent to u modulo $\bar{H}_\alpha$. Then $u \to u^*$ is an isomorphism of L onto L^* which maps S_σ onto S_σ^* $(\sigma < \alpha)$ and so we have

(3) $$T_\sigma^* = \langle S_\sigma^*, x^*\rangle \leqq L^* \; (\sigma < \alpha).$$

We now show that T_σ^* is conjugate under $\bar{H}_\alpha$ to T_σ for $\sigma < \alpha$. If $\alpha = \aleph_0$ then S_σ is finite by (ii) and so T_σ, being $\aleph_0$-bounded, is finite. Hence T_σ^* is finite and so T_σ^* is conjugate to T_σ under $\bar{H}_\alpha$ by Corollary 3.3 and the fact that $\aleph_0$ is regular. Therefore we may assume that $\alpha > \aleph_0$, in which case Corollary 3.3 is inadequate since α may be irregular.

Let $t \in T_\sigma$. Then $t^* \in T_\sigma^*$ is uniquely expressible in the form $h_t t$ with $h_t \in \bar{H}_\alpha$. Let $\lambda = |\operatorname{supp} y_\sigma|$, $\mu = |\operatorname{supp} h_x|$ and $\beta = \max(\lambda, \mu, \aleph_0)$. Then β is infinite and $\beta < \alpha$. By (3) we have, if $t \in T$,

$$h_t t = t^* = s_1^{y_\sigma} x^{*\varepsilon_1} \cdots s_n^{y_\sigma} x^{*\varepsilon_n},$$

where $s_i \in S_\sigma$, $\varepsilon_i = \pm 1$ and $n \geqq 0$. We show by induction on n that $|\operatorname{supp} h_t| \leqq \beta$. To do this it suffices to show that if $|\operatorname{supp} h_t| \leqq \beta$ and $h_{t'}t'$ has either of the forms $h_t t s^{y_\sigma}$ $(s \in S_\sigma)$ or $h_t t\, x^{*\varepsilon}$ $(\varepsilon = \pm 1)$ then $|\operatorname{supp} h_{t'}| \leqq \beta$. In the first case we have

$$h_{t'} t' = h_t y_\sigma^{-t^{-1}} y_\sigma^{s^{-1}t^{-1}} ts.$$

Then

$$h_{t'} = h_t y_\sigma^{-t^{-1}} y_\sigma^{s^{-1}t^{-1}};$$

the support of this is contained in the union of three sets each of cardinal at most β, and so $|\operatorname{supp} h_{t'}| \leqq \beta$ in this case. In the second case, $h_{t'}$ is either

$$h_t h_x^{t^{-1}} \text{ or } h_t h_x^{-xt^{-1}}$$

and similar considerations apply.

We now have that if $S(t) = \operatorname{supp} h_t$ then $|S(t)| \leqq \beta$ for each $t \in T_\sigma$. Let $X = \bigcup_{t \in T_\sigma} S(t) \cup T_\sigma$. Then $|X| \leqq \beta\, |T_\sigma| + |T_\sigma| = \max(\beta, |T_\sigma|) < \alpha$. Hence, if $Y = \langle X\rangle$, then $|Y| < \alpha$. Therefore $\bar{H}_\alpha$ contains every function from K to H with support contained in Y; the group generated by these functions and Y is evidently isomorphic to $H \bar{\wr} Y$ and contains T_σ^*. Lemma 3.2(i) now shows that T_σ^* is conjugate in this group to a subgroup of Y. Hence T_σ^* is conjugate to T_σ under $\bar{H}_\alpha$ and we have

$$(4) \qquad T_\sigma^* = T_\sigma^{z_\sigma} \; (z_\sigma \in \bar{H}_\alpha, \sigma < \alpha).$$

Now $T_\sigma^{z_\sigma} \leqq T_\tau^{z_\tau}$ if $\sigma < \tau$ and so

$$(5) \qquad z_\tau z_\sigma^{-1} \in C_{\bar{H}_\alpha}(T_\sigma) \; (\sigma < \tau),$$

by Lemma 3.1. Also $S_\tau^{y_\tau} \leqq T_\tau^{z_\tau}$ and so the same lemma gives

$$(6) \qquad z_\tau y_\tau^{-1} \in C_{\bar{H}_\alpha}(S_\tau) \; (\tau < \alpha).$$

Since $x \in T_\sigma$ for all $\sigma \geqq 1$ we have from (5) that $z_\tau z_\sigma^{-1} \in C_{\bar{H}_\alpha}(x)$ for $1 \leqq \sigma < \tau$, and consequently, using (6), we can write

$$(7) \qquad z_1 = c_\tau d_\tau y_\tau,$$

for any $1 \leqq \tau < \alpha$, where $c_\tau \in C_{\bar{H}_\alpha}(x)$ and $d_\tau \in C_{\bar{H}_\alpha}(S_\tau)$. We shall deduce from this that, for each $\sigma < \alpha$, the support of z_1 contains at least one point from the set $B_\sigma = \{u_\sigma, u'_\sigma, u_\sigma x, u'_\sigma x\}$. Since $S \cap Sx = \emptyset$ it is easy to see that these sets are pairwise disjoint, from which it follows that the support of z_1 has cardinal at least α, contradicting the fact that $z_1 \in \bar{H}_\alpha$.

Consider than an ordinal $\sigma < \alpha$. Now by assumption we have $U^x \leqq S$ and so we can write

$$(8) \qquad u_\sigma x = x v_\sigma, \; u'_\sigma x = x v'_\sigma$$

for suitable $v_\sigma, v'_\sigma \in S$. Choose $\tau < \alpha$ such that $\langle S_{\sigma+1}, v_\sigma, v'_\sigma \rangle \leqq S_\tau$ and express z_1 in the form (7). Then using Lemma 3.4, we have that c_τ is constant on each right coset of $\langle x \rangle$ in K and d_τ is constant on each right coset of S_τ. From the definition of y_τ we have that $y_\tau(u_\sigma) = b \neq 1 = y_\tau(u'_\sigma)$. Also (8) gives that $u_\sigma x$ and $u'_\sigma x$ lie in xS_τ; since supp $y_\tau \leqq S_\tau$ this coset does not meet the support of y_τ. Therefore Lemma 4.1 gives that $z_1(w) \neq 1$ for some $w \in B_\sigma = \{u_\sigma, u'_\sigma, u_\sigma x, u'_\sigma x\}$, concluding the proof of Lemma 4.2.

