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# Differentiable mappings in the Schoenflies theorem

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# Differentiable mappings in the Schoenflies theorem

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## § 0. Introduction

The recent remarkable contributions of Mazur of Princeton University to the topological theory of the Schoenflies Theorem lead naturally to questions of fundamental importance in the theory of differentiable mappings. In particular, if the hypothesis of the Schoenflies Theorem is stated in terms of regular differentiable mappings, can the conclusion be stated in terms of regular differentiable mappings of the same class? This paper gives an answer to this question.

Further reference to Mazur's method will be made later when the appropriate technical language is available. Ref. 1.

An abstract $r$-manifold $\Sigma_r$, $r > 1$, of class $C^m$ is understood in the usual sense except that we do not require that $\Sigma_r$ be connected. Refs. 2 and 3. For the sake of notation we shall review the definition of $\Sigma_r$ and of the *determination* of a $C^m$-structure on $\Sigma_r$, $m > 0$.

We suppose that $\Sigma_r$ is a topological space which satisfies the Hausdorff condition and which is an $r$-manifold in the sense that each point $p \in \Sigma_r$ has a neighborhood which is the homeomorph of a euclidean $r$-disc.

We consider local representations $F$ of $\Sigma_r$ of the form

$$F : U \to X, \tag{0.0}$$

in which $U$ is an *open* subset of a euclidean $r$-space, and $X$ an open subset of $\Sigma_r$, homeomorphic to $U$ under $F$. We term $X$ a *coordinate domain* on $\Sigma_r$ and $U$ a corresponding *coordinate range.* The euclidean coordinates $(u_1, \ldots, u_r)$ of a point $(u) \in U$ are called *local coordinates* of the corresponding point $F(u) \in X$.

Let $U$ and $V$ be open subsets of euclidean $r$-spaces. A homeomorphism of $U$ onto $V$ will be said to be a $C^m$-diffeomorphism of $U$ onto $V$ if $f$ has the form

$$v_i = f_i(u_1, \ldots, u_r) = f_i(u), \qquad (i = 1, \ldots, r) \tag{0.1}$$

where $(u)$ is a point in $U$, and $(f(u))$ is the image of $(u)$ under $f$, where each $f_i$ is of class $C^m$ over $U$, and where the functional matrix

$$\left\| \frac{\partial f_i}{\partial u_j} \right\| \qquad (i, j = 1, \ldots, r)$$

is of rank $r$ at each point of $U$.

A set $[F]$ of local representations $F$ of $\Sigma_r$ of the form (0.0) will be said to *determine* a $C^m$-structure on $\Sigma_r$ if $[F]$ together with the associated sets $[U]$ and $[X]$ of euclidean coordinate ranges $U$ and coordinate domains $X$ on $\Sigma_r$ satisfy the following two conditions.

I. *Covering Condition.* The sets of $[X]$ shall have $\Sigma_r$ as a union.

II. *$C^m$-compatibility Condition.* If

$$F_1 : U_1 \to X_1 \qquad F_2 : U_2 \to X_2$$

are arbitrary mappings in $[F]$ such that

$$X_1 \cap X_2 = X \neq \emptyset, \tag{0.2}$$

then the mapping $(u) \to (v)$ defined by the condition

$$F_1(u) = F_2(v)$$

for $(u) \in F_1^{-1}(X)$ and $(v) \in F_2^{-1}(X)$ shall be a $C^m$-diffeomorphism

$$f : F_1^{-1}(X) \to F_2^{-1}(X) \tag{0.3}$$

in the sense just defined.

*Local representations of class $C^m$, admissible in $H$.* Given a $C^m$-structure $H$ on $\Sigma_r$ determined by $[F]$, a local representation

$$F_1 : U \to X$$

of $X \subset \Sigma_r$, satisfying the conditions imposed on $F_1$ in (0.0), will be *admitted* in $H$ as a representation of class $C^m$, if $F_1$ and an arbitrary $F_2 \in [F]$ satisfy the above compatibility condition defined for $F_1$ and $F_2$. We do not demand that $F_1$ be in $[F]$. If $[F']$ is a second set of local representations of $\Sigma_r$ such that $[F]$ and $[F']$ both "determine" a $C^m$-structure on $\Sigma_r$, we understand that these structures are the *same* if the ensemble of admissible representations of $\Sigma_r$ of class $C^m$, associated as above with $[F]$, is the ensemble associated with $[F']$.

*$C^m$-diffeomorphisms of $\Sigma_r$ onto $\Sigma_r'$.* Suppose that $\Sigma_r$ and $\Sigma_r'$ are given with $C^m$-structures ($m > 0$). Let $\varphi$ be a homeomorphism of $\Sigma_r$ onto $\Sigma_r'$. We say that $\varphi$ is a $C^m$-diffeomorphism of $\Sigma_r$ onto $\Sigma_r'$ if whenever $F : U \to X$ is an admissible local representation of $\Sigma_r$ of class $C^m$ then

$$\varphi F : U \to \varphi X \tag{0.4}$$

is an admissible local representation of $\Sigma_r'$ of class $C^m$.

*The euclidean n-spaces $E$ and $\mathscr{E}$.* Let $n > 1$ be a fixed integer. Let $E$ and $\mathscr{E}$ be two euclidean $n$-spaces with euclidean coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively. We shall be concerned with an $(n-1)$-sphere $S_{n-1}$ in $E$ of unit radius, and a compact $(n-1)$-manifold $\mathscr{M}_{n-1}$ in $\mathscr{E}$ of class $C^m$ such that $\mathscr{M}_{n-1}$ is the image of $S_{n-1}$ under a $C^m$-diffeomorphism $\varphi$.

We suppose that $S_{n-1}$ and $\mathscr{M}_{n-1}$ have $C^m$-structures *derived* from those of $E$ and $\mathscr{E}$ respectively. That is we suppose that $\mathscr{M}_{n-1}$ is the union of a finite set of coordinate domains each with a representation of class $C^m$ of the form

$$y_i = F_i(u_1, \ldots, u_{n-1}) \qquad (i = 1, \ldots, n) \tag{0.5}$$

where $(y_1, \ldots, y_n)$ are the euclidean coordinates of the point of $\mathscr{M}_{n-1}$ represented and

$$(u_1, \ldots, u_{n-1}) = (y_1, \ldots, \hat{y}_r, \ldots, y_n).$$

Here $\hat{y}_r$ is to be deleted from the set $(y_1, \ldots, y_n)$ corresponding to an integer $r$ which depends on the given coordinate range. The $(n-1)$-sphere $S_{n-1}$ shall have an analytic structure similarly related to the structure of $E$.

*$C_0^m$-diffeomorphisms.* Let $\Sigma_r$ and $\Sigma_r'$ be $r$-manifolds of class $C^m$. Let $P$ and $P'$ be points respectively of $\Sigma_r$ and $\Sigma_r'$. Then $\Sigma_r - P$ and $\Sigma_r' - P'$ are $r$-manifolds to which $C^m$-structures derived from those of $\Sigma_r$ and $\Sigma_r'$ respectively will be assigned. A homeomorphism

$$\Lambda : \Sigma_r \to \Sigma_r'$$

of $\Sigma_r$ onto $\Sigma_r'$ will be termed a $C_0^m$-diffeomorphism if for some $P \in \Sigma_r$ and $P' \in \Sigma_r'$, $\Lambda(P) = P'$ and the restriction of $\Lambda$ to $\Sigma_r - P$ is a $C^m$-diffeomorphism of $\Sigma_r - P$ onto $\Sigma_r' - P'$. We term $P$ the *exceptional point* of $\Lambda$.

Let $JS_{n-1}$ and $J\mathscr{M}_{n-1}$ be respectively the closures of the interiors of $S_{n-1}$ and $\mathscr{M}_{n-1}$ in $E$ and $\mathscr{E}$. Open sets in $E$ and $\mathscr{E}$ are $n$-manifolds to which the differential structures of $E$ and $\mathscr{E}$ respectively will be assigned. When we refer to a $C_0^m$-diffeomorphism of a neighborhood of $JS_{n-1}$ relative to $E$ we shall always mean a $C_0^m$-diffeomorphism in which the exceptional point (if any exists) is on the interior of $S_{n-1}$. With this understood the fundamental theorem of this paper is as follows.

THEOREM 0.1. *Given a $C^m$-diffeomorphism* $(m > 0)$

$$\varphi : S_{n-1} \to \mathscr{M}_{n-1} \tag{0.6}$$

*of $S_{n-1}$ onto $\mathscr{M}_{n-1}$ there exists a $C_0^m$-diffeomorphism $\Lambda_\varphi$ of some open* *neighborhood of* $JS_{n-1}$ *relative to* $E$ *onto some open neighborhood of* $JM_{n-1}$ *relative to* $\mathscr{E}$, *where* $A_\varphi$ *is such that*

$$A_\varphi | S_{n-1} = \varphi. \tag{0.7}$$

There are indications that Theorem 0.1 would be false if $C_0^m$ were replaced by $C^m$ in its statement, at least false for some integers $n$. See ref. 5.

Mazur has established an Embedding Theorem under certain restrictions on $\mathscr{M}_{n-1}$, without affirming any differential structure for the mapping function $A_\varphi$. If $\mathscr{M}_{n-1}$ is an $(n-1)$-manifold of class $C^m$, $m > 0$, and if $\varphi$ is a homeomorphism, Mazur's process is applicable to prove that $A_\varphi$ exists as a homeomorphism. The essence of this paper is that the construction of Mazur can be so modified and extended that when $\varphi$ is a $C^m$-diffeomorphism, a mapping $A_\varphi$ exists which is more than a homeomorphism, which is in fact a $C_0^m$-diffeomorphism satisfying Theorem 0.1.

*Methods.* We begin in § 1 and § 2 by setting up a theory of the composition by partial identification of two $n$-manifolds of class $C^m$ to form a new $n$-manifold $\Sigma$ of class $C^m$. In § 3 we introduce a fundamental theorem on a modification of a $C^m$-diffeomorphism given initially as a $C^m$-diffeomorphism of a neighborhood of the origin in $E$ onto a neighborhood of the origin in $\mathscr{E}$. The modification is a $C^m$-diffeomorphism of $E$ onto $\mathscr{E}$.

In the next five sections we successively introduce three different classes of problems arising out of our initial Schoenflies problem: given $\varphi$ to find $A_\varphi$. We say that a class $(A)$ of problems $A$ is *effectively* mapped into a class $(B)$ of problems $B$ if to each problem $A$ corresponds at least one problem $B$ such that the solution of $B$ implies the solution of $A$. We show that each of our classes of problems (excepting the fourth) can be effectively mapped onto the succeeding class of problems.

(I). The first class of problems is to find $A_\varphi$ satisfying Theorem 0.1, given $\varphi$ as in Theorem 0.1.

(II). The second class of problems is a subset of the first class in which $\varphi$ is a "sense preserving" $C^\infty$-diffeomorphism (§ 4) such that for some neighborhood $N_Q$ of the "$x_n$-pole" $Q$ of $S_{n-1}$

$$\varphi | N_Q = I | N_Q,$$

where $I$ is the mapping $x_i = y_i (i = 1, \ldots, n)$ of $E$ onto $\mathscr{E}$.

(III). In the third class of problems $\varphi$ in (II) is replaced by a $C^\infty$-diffeomorphism $\Phi$ of a neighborhood of $S_{n-1}$ relative to $E$. In some neighborhood of the $x_n$-pole of $S_{n-1}$, relative to $E$, $\Phi$ is is given by $I$.

(IV). Finally we reflect $E$ in a suitable $(n-1)$-sphere to lead to a problem which concerns a $C_0^\infty$-diffeomorphism of a rectangular subregion of an $n$-cube $K$ in $E$ onto a special subregion of an $n$-cube $\mathscr{K}$ in $\mathscr{E}$. Cf. § 7.

We eventually reduce the problem to one involving mappings of class $C^\infty$. Various theorems on the possibility of *$C^m$-extensions* over $E$ of $C^m$-diffeomorphisms given locally are established of general character.

In the last sections we return to the theory of the partial identification of $n$-manifolds as developed in § 3. We thereby solve an arbitrary problem of our fourth class, implying a solution of an arbitrary problem of the first class, that is the existence of a mapping $\Lambda_\varphi$ which satisfies Theorem 0.1.

# PART I. TRANSFORMATIONS OF THE PROBLEM

## § 1. Composite n-manifolds $\Sigma$.

Let $M$ and $\mathscr{M}$ be respectively two $n$-manifolds abstractly given, without points in common. Let $W$ and $\mathscr{W}$ be fixed open subsets of $M$ and $\mathscr{M}$ respectively. We admit the possibility that $M$ or $\mathscr{M}$ may be empty. We presuppose the existence of a homeomorphism

$$\mu : W \to \mathscr{W}. \quad \text{(onto } \mathscr{W}) \tag{1.1}$$

Let each point $p \in W$ be identified with its image $\mu(p)$, to form a "point" which we denote by $[p : \mu(p)]$. Subject to this identification let $\Sigma$ be the ensemble of all points of $M$ and $\mathscr{M}$.

*The #-mappings $\pi$, $\pi_1$, $\pi_2$.* To each point $p \in M$ corresponds a point $\pi(p) \in \Sigma$ represented by $p$ if $p \notin W$ and by $[p : \mu(p)]$ if $p \in W$. To each point $q \in \mathscr{M}$ corresponds a point $\pi(q) \in \Sigma$ represented by $q$ if $q \notin \mathscr{W}$, and by $[\mu^{-1}(q) : q]$ if $q \in \mathscr{W}$. Set

$$\pi_1 = \pi|M \qquad \pi_2 = \pi|\mathscr{M}. \tag{1.2}$$

The mappings

$$\pi_1 : M \to \Sigma$$

$$\pi_2 : \mathscr{M} \to \Sigma$$

are biunique but not in general onto. The mapping $\pi$ is onto $\Sigma$ but not in general biunique. For the purpose of future identification we shall refer to $\pi$, $\pi_1$, $\pi_2$ as the #-mappings *associated* with $\Sigma$.

Observe that

$$\pi(M' \cup \mathscr{M}) = \pi_1(M) \cup \pi_2(\mathscr{M}) = \Sigma \tag{1.3}$$

$$\pi_1(M) \cap \pi_2(\mathscr{M}) = \pi_1(W) = \pi_2(\mathscr{W}) \tag{1.4}$$

and that for $p \in W$ or $q \in \mathscr{W}$

$$(1.5) \qquad \pi_1(p) = \pi_2(\mu(p)) \qquad \pi_2(q) = \pi_1(\mu^{-1}(q))$$

*Notational conventions.* In the future we shall ordinarily set $\pi_2(\mu(p)) = \pi_2 \cdot \mu(p)$; $\pi_1(\mu^{-1}(q)) = \pi_1 \cdot \mu^{-1}(q)$.
We have found such a simplification of notation necessary in more complex cases. For example, if $f_1, f_2, f_3, f_4$ are mappings such that

$$(1.6) \qquad f_1(f_2(f_3(f_4(A))))$$

is well-defined for a set $A$, we shall write (1.6) as

$$(1.7) \qquad f_1 \cdot f_2 \cdot f_3 f_4(A).$$

Thus each . replaces a parenthesis ( ). We admit the notation $f_1 f_2$ for a composite function only when the range of values of $f_2$, or a restiction of $f_2$ indicated by a side condition, is included in the domain of definition of $f_1$. If $G$ is a subset of a topological space $H$, the set theoretic boundary of $G$ relative to $H$, and the closure of $G$ relative to $H$ will be respectively denoted by

$$\beta_H G \quad Cl_H G.$$

*$\Sigma$ topologized.* Let $X$ and $\mathscr{X}$ be arbitrary open subsets of $M$ and $\mathscr{M}$ respectively. Let $\Omega$ be the ensemble of subsets of $\Sigma$ of the form $\pi_1(X)$ or $\pi_2(\mathscr{X})$, together with the intersection of any finite number of these subsets of $\Sigma$. Each union of a collection of sets of $\Omega$ shall be an *open subset* of $\Sigma$, and each open subset of $\Sigma$ shall be of this character. The space $\Sigma$ is thereby topologized. So topologized $\Sigma$ is not in general a Hausdorff space. To remedy this we shall impose Condition ($\alpha$) on $\Sigma$.

In Condition ($\alpha$) and in the proof of Lemma 1.1 a neighborhood of $p \in M$ and of $q \in \mathscr{M}$, relative to $M$ and $\mathscr{M}$ respectively, will be denoted by $N_p$ and $\mathscr{N}_q$.

*Condition* ($\alpha$). *For arbitrary points $p \in \beta_M W$ and $q \in \beta_{\mathscr{M}} \mathscr{W}$ there shall exist neighborhoods $N_p$ and $\mathscr{N}_q$ relative to $M$ and $\mathscr{M}$ respectively, such that*

$$(1.8) \qquad \pi_1(N_p \cap W) \cap \pi_2(\mathscr{N}_q \cap \mathscr{W}) = \emptyset$$

*or equivalently*

$$(1.9) \qquad \mu(N_p \cap W) \cap (\mathscr{N}_q \cap \mathscr{W}) = \emptyset.$$

Since $M$ and $\mathscr{M}$ have no point in common, $p \neq q$. For the same reason

$$(1.10) \qquad N_p \cap \mathscr{N}_q = \emptyset, \; Cl_M W \cap Cl_{\mathscr{M}} \mathscr{W} = \emptyset.$$

LEMMA 1.1. *Under Condition* ($\alpha$), $\Sigma$ *is a Hausdorff space.*

To establish this lemma let $a$ and $b$ be distinct points of $\Sigma$. Let $r$ and $s$ be point antecedents of $a$ and $b$ respectively under $\pi$. Then $r$ and $s$ are in $M \cup \mathscr{M}$. We shall take both $r$ and $s$ in $M$, or in $\mathscr{M}$, if possible. The proof is divided into cases, not in general disjoint, but covering all the possibilities.

CASE I. $r, s \in M$. In this case disjoint neighborhoods $N_r$ and $N_s$ exist in $M$, and give rise to disjoint neighborhoods $\pi_1(N_r)$ and $\pi_1(N_s)$ of $a$ and $b$, respectively, in $\Sigma$.

CASE II. $r, s \in \mathscr{M}$. The proof in this case is similar to that under Case I.

CASE III. $r \in M - W$, $s \in \mathscr{M} - \mathscr{W}$. By proper choice of $r$ and $s$, including an interchange of $r$ and $s$, all possibilities fall under Case I, II, or III. If $r \in M - W$, then either $r \in \beta_M W$ or else $r \in M - \overline{W}$, where $\overline{W} = Cl_M W$. Similarly if $s \in \mathscr{M} - \overline{\mathscr{W}}$, where $\overline{\mathscr{W}} = Cl_{\mathscr{M}} \mathscr{W}$, then either $s \in \beta_{\mathscr{M}} \mathscr{W}$ or else $r \in \mathscr{M} - \overline{\mathscr{W}}$. Thus Case III may be partitioned into the following four subcases.

CASE III (1). $r \in M - \overline{W}$, $s \in \mathscr{M} - \overline{\mathscr{M}}$. In this case there exists a neighborhood $N_r$ which does not meet $W$ and a neighborhood $\mathscr{N}_s$ which does not meet $\mathscr{W}$. No point of $N_r$ is identified under $\mu$ with a point of $\mathscr{N}_s$. Hence $\pi_1(N_r)$ does not meet $\pi_2(\mathscr{N}_s)$.

CASE III (2). $r \in \beta_M W$, $s \in \beta_{\mathscr{M}} \mathscr{W}$. By virtue of Condition ($\alpha$) there exist neighborhoods $N_r$ and $\mathscr{N}_s$ such that

$$\pi_1(N_r \cap W) \cap \pi_2(\mathscr{N}_s \cap \mathscr{W}) = \emptyset.$$

It follows that $\pi_1(N_r)$ does not meet $\pi_2(\mathscr{N}_s)$.

