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# COHOMOLOGY OF NON-COMPLETE ALGEBRAIC VARIETIES

by

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## Introduction

In developing the notion of an ample subvariety of an algebraic variety, one approach is to consider the coherent sheaf cohomology of the complement of the subvariety. For example, if $Y$ is an ample divisor on a complete algebraic variety $X$, then $U = X - Y$ is affine. Hence $H^i(U, F) = 0$ for all $i > 0$ and all coherent sheaves $F$. More generally, if $Y$ is a complete intersection of ample divisors, then $H^i(U, F) = 0$ for all $i \geqq$ codim $(Y, X)$ and all coherent sheaves $F$.

Thus we are led to ask for conditions on a subvariety $Y$ of a complete variety $X$ such that the cohomology groups $H^i(U, F)$ of the complement $U = X - Y$ are finite-dimensional, or zero, for all coherent sheaves $F$ and for all $i \geqq q$ for some integer $q$.

In this paper we give a summary of recent results about this problem. Some proofs have appeared in the lecture notes [4]. The other proofs will appear in our forthcoming paper [5].

I wish to thank Wolf Barth, whose ideas inspired much of the work in section two.

## 1. $H^{n-1}$-vanishing and $G3$

Let $X$ be a complete irreducible non-singular algebraic variety of dimensional $n$ over an algebraically closed field $k$. Let $Y$ be a non-empty closed subvariety, and let $U = X - Y$. By Lichtenbaum's theorem, $H^n(U, F) = 0$ for all coherent sheaves $F$. In this section we investigate the vanishing of $H^{n-1}(U, F)$ for all coherent sheaves $F$, and we show its relation to the formal-rational functions of Hironaka and Matsumura.

Let $\hat{X}$ be the formal completion of $X$ along $Y$. Let $K(\hat{X})$ be the ring of formal-rational functions on $\hat{X}$: $K(\hat{X})$ is the ring of global sections of the sheaf of total quotient rings of the structure sheaf $O_{\hat{X}}$ of the formal scheme $\hat{X}$. Following Hironaka and Matsumura [6], we say that $Y$ is $G2$ in

$X$ if $K(\hat{X})$ is a finite module over the function field $K(X)$ of $X$, and we say that $Y$ is $G3$ in $X$ if $K(\hat{X}) = K(X)$.

We recall a definition of Grothendieck [2]: we say that the pair $(X, Y)$ satisfies the Lefschetz condition, written Lef $(X, Y)$, if for every open neighborhood $V \supseteq Y$ and for every locally free sheaf $E$ on $V$, there is a smaller neighborhood $V \supseteq V' \supseteq Y$ such that the natural map

$$H^0(V', E|_{V'}) \to H^0(\hat{X}, \hat{E})$$

is an isomorphism, where $\hat{E}$ is the completion of $E$ along $Y$.

Now we can state our results.

PROPOSITION 1.1. [4, IV. 1.1] *Let $X$ be a non-singular projective variety of dimension $n$, let $Y$ be a closed subset of $X$, and let $U = X-Y$. Then $H^{n-1}(U, F) = 0$ for all coherent sheaves $F$ if and only if* Lef $(X, Y)$ *and $Y$ meets every subvariety of codimension* 1 *in $X$.*

PROPOSITION 1.2. [4, V. 2.1] *Let $X$ be a complete non-singular variety, and $Y$ a closed subset. Then $Y$ is $G3$ in $X$ if and only if $Y$ is $G2$ in $X$ and* Lef $(X, Y)$.

For projective space and for abelian varieties, Hironaka and Matsumura [6] have given criteria for a subvariety to be $G2$ or $G3$. They prove that any positive-dimensional connected subvariety of a projective space $\boldsymbol{P}^n$ is $G3$. If $Y$ is a subvariety of an abelian variety $A$, which contains 0 and generates $A$, then $Y$ is $G2$ in $A$. If furthermore the natural map Alb $Y \to A$ has connected fibres, then $Y$ is $G3$ in $A$. Using these results, we obtain criteria for $H^{n-1}$-vanishing.