Conclusion of the proof of Theorem A. We now have to consider the baseless α-bounded N-subgroups of $W_\alpha = H \wr_\alpha K$, where $\alpha \leqq |K|$ and α is regular.

Let S^* be such a subgroup. Then as usual we have $\bar{H}_\alpha S^* = \bar{H}_\alpha S$, where $S = \bar{H}_\alpha S^* \cap K$. Let $\boldsymbol{U}$ be the set of all subgroups of cardinal $< \alpha$ of S. Then since S is α-bounded, it is the union of the members of $\boldsymbol{U}$, and any two members of $\boldsymbol{U}$ generate a third. By Corollary 3.3 there exists for each $U \in \boldsymbol{U}$ an element $y_U \in \bar{H}_\alpha$ such that $U^{y_U} \leqq S^*$. Choosing such a y_U for each $U \in \boldsymbol{U}$, we have

(9) $$S^* = \bigcup_{U \in \boldsymbol{U}} U^{y_U},$$

and from Lemma 3.1

(10) $$y_U y_V^{-1} \in C_{\bar{H}_\alpha}(V) \text{ if } V \leqq U.$$

Since the elements y_U may be varied at will by premultiplying by elements of $C_{\bar{H}_\alpha}(U)$, that is by elements of $\bar{H}_\alpha$ constant on right cosets kU of U in K, we may further assume that

(11) $$y_U(c) = 1 \text{ for all } c \in C$$

if C is a right coset of U on which y_U is constant.

Let R be a right transversal to S in K; thus $K = \bigcup_{r \in R} rS$ and $rS \cap r'S = \emptyset$ if $r \neq r'$.

We distinguish two cases.

CASE 1. *There is a subgroup* $F \in \boldsymbol{U}$ *such that, for every* $F < L \in \boldsymbol{U}$, *there exists an element* $z_L \in \bar{H}_\alpha$ *with* $y_L z_L^{-1} \in C_{\bar{H}_\alpha}(L)$ *and* $\operatorname{supp} z_L \leqq \bigcup_{r \in R} rF$.

In this case we may suppose that $\operatorname{supp} y_L \leqq \bigcup_{r \in R} rF$ whenever $F < L \in \boldsymbol{U}$. We may also assume that $F < S$, since otherwise S^* is conjugate to a subgroup of K by (9) and so we may choose E so that $F < E \in \boldsymbol{U}$. Let $E \leqq L \in \boldsymbol{U}$. Then by (10) we have $y_L = cy_E$, where c is constant on each right coset of E in K. Let $r \in R$. Then both y_L and y_E take the value 1 at each point of $rE - rF$; therefore c takes the value 1 at each such point, and hence c takes the value 1 at each point of rE. Therefore the functions y_L and y_E coincide on rE, and since each of them has support lying in $\bigcup_{r \in R} rF \leqq \bigcup_{r \in R} rE$, we obtain $y_E = y_L$ for all $E \leqq L \in \boldsymbol{U}$.

Now if $U \in \boldsymbol{U}$ and $L = \langle U, E \rangle$, then α-boundedness gives $L \in \boldsymbol{U}$. Then using (10) we have

$$U^{y_U} = U^{y_L y_U^{-1} y_U} = U^{y_L} = U^{y_E} \leqq K^{y_E}.$$

Hence by (9) we have $S^* \leqq K^{y_E}$ in this case.

CASE 2. *No subgroup* $F \in \boldsymbol{U}$ *satisfies the hypothesis of Case* 1.

In this case, thinking of α as an ordinal which is not equivalent to any of its predecessors, we construct a strictly ascending tower $\{T_\sigma : \sigma < \alpha\}$ of subgroups of S satisfying

(i) $T_\sigma \in \boldsymbol{U}$ $(\sigma < \alpha)$,

(ii) $\operatorname{supp} y_\sigma \leqq \bigcup_{r \in R} rT_{\sigma+1}$ $(\sigma < \alpha)$,

where $y_\sigma = y_{T_\sigma}$, and

(iii) if $\sigma < \alpha$, $z \in \bar{H}_\alpha$ and $y_{\sigma+1} z^{-1} \in C_{\bar{H}_\alpha}(T_{\sigma+1})$,

then $\operatorname{supp} z \nleqq \bigcup_{r \in R} rT_\sigma$.

We begin by putting $T_0 = 1$. Let $0 < \tau < \alpha$ and suppose we have the subgroups T_σ for $\sigma < \tau$. If τ is a limit ordinal we put $T_\tau = \bigcup_{\sigma<\tau} T_\sigma$. Then $|T_\tau| < \alpha$ since $|\tau| < \alpha$ and α is regular. If τ has the form $\sigma+1$ then, by α-boundedness, we can choose a subgroup $\overline{T}_\sigma \in U$ such that $T_\sigma \leqq \overline{T}_\sigma$ and $\operatorname{supp} y_\sigma \leqq \bigcup_{r\in R} r\overline{T}_\sigma$. By the hypotheses of Case 2, $\overline{T}_\sigma$ will not serve as F in the hypothesis of Case 1, and so $\overline{T}_\sigma$ is properly contained in a subgroup $T_{\sigma+1} \in U$ satisfying (iii). Also (ii) holds by construction, and so the tower can be obtained.

Let $T = \bigcup_{\sigma<\alpha} T_\sigma$. Then we have $|T| = \alpha$. In fact $|T| \geqq \alpha$ since the tower $\{T_\sigma\}$ is strictly ascending; on the other hand, it follows from (i) that $|T| \leqq \alpha^2 = \alpha$.