CASE III (3). $r \in \beta_M W$, $s \in \mathscr{M} - \overline{\mathscr{W}}$. In this case there exists an $\mathscr{N}_s$ which does not meet $\mathscr{W}$, so that $\pi_2(\mathscr{N}_s)$ will not meet $\pi_1(N_p)$ whatever the choice of $N_p$ in $M$.

CASE III (4). $r \in M - \overline{W}$, $s \in \beta_{\mathscr{M}} \mathscr{W}$. The proof is as under Case III (3).

The lemma is thereby established.

The reader will note that $\pi_1$ and $\pi_2$ are homeomorphisms into $\Sigma$.

COROLLARY 1.1. *Under Condition* ($\alpha$) $\Sigma$ *is an n-manifold.*

It remains to show that each point $a \in \Sigma$ has a neighborhood relative to $\Sigma$ which is the homeomorph of an $n$-disc. Now $a = \pi(r)$ for some point $r \in M$ (or $\mathscr{M}$), and $r$ has a neighborhood $N_r$(or $\mathscr{N}_r$) which is the homeomorph of an $n$-disc. Hence $\pi_1(N_r)$, or alternately $\pi_2(\mathscr{N}_r)$, is the homeomorph of a disc, and a neighborhood of $a$.

COROLLARY 1.2. *In the special case in which* $M = W$, $\Sigma$ *is an n-manifold, and the mapping*

$$\pi_2 : \mathscr{M} \to \Sigma \tag{1.11}$$

*is a homeomorphism onto* $\Sigma$.

If $M = W$ then $\beta_M W = \emptyset$, so that Condition ($\alpha$) is always satisfied. Since $\pi_2$ is always a homeomorphism of $\mathscr{M}$ into $\Sigma$, one has merely to note that when $M = W$, $\pi_2$ is *onto* $\Sigma$.

There exists a corollary similar to Corollary 2 in which one supposes that $\mathscr{M} = \mathscr{W}$.

*A canonical representation of* $\Sigma$. The composite $n$-manifold defined and topologized as above in terms of the $n$-manifolds $M$ and $\mathscr{M}$, their open subsets $W$ and $\mathscr{W}$ and the homeomorphism $\mu$, will be said to have the *canonical form*

$$\Sigma = [M, \mathscr{M}, \mu, W, \mathscr{W}]. \tag{1.12}$$

*Subsets of* $\Sigma$ *$\mu$-represented.* Let $A$ and $\mathscr{A}$ be subsets respectively of $M$ and $\mathscr{M}$, and set

$$\pi_1(A) \cup \pi_2(\mathscr{A}) = [A, \mathscr{A}, \Sigma]. \tag{1.13}$$

This subset of $\Sigma$ will be said to be *$\mu$-represented* if

$$\mu(A \cap W) = \mathscr{A} \cap \mathscr{W}. \tag{1.14}$$

If a subset $Z$ of $\Sigma$ is *$\mu$-represented* as in (1.13), then $A$ and $\mathscr{A}$ are uniquely determined as the sets

$$A = \pi^{-1}(Z) \cap M \qquad \mathscr{A} = \pi^{-1}(Z) \cap \mathscr{M}. \tag{1.15}$$

We shall refer to $A$ and $\mathscr{A}$ as the first and second component, respectively, of $[A, \mathscr{A}, \Sigma]$.

Whether $\mu$-represented or not, $[A, \mathscr{A}, \Sigma] = \emptyset$ if and only if $A = \emptyset$ and $\mathscr{A} = \emptyset$.

The use of $\mu$-representations of subsets of $\Sigma$ entails the validity of Lemma 1.2. It is particularly convenient when mappings of subsets $Z$ of $\Sigma$ are to be defined in terms of mappings of $Z$'s two components. In order that such mappings lead to uniquely defined mappings of $Z$ it is necessary to know what points of $A$ and of $\mathscr{A}$ are identified. When $Z$ is $\mu$-represented $A \cap W$ is identified with $\mathscr{A} \cap \mathscr{W}$.

LEMMA 1.2. *Two $\mu$-represented subsets of* $\Sigma$,

$$[A, \mathscr{A}, \Sigma] \qquad [B, \mathscr{B}, \Sigma], \tag{1.16}$$

*have a $\mu$-represented intersection,*

$$[A \cap B, \mathscr{A} \cap \mathscr{B}, \Sigma], \tag{1.17}$$

*and a $\mu$-represented union,*

$$[A \cup B, \mathscr{A} \cup \mathscr{B}, \Sigma]. \tag{1.18}$$

We first show that (1.17) and (1.18) are $\mu$-representations of subsets of $\Sigma$. This is a consequence of the fact that the relations,

(1.19) $$\mu(A \cap W) = \mathscr{A} \cap \mathscr{W} \qquad \mu(B \cap W) = \mathscr{B} \cap \mathscr{W},$$

imply the relations

$$\mu[(A \cap B) \cap W] = (\mathscr{A} \cap \mathscr{B}) \cap \mathscr{W}$$
$$\mu[(A \cup B) \cap W] = (\mathscr{A} \cup \mathscr{B}) \cap \mathscr{W}.$$

To prove that (1.17) gives the intersection of the sets (1.16), we use the definition (1.13), and show that the intersection of the sets (1.16),

(1.20) $$[\pi_1(A) \cup \pi_2(\mathscr{A})] \cap [\pi_1(B) \cup \pi_2(\mathscr{B})] = \pi_1(A \cap B) \cap \pi_2(\mathscr{A} \cap \mathscr{B}),$$

that is the set (1.17). In verifying (1.20) one uses the inclusions,

$$\pi_1(A) \cap \pi_2(\mathscr{B}) \subset \pi_1(A \cap B); \quad \pi_2(\mathscr{A}) \cap \pi_1(B) \subset \pi_2(\mathscr{A} \cap \mathscr{B}),$$

consequences of the $\mu$-representation of the sets (1.16). The proof of (1.18) presents no difficulty.

COROLLARY 1.3. *A necessary and sufficient condition that the subsets* (1.16) *of $\Sigma$ have $\emptyset$ as intersection in $\Sigma$ is that*

(1.21) $$A \cap B = \emptyset, \quad \mathscr{A} \cap \mathscr{B} = \emptyset.$$

COROLLARY 1.4. *The complement in $\Sigma$ of a $\mu$-represented subset* (1.13) *of $\Sigma$ is the $\mu$-represented subset of $\Sigma$,*

(1.22) $$[M - A, \mathscr{M} - \mathscr{A}, \Sigma].$$

This is a $\mu$-representation since the given relations

$$\mu(A \cap W) = \mathscr{A} \cap \mathscr{W}, \quad \mu(W) = \mathscr{W}$$

imply the relations

$$\mu[(M - A) \cap W] = (\mathscr{M} - \mathscr{A}) \cap \mathscr{W}.$$

That the set (1.22) is the complement of the set (1.13) follows from the fact that the set (1.22) does not intersect the set (1.13), and that the union of the sets (1.13) and (1.22) *is the set* $[M, \mathscr{M}, \Sigma] = \Sigma$.

LEMMA 1.3. *A sequence*

(1.23) $$[A_i, \mathscr{A}_i, \Sigma] \qquad [i = 0, 1, \ldots]$$

*of $\mu$-represented subsets of $\Sigma$ has a $\mu$-represented union,*

(1.24) $$[\bigcup_i A_i, \bigcup_i \mathscr{A}_i, \Sigma]$$

*and a $\mu$-represented complement in $\Sigma$,*

(1.25) $$[M - \bigcup_i A_i, \mathscr{M} - \bigcup_i \mathscr{A}_i, \Sigma].$$

The proof of the first affirmation of the lemma is similar to that of the corresponding affirmation in Lemma 1.2. The second affirmation of the lemma is a consequence of Corollary 1.4.

We shall make repeated use of Lemma 1.4.

LEMMA 1.4. *If $\Sigma$ is a composite n-manifold with the canonical form* (1.12) *then any $\mu$-represented subset*

$$Z = [A, \mathscr{A}, \Sigma]$$

*of $\Sigma$ in which $A$ is an open subset of $M$, and $\mathscr{A}$ an open subset of $\mathscr{M}$, is a composite n-submanifold of $\Sigma$.*

As an open subset of $M$, $A$ is itself an $n$-manifold, as is $\mathscr{A}$, as an open subset of $\mathscr{M}$. If $Z$ is $\mu$-represented it may be identified with the composite $n$-manifold,

$$[A, \mathscr{A}, \mu|(A \cap W), A \cap W, \mathscr{A} \cap \mathscr{W}]. \tag{1.26}$$

If $\pi_1$ and $\pi_2$ are $\#$-mappings associated with $\Sigma$, $Z$ is the union

$$\pi_1(A) \cup \pi_2\mathscr{A}$$

of two open subsets of $\Sigma$, accordingly an open subset of $\Sigma$. Its topology as a composite $n$-manifold (1.26) is readily seen to be its topology as derived from $\Sigma$.

The following will be readily verified by the reader, recalling that $X-Y = X \cap (\Sigma - Y)$.

LEMMA 1.4. *If $X = [A, \mathscr{A}, \Sigma]$ and $Y = [B, \mathscr{B}, \Sigma]$ are two $\mu$-represented subsets of $\Sigma$ then $X-Y$ is the $\mu$-represented subset of $\Sigma$*

$$[A-B, \mathscr{A}-\mathscr{B}, \Sigma].$$

## § 2. $\Sigma$ as an $n$-manifold of class $C^m$, $m > 0$

Suppose that a composite $n$-manifold has a canonical representation

$$\Sigma = [M, \mathscr{M}, \mu, W, \mathscr{W}]$$

where $M$ and $\mathscr{M}$ are $n$-manifolds of class $C^m$, $m > 0$, and $\mu$ a $C^m$-diffeomorphism of $W$ onto $\mathscr{W}$. We refer to the $\#$-mappings $\pi$, $\pi_1$, $\pi_2$ associated with $\Sigma$ as in § 1. In terms of these elements we shall assign $\Sigma$ a unique structure as an $n$-manifold of class $C^m$.

As in § 1 let $[F]$ and $[\mathscr{F}]$ be respectively sets of local representations

$$F : U \to X \qquad \mathscr{F} : \mathscr{U} \to \mathscr{X} \tag{2.1}$$

of $M$ and $\mathscr{M}$ which "determine" the given $C^m$-structure of $M$ and $\mathscr{M}$. A system $[\mathbf{F}]$ of local representations of $\Sigma$ adequate to determine a $C^m$-structure on $\Sigma$ may be defined as follows.

To each $F \in [F]$ defining a mapping $U \to X$ we make correspond a mapping

$$\mathbf{F} : U \to \pi_1(X) \qquad \text{(onto } \pi_1(X)) \tag{2.2}$$

such that for $(u) \in U$

$$(u) \to \mathbf{F}(u) = \pi_1 \cdot F(u).$$

Similarly to each $\mathscr{F} \in [\mathscr{F}]$ defining a mapping $\mathscr{U} \to \mathscr{X}$ we make correspond a mapping

$$\mathbf{F} : \mathscr{U} \to \pi_2(\mathscr{X}) \tag{2.3}$$

such that for $(u) \in \mathscr{U}$

$$(u) \to \mathbf{F}(u) = \pi_2(\mathscr{F}(u)).$$

The mappings (2.2) and (2.3) are homeomorphisms of the respective coordinate domains $U$ and $\mathscr{U}$ onto the open subsets $\pi_1(X)$ and $\pi_2(\mathscr{X})$ of $\Sigma$. They are thus local representations of $\Sigma$ in the sense of § 1. It remains to prove the following.

LEMMA 2.1. *The above set* $[\mathbf{F}]$ *of local representations of* $\Sigma$ *satisfy the Covering Condition I and the* $C^m$*-Compatibility Condition II of* § 1.

*The Covering Condition.* The set $[X]$ of coordinate domains associated with the system $[F]$ have $M$ as a union, while the set $[\mathscr{X}]$ of coordinate domains associated with the system $[\mathscr{F}]$ have $\mathscr{M}$ as a union. Taking into account the fact that

$$\pi_1(M) \cup \pi_2(\mathscr{M}) = \Sigma$$

we see that the union of the coordinate domains

$$\pi_1(X) \quad \pi_2(\mathscr{X})$$

associated with the system $[\mathbf{F}]$ is $\Sigma$.

*The* $C^m$*-Compatibility Condition.* We consider first the compatibility of two local representations of $M$

$$F_i : U_i \to X_i \qquad (i = 1, 2)$$

in $[F]$. To these mappings correspond two local representations

$$\mathbf{F}_i : U_i \to \pi_1(X_i)$$

in $[\mathbf{F}]$. Set $X_1 \cap X_2 = X$. Then

$$\pi_1(X_1) \cap \pi_1(X_2) = \pi_1(X).$$

The compatibility condition for $\mathbf{F}_1$ and $\mathbf{F}_2$ involves the two sets

$$U' = \mathbf{F}_1^{-1} \cdot \pi_1(X) \subset U_1$$
$$U'' = \mathbf{F}_2^{-1} \cdot \pi_1(X) \subset U_2.$$

Taking into account the relations $\pi_1 F_i = \mathbf{F}_i$ we see that

$$U' = F_1^{-1}(X), \qquad U'' = F_2^{-1}(X). \tag{2.4}$$

If $X \neq \emptyset$ these sets are not empty and the compatibility condition on $\mathbf{F}_1$ and $\mathbf{F}_2$ is satisfied if for $(u) \in U'$ and $(v) \in U''$ the condition,

$$\mathbf{F}_1(u) = \mathbf{F}_2(v), \tag{2.5}$$

implies a $C^m$-diffeomorphism of $U'$ onto $U''$. Since the condition (2.5) is equivalent to the condition

$$F_1(u) = F_2(v) \qquad [(u) \in U', \ (v) \in U'']$$

and since $U'$ and $U''$ are the sets (2.4), this compatibility condition on $\mathbf{F}_1$ and $\mathbf{F}_2$ reduces to a compatibility condition on $F_1$ and $F_2$, that is to a condition satisfied by hypothesis.

The case in which $M$ is replaced by $\mathscr{M}$ is similar.

There remains the question of the compatibility of two local representations in $[\mathbf{F}]$ arising from two local representations

$$F : U \to X \qquad \mathscr{F} : \mathscr{U} \to \mathscr{X}$$

in $[F]$ and $[\mathscr{F}]$ respectively. These two representations in $[\mathbf{F}]$ have the form

$$\pi_1 F = \mathbf{F}_1 : U \to \pi_1(X) \quad \pi_2 \mathscr{F} = \mathbf{F}_2 : \mathscr{U} \to \pi_2(\mathscr{X}).$$

Set

$$\pi_1(X) \cap \pi_2(\mathscr{X}) = A, \qquad U' = \mathbf{F}_1^{-1}(A), \qquad U'' = \mathbf{F}_2^{-1}(A).$$

If $A \neq \emptyset$ the requirement (C) of compatibility on $\mathbf{F}_1$ and $\mathbf{F}_2$ is the following.

(C) *For $(u) \in (U')$ and $(v) \in U''$ the condition*

$$\mathbf{F}_1(u) = \mathbf{F}_2(v) \tag{2.6}$$

*shall imply a $C^m$-diffeomorphism of $U'$ onto $U''$.*

We shall transform this condition into an equivalent form which is known to be satisfied. To this end recall that $X$ is a subset of $M$, and $\mathscr{X}$ a subset of $\mathscr{M}$ so that $A$ has the form

$$A = \pi_1(Y) = \pi_2(\mathscr{Y})$$

with $Y \subset W$, $\mathscr{Y} \subset \mathscr{W}$ and $\mathscr{Y} = \mu Y$. Consequently

$$\begin{aligned} U' &= \mathbf{F}_1^{-1} \cdot \pi_1(Y) = F^{-1}(Y) \\ U'' &= \mathbf{F}_2^{-1} \cdot \pi_2(\mathscr{Y}) = \mathscr{F}^{-1}(\mathscr{Y}). \end{aligned}$$

Rewrite (2.6) in the form $\pi_1 \cdot F(u) = \pi_2 \cdot \mathscr{F}(v)$. Since $(u)$ is required to be in $U' = F^{-1}(Y)$ we infer that $F(u) \in Y \subset W$. On $W$, $\pi_1 = \pi_2 \mu$ so that (2.6) can be rewritten in the form

$$\pi_2 \cdot \mu \cdot F(u) = \pi_2 \cdot \mathscr{F}(v)$$

or equivalently

$$\mu \cdot F(u) = \mathscr{F}(v). \tag{2.7}$$

Thus $(C)$ is equivalent to the following.

$(C^0)$. *For $(u) \in U'$ and $(v) \in U''$ the condition* (2.7) *shall imply a $C^m$-diffeomorphism of $U'$ onto $U''$.*

With respect to the given $C^m$-structure of $M$

$$F : U' \to F(U')$$

is an admissible local representation of the $n$-manifold $W$. Moreover $\mu$ is given as a $C^m$-diffeomorphism of $W$ onto $\mathscr{W}$. According to the definition of such a $C^m$-diffeomorphism (Cf. 0.4)

$$\mu F : U' \to \mu \cdot F(U') \tag{2.8}$$

is an admissible local representation of $\mathscr{W}$ of class $C^m$. The mapping

$$\mathscr{F} : U'' \to \mathscr{F}(U'') \tag{2.9}$$

is also an admissible local representation of $\mathscr{W}$ of class $C^m$. The coordinate domains $\mu \cdot F(U')$ and $\mathscr{F}(U'')$ in $\mathscr{W}$ are identical, in fact are $\mu Y$ and $\mathscr{Y}$. Hence $\mu F$ and $\mathscr{F}$ must satisfy the compatibility condition $(C^0)$. Thus $(C)$ is satisfied.

This completes the proof of the lemma.

*Composite n-manifolds $\Sigma$ based on $(E, \mathscr{E})$*. As in § 1 let $E$ and $\mathscr{E}$ be two euclidean $n$-spaces. We regard $E$ and $\mathscr{E}$ as $n$-manifolds with differential structures determined by the assumption that their cartesian coordinates are admissible local parameters. We shall consider the special case of a composite $n$-manifold.

$$\Sigma = [M, \mathscr{M}, \mu, W, \mathscr{W}] \tag{2.10}$$

in which $M$ and $\mathscr{M}$ are open subsets of $E$ and $\mathscr{E}$ respectively, with differential structures derived from those of $E$ and $\mathscr{E}$. The sets $W$ and $\mathscr{W}$ are open subsets of $M$ and $\mathscr{M}$ respectively, and $\mu$ is a $C^m$-diffeomorphism of $W$ onto $\mathscr{W}$. In such a case we say that $\Sigma$ is *based on* $(E, \mathscr{E})$.

We shall define a condition $(\gamma)$ on $\Sigma$ sufficient that $\Sigma$ be an $n$-manifold.

*Condition $\gamma$. Let the composite n-manifold* (2.10) *be based on $(E, \mathscr{E})$. Under Condition $(\gamma)$, $\mu$ as defined over $W$, shall be continuously extensible over $Cl_M W$ as a mapping $\nu$ into $\mathscr{E}$, and the subset*

$$\nu(\beta_M W) \tag{2.11}$$

*of $\mathscr{E}$ shall not meet $\mathscr{M}$.*

LEMMA 2.2. *Under Condition* $(\gamma)$ $\Sigma$ *is an n-manifold.*

Lemma 2.2 will follow from Corollary 1.1 provided we show that $\Sigma$ satisfies Condition $(\alpha)$.

As in Condition $(\alpha)$, let $p \in \beta_M W$ and $q \in \beta_{\mathscr{M}} \mathscr{W}$ be given. Since $q$ is in $\mathscr{M}$ and $\mathscr{M}$ is open relative to $\mathscr{E}$, $\mathscr{M}$ includes a neighborhood $\mathscr{N}_q$ of $q$ relative to $\mathscr{E}$, closed relative to $\mathscr{E}$. The point $\nu(p)$ is in the set (2.11) and so by hypothesis not in $\mathscr{M}$, and in particular not in $\mathscr{N}_q$.