THEOREM 1.3. [3, *Thm.* 7.5] *Let $Y$ be a connected, positive-dimensional subvariety of a projective space $X = \boldsymbol{P}_k^n$, and let $U = X-Y$. Then $H^{n-1}(U, F) = 0$ for all coherent sheaves $F$.*

THEOREM 1.4. (*Speiser* [9]) *Let $Y$ be a subvariety of an abelian variety $A$, containing* 0, *such that $Y$ generates $A$, and the natural map* Alb $Y \to A$ *has connected fibres. Let $U = A-Y$, and let $n =$* dim $A$. *Then $H^{n-1}(U, F) = 0$ for all coherent sheaves $F$.*

## 2. The Main Theorem and its consequences

We now turn to subvarieties of projective space. Let $k$ be a field, let $X = \boldsymbol{P}_k^n$ be the projective $n$-space over $k$, and let $Y$ be a closed subvariety of $X$. We will study the finite-dimensionality and vanishing of the cohomology groups $H^i(X-Y, F)$ for large $i$, in terms of various properties of $Y$. By duality we are led to study the cohomology groups $H^i(\hat{X}, \mathcal{O}_{\hat{X}}(\nu))$ for small $i$ and for all $\nu \in \boldsymbol{Z}$, where $\hat{X}$ is the formal completion of $X$ along

$Y$. In this section we will state the main theorem about these cohomology groups, and its consequences.

Let $S = k[x_0, \cdots, x_n]$ be the homogeneous coordinate ring of the projective space $X$. Then for any $i$, we consider the graded $S$-module

$$M^i = \sum_{\nu \in \mathbf{Z}} H^i(\hat{X}, O_{\hat{X}}(\nu)).$$

The $S$-module structure is via cup-product and the natural identification $S \cong \sum H^0(X, O_X(\nu))$. For an integer $s$, we consider the condition

($I_s^*$) For all $i < s$, $M^i$ is a free finite-type $S$-module, generated by the component $(M^i)_0$ in degree 0.

If $k = \mathbf{C}$, we have an analogue using cohomology of analytic sheaves in the usual complex topology. Let $O_X^h$ be the sheaf of germs of holomorphic functions on $X$. For any coherent analytic sheaf $\mathfrak{F}$ on $X$, let $\mathfrak{F}|_Y$ be the topological restriction of $\mathfrak{F}$ to $Y$. Now for any $i$, we consider the graded $S$-module

$$M_{an}^i = \sum_{\nu \in \mathbf{Z}} H_{an}^i(Y, O_X^h(\nu)|_Y).$$

For an integer $s$, we consider the condition

($I_{s,an}^*$) For all $i < s$, $M_{an}^i$ is a free finite-type $S$-module, generated by the component $(M_{an}^i)_0$ in degree 0.

THEOREM 2.1 (*Main Theorem*). *Let $k$ be any field, let $X = \mathbf{P}_k^n$, and let $Y$ be a purely $s$-dimensional subvariety of $X$.*

a) *Assume* char $k = 0$ *and $Y$ is absolutely non-singular (not necessarily irreducible). Then condition ($I_s^*$) holds.*

b) *Assume* char $k = p > 0$ *and $Y$ is Cohen-Macaulay. Then condition ($I_s^*$) holds.*

c) *Assume $k = \mathbf{C}$ and $Y$ is non-singular. Then condition ($I_{s,an}^*$) holds.*

REMARKS. 1. In the analytic case, the main theorem follows immediately from the methods of Barth [1]. In the algebraic case (char. $k = 0$), one can reduce to $k = \mathbf{C}$. Then the proof proceeds by a modified version of Barth's technique, using algebraic De Rham cohomology in place of complex cohomology. In characteristic $p$, the proof rests on an application of the Frobenius morphism and its iterates. This proof can be found in [4, Ch. III, § 6].

2. The hypotheses in this theorem are certainly not the best possible. For example, even if $Y$ has singularities, one can expect that the condition ($I_s^*$) will hold for suitable $s < \dim Y$. So we emphasize that all the consequences we will draw below depend only on the condition ($I_s^*$) or ($I_{s,an}^*$), and not on the proof of the main theorem.