We shall now show that $T = S$, thus showing that $|S| = \alpha$ and completing the proof of Theorem A. Suppose if possible that $T < S$. Then since S satisfies the normalizer condition there is an element $x \in N_S(T) - T$. For each $\sigma < \alpha$ let $U_\sigma = \langle T_\sigma, x\rangle$ and $z_\sigma = y_{U_\sigma}$. Then from (10) we have

$$y_\sigma z_\sigma^{-1} \in C_{\overline{H}_\alpha}(T_\sigma) \quad (\sigma < \alpha) \tag{12}$$

and

$$z_\tau z_\sigma^{-1} \in C_{\overline{H}_\sigma}(U_\sigma) \quad (\sigma \leqq \tau < \alpha). \tag{13}$$

Hence a fortiori $z_\tau z_1^{-1} \in C_{\overline{H}_\alpha}(x)$ if $1 \leqq \tau < \alpha$, and so from (12), we can write

$$z_1 = c_\tau d_\tau y_\tau, \tag{14}$$

where $c_\tau \in C_{\overline{H}_\alpha}(x)$, $d_\tau \in C_{\overline{H}_\alpha}(T_\tau)$, for any $1 \leqq \tau < \alpha$.

Let $\sigma < \alpha$ be an ordinal of the form $\mu+m$, where μ is a limit ordinal or zero and m is an odd positive integer. Ordinals of this form will be called 'odd', and the number of odd ordinals $< \alpha$ is clearly α. We claim that there exist elements $u_\sigma, u'_\sigma \in T_{\sigma+1} - T_{\sigma-1}$ and an element r_σ belonging to the transversal R such that

$$u_\sigma^{-1} u'_\sigma \in T_\sigma, \tag{15}$$

$$y_\sigma(r_\sigma u_\sigma) \neq y_\sigma(r_\sigma u'_\sigma). \tag{16}$$

Indeed, in the contrary case, y_σ takes a constant value on each right coset of T_σ lying in $rT_{\sigma+1}$ with the possible exception of rT_σ itself, and on rT_σ, y_σ is constant outside $rT_{\sigma-1}$. This holds for all $r \in R$. Therefore by (ii) there is an element $e_\sigma \in \overline{H}_\alpha$ which is constant on right cosets of T_σ, and is such that $\operatorname{supp} e_\sigma^{-1} y_\sigma \leqq \bigcup_{r\in R} rT_{\sigma-1}$. However this contradicts (iii) above, and so the desired elements $u_\sigma, u'_\sigma, r_\sigma$ indeed exist.

Now if $\tau \geqq \sigma$ then we have from (10) that $y_\tau = f_\sigma y_\sigma$, where f_σ belongs to the centralizer in $\overline{H}_\alpha$ of T_σ and so is constant on right cosets of T_σ; hence from (15) and (16) we obtain

$$y_\tau(r_\sigma u_\sigma) \neq y_\tau(r_\sigma u'_\sigma) \text{ if } \tau \geqq \sigma. \tag{17}$$

For each odd ordinal $\sigma < \alpha$ let $B_\sigma = \{u_\sigma\, u'_\sigma,, u_\sigma x, u'_\sigma x\}$ and let $B^*_\sigma = \bigcup_{r \in R} rB_\sigma$.

Since $x \notin T$ and $u_\sigma, u'_\sigma \in T_{\sigma+1} - T_{\sigma-1}$, the sets B_σ are pairwise disjoint; hence so are the sets B^*_σ. We shall show that the support of z_1 contains at least one point from each B^*_σ, thereby establishing that $|\text{supp } z_1| \geqq \alpha$ and obtaining a contradiction.

Now x normalizes T and so we have

$$(18) \qquad u_\sigma x = xv_\sigma, u'_\sigma x = xv'_\sigma$$

with $v_\sigma, v'_\sigma \in T$. We choose $\tau < \alpha$ such that $T_\tau \geqq \langle T_{\sigma+1}, v_\sigma, v'_\sigma\rangle$ express z_1 in the form (14) and evaluate the result on the set $r_\sigma B_\sigma = \{r_\sigma u_\sigma, r_\sigma u'_\sigma, r_\sigma u_\sigma x, r_\sigma u'_\sigma x\}$. Now $c_\tau(r_\sigma u_\sigma) = c_\tau(r_\sigma u_\sigma x)$, $c_\tau(r_\sigma u'_\sigma) = c_\tau(r_\sigma u'_\sigma x)$, and by (18), $d_\tau(r_\sigma u_\sigma) = d_\tau(r_\sigma u'_\sigma)$ and $d_\tau(r_\sigma u_\sigma x) = d_\tau(r_\sigma u'_\sigma x)$. From (17), $y_\tau(r_\sigma u_\sigma) \neq y_\tau(r_\sigma u'_\sigma)$ and from (ii), since $x \in N_S(T) - T$, we have $y_\tau(r_\sigma u_\sigma x) = y_\tau(r_\sigma u'_\sigma x) = 1$. Putting these facts together as in Lemma 4.1 we easily find that the support of z_1 meets $r_\sigma B_\sigma$ non-trivially, as required to complete the proof.

5. Proof of Theorem C

The arguments here are similar in spirit to those of Theorems A and B, but differ somewhat in the technical details. The following lemma plays the part previously occupied by Lemma 4.1.

LEMMA 5.1. *Let $G = \langle U, x\rangle$ be a finite nilpotent group generated by a π-subgroup $U < G$ and an element x. Suppose that X is a proper normal subgroup of G containing x. Let A be an abelian group and let c, d, y be functions from G to A satisfying the following conditions*:

(i) *c is constant on each right coset of $\langle x\rangle$ in G,*

(ii) *d is constant on each right coset of U in G,*

(iii) *supp $y \leqq U$ but y is not constant on the set $U - (U \cap X)$.*

(iv) *$y(U)$ is contained in a subgroup B of A which has no non-trivial π-elements.*

Let $f = cdy$. Then $f(w) \neq 1$ for some $w \notin X$.

PROOF. The proof is by induction on $|G|$. In making the inductive step there are two cases to consider, and the first of them also starts the induction.