Set $Cl_M W = \overline{W}$. By hypothesis $\nu$ maps $\overline{W}$ continuously into $\mathscr{E}$, extending $\mu$. In particular $\nu$ maps $p$ into a point $\nu(p)$ which does not meet the closed set $\mathscr{N}_q$. If $N_p$ is a sufficiently restricted neighborhood of $p$ relative to $M$, $N_p \cap \overline{W}$ will be so restricted a neighborhood of $p$ relative to $\overline{W}$ that

$$\nu(N_p \cap \overline{W}) \cap \mathscr{N}_q = \emptyset.$$

For such a choice of $N_p$

$$\mu(N_p \cap W) \cap (\mathscr{N}_q \cap \mathscr{W}) = \emptyset.$$

Thus (1.9) of Condition $(\alpha)$ holds. Lemma 2.2 follows from Corollary 1.1.

*$C^m$-diffeomorphisms of $\Sigma$.* Let $\Sigma$ be represented canonically as in (2.10). Let $\Sigma'$ be a differential $n$-manifold of the same class $C^m$, $m > 0$, as $\Sigma$. We shall establish a fundamental lemma.

LEMMA 2.3. *If $\Sigma$ has the canonical form* (2.10), *necessary and sufficient conditions that there exist a $C^m$-diffeomorphism $\alpha$ of $\Sigma$ onto $\Sigma'$ are that there exist a $C^m$-diffeomorphism $f$ of $M$ into $\Sigma'$ and a $C^m$-diffeomorphism $\ell$ of $\mathscr{M}$ into $\Sigma'$ such that*

(i) $(\ell\mu)|W = f|W$,

(ii) $f(M-W) \cap \ell(\mathscr{M}) = \emptyset$,

(iii) $f(M) \cup \ell(\mathscr{M}) = \Sigma'$.

*When these conditions are satisfied the $C^m$-diffeomorphism $\alpha$ is uniquely defined by the conditions*

$$(2.11)' \qquad \alpha \cdot \pi_1(p) = f(p) \qquad \alpha \cdot \pi_2(q) = \ell(q) \qquad [p \in M, q \in \mathrm{W}]$$

If a $C^m$-diffeomorphism $\alpha$ of $\Sigma$ onto $\Sigma'$ exists, the mappings $f = \alpha\pi_1$ and $\ell = \alpha\pi_2$ are $C^m$-diffeomorphisms of $M$ into $\Sigma'$, and of $\mathscr{M}$ into $\Sigma'$ respectively. For $p \in W$

$$\alpha \cdot \pi_2 \cdot \mu(p) = \alpha \cdot \pi_1(p)$$

so that (i) is satisfied. The condition (ii) is necessary if $\alpha$ is to be biunique, while the condition (iii) is necessary if $\alpha$ is to map $\Sigma$ *onto* $\Sigma'$.

The conditions are sufficient. Given $f$ and $\not{f}$ satisfying these conditions one defines $\alpha$ by (2.11)′, noting that $\alpha$ is single-valued on $\Sigma$ by (i), and biunique by (ii). That $\alpha$ is a homeomorphism follows from the existence and continuity of the inverses $f^{-1}$, $\not{f}^{-1}$, $(\pi_1)^{-1}$, $(\pi_2)^{-1}$. That $\alpha$ maps $\Sigma$ onto $\Sigma'$ follows from (iii). Since $f$, $\pi_1$, and $\pi_2$ are $C^m$-diffeomorphisms, $\alpha$, as defined by (2.11)′, is a $C^m$-diffeomorphism.

This establishes the lemma.

*A special procedure for defining a $C^m$-diffeomorphism $\alpha$.* Suppose $\Sigma$, as given by (2.10), partitioned into a countable ensemble of disjoint subsets,

$$X_{-1}, X_0, X_1, \ldots, \tag{2.12}$$

of which $X_0, X_1, \ldots$ shall be open subsets of $\Sigma$. Suppose further that there exists a sequence,

$$t_{-1}, t_0, t_1, \ldots, \tag{2.13}$$

of homeomorphisms of the respective sets (2.12) onto disjoint subsets of $\Sigma'$,

$$t_{-1}(X_{-1}),\ t_0(X_0),\ t_1(X_1), \ldots, \quad [t_i(X_i) \text{ open in } \Sigma',\ i \geqq 0] \tag{2.14}$$

whose union is $\Sigma'$. Suppose moreover that there exists an open subset $X_+$ of $\Sigma$ such that $X_+ \supset X_{-1}$ and a $C^m$-diffeomorphism $t_+$ of $X_+$ into $\Sigma'$ such that

$$t_+|(X_+ \cap X_i) = t_i|(X_+ \cap X_i). \quad (i = -1, 0, 1, 2, \ldots)$$

Suppose finally that for $i \geqq 0$, $t_i$ is a $C^m$-diffeomorphism.

LEMMA 2.4. *Then the mapping* $\mathbf{t} : \Sigma \to \Sigma'$ *defined by setting*

$$\mathbf{t}|X_i = t_i|X_i \qquad (i = -1, 0, 1, 2. \ldots)$$

*is a $C^m$-diffeomorphism of $\Sigma$ onto $\Sigma'$.*

It is clear that $\mathbf{t}$ is biunique since each mapping $t_i$ is biunique, since the sets (2.12) are disjoint and have $\Sigma$ as union, and the image sets (2.14) are disjoint and have $\Sigma'$ as union. It remains to show that $\mathbf{t}$ and its inverse are locally $C^m$-diffeomorphisms. This is clear for $\mathbf{t}$. For an arbitrary point in $\Sigma$ has an open neighborhood $N$ in at least one of the sets

$$X_+, X_0, X_1, \ldots$$

and $\mathbf{t}|N$ is then a $C^m$-diffeomorphism into $\Sigma'$. It is true for $\mathbf{t}^{-1}$, since an arbitrary point in $\Sigma'$ has an open neighborhood $N^1$ in at least one of the sets

$$(t_+(X_+),\ t_0(X_0),\ t_1(X_1), \ldots$$

and $\mathbf{t}^{-1}|N^1$ is then a $C^m$-diffeomorphism into $\Sigma$.

## § 3. Modifications of mappings

In this section we establish Lemma 3.2, a technical lemma of fundamental importance in transforming the Schoenflies problem.

As previously, let $E$ and $\mathcal{E}$ be euclidean $n$-spaces with Cartesian coordinates $(x) = (x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively. Let $f_i$, $i = 1, \ldots, n$, be a function of class $C^m$, $m > 0$, mapping an open neighborhood of the origin in $E$ into the axis of reals. We suppose that $f_i(0) = 0$ for $i = 1, \ldots, n$, and that the origin is a critical point of each $f_i$. The transformation

$$y_i = x_i + f_i(x) \qquad (i = 1, \ldots, n) \tag{3.1}$$

defines a $C^m$-diffeomorphism of some spherical neighborhood $N$ of the origin in $E$ onto a neighborhood of the origin in $\mathcal{E}$.

Let $t \to \lambda(t)$ be a mapping of the $t$-axis into the interval $[0, 1]$, of Class $C^\infty$, such that $\lambda(t) = \lambda(-t)$ and

$$\begin{aligned} \lambda(t) &= 1, \qquad (0 \leqq t \leqq 1) \\ \lambda(t) &= 0, \qquad (t \geqq 4). \end{aligned}$$

Set $r^2 = x_1^2 + \ldots + x_n^2$. We suppose $r \geqq 0$. Set $f_i(x) = 0$ when $f_i(x)$ is not already defined. These new values of $f_i(x)$ will enter at most formally.

LEMMA 3.1. *If $e$ is a sufficiently small positive constant the mapping*

$$y_i = x_i + \lambda\left(\frac{r^2}{e^2}\right) f_i(x) \qquad (i = 1, \ldots, n) \tag{3.2}$$

*is a $C^m$-diffeomorphism of $E$ onto $\mathcal{E}$, reducing to the mapping* (3.1) *for $r \leqq e$ and to the identity,*

$$I : y_i = x_i, \qquad (i = 1, 2, \ldots, n) \tag{3.3}$$

*for $r \geqq 2e$.*

The proof of this lemma will be reduced to the verification of statements (I) to (IV).

(I). *The mapping* (3.2) *reduces to the form* (3.1) *for $r \leqq e$ and to $I$ for $r \geqq 2e$.*

Statement (I) is immediate. We continue by setting

$$\lambda\left(\frac{r^2}{e^2}\right) f_i(x) = R_i(x) \qquad (i = 1, \ldots, n) \tag{3.4}$$

and note that the partial derivative $R_{ix_j}$ exists and that

$$R_{ix_j}(x) = \lambda\left(\frac{r^2}{e^2}\right) f_{ix_j}(x) + 2\lambda'\left(\frac{r^2}{e^2}\right) e^{-2} x_j f_i(x) \tag{3.5}$$

for $(x) \in N$, and $i, j = 1, \ldots, n$. Now $\lambda(r^2/e^2)$ and $\lambda'(r^2/e^2)$ are bounded for $r \leqq 2e$. Moreover for $j = 1, \ldots, n$, the integral form of the Law of the Mean shows that, with $i$ summed from 1 to $n$,

$$f_j(x) = x_i a_{ij}(x) \qquad [(x) \in N] \tag{3.6}$$

where $a_{ij}(x)$ is continuous on $N$ and $a_{ij}(0) = 0$. Let $\eta$ be an arbitrary positive constant. It follows from (3.5) that for a suitable choice $e(\eta)$ of $e$ and for $(x) \in E$

$$|R_{ix_j}(x)| \leqq \eta \qquad (i, j = 1, \ldots, \eta). \tag{3.7}$$

*The choice of $\eta$ and $e$.* We put two conditions on $e$. The first is that $e$ be so small a positive constant that the subset of $E$ on which $r \leqq 2e$ is interior to the spherical neighborhood $N$. Under this condition on $e$ the right members of (3.2) are functions of class $C^m$ over $E$. We now put two conditions on $\eta$. We choose $\eta$ so small that when (3.7) holds the jacobian

$$\frac{D(y_1, \ldots, y_n)}{D(x_1, \ldots, x_n)} > \tfrac{1}{2} \tag{3.8}$$

for the mapping (3.2). A second condition on $\eta$ which will be used presently is that for the given fixed $n$

$$2n^2\eta + n^1\eta^3 < \tfrac{1}{2}. \tag{3.9}$$

A final condition on $e$ is that $e < e(\eta)$. With this choice of $e$, (3.7) and (3.8) hold.

(II). *With $e$ so chosen let $(x)$ and $(a)$ be arbitrary distinct points in $E$ with images $(y)$ and $(b)$, respectively, under (3.2). Then $(y) \neq (b)$.*

To prove this let $d(x, a)$ be the euclidean distance between $(x)$ and $(a)$ in $E$, and $d(y, b)$ the euclidean distance between $(y)$ and $(b)$ in $\mathscr{E}$. We shall show that

$$\frac{d(y, b)}{d(x, a)} > \tfrac{1}{2} \tag{3.10}$$

independently of the choice of $(x)$ and $(a)$ in $E$ with $(x) \neq (a)$.

Note that

$$y_i - b_i = x_i - a_i + R_i(x) - R_i(a) \qquad (i = 1, \ldots, n) \tag{3.11}$$

We shall adopt the convention that a repeated index $j, h, k$, etc. is summed from 1 to $n$. Using the integral form of the Law of the Mean we infer that

$$R_i(x) - R_i(a) = A_{ij}(x)(x_j - a_j) \quad (i, j = 1, \ldots, n), \tag{3.12}$$

where $A_{ij}$ is continuous over $E$ and

$$(3.13) \qquad |A_{ij}(x)| \leqq \eta,$$

where $\eta$ is the constant appearing in (3.7). It follows from (3.11) and (3.12) that

$$(3.14) \quad d^2(y, b) = d^2(x, a) \\ +2A_{ij}(x)(x_i-a_i)(x_j-a_j)+A_{ih}(x)A_{ik}(x)(x_h-a_h)(x_k-a_k).$$

From (3.14), (3.13) and (3.9) we see that

$$\frac{d^2(y, b)}{d^2(x, a)} > 1-2n^2\eta-n^3\eta^2 > \tfrac{1}{2}.$$

Relation (3.10) thus holds and (II) is established.

(III). *The mapping* (3.2) *is onto* $\mathscr{E}$. Let $D$ be the subset of $E$ on which $d(x, 0) \leqq 2e$, and $\mathscr{D}$ the subset of $\mathscr{E}$ on which $d(y, 0) \leqq 2e$. It is clear that the image $\mathscr{E}'$ of $E$ under (3.2) is open and includes $\mathscr{E}-\mathscr{D}$ since (3.2) reduces to the identity $I$ on $E-D$. Suppose that $\mathscr{D}$ is not wholly included in $\mathscr{E}'$. There would then exist a sequence of points $q_1, q_2, \ldots$ in $\mathscr{D}$ with antecedents $p_1, p_2, \ldots$ in $D$ such that $(q_n)$ converges in $\mathscr{E}$ to a point $q$ not in $\mathscr{E}'$. A suitable subset of $(p_n)$ will converge to a point $p \in D$, since $D$ is compact. Under (3.2) some open neighborhood of $p$ is mapped homeomorphically onto an open set in $\mathscr{E}$ which must necessarily include $q$. Hence $q$ is in $\mathscr{E}'$. From this contradiction we infer the truth of (III).

(IV). *The mapping* (3.2) *is* $1-1$ *and bicontinuous.*

That the mapping is $1-1$ follows from (II). The mapping (3.2), restricted to the subset $[r \geqq 2e]$, of $E$ reduces to $I$, and so is bicontinuous. Restricted to the subset $[r \leqq 2e]$ the mapping (3.2) is also bicontinuous. Cf. Bourbaki ref. 4, Cor. 2, p. 96. For the subset $[r \leqq 2e]$ of $E$ is compact and the restricted mapping is $1-1$ and continuous onto an image in $\mathscr{E}$ which is a Hausdorff space.

This establishes the lemma. We shall prove a fundamental theorem.

LEMMA 3.2. *Let* $\mathfrak{g}$ *be a* $C^m$*-diffeomorphism,* $m > 0$, *of some neighborhood of* $(x) = (0)$ *in* $E$ *onto a neighborhood of* $(y) = (0)$ *in* $\mathscr{E}$, *where* $\mathfrak{g}$ *has the form*

$$(3.15) \qquad y_i = g_i(x), \quad g_i(0) = 0 \qquad (i = 1, \ldots, n).$$

*Set*

$$(3.16) \qquad L_i(x) = g_{ix_j}(0)x_j \quad \text{(summing as to } j).$$

*For a sufficiently small positive constant e there exists a* $C^m$*-diffeomor-*

*phism* $\mathbf{T}$ *of* $E$ *onto* $\mathscr{E}$ *such that* $\mathbf{T}$ *reduces to* $\mathfrak{g}$ *for* $r \leqq e$ *and to the mapping*

$$T_1 : y_i = L_i(x) \qquad (i = 1, \ldots, n) \tag{3.17}$$

*for* $r \geqq 2e$.

We shall obtain $\mathbf{T}$ as a mapping $T_1 I^{-1} T_2$ where $T_2$ is defined as follows.

The mapping $\mathfrak{g}$ can be represented in a neighborhood of the origin in the form

$$y_i = L_i(x) + r_i(x) \qquad (i = 1, \ldots, r) \tag{3.18}$$

where $r_i$ is of class $C^m$ in some neighborhood of the origin, where $r_i(0) = 0$, and the origin is a critical point of $r_i$. The mapping $IT_1^{-1}\mathfrak{g} = T_0$ has the form

$$y_i = x_i = \rho_i(x) \qquad (i = 1, \ldots, n) \tag{3.19}$$

where $\rho_i(x)$ has the general properties attributed to $r_i(x)$. In accordance with Lemma 3.1, for $e > 0$ and sufficiently small, there exists a $C^m$-diffeomorphism $T_2$ of $E$ onto $\mathscr{E}$ such that $T_2$ reduces to $T_0$ for $r < e$, and to $I$ for $r \geqq 2e$. Set $\mathbf{T} = T_1 I^{-1} T_2$. Then $\mathbf{T}$ reduces to $\mathfrak{g}$ for $r < \varepsilon$ and to $T_1$ for $r \geqq 2\varepsilon$. Thus $\mathbf{T}$ satisfies the lemma.

## § 4. The second class of problems

By a problem of the first class we mean a problem of finding a mapping $\Lambda_\varphi$ which satisfies Theorem 0.1 when a $C^m$-diffeomorphism

$$\varphi : S_{n-1} \to \mathscr{M}_{n-1} \tag{4.1}$$

is given as in Theorem 0.1. We shall designate such a problem by

$$[\varphi; S_{n-1}, \mathscr{M}_{n-1}]_1. \tag{4.2}$$

We term $m$ the *index* of the problem. In the first class of problems $m$ may be any integer from 1 to $\infty$ or $\infty$. We shall introduce a second and equivalent class of problems.

$$[\varphi, S_{n-1}, \mathscr{M}_{n-1}]_2. \tag{4.3}$$

*Sense preserving* $\varphi$. In this second class of problems we shall start with a $C^m$-diffeomorphism $\varphi$ of the nature of $\varphi$ in (4.1), but with $\varphi$ restricted in two ways. The mapping $\varphi$ shall be *sense preserving* in the following sense.

Let $p$ be an arbitrary point of $S_{n-1}$ and let

$$(a_1, \ldots, a_n) \tag{4.4}$$

be a set of independent vectors at $p$ of which $a_1, \ldots, a_{n-1}$ shall be tangent to $S_{n-1}$ at $p$ and $a_n$ shall be a vector which has the direction of the interior normal to $S_{n-1}$ at $p$. At the point $\varphi(p)$ in $\mathscr{E}$ let

$$(b_1, \ldots, b_n)$$

be a set of independent vectors of which $b_1, \ldots, b_{n-1}$ shall be the vectors tangent to $\mathscr{M}_{n-1}$ at $\varphi(p)$ which are the transforms under $\varphi$ of the vectors $a_1, \ldots, a_{n-1}$, and let $b_n$ be a vector whose direction is that of the interior normal to $\mathscr{M}_{n-1}$ at $\varphi(p)$. We say that $\varphi$ is sense preserving if the determinant of the direction numbers of the vectors $(a_1, \ldots, a_n)$ has the sign of the determinant of the direction numbers of the vectors $(b_1, \ldots, b_n)$. It is clear that this characterization of a sense preserving $\varphi$ is independent of the choice of $p \in S_{n-1}$ and of the vectors $a_1, \ldots, a_{n-1}$ tangent to $S_{n-1}$ at $p$.

*The sense index* $\sigma(\varphi)$ *of* $\varphi$. We shall assign $\varphi$ of (4.1) a sense index $\sigma(\varphi)$, equal to 1 or $-1$ according as $\varphi$ is sense preserving or not. A $C^m$-diffeomorphism $f$ of $\mathscr{E}$ onto $\mathscr{E}$ of the form

$$y_i^1 = f_i(y) \qquad (i = 1, \ldots, n)$$

will be said to be *sense preserving* if

$$\Delta = \frac{D(y_1^1, \ldots, y_n^1)}{D(y_1, \ldots, y_n)} > 0.$$

We note that $f\varphi$ is a $C^m$-diffeomorphism

$$f\varphi : S_{n-1} \to f(\mathscr{M}_{n-1}) \qquad [\text{onto } f(\mathscr{M}_{n-1})]$$

We note further that

$$\sigma(f\varphi) = \sigma(\varphi)(\text{sign } \Delta). \tag{4.5}$$

Let $Q$ be the "$x_n$-pole" of $S_{n-1}$, that is the point on $S_{n-1}$ at which $x_n$ attains its maximum value.

