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Grothendieck universes and the super-complete models of Shepherdson

by

A. H. Kruse

In this note we are working in Mostowski set theory (axiomatized in [3]) in which we assume neither the axiom of foundation nor its negation, in which we assume the axiom of choice for sets (thus each set is well-ordered by some relation), and in which the existence of Urelemente is not ruled out. For each set X, $P(X)$ is the set of all subsets of X. For all elements x and y, $[x, y] = \{\{x\}, \{x, y\}\}$. A function is a class of pairs satisfying the usual conditions. The indexed family notation "$\{x_i\}_{i \in I}$" is used for a function with domain I and with value x_i at each $i \in I$, whereas $\{x_i : i \in I\}$ is the image of this function. Ordinals are taken in the sense of von Neumann. A cardinal is an ordinal not equipotent with any of its members. A cardinal α is *inaccessible* if and only if $2^\beta < \alpha$ for each cardinal $\beta < \alpha$ and card $(\cup\{X_i : i \in I\}) < \alpha$ for each family $\{X_i\}_{i \in I}$ of sets with card $(I) < \alpha$ and with card $(X_i) < \alpha$ for each $i \in I$. Thus $\aleph_0$ is the first inaccessible cardinal. The existence of an inaccessible cardinal $> \aleph_0$ cannot be proved in the set theory in which we are working if this set theory is consistent (cf. [4, Part II]), and it is suspected with conviction that the consistency of this set theory is not impaired by adding as an axiom that for each cardinal number α there is a cardinal number $> \alpha$ which is inaccessible.

The primitive notions of the set theory used herein may be taken to be represented by atomic statements of the form "α is an element", "α is a class", "α is a member of β", and "$\alpha = \beta$". An *Urelement* is an element which is not a class. A class is *small* or *big* according as it is or is not an element. A *set* is a small class. An *object* is any element or class. The set theory used herein is based on finitely many axioms each of which may and will be taken as a formula built up from primitive statements by use of the usual logical connectives of the first order predicate calculus, including quantifications, and in which each variable appearing is bound. We may so write each axiom that all quantifications are

taken over all objects. One of the axioms is the axiom of infinity, which asserts the existence of an infinite set. We shall let Σ^- be the conjunction of the axioms other than the axiom of infinity. We shall let Σ be the conjunction of Σ^- and the axiom of infinity.

Consider any class U. A *U-element* is any member of U. A *U-class* is any subclass of U. A *U-object* is any U-element or U-class. The *U-transform* of Σ^- (resp., Σ) is Σ_U^-(resp., Σ_U) obtained by changing "element" and "class" to "U-element" and "U-class" respectively and changing each quantification "$(\exists\alpha)(\ldots)$" to "$(\exists\alpha)($(α is a U-object) &$(\ldots))$" and each quantification "$(\forall\alpha)(\ldots)$.. to "$(\forall\alpha)$ $((\alpha$ is a U-object) $\Rightarrow (\ldots))$".

A *Grothendieck universe* (cf. [1, p. 328]) is any nonvoid set U such that (i)—(v) below hold (we depart from [1] in adding the word *nonvoid*).

(i) For each family $\{X_i\}_{i\in I}$ such that $I \in U$ and such that X_i is a set and $X_i \in U$ for each $i \in I$, $\cup \{X_i : i \in I\} \in U$.

(ii) For each $x \in U$, $\{x\} \in U$.

(iii) For each set $X \in U$, $X \subset U$.

(iv) For each set $X \in U$, $P(X) \in U$.

(v) For all elements x and y, $[x, y] \in U$ if and only if $x \in U$ and $y \in U$.

Since we are not taking *ordered pair* as primitive and have used the usual definition for the notion, we may show as follows that (v) follows from (i)—(iv). If (i), (ii), and (iv) hold and $x, y \in U$, then $P(\{x\}) \in U$ by (ii) and (iv) and has cardinal 2, whence $\{x, y\} = \{x\} \cup \{y\} \in U$ by (ii) and (i) and $[x, y] = \{\{x\}\} \cup \{\{x, y\}\}$ $\in U$ by (ii) and (i). If (iii) holds and $[x, y] \in U$, then $x, y \in \{x, y\} \in [x, y] \in U$, whence $x, y \in U$ by two applications of (iii).