CASE 1. $U \lhd G$. Let u, v be points in $U - (U \cap X)$ such that $y(u) \neq y(v)$. Then u, v, ux, vx are points not lying in X and satisfying the hypotheses of Lemma 4.1. Therefore $f(w) \neq 1$, where w is one of these points.

Case 2. *U is not normal in G.* Now $x \neq 1$ as $U < G$; hence $X \neq 1$ and so X contains a non-trivial element z of prime order p belonging to the centre of G. Let $t \to \bar{t}$ be the natural homomorphism of G onto $\bar{G} = G/\langle z \rangle$. Then $\bar{U} < \bar{G}$ since otherwise $G = U\langle z \rangle$ and $U \lhd G$; also $\bar{U}$ is a π-group. Evidently $\bar{X}$ is a proper normal subgroup of $\bar{G}$ containing $\bar{x}$.

Let $\varphi \in A^G$, the multiplicative group of all functions from G to A under pointwise multiplication, and let $\bar{\varphi}$ be the element of $A^{\bar{G}}$ defined by $\bar{\varphi}(\bar{t}) = \prod_{u \in \bar{t}} \varphi(u)$ $(t \in G)$. The map $\varphi \to \bar{\varphi}$ is a homomorphism of A^G into $A^{\bar{G}}$ satisfying

$$\overline{\varphi t} = \bar{\varphi}\bar{t} \quad (\varphi \in A^G,\, t \in G), \tag{1}$$

where φt is the element of A^G defined by $\varphi t(u) = \varphi(ut^{-1})$ $(u \in G)$ and $\bar{G}$ acts on $A^{\bar{G}}$ in a similar way. If $L \leqq G$ then the centralizer of L in A^G consists precisely of the functions constant on the right cosets of L in G; furthermore (1) shows that $\overline{C_{A^G}(L)} \leqq C_{A^{\bar{G}}}(\bar{L})$.

We therefore have that $\bar{c}$ and $\bar{d}$ are constant on the right cosets of $\langle \bar{x} \rangle$ and $\bar{U}$ respectively in $\bar{G}$; this can in any case be verified directly without difficulty. Evidently $\bar{y}(\bar{U}) \leqq B$ and so in order to apply induction to $\bar{G}$ it remains to verify (ii) for $\bar{G}$. We now subdivide Case 2 further.

Case 2a. $z \notin U$. Clearly $\bar{y}(\bar{t}) = 1$ unless $\bar{t} \cap U \neq \emptyset$, that is unless $\bar{t} \leqq U\langle z \rangle$. Thus supp $\bar{y} \leqq \bar{U}$. Let u, v be points in $U - (U \cap X)$ such that $y(u) \neq y(v)$. Then $\bar{u}, \bar{v} \notin \bar{X}$ since $z \in X$, and $\bar{y}(\bar{u}) = y(u)$, $\bar{y}(\bar{v}) = y(v)$, since u and v are now the unique points of U in $\bar{u}$, $\bar{v}$ respectively. Therefore $\bar{y}(\bar{u}) \neq \bar{y}(\bar{v})$. It now follows by induction, since $\bar{f} = \bar{c}\,\bar{d}\,\bar{y}$, that there exists an element $\bar{t} \notin \bar{X}$ such that $\bar{f}(\bar{t}) \neq 1$. Therefore $f(w) \neq 1$ for some $w \in \bar{t}$, and clearly $w \notin X$.

Case 2b. $z \in U \cap X$. Suppose first that there is a coset $\bar{u}$ of $\langle z \rangle$ in $U - (U \cap X)$ on which y is not constant, and let u, v be points of $\bar{u}$ such that $y(u) \neq y(v)$. Now $v = uz^{\lambda}$ for some integer λ and so $vx = uz^{\lambda}x = uxz^{\lambda} \in uxU$. Thus ux and vx lie in the right coset uxU of U, which does not meet supp $y \leqq U$. Therefore by Lemma 4.1 we have $f(w) \neq 1$, where w is one of the points u, v, ux, vx; since none of these points lies in X the result follows in this case.

Therefore we may suppose that y is constant on each coset of $\langle z \rangle$ lying in $U - (U \cap X)$. Now as in case 2a we find that supp $\bar{y} \leqq \bar{U}$. Furthermore let u, v be points of $U - (U \cap X)$ such that $y(u) = \alpha \neq \beta = y(v)$. Then $\bar{y}(\bar{u}) = \alpha^p$, $\bar{y}(\bar{v}) = \beta^p$ and $\alpha^p \neq \beta^p$ since $p \in \pi$ and B has no non-trivial π-elements. Evidently $\bar{u}, \bar{v} \notin \bar{X}$ and so induction yields an element $\bar{t} \notin \bar{X}$ such that $\bar{f}(\bar{t}) \neq 1$. Hence $f(w) \neq 1$ for some $w \in \bar{t}$, and certainly $w \notin X$. This establishes Lemma 5.1.

PROOF OF THEOREM C. We have a countably infinite periodic locally nilpotent π-subgroup S of K and have to construct a maximal member S^* of the set of baseless periodic locally nilpotent subgroups of $W = H \wr K$ satisfying $\bar{H}S = \bar{H}S^*$, under the assumption that H is abelian and satisfies certain other conditions to be found in the statement of the theorem.

CASE 1. *The Sylow* 2*-subgroup of* S *is finite and* H *is not a* π*-group.* Let S_1 be the Sylow 2-subgroup of S. Then by constructing an arbitrary tower of finite subgroups from S_1 to S and refining it suitably we can write $S = \bigcup_{i=0}^{\infty} S_i$, where

$$(2) \qquad 1 = S_0 < S_1 < \cdots$$

are finite subgroups of S such that S_i is maximal and of index at least three in S_{i+1} for $i \geqq 1$. For each $i \geqq 0$ let $D_i \neq S_i$ be a right coset of S_i in S_{i+1}. Let $\langle t \rangle$ be a non-trivial cyclic subgroup of H containing no non-trivial π-element and let y_i be the element of $\bar{H}$ which takes the value t at each point of $\bigcup_{j<i} D_j$ and 1 elsewhere. Then $y_{i+1}y_i^{-1} \in C_{\bar{H}}(S_i)$ and so if we define $S_i^* = S_i^{y_i}$ $(i \geqq 0)$ then $S^* = \bigcup_{i=0}^{\infty} S_i^*$ is a subgroup of W satisfying $\bar{H}S = \bar{H}S^*$, $\bar{H} \cap S^* = 1$.