*The second class of problems. A problem* (4.3) *of the second class shall be a problem* (4.2) *of the first class in which* $\varphi$ *is a sense preserving diffeomorphism of class* $C^\infty$ *of* $S_{n-1}$ *onto* $\mathscr{M}_{n-1}$ *such that for some neighborhood,* $R_Q$, *relative to* $S_{n-1}$, *of the* $x_n$*-pole* $Q$ *of* $S_{n-1}$

$$\varphi | R_Q = I | R_Q \tag{4.6}$$

That the first class of problems is equivalent to the second class of problems will follow from Lemmas 4.1 to 4.3.

LEMMA 4.1. *Given a* $C^m$*-diffeomorphism* $\varphi$ *of the form* (4.1) *there exists a* $C^m$*-diffeomorphism* $\mathbf{f}$ *of* $\mathscr{E}$ onto $\mathscr{E}$ *such that* $\mathbf{f}\varphi$ *is a sense preserving* $C^m$*-diffeomorphism of* $S_{n-1}$ *onto* $\mathbf{f} \cdot \varphi(S_{n-1})$, *and such that for some neighborhood* $R_Q$ *relative to* $S_{n-1}$ *of the* $x_n$*-pole* $Q$ *of* $S_{n-1}$

$$(\mathbf{f}\varphi)|R_Q = I|R_Q. \tag{4.7}$$

Let $L$ and $\mathscr{L}$ be the coordinate $(n-1)$-planes in $E$ and $\mathscr{E}$ on which $x_n = 0$ and $y_n = 0$ respectively. Let O and $\mathscr{O}$ be respectively the origin in $E$ and $\mathscr{E}$. Set $\varphi(Q) = \mathscr{Q}$. We begin by proving (a).

(a). *There exists a sense preserving $C^m$-diffeomorphism $\mathscr{F}$ of $\mathscr{E}$ onto $\mathscr{E}$ such that $\mathscr{F}(\mathscr{Q}) = \mathscr{O}$ and $\mathscr{F}(\mathscr{M}_{n-1})$ intersects $\mathscr{L}$ in a neighborhood of $\mathscr{O}$ relative to $\mathscr{L}$.*

The mapping $\mathscr{F}$ will have the form $hk$, where $h$ and $k$ are defined as follows.

$k$. The mapping $k$ shall be a rigid motion of $\mathscr{E}$ such that $k(\mathscr{Q})=\mathscr{O}$ and such that the $(n-1)$-plane tangent to $k(\mathscr{M}_{n-1})$ at $\mathscr{O}$ is $\mathscr{L}$. Note that $k$ is sense preserving.

$h$. A sufficiently small open set on $\mathscr{L}$ containing $\mathscr{O}$ will serve as a coordinate range for an admissible local representation

$$y_n = Y(y_1, \ldots, y_{n-1})$$

of $k(\mathscr{M}_{n-1})$ of class $C^m$. The function $Y$ has a critical point when $y_1 = y_2 = \ldots = y_{n-1} = 0$. The equations

$$\begin{aligned} y_i^1 &= y_i \qquad\qquad\qquad (i = 1, \ldots, n-1) \\ y_n^1 &= y_n - Y(y_1, \ldots, y_{n-1}) \end{aligned}$$

define a $C^m$-diffeomorphism $\lambda$ of a sufficiently restricted neighborhood of $\mathscr{O}$ relative to $\mathscr{E}$, onto a similar neighborhood of $\mathscr{O}$. It follows from Lemma 3.1 that there exists a $C^m$-diffeomorphism $h$ of $\mathscr{E}$ onto $\mathscr{E}$ which reduces to $\lambda$ in some neighborhood of $\mathscr{O}$ relative to $\mathscr{E}$. The image $(hk)(\mathscr{M}_{n-1})$ accordingly intersects $\mathscr{L}$ in a neighborhood of $\mathscr{O}$ relative to $\mathscr{L}$. Note that $\lambda$ and hence $h$ is sense preserving.

The $C^m$-diffeomorphism $\mathscr{F} = hk$ of $\mathscr{E}$ onto $\mathscr{E}$ satisfies (a).

The mapping **f** required in Lemma 4.1 will be set up in terms of $I$ and $\mathscr{F}$, already defined, and mappings $F$ and **T** now to be defined.

$F$. Statement (a) admits a parallel statement as follows. There exists a sense preserving $C^m$-diffeomorphism $F$ of $E$ onto $E$ such that $F(Q) = \mathrm{O}$, and $F(S_{n-1})$ intersects the coordinate $(n-1)$-plane $L$ in a neighborhood of O relative to $L$.

**T**. If $R_0$ is a sufficiently small open neighborhood of O, relative to $L$, then for $p \in R_0$, the mapping

$$(4.8) \qquad (y) \to \mathscr{F} \,.\, \varphi \,.\, F^{-1}(p); \;\; R_0 \to \mathscr{L}$$

is a $C^m$-diffeomorphism of $R_0$ onto a neighborhood of $\mathscr{O}$ relative to the coordinate $(n-1)$-plane $\mathscr{L}$. The mapping (4.8) carries O into $\mathscr{O}$. It admits an obvious extension

$$\mathbf{g} : N_0 \to \mathscr{E}$$

over a neighborhood $N_0$ of O *relative to* $E$, an extension which is a $C^m$-diffeomorphism of $N_0$ onto a neighborhood of $\mathcal{O}$ relative to $\mathcal{E}$. Suppose $\mathbf{g}$ represented in the form (3.15). By proper choice of $\mathbf{g}$ the sign of the jacobian

$$\Delta_1 = \frac{D(g_1, \ldots, g_n)}{D(x_1, \ldots, x_n)}$$

can be made positive or negative at pleasure. We choose the sign of $\Delta_1$ so that

$$\sigma(\varphi)(\text{sign } \Delta_1) > 0. \tag{4.9}$$

In accord with Lemma 3.2 there exists a $C^m$-diffeomorphism $\mathbf{T}$ of $E$ onto $\mathcal{E}$ which reduces to $\mathbf{g}$ on some neighborhood $N$ of O relative to $E$. According to this choice of $\mathbf{T}$

$$\mathbf{T}^{-1} . \mathscr{F} . \varphi . F^{-1}(p) = p \qquad (p \in N \cap R_0). \tag{4.10}$$

*The choice of* $\mathbf{f}$. We now introduce the mapping

$$\mathbf{f} = IF^{-1}\mathbf{T}^{-1}\mathscr{F}; \ \mathcal{E} \to \mathcal{E}. \tag{4.11}$$

For $p \in N \cap R_0$

$$\begin{aligned}(\mathbf{f}\varphi)(F^{-1}(p)) &= I . F^{-1} . \mathbf{T}^{-1} . \mathscr{F} . \varphi . F^{-1}(p) \quad &&[\text{by } (4.11)] \\ &= I(F^{-1}p) &&[\text{by } (4.10)]\end{aligned}$$

so that (4.7) is satisfied for $R_Q = F^{-1}(N \cap R_0)$. The mapping $\mathbf{f}$ is sense preserving or not, according as $\mathbf{T}$ is sense preserving or not. If $\Delta_2$ is the jacobian associated with $\mathbf{f}$, sign $\Delta_2 = \Delta_1$ and

$$\sigma(\mathbf{f}\varphi) = \sigma(\varphi)(\text{sign } \Delta_2) = \sigma(\varphi)(\text{sign } \Delta_1) > 0$$

in accord with (4.9). Thus $\mathbf{f}\varphi$ is sense preserving.

This completes the proof of Lemma 4.1.

We shall now state a lemma whose proof will be given in a separate paper. Ref. 6. If $\varphi$ were initially of class $C^\infty$ this lemma would not be needed.

LEMMA 4.2. *If the $C^m$-diffeomorphism $\varphi$ is given as in* (4.1) *there exists a sense preserving $C^m$-diffeomorphism* $\mathbf{J}$ *of $\mathcal{E}$ onto $\mathcal{E}$ such that* $\mathbf{J}\varphi$ *is a $C^\infty$-diffeomorphism,*

$$\mathbf{J}\varphi : S_{n-1} \to \mathbf{J}(\mathscr{M}_{n-1}) \qquad [\text{onto } \mathbf{J}(\mathscr{M}_{n-1})].$$

Without this lemma the procedures of this paper would lead to a proof of Theorem 0.1 modified by supposing that $m > 1$, and affirming that $\Lambda_\varphi$ was a $C_0^{m-1}$-diffeomorphism and not necessarily a $C_0^m$-diffeomorphism. When $\varphi$ is merely of class $C^1$ the family of normals to $\mathscr{M}_{n-1}$ to be introduced in § 5 would in general form a "field" on no neighborhood of $\mathscr{M}_{n-1}$ relative to $\mathcal{E}$. Our use of this family of normals reduces the class of the mapping functions by 1 when $m$ is finite, but not at all when $m = \infty$. Lemma 4.2 enables us to prove that the first and second classes of problems are equivalent. In the second class the given $\varphi$ is of class $C^\infty$.

Starting with the $C^m$-diffeomorphism $\varphi$, given in (4.1), we can apply Lemma 4.2 and obtain the $C^\infty$-diffeomorphism $\mathbf{J}\varphi$. We now apply Lemma 4.1 to $\mathbf{J}\varphi$ in place of $\varphi$, and obtain thereby a $C^\infty$-diffeomorphism $\mathbf{f}$ and a new boundary mapping $\mathbf{fJ}\varphi$. Setting $\mathbf{F} = \mathbf{fJ}$ we state the following corollary.

Corollary 4.1. *Given a $C^m$-diffeomorphism $\varphi$, $m > 0$ of the form 4.1, there exists a $C^m$-diffeomorphism $\mathbf{F}$ of $\mathscr{E}$ onto $\mathscr{E}$ such that $\mathbf{F}\varphi$ is a sense preserving $C^\infty$-diffeomorphism of $S_{n-1}$ onto $\mathbf{F} \cdot \varphi(S_{n-1})$, and*

$$(\mathbf{F}\varphi)|R_Q = I|R_Q$$

*for some neighborhood $R_Q$ relative to $S_{n-1}$ of the $x_n$-pole $Q$ of $S_{n-1}$.*

To employ this corollary we need the following general lemma.

Lemma 4.3. *Let $\mathbf{F}$ be a $C^m$-diffeomorphism of $\mathscr{E}$ onto $\mathscr{E}$. To a problem (4.2) in the first class, of index $m > 0$, corresponds a problem*

$$[\varphi^1, S_{n-1}, \mathscr{M}^1_{n-1}]_1 \tag{4.12}$$

*also in the first class with index $m^1 \geqq m$ such that*

$$\varphi^1 = \mathbf{F}\varphi, \quad \mathscr{M}^1_{n-1} = \mathbf{F}\mathscr{M}_{n-1}.$$

*If $\Lambda_{\varphi 1}$ is a solution of problem (4.12) then*

$$\Lambda_\varphi = \mathbf{F}^{-1}\Lambda_{\varphi 1} \tag{4.13}$$

*is a solution of problem (4.1).*

Since $\Lambda_{\varphi 1}$ is a $C_0^{m^1}$-diffeomorphism of a neighborhood of $JS_{n-1}$ onto a neighborhood of $J\mathscr{M}^1_{n-1}$, we infer that $\Lambda_\varphi$ is a $C_0^m$-diffeomorphism of a neighborhood of $JS_{n-1}$ onto a neighborhood of

$$\mathbf{F}^{-1}(J\mathscr{M}^1_{n-1}) = J\mathbf{F}^{-1}(\mathscr{M}^1_{n-1}) = J\mathscr{M}_{n-1}.$$

As for the boundary condition we have

$$\Lambda_{\varphi 1}|S_{n-1} = \varphi^1 = \mathbf{F}\varphi$$

by hypothesis, so that

$$\Lambda_\varphi|S_{n-1} = (\mathbf{F}^{-1}\Lambda_{\varphi 1})|S_{n-1} = \mathbf{F}^{-1}\mathbf{F}\varphi = \varphi.$$

Thus $\Lambda_\varphi$ is a solution of problem (4.2).

Corollary 4.2. *The first and second classes of problems are equivalent.*

Since each problem in the second class is by definition a problem in the first class, it remains only to show that to each problem (4.2) in the first class corresponds a problem

(4.14) $$[\varphi^1, S_{n-1}, \mathscr{M}^1_{n-1}]_2$$

in the second class whose solution $\Lambda_{\varphi 1}$ implies a solution $\Lambda_\varphi$ of the given problem (4.2). To $\varphi$, given in the form (4.1), corresponds a mapping **F** of Corollary 4.1. Setting $\varphi^1 = \mathbf{F}\varphi$ we make the problem (4.2) correspond to the problem (4.12) of Lemma 4.3. The new index $m^1 = \infty$, and $\varphi^1$ is sense preserving. The mapping $\varphi^1$ is thus of the type admitted in a problem of the second class. By Lemma 4.3, $\Lambda_\varphi$, as given in (4.13), affords a solution to problem (4.2).

The corollary follows.

## § 5. The "bands" $B_a S_{n-1}$ and $B_a \mathscr{M}_{n-1}$.

In this section we shall define "band" or "shell" neighborhoods $B_a S_{n-1}$ and $B_a \mathscr{M}_{n-1}$ of $S_{n-1}$ and $\mathscr{M}_{n-1}$ respectively, and establish a $C^{m-1}$-diffeomorphism, $m > 1$, of the form

(5.1) $$\varphi_a : B_a S_{n-1} \to B_a \mathscr{M}_{n-1} \qquad [\text{onto } B_a \mathscr{M}_{n-1}].$$

*Fields of normals to* $\mathscr{M}_{n-1}$. Let $q$ be a point on $\mathscr{M}_{n-1}$ and $(y_1, \ldots, y_n)$ coordinates in $\mathscr{E}$. Let

(5.2) $$\mathscr{F} : \mathscr{U} \to \mathscr{X};\ y_i = \mathscr{F}_i(u);\ (u) \in \mathscr{U} \qquad (i = 1, \ldots, n)$$

be an arbitrary admissible local representation of $\mathscr{M}_{n-1}$ with a coordinate domain $\mathscr{X}$ which contains $q$. We are supposing that the $\mathscr{F}_i$ are of class $C^m$, $m > 1$, and that the functional matrix

(5.3) $$\left\| \frac{\partial \mathscr{F}_i}{\partial u_j} \right\|$$

with columns $i = 1, \ldots, n$, and rows $j = 1, \ldots, n-1$, has the rank $n-1$ at each point of $\mathscr{U}$. Let $a_i(u)$, $i = 1, \ldots, n$, be the determinant of the submatrix of (5.3) obtained by deleting the $i$-th column of this matrix. The set of numbers

(5.4) $$(a_1(u), \ldots, a_n(u)) = (a_i(u))$$

define a vector $\mathscr{Y}_{(u)}$ normal to $\mathscr{M}_{n-1}$ at the point $(\mathscr{F}_i(u))$. Without loss of generality we can suppose that $(a_i(u))$ has the direction of the exterior normal to $\mathscr{M}_{n-1}$ at $(\mathscr{F}_i(u))$. Were this not the case an interchange of the parameters $u_1$ and $u_2$ and the corresponding interchange of the first two columns of the matrix (5.4) would bring this about.

Let $\mathscr{Y}_{(u)}$ be regarded as a directed axis with origin at $(F_i(u))$. The point on $\mathscr{Y}_{(u)}$ with algebraic coordinate $s$ will have euclidean coordinates

(5.5) $$y_i = \mathscr{F}_i(u)+sa_i(u) \qquad [(u)\in\mathscr{U}].$$

Since $m > 1$, each mapping $a_i$ is of class $C^1$ at least. Set

$$\frac{D(y_1, \ldots, y_n)}{D(s, u_1, \ldots, u_{n-1})} = \Delta(u, s)$$

for each value of $s$ and each $(u)\in\mathscr{U}$. Recall that $\Delta(u, 0)$ is the determinant of an $n$-square matrix obtained by adding (5.4) as a first row to the matrix (5.3). Hence

$$\Delta(u, 0) = a_1^2(u)+ \quad +a_n^2(u) \neq 0 \qquad (u\in\mathscr{U}).$$

Let $\mathscr{U}_1$ be an open subset of $\mathscr{U}$ which contains $F^{-1}(q)$ and whose closure relative to the euclidean $(n-1)$-plane of $\mathscr{U}$ is in $\mathscr{U}$. Then on $\mathscr{U}_1$, $\Delta(u, 0)$ is bounded from zero. We infer that there exists a positive constant $s_0$ such that $\Delta(u, s)$ is bounded from zero for $(u)\in\mathscr{U}_1$ and $s$ on the interval $(-s_0, s_0)$. Let $J_a$ designate the interval $(-a, a)$. It follows from the usual implicit function analysis that if $a$ is a sufficiently small positive constant, and if

$$[(u), s]\in\mathscr{U}_1\times J_a,$$

then the mapping of $\mathscr{U}_1\times J_a$ into $\mathscr{E}$ defined by (5.5) is a homeomorphism. Now $\mathscr{M}_{n-1}$ is compact and so is the union of a finite ensemble of open sets each of the character of $\mathscr{U}_1$.

*The band* $B_a\mathscr{M}_{n-1}$. We draw the following conclusions. Let $q$ be an arbitrary point of $\mathscr{M}_{n-1}$ and $\mathscr{V}_q$ the unit vector normal to $\mathscr{M}_{n-1}$ at $q$ with the direction of the exterior normal. Let $\mathbf{y}(q, s)$ be the point $(y)$ on $\mathscr{V}_q$ with algebraic coordinate $s$. Let

(5.6) $$\mathscr{H}_a : \mathscr{M}_{n-1}\times J_a \to \mathscr{E}$$

be a map in which $(q, s) \to \mathbf{y}(q, s)\in\mathscr{E}$. If $a_0$ is a sufficiently small positive constant $\mathscr{H}_a$ is readily seen to be a *homeomorphism* of

(5.7) $$\mathscr{M}_{n-1}\times J_a \qquad [0 < a < a_0]$$

onto a neighborhood

(5.8) $$B_a\mathscr{M}_{n-1}$$

of $\mathscr{M}_{n-1}$ relative to $\mathscr{E}$. As an open subset of $\mathscr{E}$, $B_a\mathscr{M}_{n-1}$ derives a differential structure from $\mathscr{E}$. We term $B_a\mathscr{M}_{n-1}$ a *band neighborhood* of $\mathscr{M}_{n-1}$ *of width* $2a$.

One can regard the product $\mathscr{M}_{n-1}\times J_a$ as an $n$-manifold of class $C^m$ since it is the product of two manifolds each of class $C^m$. Given the mappings (5.2), the corresponding set of mappings

(5.9) $$\mathscr{U}\times J_a \to \mathscr{F}(\mathscr{U})\times J_a; \quad [(u)\times s] \to [F_i(u)\times s]$$

regarded as local representations of $\mathcal{M}_{n-1} \times J_a$, define a $C^m$-structure on $\mathcal{M}_{n-1} \times J_a$. The mapping $\mathcal{H}_a$ is a $C^{m-1}$-diffeomorphism of $\mathcal{M}_{n-1} \times J_a$ into $B_a\mathcal{M}_{n-1}$, [Cf. (10.4)]. For a mapping (5.9), followed by $\mathcal{H}_a$, leads to an admissible local representation (5.5) of $B_a M_{n-1}$ of class $C^{m-1}$.

*The band* $B_a S_{n-1}$. Corresponding to $p \in S_{n-1}$ let $V_p$ be an exteriorly directed unit vector, normal to $S_{n-1}$ at $p$, and let $s$ be the signed coordinate of an arbitrary point on $V_p$ with $s = 0$ at $p$. If $0 < a < 1$ there clearly exists, as in the case of $\mathcal{M}_{n-1}$, a $C^\infty$-diffeomorphism

$$H_a : S_{n-1} \times J_a \to E, \tag{5.10}$$

in which the point $(p, s) \in S_{n-1} \times J_a$ corresponds to the point on $V_p$ with algebraic coordinate $s$. The image of $S_{n-1} \times J_a$ under $H_a$ is denoted by $B_a S_{n-1}$ and termed the *band neighborhood* of $S_{n-1}$ *of width* $2a$.