Adapting Shepherdson's definition of *super-complete model* in [4, Part I] to the set theory assumed herein and to the purpose of this note, we define a *super-complete model* to be any class U such that $X \subset U$ for each set $X \in U$ and such that Σ_U holds. Furthermore, we define a *super-complete near-model* to be any class U such that $X \subset U$ for each set $X \in U$ and such that Σ_U^- holds. It is easy to show that for each set U, U is a super-complete model if and only if U is a super-complete near-model and some member of U is an infinite set (use the methods of [4, Part I] to show the absoluteness of *infinite*).

The main purpose of this paper is to point out that the notions of *Grothendieck universe* and *small super-complete near-model* are equivalent.

THEOREM 1. *For each set U,* (a)—(c) *below are mutually equivalent.*

(a) *U is a Grothendieck universe.*

(b) *There is an inaccessible cardinal number α such that for each set X, $X \in U$ if and only if $X \subset U$ and card $(X) < \alpha$.*

(c) *U is a super-complete near-model.*

Outline of proof. Consider any set U.

Suppose (a). Let α be the smallest cardinal number β such that card $(X) < \beta$ for each set $X \in U$. It may be proved from (i) and (iv) that α is inaccessible. For each set X, if $X \in U$, then $X \subset U$ by (iii). For each nonvoid set $X \subset U$, if card $(X) < \alpha$, then there is a set I and a family $\{x_i\}_{i \in I}$ such that card $(X) \leqq$ card (I), $I \in U$, $X = \{x_i : i \in I\}$, and hence $X = \cup\{\{x_i\} : i \in I\} \in U$ by (ii) and (i). Finally, the void set is a member of U by (iv). Thus (b) holds.

If (b) holds, then (c) holds by a straightforward check of the axioms of set theory.

Suppose (c). The concepts appearing in the definition of *Grotendieck universe* may be shown to be absolute by the methods of [4, Part I]. Then (a) follows easily.

REMARK 1. *If* (b) *of Theorem* 1 *holds, then α is uniquely determined by U in* (b).

For each set X, define $\bigcup^0 X = X$, $\bigcup^{n+1} X = \bigcup^n X$ for each $n \in \omega$ (thus $\bigcup^1 X = \cup X = \{y : y \in x$ for some $x \in X\}$), and $\bigcup^\omega X = \cup\{\bigcup^n X : n \in \omega\}$. For each set X and each cardinal γ, let $P_\gamma(X)$ be the set of all $Z \subset X$ with card $(Z) < \gamma$. For each set X and each cardinal γ, let

$$S^\nu_\gamma(X) = X \cup P_\gamma(\bigcup\{S^\mu_\gamma(X) : \mu \in \nu\})$$

for each ordinal ν (thus $S^0_\gamma(X) = X \cup P_\gamma(0)$).

THEOREM 2. *Consider any set A and any inaccessible cardinal γ, and suppose that* card $(Z) < \gamma$ *for each $Z \in \bigcup^\omega A$. Let $U = \cup\{S^\mu_\gamma(\bigcup^\omega A) : \mu \in \gamma\}$. Then U is a Grothendieck universe, and $A \subset U$. If also γ is the smallest inaccessible cardinal δ such that* card $(Z) < \delta$ *for each $Z \in \bigcup^\omega A$, then U is the smallest Grothendieck universe V such that $A \subset V$.*

Outline of proof. It is easily seen that (b) of Theorem 1 holds (take $\alpha = \gamma$). Hence U is a Grothendieck universe by Theorem 1. Now suppose that γ is the smallest inaccessible cardinal δ such that card $(Z) < \delta$ for each $Z \in \bigcup^\omega A$. Consider any Grothendieck universe V, and suppose that $A \subset V$. Let β be given for V as α is given for U in (b) of Theorem 1. Then $\gamma \leqq \beta$. Suppose that

$U \not\subset V$. Let λ be the smallest $\mu \in \gamma$ such that $S^{\mu}_{\gamma}(\bigcup^{\omega} A) \not\subset V$. One obtains a contradiction easily after showing without difficulty that $\bigcup^{\omega} A \subset V$. Theorem 2 follows.

For each set X and each ordinal γ, let $P^{0}_{\gamma}(X) = X$, $P^{\mu+1}_{\gamma}(X) = P_{\gamma}(P^{\mu}_{\gamma}(X))$ for each ordinal number μ, and $P^{\nu}_{\gamma}(X) = \cup \{P^{\mu}(X): \mu \in \nu\}$ for each ordinal $\nu > 0$ without a maximum member.

REMARK 2. *In Theorem* 2, *if* $\aleph_0 < \gamma$, *then* $U = P^{\gamma}_{\gamma}(\bigcup^{\omega} A)$.

The writer is under the impression that Corollaries 1 and 2 below are well-known to Grothendieck and others.