Notice that

$$(3) \qquad \operatorname{supp} y_i \leqq S_i \quad (i \geqq 0),$$

and

(4) $\quad$ *y_i is not constant outside any proper subgroup of S_i $(i \geqq 2)$.*

Indeed if u is either 1 or t, then since $|S_i : S_{i-1}| \geqq 3$ the set of points $s \in S_i$ at which $y_i(s) \neq u$ contains a right coset of S_{i-1} other than S_{i-1} itself and so generates a subgroup of S_i properly containing S_{i-1}. Since S_{i-1} is maximal in S_i this subgroup must be S_i itself, and so y_i cannot take the value u at all points of the complement of a proper subgroup of S_i.

CASE 2. *The Sylow* 2*-subgroup of* S *is infinite but* H *contains a subgroup* B *of order at least* 4 *containing no non-trivial* π*-elements.* In this case we proceed rather differently to obtain conditions (3) and (4). We construct a tower (2) of finite subgroups of S such that, for $i \geqq 1$, either $|S_i : S_{i-1}| = 4$ or S_{i-1} is maximal and of index at least three in S_i. This is obviously possible. Let $1, t_1, t_2, t_3$ be distinct elements of B. If $|S_{i+1} : S_i| = 4$ let w_i be the element of $\bar{H}$ which takes the values $1, t_1, t_2, t_3$ on the respective right cosets of S_i in S_{i+1}, and 1 elsewhere. Otherwise let w_i be the element of $\bar{H}$ taking the value t_1 on some right coset $D_i \neq S_i$ of S_i in S_{i+1} and 1 elsewhere. Then if $y_i = \prod_{j<i} w_j$ $(i \geqq 1)$,

we have $y_{i+1}y_i^{-1} \in C_{\bar{H}}(S_i)$ and so, defining $S_i^* = S_i^{y_i}$ and $S^* = \bigcup_{i=0}^{\infty} S_i^*$, we obtain $\bar{H}S^* = \bar{H}S$, $\bar{H} \cap S^* = 1$ as before. It is not difficult to see that (3) and(4) hold.

Suppose now that there exists a periodic locally nilpotent subgroup $T^* > S^*$ satisfying $\bar{H} \cap T^* = 1$. We may suppose that T^* has the form $\langle S^*, x^* \rangle$ where $x^* \notin S^*$. Let x be the unique element of K which is congruent to x^* modulo $\bar{H}$ and let $T = \langle S, x \rangle$, $T_i = \langle S_i, x \rangle$ $(i \geqq 2)$. Then $\bar{H}T^* = \bar{H}T$ and $\bar{H}T_i^* = \bar{H}T_i$; also $T = \bigcup_{i=0}^{\infty} T_i$ and T is locally nilpotent. Since T_i is finite we have from Corollary 3.3 that $T_i^* = T_i^{z_i}$ for some $z_i \in \bar{H}$. Arguing as in the proofs of Theorems A and B, we obtain $z_i y_i^{-1} \in C_{\bar{H}}(S_i)$ and $z_i z_1^{-1} \in C_{\bar{H}}(x)$ for $i \geqq 1$, whence we can write

$$z_1 = c_i d_i y_i \tag{5}$$

for any $i \geqq 1$, where $c_i \in C_{\bar{H}}(x)$ and $d_i \in C_{\bar{H}}(S_i)$.

Suppose now that we have n points $u_1, \cdots, u_n$ of T lying in the support of z_1. We shall show how to construct a further such point u_{n+1}, thereby showing that the support of z_1 is infinite. This contradiction will show that the assumption $T^* > S^*$ is false and establish the theorem. Let $W = \langle S_1, x, u_1, \cdots, u_n \rangle$, which is finite since K is locally finite. Let i be the first integer $\geqq 2$ such that $W < T_i$, and let X be the normal closure of W in T_i. Then $X < T_i$ as T_i is nilpotent. Consequently since $T_i = \langle S_i, x \rangle$ and $x \in X$ we have $X \cap S_i < S_i$. We now express z_1 in the form (5), and verify the hypotheses of Lemma 5.1 with $G = T_i$, $U = S_i$, $A = H$. These are all immediate except perhaps for conditions (iii) and (iv) of the lemma; these follow from (3) and (4) above and the construction of y_i. Lemma 5.1 therefore gives that $z_1(w) \neq 1$ for some $w \notin X$, as required to complete the argument.

6. Non-periodic baseless subgroups of *W*

In this section we consider only the ordinary restricted wreath product $W = H \wr K$, and show that under fairly general conditions the presence of sufficiently many elements of infinite order in a baseless subgroup of W will ensure that it is W-contained in K.

The main result of this section, stated in the introduction, is Theorem D. It falls into two parts, the first of which is an immediate consequence of

LEMMA 6.1 *Let L^* be a baseless subgroup of W and suppose that the Hirsch-Plotkin radical of L^* is neither periodic nor finite-by-cyclic. Then $L^* \leqq_W K$.*

To obtain the first part of Theorem D, suppose that U is a baseless radical subgroup of W with non-periodic Hirsch-Plotkin radical R, and

$U \nleqq_W K$. Then by Lemma 6.1 we have that, if T is the torsion subgroup of R, then T is finite and R/T is infinite cyclic. Let $C_1 = C_U(T)$, $C_2 = C_U(R/T)$. Then U/C_i is finite ($i = 1, 2$) and so $U/C_1 \cap C_2$ is finite. But $C = C_1 \cap C_2 \leqq R$. For otherwise $C > C \cap R$ and so $C/C \cap R$ contains a non-trivial characteristic locally nilpotent subgroup $X/C \cap R$. Then $X \lhd U$ and we have $[X, C \cap R] \leqq C \cap T$ and $[X, C \cap T] = 1$. From this it follows easily that X is locally nilpotent and hence that $X \leqq R$, a contradiction. Therefore we have that U/R is finite, and so U is polycyclic and of Hirsch number one.