LEMMA 5.1. *If* $\varphi$ *is a* $C^m$*-diffeomorphism given by* (4.1) *with* $m > 1$, *and if* $a$ *is a sufficiently small positive constant there exists a* $C^{m-1}$*-diffeomorphism* $\varphi_a$ *of the band* $B_a S_{n-1}$ *onto the band* $B_a \mathcal{M}_{n-1}$ *such that*

$$\varphi_a | S_{n-1} = \varphi,$$

*and such that the point on the vector* $V_p$ *exteriorly normal to* $S_{n-1}$ *at* $p$ *with algebraic coordinate* $s \in J_a$, *corresponds to the point on the vector* $\mathcal{V}_{\varphi(p)}$, *exteriorly normal to* $\mathcal{M}_{n-1}$ *at* $\varphi(p)$ *with algebraic coordinate* $s$.

To set up such a $C^{m-1}$-diffeomorphism note that the homeomorphism

$$G_a : S_{n-1} \times J_a \to \mathcal{M}_{n-1} \times J_a$$

in which $(p, s) \in S_{n-1} \times J_a$ corresponds to $(\varphi(p), s) \in \mathcal{M}_{n-1} \times J_a$, is a $C^m$-diffeomorphism. If one sets

$$\varphi_a = \mathcal{H}_a G_a H_a^{-1} \qquad [0 < a < \min(a_0, 1)]$$

one infers that $\varphi_a$ is a $C^{m-1}$-diffeomorphism of $B_a S_{n-1}$ onto $B_a \mathcal{M}_{n-1}$.

NOTE. If $m = \infty$ in Lemma 5.1 $\varphi_a$ is of class $C^\infty$, as the proof shows. If $m = 1$ Lemma 5.1 does not apply.

## § 6. Problems of the third class

As previously, we suppose that $\mathscr{E}$ is assigned its conventional differential structure.

*Problems of the third class. Let* $\Phi$ *be a* $C^\infty$*-diffeomorphism of an open neighborhood of* $S_{n-1}$ *into* $\mathscr{E}$ *such that in some open neighborhood* $N_Q$ *relative to* $E$ *of the* $x_n$*-pole* $Q$ *of* $S_{n-1}$

(6.1) $$\Phi | N_Q = I | N_Q.$$

*Setting* $\mathscr{M}_{n-1} = \Phi(S_{n-1})$ *a problem*

(6.2) $$[\Phi, S_{n-1}, \mathscr{M}_{n-1}]_3$$

*of the third class is to find a* $C_0^\infty$*-diffeomorphism* $\Lambda_\Phi$ *of some neighborhood of* $JS_{n-1}$ *onto a neighborhood of* $J\mathscr{M}_{n-1}$ *relative to* $\mathscr{E}$, *such that for some neighborhood* $Z$ *of* $S_{n-1}$ *relative to* $E$

(6.3) $$\Lambda_\Phi | Z = \Phi | Z.$$

As in § 1, we understand that a class $(A)$ of problems $A$ is "effectively mapped" into a class $(B)$ of problems $B$ if to each problem $A$ corresponds at least one problem $B$ whose solution implies a solution of $A$. In this sense we state the following lemma.

LEMMA 6.1. *The second class of problems can be effectively mapped into the third class of problems.*

Our first task is to assign to each problem (4.3) of the second class a problem of the third class.

Given $\varphi$ in problem (4.3), Lemma 5.1 implies the existence of a constant $a > 0$ and a $C^\infty$-diffeomorphism

(6.4) $$\varphi_a : B_a S_{n-1} \to B_a \mathscr{M}_{n-1} \qquad (\text{onto } B_a \mathscr{M}_{n-1})$$

such that $\varphi_a | S_{n-1} = \varphi$. By definition of the problem (4.3) there exists a neighborhood $R_0$, relative to $S_{n-1}$, of the $x_n$-pole $Q$ of $S_{n-1}$ such that $\varphi | R_0 = I | \varphi$. With $a$ chosen as in Lemma 5.1 and $H_a$ as in (5.10), set

$$N_Q = H_a(R_0 \times J_a).$$

Then $N_Q$ is a neighborhood of $Q$ relative to $E$. It follows from the definition of $\varphi_a$ in Lemma 5.1 and the relation $\varphi | R_0 = I\varphi$, that

(6.5) $$\varphi_a | N_Q = I | N_Q.$$

Thus $\varphi_a$ is a mapping $\Phi$ admissible as datum in a problem of the third class. To a problem (4.3) of the second class we made correspond the problem

(6.6) $$[\Phi, S_{n-1}, \mathscr{M}_{n-1}]_3$$

in which $\Phi = \varphi_a$.

Our second task is to show that a solution $\Lambda_\Phi$ of the problem (6.6) is equally a solution of problem (4.3).

It is clear that $\Lambda_\Phi$ is a $C_0^\infty$-diffeomorphism of some neighborhood of $JS_{n-1}$ onto some neighborhood of $J\mathscr{M}_{n-1}$. Moreover the condition (6.3) is satisfied and implies the boundary condition

$$\Lambda_\Phi | S_{n-1} = \Phi | S_{n-1} = \varphi_a | S_{n-1} = \varphi$$

in accord with the definition of $\Phi$ and of $\varphi_a$. Thus the boundary condition (0.7) is satisfied with $\Lambda_\Phi$ replacing $\Lambda_\varphi$. Hence $\Lambda_\Phi$ is a solution of problem (4.3).

Thus the second class of problems has been effectively mapped into the third class of problems.

This establishes Lemma 6.1.

## § 7. Problems of type K

In this section we shall define a final class of problems, termed problems of type $K$. In § 8 we shall show that the third class of problems can be effectively mapped into the class of problems of type $K$. In later sections we shall show that any problem of type $K$ admits a solution, thereby implying a solution of our original problem of class 1.

Two notational conventions require mention. The *interior* of a subset $X$ of $E$ or of $\mathscr{E}$ will be denoted by $\mathring{X}$. The *complement* of $X$ relative to $E$ or $\mathscr{E}$ will be denoted by $CX$ or $\mathscr{C}X$ respectively.

*The sets* $K$, $K'$, $K''$, $k$, $k_0$. Let $K$ be an $n$-cube in $E$ with center at the origin, with $(n-1)$-faces parallel to the coordinate $(n-1)$-planes, and with arbitrary diameter. Let $\Pi_0$ denote the $(n-1)$-plane $[x_n = 0]$. Set

$$k = K \cap \Pi_0 \qquad k_0 = k \cap \mathring{K}. \tag{7.1}$$

The set $\mathring{K}-k_0$ is the union of two disjoint open sets $K'$ and $K''$ into which $\mathring{K}$ is separated by deletion of $k_0$. Of the two sets $K'$ and $K''$ let $K'$ be the set on which $x_n < 0$. Note that

$$\beta_E K' \cup \beta_E K'' = k_0 \cup \beta_E K. \tag{7.2}$$

*Mappings* $\omega'$ *and* $\omega$. Let $G$ be a compact subset of $K' \cup K''$ such that $\mathring{K}-G$ is arc-wise connected. Set

$$G' = G \cap K', \quad G'' = G \cap K'' \tag{7.3}$$

so that $G = G' \cup G''$. We suppose that $G''$ is not empty.

*We introduce a* $C^\infty$*-diffeomorphism*

$$\omega' : E-G \to \mathscr{E}$$

*such that* $\omega'$ *reduces to* $I$ *on some* $\varepsilon$*-neighborhood* $N_\varepsilon$ *relative to* $E$ *of* $C\mathring{K}$.

*The class* $(\omega)$. For each such $\omega'$ and choice of $G$ set

$$\omega'|(K-G) = \omega, \tag{7.4}$$

thereby defining a class $(\omega)$ of mappings $\omega$. Note that $\omega'$ is uniquely determined by its restriction $\omega$.

*The sets* $\mathscr{K}$, $\mathscr{K}'$, $\mathscr{K}''$, $\mathscr{k}$, $\mathscr{k}_0$. Observe that

$$\omega'(\beta_E K) = I(\beta_E K). \tag{7.5}$$

Set

$$\mathscr{K} = IK, \ \mathscr{k} = \omega'(k) \qquad \mathscr{k}_0 = \omega(k_0) \tag{7.6}$$

and note that $\mathring{\mathscr{K}} - \mathscr{k}_0$ is the union of two disjoint open sets $\mathscr{K}'$ and $\mathscr{K}''$. Just one of the two sets $\mathscr{K}'$ and $\mathscr{K}''$, say $\mathscr{K}'$, is such that

$$I(K' \cap N_\epsilon) \subset \mathscr{K}'. \tag{7.7}$$

*The sets* $H$, $H'$, $H''$, $\mathscr{H}$, $\mathscr{H}'$, $\mathscr{H}''$, $\mathscr{G}'$, $\mathscr{G}''$, $\mathscr{G}$. Let $d$ be so small a positive constant that the subsets $H'$ and $H''$ of $\mathring{K}$ on which $x_n < -d$ and $x_n > d$ respectively are open sets such that

$$H' \supset G' \qquad H'' \supset G''. \tag{7.8}$$

Observe that the parameter $d$ is determined by the choice of $H'$ and $H''$. Recalling that $\omega$ is defined on $\mathring{K} - G$, but not over all of $\mathring{K}$, set

$$(7.9)' \quad \mathscr{H}' = \mathscr{K}' - \omega(K' - H') \quad \mathscr{H}'' = \mathscr{K}'' - \omega(K'' - H'')$$

$$(7.9)'' \quad \mathscr{G}' = \mathscr{H}' - \omega(H' - G') \quad \mathscr{G}'' = \mathscr{H}'' - \omega(H'' - G'')$$

$$\mathscr{G} = \mathscr{G}' \cup \mathscr{G}'' \quad H = H' \cup H'' \quad \mathscr{H} = \mathscr{H}' \cup \mathscr{H}''.$$

It follows from (7.7) and (7.9) that

$$\omega(K' - H') \subset \mathscr{K}' \quad \omega(K'' - H'') \subset \mathscr{K}''$$

$$I(H' \cap N_\epsilon) \subset \mathscr{H}' \quad I(H'' \cap N_\epsilon) \subset \mathscr{H}''$$

and, since $H' - G'$ and $H'' - G''$ are arc-wise connected that

$$(7.9)''' \qquad \omega(H' - G') \subset \mathscr{H}' \quad \omega(H'' - G'') \subset \mathscr{H}''.$$

It follows then from (7.9) that

$$\mathring{\mathscr{K}} - \mathscr{H} = \omega(\mathring{K} - H), \ \mathscr{H} - \mathscr{G} = \omega(H - G), \ \mathring{\mathscr{K}} - \mathscr{G} = \omega(\mathring{K} - G). \tag{7.10}$$

*Problems of type K*. A problem of type $K$ will be denoted by

$$[\omega, H', \mathscr{H}']_K \tag{7.11}$$

and defined as the problem of finding a mapping $\lambda_\omega$ which satisfies Lemma 7.1.

LEMMA 7.1. *Corresponding to an n-cube $K$, a mapping $\omega \in (\omega)$, an n-rectangle $H' \subset K'$ and set $\mathscr{H}' \subset \mathscr{K}'$, chosen as above, there exists a $C_0^\infty$-diffeomorphism,*

$$\lambda_\omega : H' \to \mathscr{H}' \qquad [\text{onto } \mathscr{H}'] \tag{7.12}$$

*such that for some compact subset $\Omega$ of $H'$ with $\Omega \supset G'$*

$$\lambda_\omega|(H'-\Omega) = \omega|(H'-\Omega). \tag{7.13}$$

NOTE. To be assured of a solution of an arbitrary problem of type $K$ it is sufficient to be assured of a solution of all such problems for which $K$ is a given fixed $n$-cube $K$. To establish this it merely is necessary to subject $E$ and $\mathscr{E}$ to a common change of scale, defined by a suitable common change of scale on each coordinate axis in $E$ and $\mathscr{E}$. We shall make use of a special $n$-cube $K$ of the form

$$(-1 \leqq x_i \leqq 1) \qquad (i = 1, \ldots, n) \tag{7.14}$$

and solve a problem of type $K$ which is arbitrary except for this special choice of $K$.

We need the following lemma.

LEMMA 7.2. *The sets $\mathscr{H}'$, $\mathscr{H}''$, $\mathscr{H}'-\mathscr{G}'$ and $\mathscr{H}''-\mathscr{G}''$ are open relative to $\mathscr{E}$.*

To prove that $\mathscr{H}'$ is open relative to $\mathscr{E}$ set $K'-H' = A$ and $Cl_E A = B$. By (7.9)'

$$\mathscr{H}' = \mathscr{K}'-\omega'(A).$$

We shall show that

$$\mathscr{H}' = \mathscr{K}'-\omega'(B). \tag{7.15}$$

This will follow if $\omega'(B)-\omega'(A)$ does not meet $\mathscr{K}'$, a condition which is satisfied since

$$\omega'(B)-\omega'(A) = \omega'(B-A) \subset \omega'(k_0 \cup \beta_E K) \subset \beta_{\mathscr{E}}\mathscr{K} \cup \omega'(k_0).$$

With (7.15) established note that $B$ is compact. Hence $\omega'(B)$ is compact and $\mathscr{H}'$ open relative to $\mathscr{E}$. That $\mathscr{H}''$ is open relative to $\mathscr{E}$ follows similarly.

Since $H'-G'$ is open relative to $E$ and the diffeomorphism $\omega'$ maps $H'-G'$ onto $\mathscr{H}'-\mathscr{G}'$, in $\mathscr{E}$, the set $\mathscr{H}'-\mathscr{G}'$ is open relative to $\mathscr{E}$. The set $\mathscr{H}''-\mathscr{G}''$ is similarly open relative to $\mathscr{E}$.

## § 8. Final reduction to problems of type $K$

In this section we shall transform a problem

$$[\Phi, S_{n-1}, \mathscr{M}_{n-1}]_3 \tag{8.0}$$

of the third class by means of a reflection $t$ and define thereby a problem of type $K$ of the form

$$[\omega, H', \mathscr{H}']_K. \tag{8.1}$$

*The $n$-cube $K$.* The $(n-1)$-sphere $S_{n-1}$ has a unit radius. The coordinates of its center will be determined by our construction but are in fact immaterial. Let $S_c$ be an $(n-1)$-sphere in $E$ with center at the $x_n$-pole $Q$ of $S_{n-1}$, and with radius $c < 1$. We shall presently further condition the choice of $c$. The $(n-2)$-sphere $S_c \cap S_{n-1}$ lies in an $(n-1)$-plane $\Pi_c$. See Fig. 1. Let $P_c$ be the

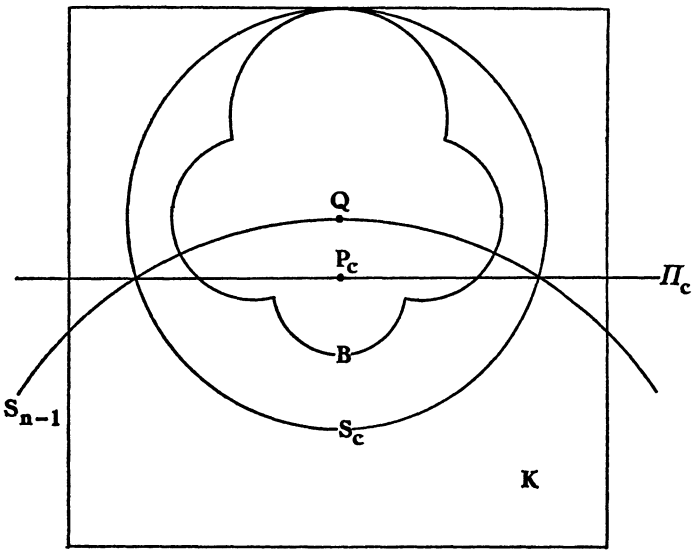


Fig. 1, $n=2$, $B = t(\beta_E K)$

intersection of $\Pi_c$ with the $x_n$-axis. Of the $n$-cubes with center at $P_c$ and with $(n-1)$-faces parallel to the coordinate $(n-1)$-planes let $K_c$ be the smallest $n$-cube which includes $S_c$. Let $N_Q$ be an open spherical neighborhood of $Q$ relative to $E$ such that

$$\Phi|N_Q = I|N_Q \tag{8.2}$$

where $\Phi$ is the mapping given in the problem (8.0). Let $c$ be so small that $K_c \subset N_Q$. We suppose $c$ so chosen and fixed hereafter. We then set $K_c = K$ and take $P_c$ as the origin of coordinates in $E$.

*The reflections* $t$ and $\tau$. Let $t$ be the reflection of $E-Q$ in $S_c$, recalling that the center of $S_c$ is the $x_n$-pole $Q$ of $S_{n-1}$. Set $IQ = \mathcal{Q}$ and let $\tau$ be the reflection of $\mathcal{E}-\mathcal{Q}$ in $IS_c$. Both $t$ and $\tau$ are involutions, and $\tau I = It$. Note that $t = t^{-1}$ and $\tau = \tau^{-1}$.

*The choice of* $\omega'$. Recall that the domain of definition of $\Phi$ in (8.0) is an open neighborhood $N$ of $S_{n-1}$ which includes the neighborhood $N_Q$ of $Q$. Without loss of generality we can suppose that $N$ is arc-wise connected. Set $\Gamma = N-Q$ and

$$\omega' = \tau\Phi t^{-1} \qquad \text{(on } t\Gamma). \tag{8.3}$$

We shall prove (a) and (b).

(a). *The mapping $\omega'$ reduces to $I$ on a neighborhood of $C\mathring{K}$ relative to $E$.*

Now $\omega'$ reduces to $I$ on the set $R = t(N_Q - Q)$. In fact if $p \in R$, then $p = t(q)$, with $q \in N_Q - Q$, so that

$$\omega'(p) = \tau.\ \Phi\ .\ t^{-1}(p) = \tau\ .\ \Phi(q) = \tau\ .\ I(q) = I\ .\ t(q) = I(p).$$

Referring to the definition of $K$, set

$$A = J[t(\beta_E K)] \quad \text{[Cf. § 0 for def of } J]. \tag{8.4}$$

and recall that $N_Q \supset K \supset A$. It follows that $N_Q$ is a neighborhood of $A$ relative to $E$, and $N_Q - Q$ a neighborhood of $A - Q$ relative to $E - Q$. We infer from (8.4) that

$$t(A-Q) = C\mathring{K}.$$

Hence $t(N_Q - Q)$ is a neighborhood of $C\mathring{K}$ and (a) is proved.

Let $K'$, $K''$, $k$, $k_0$ be subsets of $\mathring{K}$, defined as in § 7. The $t$ image of $S_{n-1} - Q$ is the $(n-1)$-plane $\Pi_c$ passing through the origin. Hence $t(S_{n-1} - Q) \supset k_0$. We continue by proving (b).

(b). *The domain of definition, $t\Gamma$, of $\omega'$, is arc-wise connected and has the form*

$$t\Gamma = E - G, \tag{8.5}$$

*where $G$ is a compact subset of $K' \cup K''$ with $Q$ contained in $G \cap K''$.*

Recall that $\Gamma \cup Q$ is the domain of definition of $\Phi$ so that

$$\Gamma \supset S_{n-1} - Q.$$

On applying $t$ to the members of this inclusion we find that

$$t\Gamma \supset k_0.$$

Since $t\Gamma$ is open, and, according to (a), includes $k_0$ and a neighborhood of $C\mathring{K}$, it follows that (8.5) holds with $G$ a compact subset of $K' \cup K''$. Since $t\Gamma$ does not contain $Q$, and $K''$ does, we infer that $Q$ is in $G \cap K''$.