COROLLARY 1. *For each set* X, $X \subset U$ *for some Grothendieck universe* U *if and only if there is an inaccessible cardinal number* α *such that* $\operatorname{card}(Z) < \alpha$ *for each set* $Z \in \bigcup^{\omega} X$.

COROLLARY 2. (A) *and* (B) *below are equivalent.*

(A) *For each cardinal number* α, *there is an inaccessible cardinal number* β *such that* $\alpha < \beta$.

(B) *For each set* X, *there is a Grothendieck universe* U *such that* $X \subset U$.

REMARK 3. *Suppose that the set theory assumed in this paper is consistent. Insofar as* (A) *of Corollary* 2 *is then unproveable, so is* (B).

In [2], a *subuniverse* is any class U such that for each set X, $X \in U$ if and only if $X \subset U$. An analogue of Theorem 2 (cf. Remark 2) for subuniverses may be found in [2]. It follows from observations in [2, § 8] that each subuniverse is a super-complete inner model. Using Theorem 1 we obtain the following two theorems.

THEOREM 3. *For each class* U, (a)—(c) *below are mutually equivalent.*

(a) U *is a Grothendieck universe or a subuniverse.*

(b) *There is an inaccessible cardinal* α (*which may be the big class of all ordinal numbers*) *such that for each set* X, $X \in U$ *if and only if* $X \subset U$ *and* $\operatorname{card}(X) < \alpha$.

(c) U *is a super-complete near-model.*

THEOREM 4. *For each class* U, (a)—(c) *below are mutually equivalent.*

(a) U *is either a Grothendieck universe with an infinite set as a member or a subuniverse.*

(b) *There is an inaccessible cardinal* $\alpha > \aleph_0$ (*which may be the big class of all ordinal numbers*) *such that for each set* X, $X \in U$ *if and only if* $X \subset U$ *and* $\operatorname{card}(X) < \alpha$.

(c) *U is a super-complete model.*

Consider the following three statements.

(I) For each nonvoid class C of sets, $X \cap C$ is void for some $X \in C$.

(II) For each set X there are a set A of Urelemente and an ordinal number μ such that $X \subset S^{\mu}_{\mu}(A)$.

(III) For each set X there are a set A of Urelemente and an ordinal number μ such that $X \subset P^{\mu}_{\mu}(A)$.

It is well-known that (I)-(III) are mutually equivalent; (I) is the axiom of foundation, and each of (II) and (III) says that each set is built ultimately from Urelemente, i.e., has a foundation of Urelemente.

THEOREM 5. *Suppose* (I). *Suppose that U is a set. Then U is a Grothendieck universe (i.e., a small super-complete near-model) if and only if $U = \cup \{S^{\mu}_{\alpha}(A) : \mu \in \alpha\}$ for some set A of Urelemente and some inaccessible cardinal α.*

PROOF. The "if" part follows from Theorem 2 since $\bigcup^{\omega} A = A$ for each set A of Urelemente. Now suppose that U is a Grotendieck universe. Let A be the set of all Urelemente $\in U$. Let α be given by (b) of Theorem 1. Let $T(Z) = \cup \{S^{\mu}_{\alpha}(Z) : \mu \in \alpha\}$ for each set Z. Then $T(A) \subset T(U) = U$ by Theorem 2 (since $\bigcup^{\omega} U = U$ by Theorem 2). Suppose $U \neq T(A)$. By (I) there is $X \in U \backslash T(A)$ such that $X \cap (U \backslash T(A))$ is void; let X be so given. Then $X \subset T(A)$ and card $(X) < \alpha$. Since also α is inaccessible, $X \subset S^{\mu}_{\alpha}(A)$ for some $\mu \in \alpha$, and $X \in T(A)$, contradiction. Thus Theorem 5 is proved.

For each class C and each (possibly big) cardinal α, let $T_{\alpha}(C) = \cup \{S^{\mu}_{\alpha}(A)$: A is a subset of C, and $\mu \in \alpha\}$.

The proof of the following theorem will be omitted.

THEOREM 6. *Suppose* (I). *Suppose that U is a class. Then U is a super-complete near-model [resp., super-complete model] if and only if there are a class C of Urelemente and a (possibly big) inaccessible cardinal α [resp., inaccessible cardinal $\alpha > \aleph_0$] such that $U = T_{\alpha}(C)$.*

REMARK 4. *The case of Theorem 6 for super-complete models in which the class of all Urelemente is void reduces to Shepherdson's characterization of super-complete models in* [4, *Part* II].

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