For the converse suppose that L is any polycyclic group with Hirsch number one, let F be the largest finite normal subgroup of L, and let S/F be the Hirsch-Plotkin radical of L/F. Then S/F contains no non-trivial finite normal subgroup and so must be infinite cyclic. Since S/F contains its centralizer in L/F it follows that $|L : S|$ is either 1 or 2 and L is an extension of F by a group which is either infinite cyclic or infinite dihedral. Therefore the proof of Theorem D is completed by our next lemma.

LEMMA 6.2. *Suppose that $L \leqq K$ is an extension of a finite group F by a group which is either infinite cyclic or infinite dihedral. Then there is a baseless subgroup L^* of W such that $\bar{H}L = \bar{H}L^*$ but $L^* \nleqq_W K$.*

PROOF. We deal first with the case when L/F is infinite dihedral. Then L/F is generated by an element xF of infinite order and an element tF such that $x^t \equiv x^{-1}$ modulo F.

Let $y = tx^{-1}$. Then $y \notin F$ but $y^2 \in F$. Choose $1 \neq h \in H$ and let u be the element of $\bar{H}$ taking the value h on F, h^{-1} on $yF = Fy$, and 1 elsewhere. Then $u \in C_{\bar{H}}(F)$ by Lemma 3.4. Since right multiplication by y interchanges the two cosets F and yF we have $u^y = u^{-1}$. Therefore $u^{tx^{-1}} = u^{-1}$ and so $(ux)^t = x^{-1}xu^t x^t \equiv x^{-1}xu^t x^{-1} \bmod F$. Hence, modulo F, we have $(ux)^t \equiv x^{-1}u^{tx^{-1}} = x^{-1}u^{-1}$. Therefore, if $x^* = ux$, then x^* normalizes F and $x^{*t} \equiv x^{*-1} \bmod F$.

Let $L^* = \langle F, x^*, t\rangle$. Then $\bar{H}L^*$ contains x and it follows that $\bar{H}L^* = \bar{H}L$. An arbitrary element of $L^* - F$ is congruent modulo F to an element of the form $x^{*n}t^{\varepsilon}$, where n is a non-zero integer and $\varepsilon = 0$ or 1. Such an element, being congruent modulo $\bar{H}$ to $x^n t^{\varepsilon}$, cannot lie in $\bar{H}$. Therefore $\bar{H} \cap L^* \leqq \bar{H} \cap F = 1$ and L^* is baseless.

Finally, if $L^* \leqq_W K$ then we have $L^* = L^w$ for some $w \in \bar{H}$. It follows that $ux = x^* = x^w = [w, x^{-1}]x$, whence $u = [w, x^{-1}]$. But $F \cup Fy = \langle F, y\rangle$ and $\langle x\rangle \cap \langle F, y\rangle = 1$ since $\langle F, y\rangle$ is finite. Therefore no right coset of $\langle x\rangle$ in K contains more than one point of supp u. Since u has the form $[w, x^{-1}]$ it follows that $u = 1$, which is a contradiction.

The case when L/F is infinite cyclic may be discussed similarly. Let $L/F = \langle xF\rangle$. Let $1 \neq h \in H$ and let u be the element of $\bar{H}$ taking the value h on F and 1 elsewhere. Finally, if $x^* = ux$, then $L^* = \langle F, x^* \rangle$ satisfies our requirements.

It therefore remains only to establish Lemma 6.1, and we do this in several stages. The first two steps give results which are probably well known.

LEMMA 6.3. *Let A be a normal subgroup of a group G complemented by each of two subgroups B and B^*, and let $C = B \cap B^*$. Then $C_A(C \cap C^b) \neq 1$ for each $b \in B - C$.*

PROOF. Let $b \in B$ and suppose that $C_A(C \cap C^b) = 1$. We have $b = b^*a$ with $b^* \in B^*$, $a \in A$. Let $c \in C \cap C^b$. Then $c^{b^{-1}} \in C$ and so C contains $[c, b^{-1}] = [c, a^{-1}b^{*-1}] = [c, b^{*-1}]\,[c, a^{-1}]^{b^{*-1}}$. Of these factors, the first lies in B^* while the second lies in A. Since their product lies in $C \leqq B^*$, we have $[c, a^{-1}] = 1$. This holds for all $c \in C \cap C^b$, whence a centralizes $C \cap C^b$ and so $a = 1$. Therefore $b = b^* \in C$.

COROLLARY 6.4. *Let L, L^* be baseless subgroups of W such that $L \leqq K$ and $\bar{H}L = \bar{H}L^*$, and let $M = L \cap L^*$. Then $M \cap M^l$ is finite for each $l \in L - M$. In particular M contains the normalizer in L of each of its infinite subgroups.*

PROOF. Since $C_{\bar{H}}(J) = 1$ for every infinite subgroup J of K, Lemma 6.3 gives immediately that $M \cap M^l$ is finite for $l \in L - M$. The rest follows.

Examples in which two distinct complements to $\bar{H}$ in W have infinite intersection seem to be fairly uncommon and we know of none in which K is locally finite. But if K is freely generated by two elements x and y and $1 \neq h \in \bar{H}$, then x and hy evidently generate a subgroup which complements $\bar{H}$ and has infinite intersection with K.

LEMMA 6.5. *Let $L \leqq K$ and let L^* be a baseless subgroup of W such that $\bar{H}L = \bar{H}L^*$. Suppose that either*

(i) *L contains a central element x of infinite order such that $L/\langle x \rangle$ is infinite, or*

(ii) *L contains an infinite locally finite normal subgroup M and an element x of infinite order such that M is the union of the finite subgroups of M normalized by x.*

Then L and L^* are conjugate under $\bar{H}$.