Set $G' = K' \cap G$, $G'' = K'' \cap G$ and define $H'$ and $H''$ as in § 7. On setting

$$\omega = \omega' | (\mathring{K} - G)$$

one can further define $\mathscr{H}$, $\mathscr{H}'$, $\mathscr{H}''$, $\mathscr{G}$, $\mathscr{G}'$, $\mathscr{G}''$ as in § 7. A problem

$$[\omega, H', \mathscr{H}']_K \tag{8.6}$$

of type $K$ is now defined in the sense of § 7. We accordingly turn to the following lemma.

LEMMA 8.1. *On making the problem* (8.0) *of the third class correspond as above to the problem* (8.6) *of type* $K$, *there is defined an effective mapping of the third class of problems into the class of problems of type* $K$.

Suppose there exists a solution

$$\lambda_\omega : H' \to \mathscr{H}'$$

of problem (8.6). In accord with (7.13) and the definition of $\omega$ as a restriction of $\omega'$

$$\text{(8.7)} \qquad \lambda_\omega|(H'-\Omega) = \omega'|(H'-\Omega).$$

Now $\omega'$ is defined on $E-G'-G''$ and $\lambda_\omega$ on $H'$. We can extend $\lambda_\omega$ by a $C_0^\infty$-diffeomorphism $\mu_\omega$, such that

$$\text{(8.7)}' \qquad \mu_\omega|H' = \lambda_\omega|H'$$

$$\text{(8.7)}'' \qquad \mu_\omega|(E-G''-\Omega) = \omega'|(E-G''-\Omega).$$

The mapping $\mu_\omega$ is thereby defined on

$$H' \cup (E-G''-\Omega) = E-G''.$$

It is defined twice on

$$(E-G''-\Omega) \cap H' = H'-\Omega,$$

but consistently because of (8.7). The sets $H'$ and $E-G''-\Omega$ are open so that $\mu_\omega$ is a $C_0^\infty$-diffeomorphism. The "exceptional" point of $\mu_\omega$ (if such exists) is in $\Omega$. Cf. (8.7)''. We now set

$$\text{(8.8)}' \qquad \Lambda_\Phi = \tau\mu_\omega t^{-1} \qquad [\text{on } t(E-G'')]$$

and complete the definition of $\Lambda_\Phi$ by setting

$$\text{(8.8)}'' \qquad \Lambda_\Phi(Q) = Q$$

*We shall prove that* $\Lambda_\Phi$, *so defined, is a solution of problem* (8.0). *To that end we establish* (c), (d), *and* (e).

(c). *The domain of definition of* $\Lambda_\Phi$ *is an open neighborhood of* $JS_{n-1}$.

Recall that $E-Q$ is the union of the disjoint sets

$$\text{(8.9)} \qquad S_{n-1}-Q,\ \mathring{J}S_{n-1},\ C(JS_{n-1})$$

where $\mathring{J}S_{n-1}$ denotes the interior of $JS_{n-1}$. The space $E$ is also the union of the sets

$$\text{(8.10)} \qquad [x_n = 0] \quad [x_n < 0] \quad [x_n > 0].$$

Moreover the images under $t$ of the sets (8.9) are the respective sets (8.10) and conversely. The domain of definition of $\Lambda_\Phi$ is

$$\text{(8.11)} \qquad Q \cup t(E-G'') = E-t(G'').$$

Since $G''$ is a compact subset of $[x_n > 0]$ we infer that $\Lambda_\Phi$ is defined on an open neighborhood of $\mathring{J}S_{n-1}$.

(d). *The mapping $\Lambda_\Phi$ is a $C_0^\infty$-diffeomorphism of its domain of definition into $\mathscr{E}$, with "exceptional point" (if any exists) in $\mathring{J}S_{n-1}$.*

That $\Lambda_\Phi$ defines a $C_0^\infty$-diffeomorphism of $t(E-G'')$ into $\mathscr{E}-\mathscr{Q}$ appears from (8.8)′. In the domain of definition of $\Lambda_\Phi$ there remains the point $Q$. Since $E-G''$ contains all points in $E$ whose distance from $Q$ is sufficiently large

$$t(E-G'') \supset (H_Q-Q)$$

provided $H_Q$ is a sufficiently small neighborhood of $Q$ relative to $E$. According to (a), $\omega'$ reduces to $I$ outside of a sufficiently large $(n-1)$-sphere in $E$ with center at the origin, and by virtue of (8.7)″, $\mu_\omega$ does likewise. Hence $\mu_\omega$ reduces to $I$ on $t^{-1}(H_Q-Q)$ if $H_Q$ is sufficiently restricted. For such $H_Q$ it follows from (8.8)′ that $\Lambda_\Phi$ reduces to $I$ on $H_Q-Q$. Since $\Lambda_\Phi(Q)$ is defined as $I(Q)$ we conclude that $\Lambda_\Phi$ is a $C_0^\infty$-diffeomorphism of its domain of definition, with exceptional point (if any exists) on

$$t\,\Omega \subset tH' \subset \mathring{J}S_{n-1}.$$

In accord with the definition in § 6 of a problem (8.0) of the third class it remains only to show that for some open neighborhood $Z$ of $S_{n-1}$ relative to $E$

$$\Lambda_\Phi|Z = \Phi|Z \tag{8.12}$$

To this end we refer to the compact subset $\Omega$ of $H'$ introduced in (7.13), and prove the following.

(e). *An open neighborhood of $S_{n-1}$ relative to $E$ on which* (8.12) *holds is afforded by the set*

$$Z = t(E-\Omega-G'') \cup Q = E-t\Omega-tG''. \tag{8.13}$$

Since $\Omega$ and $G''$ are compact subsets of $[x_n < 0]$ and $[x_n > 0]$ respectively, it follows from the fact that the $t$-images of the sets (8.10) are the respective sets (8.9) that $Z$ as defined by (8.13) is an open neighborhood of $S_{n-1}$ relative to $E$. That (8.12) holds when $Z$ is given by (8.13) is seen as follows. On $Z-Q$

$$\Lambda_\Phi = \tau\mu_\omega t^{-1} = \tau\omega' t^{-1} = \Phi$$

by virtue of (8.8)′, (8.7)″ and (8.3), respectively, provided the domains of validity of (8.8)′, (8.7)″ and (8.3) permit this application. That (8.8)′ may be so applied is clear since $E-\Omega-G'' \subset E-G''$. The application of (8.7)″ is exactly as written, while (8.3) may be applied since $E - \Omega - G'' \subset t\Gamma$ by virtue of the relations $\Omega \supset G'$ and (8.5). Finally $\Lambda_\Phi(Q) = I(Q) = \Phi(Q)$.

Thus $\Lambda_\Phi$, as defined by (8.8)′ and (8.8)″, is a solution of problem (8.0), and Lemma 8.1 is established.

# PART II. CONSTRUCTION OF A SOLUTION

## § 9. The mappings $R, \mathscr{R}, T_r, \mathscr{T}_r$

In this section we shall define certain mappings essential for the construction of a solution of a problem of type $K$. As indicated in a Note in § 7 it will be sufficient to take $K$ as the special $n$-cube

$$(9.1) \qquad (-1 \leqq x_i \leqq 1) \qquad (i = 1, \ldots, n).$$

The sets $\mathring{K}, K', K'', H, H', H'', \mathscr{K}, \mathring{\mathscr{K}}, \mathscr{K}', \mathscr{K}'', \mathscr{H}, \mathscr{H}', \mathscr{H}''$ shall be sets associated with such a $K$ in § 7. Recall that $H'$ and $H''$ are the subsets of $K'$ and $K''$ on which $x_n < -d$ and $x_n > d$ respectively.

*The radial transformation* $R$. Let $R$ be a radial transformation of $E$ onto $E$ in which a point $(x_1, \ldots, x_n)$ has an image $(y_1, \ldots, y_n)$ such that

$$(9.2) \qquad y_1 - 8 = \frac{x_1 - 8}{2}, \quad y_r = \frac{x_r}{2} \qquad (r = 2, \ldots, n).$$

The point $P = (8, 0, \ldots, 0)$ in $E$ is fixed under $R$. The image $R(K)$ of $K$ under $R$ is an $n$-cube of breadth 1, with center at the point $(4, 0, \ldots, 0)$ in $E$.

*The mapping* $T$. This mapping is essential in setting up our modification of the Mazur construction. In enumerating its characteristic properties we shall denote by Int $A$ the smallest product of $n$ subintervals of the respective coordinate axes of $E$ which contains a given bounded subset $A$ of $E$. Let $\mathbf{u}$ denote the mapping of $E$ onto $E$ by the identity. The mapping $T$ shall be a $C^\infty$-diffeomorphism of $E$ onto $E$ with the properties (9.3)—(9.8).

| | |
|---|---|
| (9.3) | $T \vert H'' = \mathbf{u} \vert H''$ |
| (9.4) | $T \vert H' = R \vert H'$ |
| (9.5) | $RT(K) \cap T(K) = \emptyset$ |
| (9.6) | $T(\mathring{K}) \subset \text{Int.}\,[\mathring{K} \cup R(\mathring{K})]$ |
| (9.7) | $T[x_n \leqq 0] \subset [x_n \leqq 0]$ |
| (9.8) | $T[x_n \geqq 0] \subset [x_n \geqq 0]$ |

We shall define such a mapping $T$ by a composition $\rho_3\rho_2\rho_1$ of three $C^\infty$-diffeomorphisms of $E$ onto $E$.

*The mapping* $\rho_1$. Let $\eta$ be a mapping of class $C^\infty$ of the $t$-axis onto the interval $[0, 1]$ such that $\eta(t) = 0$ for $t < .5$ and $\eta(t) = 1$ for $t > 1$. In terms of the above constant $d$, let $a < d^2$ be a positive constant presently to be conditioned as in (I). Set $\alpha(t) = 1$ for $t \leqq 0$ and

$$\alpha(t) = \eta\left(\frac{t^2}{a}\right) + 1 \qquad [t \geqq 0]. \tag{9.9}$$

The mapping $\rho_1$ shall have the form

$$y_i = \alpha(x_n)\frac{x_i}{2} \qquad (i = 1, \ldots, n). \tag{9.10}$$

It reduces to $\mathbf{u}$ for $x_n^2 \geqq a$ and $x_n > 0$, and to the mapping

$$y_i = \frac{x_i}{2} \qquad (i = 1, \ldots, n) \tag{9.11}$$

for $x_n \leqq 0$. We continue by proving (I).

(I). *If* $a > 0$ *is sufficiently small* $\rho_1$ *is a* $C^\infty$*-diffeomorphism of* $E$ *onto* $E$.

To establish (I) it is clearly sufficient to show that the mapping

$$y_n = \alpha(x_n)\frac{x_n}{2}$$

is a $C^\infty$-diffeomorphism of the $x_n$-axis onto itself. To that end note that

$$y_n'(x_n) = \alpha'(x_n)\frac{x_n}{2} + \frac{\alpha(x_n)}{2} \tag{9.12}$$

$$\alpha'(x_n) = 2\eta'\left(\frac{x_n^2}{a}\right)\frac{x_n}{a}. \tag{9.13}$$

From (9.13) it follows that $\alpha'(x_n) = 0$ when $x_n^2 \geqq a$, and accordingly that $|\alpha'(x_n)|$ is bounded independently of the choice of $a$. Formula (9.12) now shows that for $a > 0$ sufficiently small, $y_n'(x_n)$ has the sign of $\alpha(x_n)$, that is, is positive, Statement (I) follows.

*The mapping* $\rho_2$. This mapping is defined as the transformation

$$\begin{aligned} y_1 &= x_1 + 2 - 2\eta\left(\frac{x_n}{d}\right) \\ y_r &= x_r. \end{aligned} \qquad (r = 2, \ldots, n) \tag{9.14}$$

It is clearly a $C^\infty$-diffeomorphism of $E$ onto $E$. It reduces to the mapping

$$y_1 = x_1+2, \quad y_r = x_r \qquad (r = 2, \ldots, x_n) \tag{9.15}$$

for $x_n \leqq 0$, and to $\mathbf{u}$ for $x_n \geqq d$.

*The mapping* $\rho_3$. This mapping is defined as the transformation

$$\begin{aligned} y_1 &= x_1+2\eta\left(-\frac{2x_n}{d}\right) \\ y_r &= x_r \end{aligned} \qquad (r = 2, \ldots, n)$$

and is a $C^\infty$-diffeomorphism of $E$ onto $E$. It reduces to $\mathbf{u}$ for $x_n \geqq 0$ and to the mapping (9.15) for $x_n < -d/2$.

LEMMA 9.1. *The mapping* $T = \rho_3\rho_2\rho_1$ *has the properties* (9.3)—(9.8).

Relation (9.3) is immediate since $x_n > d$ on $H''$ and each of the mappings $\rho_1$, $\rho_2$, and $\rho_3$ reduces to $\mathbf{u}$ for $x_n > d$.

To verify (9.4) write the mapping $R$ in the form

$$y_1 = \frac{x_1}{2}+4, \quad y_r = \frac{x_r}{2} \qquad (r = 2, \ldots, n) \tag{9.16}$$

and observe that when $x_n \leqq -d$, $T$ reduces to a composition of the mappings (9.11), (9.15), and (9.15), applied in the order written, that is to (9.16).

We verify (9.5) as follows. The ranges of $x_1$ on the images of $K''$ under $\rho_1$, $\rho_2\rho_1$, $T$, $RT$ are included respectively in the intervals $(-1, 1)$, $(-1, 3)$, $(-1, 3)$, $(3.5, 5.5)$. Since the latter two intervals do not intersect we infer that

$$RT(K'') \cap T(K'') = \emptyset. \tag{9.17}$$

The ranges of $x_1$ on the images of $K'$ under $\rho_1$, $\rho_2\rho_1$, $T$, $RT$ are included respectively in the intervals $(-.5, .5)$, $(1.5, 2.5)$, $(1.5, 4.5)$, $(4.75, 6.25)$. Since the last two intervals do not intersect it follows that

$$RT(K') \cap T(K') = \emptyset. \tag{9.18}$$

The relations (9.7) and (9.8) are obviously valid. From these results, from (9.17), (9.18) and the continuity of $T$, (9.5) follows.

The validity of (9.6) is immediate. One notes that $\text{Int}[\mathring{K} \cup R\mathring{K}]$ is the $n$-interval

$$(-1 < x_1 < 4.5), \ (-1 < x_r < 1) \ \ (r = 2, \ldots, n).$$

The range of $x_1$ on the image of $\mathring{K}$ under $T$ is included in $(-1, 3) \cup (1.5, 4.5) = (-1, 4.5)$ and (9.6) follows.

*The mappings* $R^r$, $T_r$, $\mathscr{R}^r$, $\mathscr{T}_r$. The $r$-th iterate of the mapping $R$ will be denoted by $R^r (r = 1, 2, \ldots)$. When $r = 0$ we understand that $R^0 = \mathbf{u}$. The inverse of $R^r$ will be denoted by $R^{-r}$. We shall also set

$$R^r T = T_{r-1} \qquad (r = 0, 1, \ldots). \tag{9.19}$$

Observe that $T = T_1$, and that

$$(9.20)' \qquad T_{r+1}|H'' = R^r|H''$$
$$(9.20)'' \qquad T_{r+1}|H' = R^{r+1}|H' \qquad (r = 0, 1, \ldots)$$

in accord with (9.3) and (9.4).

We shall consider the sequence of subsets of $E$

$$T_1(\mathring{K}),\ T_2(\mathring{K}), \ldots. \tag{9.21}$$

As $r \uparrow \infty$ the set $T_r(\mathring{K})$ tends uniformly to the center $P = (8, 0, \ldots, 0)$ of the radial mapping $R$. Each of the sets (9.21) is included in the $n$-interval

$$(-1 < x_1 < 8),\ (-1 < x_j < 1) \qquad (j = 2, \ldots, n).$$

Moreover the sets in (9.21) are disjoint. More explicitly:

(a) *For integers* $r$ *and* $p$ *such that* $r > p \geqq 1$

$$T_r(\mathring{K}) \cap T_p(\mathring{K}) = \emptyset. \tag{9.22}$$

The relation (9.22) has already been established when $p = 1$ and $r = 2$. See (9.5). To establish (9.22) in the general case recall that the range of $x_1$ on $T(K'')$ is included in the interval $(-1, 3)$ of the $x_1$-axis. The images of this interval under $R^r$, $r = 0, 1, \ldots$ are disjoint for different integers $r$. It follows that

$$T_r(K'') \cap T_p(K'') = \emptyset \tag{9.23}$$

for arbitrary integers $r > p \geqq 1$. On recalling that the range of $x_1$ on $T(K')$ is included in the interval $(1.5, 4.5)$. one finds similarly that the images of this interval under $R^r$, $r = 0, 1, \ldots$ are disjoint for different integers $r$. It follows that

$$T_r(K') \cap T_p(K') = \emptyset \qquad (r > p \geqq 1). \tag{9.24}$$

Relation (9.22) follows from (9.23) and (9.24), taking account of (9.7) and (9.8).

We shall need the following lemma.

LEMMA 9.2. *For an arbitrary integer* $r > 0$, *and for arbitrary subsets* $A$ *and* $B$ *of* $\mathring{K}$ *and* $H$, *respectively*

$$(9.25)' \qquad \bigcup_{p=0}^{\infty} R^p(B) \cap T_r(A) = T_r(A \cap B) \qquad (r > 0).$$

*Moreover*

$$H' \cap T_r(\mathring{K}) = \emptyset \qquad (r > 0). \tag{9.25)''$$

Set $B' = H' \cap B$, $B'' = H'' \cap B$. It will be sufficient to prove (9.25)′ in the special cases $B = B'$ and $B = B''$.

Suppose that $B = B''$ and note that $R^p(B'') = T_{p+1}(B'')$ by (9.20), so that

$$R^p(B'') \cap T_r(A) = T_{p+1}(B'') \cap T_r(A).$$

This intersection is $\emptyset$ if $r \neq p+1$, by (9.22). However,

$$T_r(B'') \cap T_r(A) = T_r(A \cap B''),$$

since $T_r$ is biunique. Thus (9.25)′ holds when $B = B''$.

Suppose that $B = B'$. Observe that

$$\begin{aligned} R^p(B') \cap T_r(A) &= R^{-1}[R^{p+1}(B') \cap T_{r+1}(A)] \\ &= R^{-1}[T_{p+1}(B') \cap T_{r+1}(A)] = \emptyset \end{aligned} \tag{9.26}$$

unless $r = p$. Moreover

$$R^r(B') \cap T_r(A) = T_r(B') \cap T_r(A) = T_r(A \cap B') \quad (r > 0).$$

It follows that (9.25)′ holds as stated.

The relation (9.25)″ is a consequence of (9.26) with $p = 0$ therein.

*The radial mappings* $\mathcal{R}^r$. Set

$$I(P) = \mathcal{P} \quad \mathcal{R}I = IR \tag{9.27}$$

introducing $\mathcal{P}$ and $\mathcal{R}$. The mapping $\mathcal{R}$ is a radial transformation of $\mathcal{E}$ with center at $\mathcal{P}$ and could be defined in terms of the cartesian coordinates of $\mathcal{E}$ as $R$ was defined on $E$. The $r$-th iterate of $\mathcal{R}$ will be denoted by $\mathcal{R}^r$ understanding that $\mathcal{R}^r$ is the identity when $r = 0$. The inverse of $\mathcal{R}^r$ will be denoted by $\mathcal{R}^{-r}$.

