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Remarks on equidistribution on non-compact groups *

by

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The notion of equidistributed sequence of points in a compact group G was introduced by Eckmann [2]. We quote this definition denoting by μ the normed Haar measure in G:

A sequence $\{\alpha_n\}$ is equidistributed if for every Borel set $E \subset G$ whose boundary is of μ-measure 0

$$\lim_{N\to\infty} \frac{A(N)}{N} = \mu(E) \tag{1}$$

$A(N)$ being the number of $\alpha_i \in E$ with $i \leqq N$.

It can be easily proved by means of the individual ergodic theorem and by the $0-1$ law that almost every point of the cartesian product $G^{\aleph_0}$ represents a sequence which is equidistributed in G. It was also proved [3] that the multiples ng ($n = 1, 2, \dots$) of almost every element of a connected separable compact *abelian* group form an equidistributed sequence. A sufficient condition for a sequence $\{ng\}$ in a compact group to be equidistributed is its density and $\{ng\}$ is dense if and only if $x(g) \neq 1$ for every character x of G which is not identically 1. For a (possibly non-separable) connected compact abelian group whose topological character (weight) does not exceed $\tau = 2^{\aleph_0}$ it can still be stated that there is a $g \in G$ for which $\{ng\}$ is dense and thus equidistributed [5].

Cigler has defined equidistributed sequences of measures on the real line D [1]. We restrict ourselves to equidistributed sequences of points (i.e. to the case of point measures) but instead of D we admit an arbitrary locally compact abelian topological group G. Then $\{\alpha_n\}$ is called equidistributed if for every continuous character $x \not\equiv 1$ of G

$$\frac{x(\alpha_1) + \dots + x(\alpha_n)}{n} \to 0. \tag{2}$$

* Nijenrode lecture.

Hence, if G is compact, we obtain by approximation theorem

$$\frac{f(\alpha_1)+\ldots+f(\alpha_n)}{n} \to \int f(t)d\mu(t) \tag{3}$$

for every continuous f, which yields (1) by a routine argument.

If G is non-compact, then (2) does not imply (3) for all continuous functions but it does so for every (uniformly) almost periodic (a.p.) function. Yet we cannot immediately see how a condition analogous to (1) could be derived from (3) in this case. So the above definition of equidistribution does not refer to any sets which could be "measured" by means of an equidistributed sequence, more exactly, by means of the relative number of terms belonging to them. Our aim is now to find an analogon of (1) and so to establish a closer relation between the Cigler definition and the classical concept of equidistribution.

For this purpose we use the notion of an R-almost periodic function. So we call a Haar measurable real function f if for every $\varepsilon > 0$ there are two (uniformly) a.p. functions φ and ψ so that

$$\varphi(t) \leqq f(t) \leqq \psi(t) \text{ for all } t \in G$$

and the mean value $M(\psi-\varphi) < \varepsilon$.

In [4] I considered R-a.p. functions only on D. Since in this case we dispose of the classical expression $\lim_{T\to\infty} 1/2T \int_{-T}^{T} f(t)dt$ for the mean value, the well-known space of Besicovitch a.p. functions (B-space) can be introduced as the closure of the set of ordinary a.p. functions with respect to the B-norm $||f||_\beta = M(|f|)$ and there is a $1-1$ isometric linear map of the B-space onto the space $L(K)$ of integrable functions on the Bohr compactification K of D with the normed Haar measure ν. As was proved in [4], a B-equivalence class (i.e. a point of B-space) contains a R-a.p. function if and only if the corresponding ν-equivalence class (i.e. a point of $L(K)$) contains a Riemann integrable (i.e. ν-a.e. continuous) function. These results can be carried over to some other locally compact abelian groups in which the mean value can be expressed analogously as on D, i.e. by means of $\lim_n 1/\mu(Q_n) \int_{Q_n} f(t)d\mu(t)$, μ being the Haar measure in G and $\{Q_n\}$ a fixed increasing sequence of compact sets. This is possible especially for connected groups owing to their well-known structure; some conditions which are sufficient for the sequence $\{Q_n\}$ to be applicable for the mean value integral formula were established by Struble [8]. We do not insist on this point and we emphasize rather the possibility of interpretation of R-a.p. function by means of Bohr compactification regardless whether Besicovitch a.p. functions are defined on G or not. This can be done in the following way:

(a) if f is R-a.p. on G, φ_n and ψ_n are uniformly a.p., $\varphi_n \leqq \varphi_{n+1} \leqq f$, $\psi_n \geqq \psi_{n+1} \geqq f$ $(n = 1, 2, \ldots)$ and $M(\psi_n - \varphi_n) \to 0$, then the extended continuous functions φ_n^* and ψ_n^* on the Bohr compactification K of G form two monotonic sequences and we define f^* as their common almost everywhere continuous limit function in $L(K)$. Of course, f^* is not uniquely determined. For our purposes we choose $f^*(x) = f(x)$ if $x \in G$ and $f^*(x) = \lim_n \psi_n^*(x)$ otherwise ($\lim_n \varphi_n^*$ would do the same). Thus f^* is an extension of f over K.

Conversely, arguing as in [4], we can show that

(b) if $f(x)$ is a Riemann integrable function on K, then the restricted function f/G on G is R-a.p. if it is μ-measurable. If not, we can replace $f(x)$ by $\sup f(x)$, i.e. by $\inf_U \sup_{y \in U} f(y)$ where U runs over a complete system of neighbourhoods of x. Since $\sup f(x)$ is upper semi-continuous, so must be also the restricted function $\sup f/G$; it is therefore μ-measurable.

A set in G is called a *R-a.p-set* if its characteristic function is R-a.p. It is obvious that if the sequence $\{\alpha_n\}$ is equidistributed and if E is a R-a.p. set, then (1) holds, the right-hand side being replaced by the mean value of the characteristic function of E. If $G = D$, then this mean value is the so-called "relative measure" of E. On account of the rôle the R-a.p. sets play in the theory of equidistribution we shall establish some of their properties.

It follows from (a) that every R-a.p. set can be extended to a Jordan measurable set E^* in K, i.e. to a set whose boundary has ν-measure 0, in such a way that $E^* \cap G = E$. In fact, if we suppose f in (a) to be a $0-1$ function, then the Riemann integrable function f^* can be so modified (if necessary) as to become a $0-1$ Riemannian extension f^{**} of f, ν-equivalent to f^*. We can put for this purpose $f^{**}(x) = 0$ whenever $f^*(x) \leqq \frac{1}{2}$ and $f^{**}(x) = 1$ in the opposite case. But then the set $E^* = \{x : f^{**}(x) = 1\}$ is the required Jordan extension of the set $E = \{t : f(t) = 1\} \subset G$. Conversely, it follows from (b) that every Jordan set E in K, intersected with G, gives raise to a R-a.p. set if this intersection is μ-measurable. If not, then taking the closure $\bar{E}$ in K instead of E itself we obtain a R-a.p. set by intersection with G.

THEOREM 1. The R-a.p. sets form a (finitely additive) field.

This follows by previous remarks from the fact that Jordan sets in K form a field.

Two R-a.p. sets are called equivalent if their symmetric difference is a (R-a.p.) set whose characteristic function has mean value 0.

THEOREM 2. Every R-a.p. set E is equivalent to a closed and to an open set in G. If G is non-compact, then a μ-measurable change of E within a compact set leads to an equivalent set.

To prove the first part it is sufficient to extend E to a Jordan set E^* in K and then to take $\overline{E^*} \cap G$ or $\text{Int}(E^*) \cap G$ respectively. The equivalence follows from the obvious remark that a borelian set in K of Jordan measure 0, intersected with G, becomes a R-a.p. zero set (i.e. equivalent to the void set). To prove the second part we use the same remark and the fact that a compact set Z in G is of Jordan measure 0 in K. To see this it is enough to observe that Z is of ν-measure 0; if this were not true, then there would be no sequence of disjoint translations of Z. This is, however, impossible because in view of the non-compactness of G a compact set $Z \subset G$ always admits a translation disjoint with Z.

THEOREM 3. If the weight of a locally compact connected abelian group does not exceed $2^{\aleph_0}$, then the group contains an equidistributed sequence.

PROOF. If G satisfies the assumption, then it is a direct sum of 1-dimensional vector groups D_i ($i = 1, \ldots, r$) and of a connected compact group C whose weight is at most $2^{\aleph_0}$. There is an equidistributed sequence in each D_i [1] and there is one in C [5]. Further we need the following notion: a sequence $\alpha_1, \alpha_2, \ldots$ in a compact group Γ is called *distributed with respect to a Borel measure* m, if for every continuous function

$$\frac{f(\alpha_1)+ \ldots +f(\alpha_n)}{n} \to \int_{\Gamma} f(t)dm(t)$$

(thus it is *equi*distributed, if m is the Haar measure).