For the remainder of this section, we let $X = \mathbf{P}_k^n$, let $Y$ be a closed subset of $X$, and let $s$ be an integer (not necessarily dim $Y$). The principal technical result we use is the following. It is deduced from condition $(I_s^*)$ by resolving the coherent sheaf $F$ by sheaves of the form $\sum O(v_i)$.

PROPOSITION 2.2. *Assume condition $(I_s^*)$. Let $F$ be a coherent sheaf on $X$, and let $t = hd\, F$ be the homological dimension of $F$. Then there is a spectral sequence $(E_r^{pq}, E^m)$ with*

$$\begin{array}{lr} E_2^{pq} \simeq H^p(X, F) \otimes H^q(\hat{X}, O_{\hat{X}}) & \text{for } p+q < s-t \\ & \text{or } p = s-t,\, q = 0 \\ E^m \simeq H^m(\hat{X}, \hat{F}) & \text{for } m < s-t \\ E^m \leftarrow H^m(\hat{X}, \hat{F}) & \text{injective for } m = s-t. \end{array}$$

COROLLARY 2.3. *Assume condition $(I_s^*)$. Let $F$ be a coherent sheaf which is negative in the sense that $H^i(X, F) = 0$ for all $i < n-t$, where $t = hd F$. Then*

$$H^m(\hat{X}, \hat{F}) = 0 \text{ for } m < s-t.$$

PROPOSITION 2.4. *Assume condition $(I_s^*)$, and assume furthermore depth $O_Y \geqq s$. Then the natural maps*

$$H^i(X, O_X) \to H^i(\hat{X}, O_{\hat{X}})$$

*are isomorphisms for $i \leqq 2s-n$. (Thus $H^0(\hat{X}, O_{\hat{X}}) \cong k$ if $0 \leqq 2s-n$, and $H^i(\hat{X}, O_{\hat{X}}) = 0$ for $0 < i \leqq 2s-n$.)*

PROPOSITION 2.5. *Assume condition $(I_s^*)$, and suppose $k = \mathbf{C}$. Let $b_i =$* dim $H^i(Y, \mathbf{C})$, *and let*

$$z_i = \dim(\ker(H^i(Y, \mathbf{C}) \to H^{i+2}(Y, \mathbf{C})),$$

*where the map is given by cup-product with the class of a hyperplane section in $H^2(Y, \mathbf{C})$. Then*

$$\dim H^i(\hat{X}, O_{\hat{X}}) = b_i - b_{i-2} + z_{i-1} + z_{i-2} \text{ for } i < s.$$

COROLLARY 2.6 (*Barth* [1]). *Let $Y$ be a non-singular subvariety of dimension $s$ of $X = \mathbf{P}^n_{\mathbf{C}}$. Then the natural maps*

$$H^i(X, \mathbf{C}) \to H^i(Y, \mathbf{C})$$

*are isomorphisms for $i \leqq 2s-n$.*

PROPOSITION 2.7. *Assume condition $(I_s^*)$, and suppose $k$ is an algebraically closed field of* char. $p > 0$. *Then*

$$\dim_k H^i(\hat{X}, O_{\hat{X}}) = \dim_{\mathbf{Z}/p\mathbf{Z}} H^i_{\text{ét}}(Y, \mathbf{Z}/p\mathbf{Z}) \text{ for } i < s.$$

COROLLARY 2.8. *Let $Y$ be a Cohen-Macaulay subvariety, of pure dimension $s$, of $X = \mathbf{P}_k^n$, where $k$ is an algebraically closed field of* char. $p > 0$. *Then the natural maps*

$$H_{\text{ét}}^i(X, \mathbf{Z}/p\mathbf{Z}) \to H_{\text{ét}}^i(Y, \mathbf{Z}/p\mathbf{Z})$$

*are isomorphisms for $i \leqq 2s-n$.*

Now using duality, we obtain results about the coherent sheaf cohomology of $U = X - Y$.