*The mappings* $\mathcal{T}_r$. The relations (9.20) will have a partial analogue on $\mathcal{E}$ if we define $\mathcal{T}_p$, $p = 1, 2, \ldots$ over the subset $\mathcal{H}$ of $\mathcal{E}$ as follows:

$$\mathcal{T}_{r+1}|\mathcal{H}'' = \mathcal{R}^r|\mathcal{H}'' \tag{9.28)'$$

$$\mathcal{T}_{r+1}|\mathcal{H}' = \mathcal{R}^{r+1}|\mathcal{H}' \qquad (r = 0, 1, \ldots). \tag{9.28)''$$

The mapping $\mathcal{T}_r$ is a $C^\infty$-diffeomorphism of $\mathcal{H}$ into $\mathcal{E}$.

*An extension of* $\omega$. The $C^\infty$-diffeomorphism $\omega$ was defined in § 7 as a restriction of $\omega'$ to $\mathring{K}$—$G$ We shall here give an extension of $\omega$ to the set

$$\bigcup_{r=0}^{\infty} R^r(\mathring{K}-G) \tag{9.29}$$

by setting (for $r = 0, 1, \ldots$).

$$\omega \,.\, R^r(p) = \mathscr{R}^r \,.\, \omega(p) \qquad (p \in \mathring{K}-G). \tag{9.30}$$

This extension is of course unrelated in general to the extension $\omega'$ of $\omega$. The extended $\omega$ is a $C^\infty$-diffeomorphism of the set (9.29) onto the set

$$\bigcup_{r=0}^{\infty} \mathscr{R}^r(\mathring{\mathscr{K}}-\mathscr{G}) \qquad \text{[by (7.10)]}. \tag{9.31}$$

We add a useful lemma.

LEMMA 9.3. $\omega T_{r+1} = \mathscr{T}_{r+1}\omega$ (on $H-G$).

To establish this relation refer to the definition of $\mathscr{T}_{r+1}$ and note that in accord with (9.30)

$$\mathscr{T}_{r+1}\omega = \mathscr{R}^r\omega = \omega R^r \qquad (\text{on } H''-G'')$$

$$\mathscr{T}_{r+1}\omega = \mathscr{R}^{r+1}\omega = \omega R^{r+1} \qquad (\text{on } H'-G').$$

On adding the extreme members of these equations the relation results.

In Lemmas 9.2 and 9.3 the mapping $T$ has been taken as a $C^\infty$-diffeomorphism of $E$ onto $E$ with the properties (9.3) to (9.8). For the sake of a proof in § 16 we find it useful to suppose that $T$ is, in particular, the mapping $T = \rho_3\rho_2\rho_1$ as in Lemma 9.1. To state a lemma concerning this mapping let $\varepsilon$ be a constant such that $-1 < \varepsilon < 0$ and introduce the subinterval of $\mathring{K}$,

$$K^\varepsilon : (\varepsilon \leqq x_i < 1) \qquad (i = 1, \ldots, n).$$

We shall be concerned with the subset $\mathring{K}-K^\varepsilon$ of $\mathring{K}$.

LEMMA 9.4. *The mapping $T = \rho_3\rho_2\rho_1$ has the following property. If $(x) \in \mathring{K}$ and if $T(x) \in \mathring{K}-K^\varepsilon$, then $(x) \in \mathring{K}-K^\varepsilon$.*

Set $T(x) = (y)$. Now $(y) \in \mathring{K}-K^\varepsilon$. Hence $-1 < y_i < \varepsilon$ for at least one integer $i$ on the range $1, \ldots, n$. The form of $\rho_3\rho_2\rho_1$ shows that

$$y_i = \alpha_n(x_n)\frac{x_i}{2} + h_i(x_n) < \varepsilon$$

where $h_i(x_n) \geqq 0$. We infer that

$$\alpha_n(x_n)\frac{x_i}{2} < \varepsilon.$$

Since $1 \leqq \alpha_n(x_n) \leqq 2$ it follows that $x_i < \varepsilon$. Hence $(x)$, given in $\mathring{K}$, is in $\mathring{K}-K^\varepsilon$.

## § 10. The composite n-manifold X

Composite $n$-manifolds based on $(E, \mathscr{E})$ are defined in § 2. In this section a special manifold $\mathbf{X}$ of this character will be defined. In the next section we shall show that $\mathbf{X}$ is the $C^\infty$-diffeomorph of $\mathscr{E}-\mathscr{P}$.

In defining $\mathbf{X}$ we shall make use of the points $P$ and $\mathscr{P}$ appearing in § 9 as the centers of the radial transformations $R$ and $\mathscr{R}$ of $E$ and $\mathscr{E}$ respectively. We shall refer to the special $n$-cube $K$ of the form (9.1), to the mapping $\omega$, to $\mathscr{K}$, to subsets of $K$ and $\mathscr{K}$ defined in § 7, and to the extension of $\omega$ over the set (9.29). We suppose $\mathbf{X}$ represented in the canonical form,

(10.1) $$\mathbf{X} = [M, \mathscr{M}, \mu, W, \mathscr{W}], \qquad \text{[Cf. (1.12)]}$$

and define $M, \mathscr{M}, \mu, W, \mathscr{W}$ by setting

| | |
|---|---|
| (10.2) | $W = \bigcup_{r=0}^{\infty} R^r(H-G)$ |
| (10.3) | $M = E-P- \bigcup_{r=0}^{\infty} R^r(G)$ |
| (10.4) | $\mathscr{W} = \bigcup_{r=0}^{\infty} \mathscr{R}^r(\mathscr{H}-\mathscr{G})$ |
| (10.5) | $\mathscr{M} \bigcup_{r=0}^{\infty} \mathscr{R}^r(\mathscr{H})$ |
| (10.6) | $\mu = \omega \vert W$ |

Observe that $M$ and $\mathscr{M}$ are open subsets of $E$ and $\mathscr{E}$ respectively, that $W$ and $\mathscr{W}$ are open subsets of $M$ and $\mathscr{M}$ respectively. That $\mu(W) = \mathscr{W}$ follows readily on using the relations

$$\mathscr{H}-\mathscr{G} = \omega(H-G), \quad \omega R^r = \mathscr{R}^r \omega$$

from (7.10) and (9.30), and the consequent relations,

$$\mu(W) = \bigcup_{r=0}^{\infty} \mathscr{R}^r . \omega(H-G) = \bigcup_{r=0}^{\infty} \mathscr{R}^r(\mathscr{H}-\mathscr{G}).$$

Thus $\mu$ is a $C^\infty$-diffeomorphism of $W$ onto $\mathscr{W}$. The composite space $\mathbf{X}$ is now defined as a topological space as in § 1.

LEMMA 10.1. *The space* $\mathbf{X}$ *is an n-manifold.*

The proof of this lemma is based on Lemma 2.2. We need the relations

(10.7) $$\beta_M(H-G) \subset \beta_E H; \; \omega'(\beta_E H) = \beta_{\mathscr{E}} \mathscr{H}$$

where $\omega'$ is defined over $E-G$ in § 7 and reduces to $\omega$ on $\mathring{K}-G$. To establish the first relation in (10.7) note that

$$\beta_M(H-G) \subset \beta_E(H-G) \subset \beta_E H \cup \beta_E G.$$

Since $G$ is closed in $E$ and included in $CM$ the first relation in (10.7) follows. Continuing the proof of (10.7) note that

$$\beta_E H = (\beta_E K \cap \beta_E H) \cup (\mathring{K} \cap \beta_E H)$$
$$\beta_{\mathscr{E}} \mathscr{H} = (\beta_{\mathscr{E}} \mathscr{K} \cap \beta_{\mathscr{E}}) \mathscr{H} \cup (\mathring{\mathscr{K}} \cap \beta_{\mathscr{E}} \mathscr{H})$$
$$I(\beta_E K \cap \beta_E H) = \beta_{\mathscr{E}} \mathscr{K} \cap \beta_{\mathscr{E}} \mathscr{H}$$
$$\omega'(\mathring{K} \cap \beta_E H) = \mathring{\mathscr{K}} \cap \beta_{\mathscr{E}} \mathscr{H}.$$

The second relation in (10.7) follows. The first relation in (10.7) has the useful extension,

$$\beta_M R^r(H-G) \subset \beta_E R^r(H) = R^r(\beta_E H) \quad (r = 0, 1, \ldots). \tag{10.8}$$

Lemma 2.2 is concerned with

$$\beta_M W = \bigcup_{r=0}^{\infty} \beta_M R^r(H-G) \subset \bigcup_{r=0}^{\infty} R^r(\beta_E H) \quad \text{[by (10.8)]}. \tag{10.9}$$

By definition $\mu = \omega | W$, and in particular $\omega \,.\, R^r(p) = \mathscr{R}^r \,.\, \omega(p)$ for $p \in \mathring{K}-G$. Hence $\mu$ admits a continuous extension $\nu$ over $\beta_M W$ such that

$$\nu \,.\, R^r(p) = \mathscr{R}^r \,.\, \omega'(p) \qquad [p \in \beta_E H,\ r \geqq 0].$$

For this extension it follows from (10.9) and (10.7) respectively that

$$\nu(\beta_M W) \subset \bigcup_{r=0}^{\infty} \mathscr{R}^r \,.\, \omega'(\beta_E H) = \bigcup_{r=0}^{\infty} \mathscr{R}^r(\beta_{\mathscr{E}} \mathscr{H}). \tag{10.10}$$

To apply Lemma 2.2 we have merely to note that the right member of (10.10) does not intersect $\mathscr{M}$.

The condition ($\gamma$) in Lemma 2.2 is accordingly satisfied. We infer that **X** is an $n$-manifold.

We understand that **X** has received a *differential structure* of class $C^\infty$ in accord with the procedure defined in § 2, making use of the differential structures of class $C^\infty$ given on $M$ and $\mathscr{M}$.

**A first partition of X.** In order to define the $C^\infty$-diffeomorphism **t** of **X** onto $E-P$ in the next section we shall partition **X** into an ensemble of disjoint subsets

$$X_{-1}, X_0, X_1, \ldots \tag{10.11}$$

of which $X_0, X_1, \ldots$ shall be *open*. These sets will be $\mu$-represented in the sense of § 1.

*The sets* $X_0, X_1, \ldots$. With the notation for subsets of a composite manifold introduced in § 1, let

$$X_r = [R^r(\mathring{K}-G),\ \mathscr{R}^r(\mathscr{H}),\ \mathbf{X}] \qquad (r = 0, 1, \ldots). \tag{10.12}$$

Since

$$R^r(\mathring{K}-G) \subset M, \quad \mathscr{R}^r(\mathscr{H}) \subset \mathscr{M}$$

$X_r$ is actually a subset of $\mathbf{X}$. To prove that $X_r$ is $\mu$-represented we must show that the general condition (1.14) is satisfied in the form appropriate for $X_r$.

The verification of this condition will be clearer if we begin with the case $r = 0$. When $r = 0$ condition (1.14) takes the form,

$$\mu[(\mathring{K}-G) \cap W] = \mathscr{H} \cap \mathscr{W}. \tag{10.13}$$

Since $(\mathring{K}-G) \cap W = H-G$, this condition may be written

$$\mu(H-G) = \mathscr{H} \cap \mathscr{W}. \tag{10.14}$$

Now $\mu(H-G) = \omega(H-G)$ by virtue of the definition of $\mu$ in (10.6). We then have

$$\omega(H-G) = \mathscr{H} - \mathscr{G} = \mathscr{H} \cap \mathscr{W}$$

in accord with (7.10) and the definition of $\mathscr{W}$ in (10.4). Thus (10.13) holds.

In the case of the general $r$ the verification of condition (1.14) is similar. In fact

$$\begin{aligned}\mu(R^r(\mathring{K}-G) \cap W] &= \omega \,.\, R^r(H-G) = \mathscr{R}^r \,.\, \omega(H-G) \\ &= \mathscr{R}^r(\mathscr{H}-\mathscr{G}) = \mathscr{R}^r(\mathscr{H}) \cap \mathscr{W}\end{aligned}$$

for reasons similar to those cited when $r = 0$. Thus $X_r$ is a $\mu$-represented subset of $\mathbf{X}$.

*The set* $X_{-1}$. This set is the complement relative to $\mathbf{X}$ of the union of the sets $X_0, X_1, \ldots$. Since the second components of these subsets of $\mathbf{X}$, as represented in (10.12) have $\mathscr{M}$ as union, it follows from Corollary 1.4 that $X_{-1}$ is $\mu$-represented in the form,

$$X_{-1} = [A, \emptyset, \mathbf{X}]. \tag{10.15}$$

where

$$A = M - \bigcup_{r=0}^{\infty} R^r(\mathring{K}-G) = E-P- \bigcup_{r=0}^{\infty} R^r(\mathring{K}) \tag{10.16}$$

The subsets $X_0, X_1 \ldots$ are open subsets of $\mathbf{X}$, since their first and second components in the representation (10.12) are open subsets of $M$ and of $\mathscr{M}$ respectively. Hence each of these subsets of $\mathbf{X}$ is a submanifold of $\mathbf{X}$ in accord with Lemma 1.4. As a submanifold of $\mathbf{X}$, $X_r$, $r = 0, 1 \ldots$ will be assigned the $C^\infty$-*differential structure* derived from $\mathbf{X}$.

*The set* $X_+$. To show that the mapping $\mathbf{t} : \mathbf{X} \to E-P$ (to be defined in the next section) is a $C^\infty$-diffeomorphism, we shall need a special open subset $X_+$ of $\mathbf{X}$ such that $X_+ \supset X_{-1}$. (Cf. Proof of Lemma 2.4). Let $\pi, \pi_1, \pi_2$ be the $\#$-mappings associated with $\mathbf{X}$, as defined in § 1. Note that $X_{-1} = \pi_1(A)$. It follows from the definition of $\omega'$ in § 7 that there exists a compact subset $B$ of $K$ such that

$$\omega|(\mathring{K}-B) = I|(\mathring{K}-B) \quad (B \supset G). \tag{10.17}$$

We now set $X_+ = \pi_1(A_+)$ where

$$A_+ = \bigcup_{r=0}^{\infty} R^r(\mathring{K}-B) \cup A = E-P-\bigcup_{r=0}^{\infty} R^r(B). \tag{10.18}$$

We see that $A_+$ is an open subset of $M$ and that $A_+ \supset A$. Hence $X_+$ is open relative to $\mathbf{X}$ and

$$X_+ = \pi_1(A_+) \supset \pi_1(A) = X_{-1}. \tag{10.18$'$}$$

We assign $X_+$ its $C^\infty$-differential structure as a submanifold of $\mathbf{X}$.

**A second partition of X.** We shall here partition $\mathbf{X}$ into an ensemble of disjoint subsets

$$Y_{-1}, Y_0, Y_1, Y_2, \ldots \tag{10.19}$$

of which $Y_0, Y_1, \ldots$ shall be *open*. These sets shall be $\mu$-represented in the sense of § 1. This partition will be used in § 12 defining a $C^\infty$-diffeomorphism $\mathbf{s}$ of $\mathbf{X}$ onto $\mathbf{X}^*$.

*The subset* $Y_0$ *of* $\mathbf{X}$. Let

$$Y_0 = [H'-G', \mathscr{H}', \mathbf{X}]. \tag{10.20}$$

It is clear that $H'-G' \subset M$ and $\mathscr{H}' \subset \mathscr{M}$. The condition (1.14) that $Y_0$ be $\mu$-represented takes the form

$$\mu[(H'-G') \cap W] = \mathscr{H}' \cap \mathscr{W}$$

or equivalently

$$\omega(H'-G') = \mathscr{H}'-\mathscr{G}',$$

a condition which is satisfied by virtue of (7.9).

*The subsets* $Y_1, Y_2, \ldots,$ *of* $\mathbf{X}$. Referring to the mappings $T_r$ and $\mathscr{T}_r$ defined in § 9, set

$$Y_r = [T_r(\mathring{K}-G), \mathscr{T}_r(\mathscr{H})\ \mathbf{X}] \qquad (r = 1, 2, \ldots). \tag{10.21}$$

We shall prove (a) and (b).

(a). *The right member of* (10.21) *defines a subset* $Y_r$ *of* $\mathbf{X}$.

To establish (a) we have merely to prove that

$$\text{(i) } T_r(\mathring{K}-G) \subset M; \quad \text{(ii) } \mathscr{T}_r(\mathscr{H}) \subset \mathscr{M}. \tag{10.22}$$

Taking into account the explicit form of $M$ as given in (10.3), (i) will follow if $T_r(\mathring{K}-G)$ does not meet $P$ (obviously the case) and if

$$T_r(\overset{\circ}{K}-G) \cap \bigcup_{p=0}^{\infty} R^p(G) = \emptyset. \tag{10.23}$$

Relation (10.23) follows from Lemma 9.2. That (10.22) (ii) holds is a consequence of the definition of $\mathscr{T}_r$ in (9.28) and of the form (10.5) of $\mathscr{M}$.

(b). *$Y_r$ is $\mu$-represented in* (10.21).

The general condition (1.14) here takes the form

$$\mu[T_r(\overset{\circ}{K}-G) \cap \overset{\circ}{W}] = \mathscr{T}_r(\mathscr{H}) \cap \mathscr{W} \quad (r = 1, 2, \ldots). \tag{10.24}$$

Making use of (10.2) and of Lemma 9.2 to reduce the left member of (10.24), and of the definitions of $\mathscr{T}_r$ and of $\mathscr{W}$ to reduce the right member of (10.24), one arrives at the equivalent conditions

$$\omega[T_r(H-G)] = \mathscr{T}_r(\mathscr{H}-\mathscr{G}) \tag{10.25}$$

$$\mathscr{T}_r \,.\, \omega(H-G) = \mathscr{T}_r(\mathscr{H}-\mathscr{G}), \tag{10.26}$$

of which (10.26) is satisfied on account of (7.10).

This establishes (b).

*The subset* $Y_{-1}$ of **X**. This set is the complement with respect to **X** of the union of the sets $Y_0$, $Y_1$ .... The union of the second components of these sets is

$$\mathscr{H}' \cup [\bigcup_{r=1}^{\infty} (\mathscr{R}^r(\mathscr{H}') \cup \mathscr{R}^{r-1}(\mathscr{H}''))] = \bigcup_{p=0}^{\infty} \mathscr{R}^p(H) = \mathscr{M}.$$

It follows from Corollary 1.4 that $Y_{-1}$ is $\mu$-represented in the form

$$Y_{-1} = [L, \emptyset, \mathbf{X}] \tag{10.27}$$

where

$$L = M - \bigcup_{r=1}^{\infty} T_r(\overset{\circ}{K}-G) - (H'-G') = E - P - \bigcup_{r=1}^{\infty} T_r(\overset{\circ}{K}) - H' \tag{10.28}$$

*The subset* $Y_+$ *of* **X**. We shall need a special open subset $Y_+$ of **X** such that $Y_+ \supset Y_{-1}$. Let $B$ be a compact subset of $\overset{\circ}{K}$ such that (10.17) holds. Set

$$L_+ = M - \bigcup_{r=1}^{\infty} T_r(B-G) = E - P - \bigcup_{r=1}^{\infty} T_r(B) - G'. \tag{10.29}$$

We see that $L_+$ is open relative to $M$ and that $L \subset L_+$. Now $Y_{-1}$, as defined by (10.27), is the set $\pi_1(L)$. We set $Y_+ = \pi_1(L_+)$ and note that

$$Y_+ = \pi_1(L_+) \supset \pi_1(L) = Y_{-1} \tag{10.30}$$

as required. The subset $Y_+$ is open relative to **X** and is assigned a $C^\infty$-differential structure derived from that of **X**.