Helmberg has proved [7] that if a compact group contains a sequence $\{x_n\}$ distributed with respect to m_1 and a sequence $\{y_n\}$ distributed with respect to m_2, then it equally contains a sequence which is distributed with respect to the convolution measure $m_1 \circ m_2$ and consists of elements $x_i y_j$. Let us observe that this theorem can also be deduced from the fact that convolution of two measures is a weakly continuous operation [1]. Let K_i be the Bohr compactification of D_i. The Haar measure in the direct sum $\sum_{i=1}^{r} K_i + C$ is the product measure, and so the convolution, of Haar measures in K_i and C respectively. Any sequence which is equidistributed in D_i is equidistributed in K_i – this follows from

[1] This remark is due to K. Urbanik.

the definition of equidistribution and from the properties of the Bohr compactification. Then, according to Helmberg's theorem, there exists a sequence $\{\alpha_n\}$ equidistributed in $\sum_{i=1}^{r} K_i + C$ and consisting of elements of $\sum_{i=1}^{r} D_i + C = G$. But since $\sum_{i=1}^{r} K_i + C$ is the Bohr compactification of G, $\{\alpha_n\}$ is equidistributed in G.

A set of a.p. functions on an abelian locally compact group G will be called a *countable modulus* if it consists exactly of functions which can be approximated by linear aggregates of characters from a given countable subgroup of the dual $\hat{G}$.

A point sequence $\{\alpha_n\}$ in G is called relatively equidistributed with respect to the countable modulus S if

$$\frac{f(\alpha_1) + \ldots + f(\alpha_n)}{n} \to M(f)$$

holds for every $f \in S$. An appropriate definition of R-a.p. functions and of R-a.p. sets with respect to S is obvious. All R-a.p. sets of such kind form a field and they can be "measured" by the sequence $\{\alpha_n\}$.

THEOREM 4. Every discrete abelian group H contains a relatively equidistributed sequence with respect to any given countable modulus S.

PROOF. We consider the Bohr compactification of H relatively to S, i.e. such compact group K_S that all functions of S and only these can be uniquely and continuously extended over K_S.[2] It follows from the countability of the modulus S that K_S is a separable group. It thus contains an equidistributed sequence $\{\alpha_n\}$. The terms of this sequence are not necessarily in H but since K_S is metric and H is dense in K_S we can choose points β_n in H so as to have $\rho(\beta_n, \alpha_n) \to 0$ ρ denoting the distance. Obviously, the sequence $\{\beta_n\}$ is also equidistributed in K_S and so it is relatively equidistributed in H with respect to S.

Theorem 4 can be applied e.g. to the additive group of rationals if S is generated by any countable set of characters. It is well-known that characters of this group need not be continuous in the ordinary topology although they can all be effectively computed.

Another situation appears when the group of reals is considered. Then no discontinuous character can be effectively constructed, yet we obtain from Theorem 4 the following

[2] If S does not separate points of H, then there is only a homomorphic and not an isomorphic imbedding of H into K_S.

COROLLARY. If $h_1(t)$, $h_2(t)$, ... are Hamel additive functions then there exists a sequence of real numbers $\alpha_1, \alpha_2, \ldots$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e^{ih_k(\alpha_j)} = 0 \qquad (k = 1, 2, \ldots).$$

THEOREM 5. If the discrete abelian group H is divisible and f_1, $f_2, \ldots, f_n$ are a.p. functions then for every $\varepsilon > 0$ there is an $\alpha \in H$ and an integer $N > 0$ such that

$$\left|\frac{f_i(\alpha)+f_i(2\alpha)+\ldots+f_i(N\alpha)}{N} - M(f_i)\right| < \varepsilon \qquad (i = 1, \ldots, n).$$

PROOF. If S is a countable modulus containing $f_1, \ldots, f_n$ then K_S is divisible and therefore connected (see [6] e.g.). Hence it contains an equidistributed sequence of the form $\{mx\}$ ($x \in K_S$; $m = 1, 2, \ldots$). Taking N sufficiently large and an $\alpha \in H$ sufficiently near to x we obtain the assertion.

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