PROPOSITION 2.9 (*Duality*). (*See* [4, III 3.4]) *Assume condition* $(I_s^*)$. *Then $H^i(U, F)$ is finite-dimensional for all $i \geqq n-s$. Furthermore, for any integer $r \leqq s$, the following conditions are equivalent*:

(i) $H^i(U, F) = 0$ *for all $i \geqq n-r$ and all coherent sheaves $F$ on $U$.*

(ii) $H^i(U, \omega) = 0$ *for all $i \geqq n-r$, where $\omega = \Omega_{X/k}^n$.*

(iii) *The natural maps*

$$H^i(X, O_X) \to H^i(\hat{X}, O_{\hat{X}})$$

*are isomorphisms, for $i < r$.*

Combining with the main theorem above, we have the following results.

THEOREM 2.10. *Let $Y$ be a non-singular variety of pure dimension $s$ in $X = \mathbf{P}_{\mathbf{C}}^n$. Let $U = X - Y$. Then $H^i(U, F)$ is finite-dimensional for all $i \geqq n-s$ and all coherent sheaves $F$ on $U$. For any integer $r \leqq s$, the following conditions are equivalent*:

(i) $H^i(U, F) = 0$ *for all $i \geqq n-r$ and all coherent $F$*

(ii) *The natural maps*

$$H^i(X, \mathbf{C}) \to H^i(Y, \mathbf{C})$$

*are isomorphisms for $i < r$.*

*Finally, both* (i) *and* (ii) *hold for $r = 2s-n+1$.*

THEOREM 2.11. (*See* [4, III 6.8]) *Let $Y$ be a Cohen-Macaulay subvariety of pure dimension $s$ in $X = \mathbf{P}_k^n$, where $k$ is an algebraically closed field of* char. $p > 0$. *Let $U = X - Y$. Then $H^i(U, F)$ is finite-dimensional for all $i \geqq n-s$ and all coherent $F$. For any integer $r \leqq s$, the following conditions are equivalent.*

(i) $H^i(U, F) = 0$ *for all $i \geqq n-r$ and all coherent $F$.*

(ii) *The natural maps*

$$H_{\text{ét}}^i(X, \mathbf{Z}/p\mathbf{Z}) \to H_{\text{ét}}^i(Y, \mathbf{Z}/p\mathbf{Z})$$

*are isomorphisms for $i < r$.*

*Finally, both* (i) *and* (ii) *hold for $r = 2s-n+1$.*

In the analytic case, the same technique allows us to deduce corresponding analytic results (first obtained by Barth) from the condition

$(I^*_{s,an})$. In the statements, we replace $H^i(\hat{X}, \hat{F})$ by $H^i_{an}(Y, \mathfrak{F}|_Y)$, for a coherent analytic sheaf $\mathfrak{F}$ on $X$. Thus we recover the following

THEOREM 2.12. (*Barth* [1]). *Let $Y$ be a non-singular subvariety of pure dimension $s$ of $X = \boldsymbol{P}^n_{\boldsymbol{C}}$. Let $U = X - Y$. Then $H^i_{an}(U, \mathfrak{F}|_U)$ is finite-dimensional for all $i \geqq n-s$ and all coherent analytic sheaves $\mathfrak{F}$ on $X$. For any $r \geqq s$, the following conditions are equivalent*:

(i) *$H^i_{an}(U, \mathfrak{F}|_U) = 0$ for all $i \geqq n-r$ and all coherent analytic sheaves $\mathfrak{F}$ on $X$.*

(ii) *The natural maps*

$$H^i(X, \boldsymbol{C}) \to H^i(Y, \boldsymbol{C})$$

*are isomorphisms for $i < r$.*

*Finally, both* (i) *and* (ii) *hold for $r = 2s-n+1$.*

## 3. Comparison of algebraic and analytic cohomology

Let $X$ be a scheme of finite type over $\boldsymbol{C}$. Then one can define the associated complex analytic space $X^h$ (see Serre [7]). If $F$ is a coherent sheaf on $X$, one can define the associated coherent analytic sheaf $F^h$ on $X^h$. For each $i$, there is a natural map of cohomology groups

$$\alpha_i : H^i(X, F) \to H^i(X^h, F^h).$$

Serre [7] has shown that if $X$ is a projective scheme, then the $\alpha_i$ are all isomorphisms. Furthermore, the functor $F \to F^h$ induces an equivalence of the category of coherent sheaves on $X$ with the category of coherent analytic sheaves on $X^h$.