PROOF. (i) For each $t \in L$ let $t^* = h_t t$ $(h_t \in \bar{H})$ be the unique element of L^* which is congruent to t modulo $\bar{H}$. Then $t \to t^*$ is an isomorphism. Hence $x^*t^* = t^*x^*$ for all $t \in L$, whence we obtain

$$h_x h_t^{x^{-1}} = h_t h_x^{t^{-1}} \tag{1}$$

for all $t \in L$. Let $\{k_\lambda\}$ be a set of right coset representatives of L in K and let $\{t_\mu\}$ be a set of coset representatives of $\langle x \rangle$ in L. Evaluating (1) at the point $k_\lambda x^i$ and using the fact that x and t commute, we obtain

$$h_x(k_\lambda x^i)h_t(k_\lambda x^{i+1}) = h_t(k_\lambda x^i)h_x(k_\lambda t x^i)$$

or

$$(2) \qquad h_x(k_\lambda t x^i) = h_t(k_\lambda x^i)^{-1}h_x(k_\lambda x^i)h_t(k_\lambda x^{i+1}).$$

Now $h_t(k_\lambda x^i) = h_x(k_\lambda x^i) = 1$ when i is sufficiently large or small, and so, for a fixed λ and t, we may multiply the equations (2) in order of increasing i to obtain

$$\prod_{i=-\infty}^{\infty} h_x(k_\lambda t x^i) = \prod_{i=-\infty}^{\infty} h_x(k_\lambda x^i).$$

It follows that the value of the product

$$\prod_{i=-\infty}^{\infty} h_x(k_\lambda t_\mu x^i)$$

is independent of μ. Now since $|L : \langle x \rangle| = \infty$ there must be a right coset of $\langle x \rangle$ contained in $k_\lambda L$ on which h_x takes the value 1 identically. We therefore must have

$$(3) \qquad \prod_{i=-\infty}^{\infty} h_x(k_\lambda t_\mu x^i) = 1$$

for all λ and μ.

We now define an element $w \in \bar{H}$ by

$$(4) \qquad w(k_\lambda t_\mu x^i) = \Big(\prod_{j=i}^{\infty} h_x(k_\lambda t_\mu x^j)\Big)^{-1},$$

the product being taken in order of increasing j. The equations (3), together with the fact that the support of h_x only meets finitely many of the cosets $k_\lambda t_\mu \langle x \rangle$, ensure that the support of w is finite. A straightforward calculation shows that $[w, x^{-1}] = h_x$. Hence

$$x^* = h_x x = [w, x^{-1}]x = x^w$$

and so $\langle x \rangle \leqq L \cap L^{*w^{-1}}$. Since $\langle x \rangle$ is infinite and central in L, Corollary 6.4 now gives $L = L^{*w^{-1}}$, as required.

(ii) Let $\boldsymbol{F}$ be the set of all finite subgroups of M normalized by x. Then $M = \bigcup_{F \in \boldsymbol{F}} F$, and since M is locally finite, any two members of $\boldsymbol{F}$ generate a third. Let $t \to t^*$ be the usual isomorphism of L onto L^*. Then $M^* = \bigcup_{F \in \boldsymbol{F}} F^*$, and by Corollary 3.3 there exists, for each $F \in \boldsymbol{F}$, an element $h_F \in \bar{H}$ such that

$$F^{*h_F} = F.$$

Let $x^* = hx$ ($h \in \bar{H}$). Now x^* normalizes F^* ($F \in \boldsymbol{F}$) and so x^{*h_F} normalizes F. Hence $[F, x^{*h_F}] \leqq F$ and so, if $f \in F$, then F contains

$$[f, (hx)^{h_F}] = [f, h^{h_F}[h_F, x^{-1}]x] = [f, x][f, h^{h_F}[h_F, x^{-1}]]^x.$$

It follows that

$$[f, h^{h_F}[h_F, x^{-1}]] = 1$$

and so

$$h^{h_F}[h_F, x^{-1}] \in C_F = C_{\bar{H}}(F) \tag{6}$$

if $F \in \boldsymbol{F}$.

Let T be a right transversal to $M\langle x\rangle = N$ in K, so that $K = \bigcup_{t \in T} tN$ and $tN \cap t'N = \emptyset$ if $t \neq t'$. We now distinguish two cases:

CASE 1. *There exists a subgroup $F \in \boldsymbol{F}$ with the following property: given $F < E \in \boldsymbol{F}$, there exists an element $h'_E \in \bar{H}$ such that $h'_E h_E^{-1} \in C_E$ and*

$$\operatorname{supp} h'_E \leqq X_F = \bigcup_{i=-\infty}^{\infty} \bigcup_{t \in T} tx^iF.$$

In this case we may suppose without loss of generality that $\operatorname{supp} h_E \leqq X_F$ for all $F < E \in \boldsymbol{F}$. Choose such an E and let $E \leqq D \in \boldsymbol{F}$. Then by a now familiar argument we can write $h_D = ch_E$, where c is constant on each right coset of E in K. Now no right coset of E is contained in X_F and so any such right coset contains a point at which both h_D and h_E take the value 1. Hence c takes the value 1 at such a point, and therefore at every point of the right coset in question. Therefore $c = 1$ and $h_D = h_E$ for all $E \leqq D \in \boldsymbol{F}$. Then

$$M^{*h_E} = \bigcup_{D \geqq E} D^{*h_E} = \bigcup_{D \geqq E} D^{*h_D} = \bigcup_{D \geqq E} D = M.$$

Therefore $M \leqq L \cap L^{*h_E}$. Since $M \lhd L$, Corollary 6.4 now gives $L = L^{*h_E}$, as required.

CASE 2. *Case* 1 *does not hold.* Under this assumption we shall obtain a contradiction, thereby showing that Case 2 does not in fact arise. To do this we assume that we have a finite subset A of $\operatorname{supp} h$ and show that a further point of $\operatorname{supp} h$ can always be found, thereby contradicting the finiteness of $\operatorname{supp} h$.