In this section, we ask to what extent do these results carry over to non-complete varieties? We will see that the analogous results hold in certain cases when the cohomology groups in question are all finite-dimensional.

THEOREM 3.1 [4, VI 2.1]. *Let $X$ be a complete non-singular variety of dimension $n$ over $\boldsymbol{C}$. Let $Y$ be a closed subvariety (maybe singular) of dimension $s$. Let $U = X - Y$. Then the natural maps*

$$\alpha_i : H^i(U, F) \to H^i(U^h, F^h)$$

*are isomorphisms for all $i < n-s-1$, and for all locally free sheaves $F$ on $U$. The cohomology groups in question are all finite-dimensional. Furthermore, if $s \leqq n-2$, the functor $F \to F^h$ is fully faithful on the category of locally free sheaves on $U$. If $s \leqq n-3$, the functor $F \to F^h$ induces an equivalence of the category of locally free sheaves on $U$ with the category of locally free analytic sheaves on $U^h$.*

The finite-dimensionality follows from Grothendieck [2] in the algebraic case, and from Trautmann [10] in the analytic case. The comparison of cohomology and locally free sheaves follows from results of Siu [8].

THEOREM 3.2. *Let $X = \boldsymbol{P}^n_{\boldsymbol{C}}$, let $Y$ be a closed subvariety, let $U = X - Y$, and let $s$ be an integer. Assume conditions $(I_s^*)$ and $(I_{s,an}^*)$ of the previous section. Then the natural maps*

$$\alpha_i : H^i(U, F) \to H^i(U^h, F^h)$$

*are isomorphisms for all $i > n-s$, surjective for $i = n-s$, for all coherent sheaves $F^0$ on $U$. (We have seen above that these groups are finite-dimensional.) In particular, this result holds if $Y$ is non-singular and $s = \dim Y$.*

A slightly weaker version of this theorem is proved in [4, VI 2.2].

EXAMPLE [4, VI 3.2]. To emphasize that such comparison theorems do not always hold, there is an example of Serre of a complete non-singular surface $X$ over $\boldsymbol{C}$, and an irreducible non-singular curve $Y$ on $X$, such that $U = X - Y$ is not affine, but $U^h$ is Stein. Thus there are coherent sheaves $F$ on $U$ with $H^1(U, F) \neq 0$ but $H^1(U^h, F^h) = 0$.

## REFERENCES

W. BARTH

[1] Transplanting cohomology classes in complex-projective space. Amer. J. Math. 92 (1970), 951–967.

A. GROTHENDIECK

[2] Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). North-Holland (1968).

R. HARTSHORNE

[3] Cohomological dimension of algebraic varieties. Annals of Math. 88 (1968), 403–450.

R. HARTSHORNE

[4] Ample subvarieties of algebraic varieties. Springer Lecture Notes 156 (1970).

R. HARTSHORNE

[5] Cohomological dimension of algebraic varieties, II. (In preparation).

H. HIRONAKA and H. MATSUMURA

[6] Formal functions and formal embeddings. J. Math. Soc. Japan 20 (1968), 52–82.

J.-P. SERRE

[7] Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier 6 (1956), 1–42.

Y.-T. SIU

[8] Analytic sheaves of local cohomology. Bull. Amer. Math. Soc. 75 (1969), 1011–1012.

R. SPEISER

[9] Cohomological dimension of abelian varieties, thesis, Cornell University (1970).

G. Trautmann
[10] Ein Endlichkeitsatz in der analytischen Geometrie. Invent. math. 8 (1969), 143–174.

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