Now the elements tx^i form a right transversal to M in K and so we have $A \leqq X_F$ for some $F \in \boldsymbol{F}$. By hypothesis there is a subgroup $E \in \boldsymbol{F}$ with $F < E$ and such that h_E is not congruent modulo C_E to any element with support in X_F. Therefore there exists a right coset C of E in K such that h_E is not constant on the set $C - (C \cap X_F)$. Now since x normalizes E but no non-trivial power of x lies in E, the sets Cx^i ($i = 0, \pm 1, \cdots$) are distinct right cosets of E in K. Only finitely many of them meet the support of h_E and so, by considering a suitable Cx^i instead of C, we may suppose that h_E is not constant on $C-(C \cap X_F)$, but is constant on $Cx^{-1}-(Cx^{-1} \cap X_F)$. Since X_F is invariant under right multiplication by x, it follows that there exist points $c_1, c_2 \in C-(C \cap X_F)$ such that $h_E(c_1) \neq h_E(c_2)$, but $h_E(c_1x^{-1}) = h_E(c_2x^{-1})$. It follows that $[h_E, x^{-1}]$ takes distinct values at c_1x^{-1} and c_2x^{-1}. From (6) we have

$$h^{h_E} = c_E[h_E, x^{-1}]^{-1},$$

where c_E is constant on each right coset of E and in particular on Cx^{-1}. Therefore one of c_1x^{-1} and c_2x^{-1} lies in the support of h^{h_E} and hence in the support of h. Both of these points lie outside X_F and hence outside A, and so the argument is complete.

Let $\mathfrak{X}$ denote the class of all groups which have a central infinite cyclic subgroup with infinite factor group. We define two $\mathfrak{X}$-subgroups X and Y of a group G to be *connected* if there is a finite chain $X = X_1, X_2, \cdots, X_n = Y$ of $\mathfrak{X}$-subgroups of G such that $X_i \cap X_{i+1}$ is infinite for $1 \leqq i \leqq n-1$. Connectedness is evidently an equivalence relation and so any $\mathfrak{X}$-subgroup of G will have a connected component in the set of all $\mathfrak{X}$-subgroups of G. Now if $M \leqq K$ and M^* are baseless subgroups of W such that $\bar{H}M = \bar{H}M^*$ and if X and Y are $\mathfrak{X}$-subgroups of M such that $X \leqq M \cap M^*$ and $X \cap Y$ is infinite, then Corollary 6.4 shows that $Y \leqq M \cap M^*$. For Y contains a central element y of infinite order. Since y normalizes the infinite subgroup $X \cap Y$ of $M \cap M^*$, we obtain $y \in M \cap M^*$, and hence, since $\langle y \rangle$ is infinite and central in Y, we obtain $Y \leqq M \cap M^*$. It follows that $M \cap M^*$ contains the connected component of X in M. A little more argument yields

LEMMA 6.6. *Let $L \leqq K$ and suppose that L contains a non-trivial normal subgroup M which is generated by a connected set of $\mathfrak{X}$-subgroups. Let L^* be a baseless subgroup of W such that $\bar{H}L^* = \bar{H}L$. Then $L^{*h} = L$ for some $h \in \bar{H}$.*

PROOF. Let X be an $\mathfrak{X}$-subgroup belonging to the connected set generating M. Then, in the usual notation, we have $\bar{H}X = \bar{H}X^*$, and Lemma 6.5(i) gives that $X^{*h} = X$ for some $h \in \bar{H}$. Therefore $M \cap M^{*h} \geqq X$. The remarks above give $M = M^{*h} \leqq L^{*h}$, and finally, as $M \lhd L$, Corollary 6.4 gives $L = L^{*h}$.

The relevance of these concepts to the proof of Lemma 6.1 is that most non-periodic locally nilpotent groups can be generated by a connected set of $\mathfrak{X}$-subgroups.

LEMMA 6.7. *Let G be a locally nilpotent group with torsion subgroup T.*

Then G is generated by a connected set of $\mathfrak{X}$-subgroups unless either G is finite-by-cyclic or T is infinite and G/T is locally cyclic.

PROOF. Suppose first that G/T is not locally cyclic. Let $\boldsymbol{S}$ be the set of all finitely generated subgroups S of G such that $S/S \cap T$ is not cyclic. Then clearly $G = \bigcup_{S \in \boldsymbol{S}} S$. Now a finitely-generated nilpotent group with finite centre is necessarily finite and so, if $S \in \boldsymbol{S}$, then S contains a central element x_S of infinite order. Since $S/S \cap T$ is not cyclic, $S/\langle x_S \rangle$ cannot be finite, and so $S \in \mathfrak{X}$. Since $\langle S_1, S_2 \rangle \in \boldsymbol{S}$ if $S_1, S_2 \in \boldsymbol{S}$, it follows that $\boldsymbol{S}$ is a connected set of $\mathfrak{X}$-subgroups generating G.

Now suppose that G/T is locally cyclic. We may suppose that T is finite and $T < G$. Let x be an element of infinite order in G and $t \in T$. Then x^k centralizes T for some $k > 0$ and so, if $y \in G$, we have $1 = [y, x^k]^t = [y, x^{kt}]$ by the usual commutator identities. Therefore $z = x^{kt}$ is in the centre of G. If $G/\langle z \rangle$ is finite then we find that G/T is cyclic. Therefore $G \in \mathfrak{X}$ unless G is finite-by-cyclic.

PROOF OF LEMMA 6.1. We have $\bar{H}L = \bar{H}L^*$, where $L = \bar{H}L^* \cap K$. Let R be the Hirsch-Plotkin radical of L, and let T be the torsion subgroup of R. Then by hypothesis $T < R$ and R is not finite-by-cyclic. Therefore, by Lemma 6.7, either R is generated by a connected set of $\mathfrak{X}$-subgroups or T is infinite and R/T is locally cyclic. In the first case the result follows from Lemma 6.6. In the second case let x be an element of infinite order in R. Then since R is the union of its finitely-generated subgroups containing x, and each of these has finite intersection with T, we see that the hypotheses of Lemma 6.5 (ii) hold with $M = T$. Therefore the result follows in this case from Lemma 6.5 (ii).

Added in proof: Since submitting this paper, I have been informed that some of the results it contains have been obtained independently by Dr. D. Segal.

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