## § 11. The $C^\infty$-diffeomorphism t of X onto $\mathscr{E}-\mathscr{P}$.

To define **t** we shall make use of Lemma 2.4 and follow the procedure outlined preceding this lemma. More explicitly we shall define homeomorphisms

$$t_{-1}, t_0, t_1, t_2, \ldots \tag{11.1}$$

of the disjoint subsets (§ 10)

$$X_{-1}, X_0, X_1, X_2, \ldots \tag{11.2}$$

of **X** onto disjoint subsets of $\mathscr{E}-\mathscr{P}$ whose union is $\mathscr{E}-\mathscr{P}$. Recall that $\mathscr{P} = I(P)$. The mappings $t_0, t_1, \ldots$ will be $C^\infty$-diffeomorphisms. We finally define a $C^\infty$-diffeomorphism $t_+$ of the subset $X_+$ of **X** (§ 10) into $\mathscr{E}-\mathscr{P}$, and show that

$$t_+|(X_+ \cap X_i) = t_i|(X_+ \cap X_i) \quad (i = -1, 0, 1, \ldots). \tag{11.3}$$

The mapping **t** defined by setting

$$\mathbf{t}|X_i = t_i|X_i \qquad (i = -1, 0, 1, \ldots). \tag{11.4}$$

Definition of $t_r$, $r = 0, 1, \ldots$. In defining $t_r$ use will be made of Lemma 2.3. We refer to the $\#$-mappings $\pi$, $\pi_1$, $\pi_2$ associated with **X** as in § 1. Let $\omega$ be extended as in (9.30). To define $t_r$ on $X_r$ the mappings $f$ and $\ell$ appearing in Lemma 2.3 are here denoted by $f_r$ and $\ell_r$ respectively, and defined by setting

$$f_r(p) = \omega(p) \qquad [p \in R^r(\mathring{K}-G)] \tag{11.5$'$}$$

$$\ell_r(q) = q \qquad [q \in \mathscr{R}^r(\mathscr{H})] \tag{11.5$''$}$$

The $C^\infty$-diffeomorphism $t_r : X_r \to \mathscr{E}-\mathscr{P}$ is then defined as in (2.11)$'$ by setting

$$t_r \cdot \pi_1(p) = f_r(p) \quad t_r \cdot \pi_2(q) = \ell_r(q). \tag{11.6}$$

Note that

$$f_r \cdot R^r(\mathring{K}-G) = \mathscr{R}^r \cdot \omega(\mathring{K}-G) = \mathscr{R}^r[\mathring{\mathscr{K}}-\mathscr{G}] \subset \mathscr{R}^r(\mathring{\mathscr{K}}) \tag{11.7}$$

by virtue of (9.30) and (7.10). We shall verify Conditions (i), (ii), (iii) of Lemma 2.3, taking $\Sigma'$ as $\mathscr{R}^r(\mathring{\mathscr{K}})$, and conclude that $t_r$ is a $C^\infty$-diffeomorphism of $X_r$ onto $\mathscr{R}^r(\mathring{\mathscr{K}})$.

Verification of (i). The general Condition (i) of Lemma 2.3 has the form $(\ell\mu)|W = f|W$ and must be applied here with $W$ replaced by

$$W_r = R^r(\mathring{K}-G) \cap W = R^r(H-G) \tag{11.8}$$

where $W$ is given in the canonical form of **X**. Condition (i) thus reduces to the condition

$$(11.9) \qquad (\ell_r \mu_r)|R^r(H-G) = f_r|R^r(H-G) \qquad [\mu_r = \mu|W_r]$$

and is satisfied in accord with the definition of $\mu$ in (10.6) and of $\ell_r$ and $f_r$ in (11.5).

VERIFICATION OF (ii). The condition $f(M-W) \cap \ell(\mathscr{M}) = \emptyset$ of Lemma 2.3 here takes the form

$$(11.10) \qquad \omega[R^r(\mathring{K}-G)-W_r] \cap \mathscr{R}^r(\mathscr{H}) = \emptyset$$

Replacing $W_r$ by $R^r(H-G)$ from (11.8), and using (9.30), condition (11.10) becomes

$$\mathscr{R}^r . \omega(\mathring{K}-H) \cap \mathscr{R}^r(\mathscr{H}) = \emptyset$$

By virtue of (7.10) this is equivalently

$$\mathscr{R}^r(\mathring{\mathscr{K}}-\mathscr{H}) \cap \mathscr{R}^r(\mathscr{H}) = \emptyset$$

and is clearly satisfied.

VERIFICATION OF (iii). The Condition (iii) of Lemma 2.3 has the general form $f(M) \cap \ell(\mathscr{M}) = \Sigma'$, and is here satisfied in the form

$$\mathscr{R}^r . \omega(\mathring{K}-G) \cup \mathscr{R}^r(\mathscr{H}) = \mathscr{R}^r(\mathring{\mathscr{K}}) \quad [\text{using (7.10)}.$$

*It follows from Lemma 2.3 that $t_r$ is a $C^\infty$-diffeomorphism of $X_r$ onto $\mathscr{R}^r(\mathring{\mathscr{K}})$, $r = 0, 1, \ldots$.*

DEFINITION OF $t_{-1}$. Recall that $X_{-1} = \pi_1(A)$ where $A$ is given in (10.16). We define $t_{-1}$ on $X_{-1}$ by setting

$$(11.11) \qquad t_{-1} . \pi_1(p) = I(p) \qquad [\text{for } p \in A]$$

and observe that $t_{-1}(X_{-1}) = I(A)$. It is clear that $t_{-1}$ is a homeomorphism of $X_{-1}$ onto $I(A)$. Taking into account the form of $A$ as given by (10.16) we see that

$$(11.12) \qquad I(A) = \mathscr{E}-\mathscr{P}-\bigcup_{r=0}^{\infty} \mathscr{R}^r(\mathring{\mathscr{K}}).$$

It may be concluded that

$$\bigcup_{i=-1}^{+\infty} t_i(X_i) = \mathscr{E}-\mathscr{P} \qquad (i = -1, 0, 1, \ldots)$$

and that the sets $t_i(X_i)$ are disjoint.

DEFINITION OF $t_+$. Recall that $X_+ = \pi_1(A_+)$ where $A_+$ is given in (10.18). We define $t_+$ over $X_+$ by setting

$$(11.13) \qquad t_+ . \pi_1(p) = I(p) \qquad [\text{for } p \in A_+].$$

PROOF OF (11.3). Note first that if for $p \in M$, $\pi_1(p)$ is in $X_i$, then $p$ must be in the first component of $X_i$, since $X_i$ is $\mu$-represented. Hence:

$$(11.14)\quad \begin{aligned} X_+ \cap X_r &= \pi_1(A_+) \cap \pi_1 \,.\, R^r(\mathring{K}-G) \quad (r = 0, 1, \ldots) \\ &= \pi_1 \,.\, R^r(\mathring{K}-B) \qquad [\text{by } (10.18)]. \end{aligned}$$

For $p \in R^r(\mathring{K}-B)$, $\pi_1(p)$ is thus in $X_r \cap X_+$. In accord with (11.5)', (11.6) and (9.30),

$$(11.15)\quad t_r \,.\, \pi_1(p) = \omega(p) = \omega \,.\, R^r \,.\, R^{-r}(p) = \mathscr{R}^r \,.\, \omega \,.\, R^{-r}(p) = I(p)$$

since $\omega$ reduces to $I$ on $\mathring{K}-B$. A comparison of (11.15) with (11.13) shows that

$$t_r|(X_+ \cap X_r) = t_+|(X_+ \cap X_r) \qquad (r = 0, 1, \ldots)$$

as required.

From (10.16) and (10.18) it follows that $A_+ \supset A$, so that $X_+ \cap X_{-1} = \pi_1(A)$. It follows from (11.11) and (11.13) that

$$t_{-1}|(X_+ \cap X_{-1}) = t_+|(X_+ \cap X_{-1}).$$

Thus $t_+$ is a $C^\infty$-diffeomorphism of $X_+$ into $\mathscr{E}-\mathscr{P}$ for which (11.3) holds. The conditions of Lemma 2.4 are thus satisfied, so that if $\mathbf{t}$ is defined by (11.4) we have the following lemma.

Lemma 11.1. *There exists a $C^\infty$-diffeomorphism* $\mathbf{t}$ *of* $\mathbf{X}$ *onto* $\mathscr{E}-\mathscr{P}$ *defined by the conditions*

$$\mathbf{t} \,.\, \pi_1(p) = \omega(p) \quad [p \in R^r(\mathring{K}-G),\ r \geqq 0]$$

$$\mathbf{t} \,.\, \pi_2(q) = q \quad [q \in \mathscr{R}^r(\mathscr{H}),\ r \geqq 0]$$

$$\mathbf{t} \,.\, \pi_1(p) = I(p) \quad [p \in A]$$

## § 12. The composite n-manifold X*.

In this section we shall define a composite $n$-manifold based on $(E, \mathscr{E})$. The manifold will be assigned a $C^\infty$-structure. In the next section we shall show that there exists a $C^\infty$-diffeomorphism $\mathbf{s}$ of $\mathbf{X}$ onto $\mathbf{X}^*$. Let

$$(12.1)\qquad \mathbf{X}^* = [M^*, \mathscr{M}^*, \mu^*, W^*, \mathscr{W}^*]$$

be a canonical representation of $\mathbf{X}^*$. We shall define $\mathbf{X}^*$ by defining the elements in the representation (12.1). Reference will be made to the $n$-cube $K$ of (9.1) and to the subsets $K'$, $H'$, $G'$, $\mathscr{H}'$, $\mathscr{G}'$ etc. associated with $K$ in § 7. The mapping $\omega'$ defined on $E-G$ in § 7 will also be used as well as the restriction $\omega = \omega'|(\mathring{K}-$ $-G)$. Set

| | |
|---|---|
| (12.2) | $W^* = H'-G', \quad M^* = E-P-G'$ |
| (12.3) | $\mathscr{W}^* = \mathscr{H}'-\mathscr{G}', \quad \mathscr{M}^* = \mathscr{H}'$ |
| (12.4) | $\mu^* = \omega \mid W^*$ |

The compatibility condition $\mu^*(W^*) = \mathscr{W}^*$ is satisfied since $\omega(H'-G') = \mathscr{H}'-\mathscr{G}'$ by (7.9). Thus $\mathbf{X}^*$ is a topological space as defined in § 1. We continue by proving Lemma 12.1.

LEMMA 12.1. *The space* $\mathbf{X}^*$ *is an n-manifold.*

This lemma will be proved by showing that $\mathbf{X}^*$ satisfies Condition ($\gamma$) of Lemma 2.2. The proof is similar to that of Lemma 10.1. We begin by showing that

$$\omega'(\beta_{M^*} W^*) \subset \beta_{\mathscr{E}}(\mathscr{H}') \tag{12.5}$$

As in the proof of (10.7)

$$\beta_E W^* \subset \beta_E H' \cup \beta_E G'; \quad \beta_{M^*} W^* \subset \beta_E H'$$

$$\omega'(\beta_{M^*} W^*) \subset \omega'(\beta_E H') = \beta_{\mathscr{E}} \mathscr{H}' \tag{12.6}$$

making use of (7.9)′. Relation (12.5) follows from (12.6).

Turning to Condition ($\gamma$) note that $\mu^*$ as defined by $\omega$ over $W^*$, and regarded as a map of $W^*$ into $\mathscr{E}$, admits $\omega'$ as a continuous extension over $\beta_{M^*} W^*$. Condition ($\gamma$) is satisfied if the right member of (12.6) does not meet $\mathscr{M}^*$. But the set $M^* = \mathscr{H}'$ is open in $\mathscr{E}$, and so does not meet $\beta_{\mathscr{E}} \mathscr{H}'$.

Lemma 12.1 follows from Lemma 2.2.

**A partition of $\mathbf{X}^*$.** We shall partition $\mathbf{X}$ into a sequence

$$Y^*_{-1}, Y^*_0, Y^*_1, \ldots \tag{12.7}$$

of disjoint subsets whose union is $\mathbf{X}^*$, and of which the sets $Y^*_0$, $Y^*_1$, ... are open. This partition will be used in defining the $C^\infty$-diffeomorphism $\mathbf{s}$ of $\mathbf{X}$ onto $\mathbf{X}^*$.

DEFINITION OF $Y^*_0$. Employing the representation of subsets of a composite manifold introduced in § 1, set

$$Y^*_0 = [H'-G', \mathscr{H}', \mathbf{X}^*] \tag{12.8}$$

noting first that $H'-G' \subset M^*$ and $\mathscr{H}' \subset \mathscr{M}^*$. The condition (1.14) that $Y^*_0$ be $\mu^*$-represented in (12.8), takes the form

$$\omega[(H'-G') \cap W^*] = \mathscr{H}' \cap \mathscr{W}^* \tag{12.9}$$

or equivalently

$$\omega(H'-G') = \mathscr{H}'-\mathscr{G}' \tag{12.10}$$

taking into account the definitions of $W$ and of $\mathscr{W}^*$. Condition (12.10) is satisfied in accord with (7.9)″. Moreover $Y^*_0$ is an open subset of $\mathbf{X}^*$ and as such will receive a $C^\infty$-differential structure.

DEFINITION OF $Y^*_r$, $r = 1, 2, \ldots$. Set

$$Y^*_r = [T_r(\mathring{K}),\ \emptyset,\ \mathbf{X}^*] \qquad (r = 1, 2, \ldots) \tag{12.11}$$

noting that $T_r(\mathring{K}) \subset M^*$, since $T_r(\mathring{K}) \cap G' = \emptyset$ by virtue of Lemma 9.2. The condition (1.14) that (12.11) be a $\mu^*$-representation of $Y^*_r$ takes the form,

$$T_r(\mathring{K}) \cap W^* = \emptyset,$$

or equivalently,

$$T_r(\mathring{K}) \cap (H' - G') = \emptyset,$$

and is satisfied in accord with Lemma 9.2. The subset $Y^*_r$ is open relative to $\mathbf{X}^*$ and will receive a $C^\infty$-differential structure from $\mathbf{X}^*$. The sets $Y^*_p$, $p = 0, 1, \ldots$, are disjoint since the intersection of any two of their second components is obviously $\emptyset$, and since the intersection of any two of their first components is likewise $\emptyset$, in accord with (9.23), (9.24) and (9.25)$''$.

DEFINITION OF $Y^*_{-1}$. This set is the complement with respect to $\mathbf{X}^*$ of the union of the sets $Y^*_0, Y^*_1, \ldots$ and can be $\mu$-represented using Corollary 1.4. The union of the second components of $Y^*_0, Y^*_1, \ldots$ is $\mathscr{H}' = \mathscr{M}^*$. The union of the first components of $Y^*_0, Y^*_1, \ldots$ is

$$\bigcup_{r=1}^{\infty} T_r(\mathring{K}) \cup (H' - G') \tag{12.12}$$

and the complement of this union with respect to $M^*$ is $L$, as given by (10.28). Hence by Corollary 1.4

$$Y^*_{-1} = [L,\ \emptyset,\ \mathbf{X}^*] \tag{12.13}$$

With $Y^*_{-1}$ so defined $\mathbf{X}^*$ is the union of the disjoint sets $Y^*_i$, $i = -1, 0, 1, \ldots$.

## § 13. The $C^\infty$-diffeomorphism s of X onto X*.

In defining $\mathbf{s}$ we follow the procedure preceding Lemma 2.4. We make use of the second partition of $\mathbf{X}$ into the union of the sets

$$Y_{-1}, Y_0, Y_1, Y_2, \ldots \tag{13.1}$$

defined in § 10, and define a sequence of homeomorphisms "onto" of the form

$$s_i : Y_i \to Y^*_i \qquad (i = -1, 0, 1, \ldots) \tag{13.2}$$

of which $s_0, s_1, \ldots$ will be $C^\infty$-diffeomorphisms. With $Y_+$ defined as in (10.30) we introduce a $C^\infty$-diffeomorphism

$$s_+ : Y_+ \to \mathbf{X}^* \tag{13.3}$$

such that

$$s_+|(Y_+ \cap Y_i) = s_i|(Y_+ \cap Y_i) \quad (i = -1, 0, 1, \ldots) \tag{13.4}$$

We finally set

$$\mathbf{s} = s_i|Y_i \qquad (i = -1, 0, 1, \ldots) \tag{13.5}$$

The sets $Y_+$, $Y_0$, $Y_1 \ldots$ are open and $\mathbf{X}$ is their union. The sets

$$Y^*_{-1}, Y^*_0, Y^*_1, Y^*_2, \ldots \tag{13.6}$$

of § 12 are disjoint and $\mathbf{X}^*$ is their union. It follows from Lemma 2.4 that $\mathbf{s}$ is a $C^\infty$-diffeomorphism of $X$ onto $\mathbf{X}^*$.

DEFINITION OF $s_0$. Recall that

$$Y_0 = [H'-G', \mathscr{H}', \mathbf{X}] \quad Y^*_0 = [H'-G', \mathscr{H}', \mathbf{X}^*] \tag{13.7}$$

by definition. Let $\pi$, $\pi_1$, $\pi_2$ be the $\#$-mappings associated with $\mathbf{X}$, and let $\pi^*$, $\pi^*_1$, $\pi^*_2$ be those associated with $\mathbf{X}^*$. Set

$$s_0 . \pi_1(p) = \pi^*_1(p), \; s_0 . \pi_2(q) = \pi^*_2(q) \quad [p \in H'-G', \; q \in \mathscr{H}'] \tag{13.8}$$

These two conditions on $s_0$ define a single-valued $s_0$ on $Y_0$, taking account of the identifications used in defining $Y_0$ and $Y^*_0$. In fact $H'-G'$ is identified under $\mu$ with a subset of $\mathscr{H}'$, since

$$(H'-G') \cap W^* = H'-G', \tag{13.9}$$

while $H'-G'$ is identified under $\mu^*$ with a subset of $\mathscr{H}'$ since

$$(H'-G') \cap W = H'-G' \tag{13.10}$$

If $p \in H'-G'$, then $p$ and $\omega(p)$ are identified both under $\mu$ and under $\mu^*$, and have the same image under $s_0$.

It follows from (13.9) that each point of $Y_0$ is the image of a point in $\mathscr{H}'$ under $\pi_2$, while (13.10) implies that each point of $Y^*_0$ is the image of a point in $\mathscr{H}'$ under $\pi^*_2$. Hence

$$Y_0 = \pi_2(\mathscr{H}') \quad Y^*_0 = \pi^*_2(\mathscr{H}') \tag{13.11}$$

*Thus* (13.5) *defines a* $C^\infty$ *diffeomorphism* $s_0$ *of* $Y_0$ *onto* $Y^*_0$.

DEFINITION OF $s_r$, $r = 1, 2 \ldots$. Recall that

$$Y_r = [T_r(\mathring{K}-G), \mathscr{T}_r(\mathscr{H}), \mathbf{X}] \qquad (r = 1, 2, \ldots) \tag{13.12$'$}$$

$$Y^*_r = [T_r(\mathring{K}), \emptyset, \mathbf{X}^*] \tag{13.12$''$}$$

To define a $C^\infty$-diffeomorphism $s_r$ of $Y_r$ onto $Y^*_r$ we shall follow the procedure of Lemma 2.3 and define a $C^\infty$-diffeomorphism $f_r$ of the first component of $Y_r$ into $Y^*_r$, and a $C^\infty$-diffeomorphism $\ell_r$ of the second component of $Y_r$ into $Y^*_r$. The $C^\infty$-diffeomorphism $s_r$ is then defined by setting

$$s_r . \pi_1(p) = f_r(p) \qquad s_r . \pi_2(q) = \ell_r(q) \tag{13.13}$$

for $p \in T_r(\mathring{K}-G)$ and $q \in \mathscr{T}_r(\mathscr{H})$.

Definition of $f_r$. To define $f_r$ we introduce a sequence of $C^\infty$-diffeomorphisms such that

$$\mathring{K}-G \to T_r(\mathring{K}-G) \to T_r(\mathring{K}) \to Y_r^* \qquad (r > 0) \tag{13.14}$$

and such that for $p \in \mathring{K}-G$

$$p \to T_r(p) \to T_r \,.\, I^{-1} \,.\, \omega(p) \to \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, \omega(p). \tag{13.15}$$

An arbitrary point $x$ in the first component of $Y_r$ has the form $T_r(p)$, and by virtue of the mapping (13.15) has an image in $Y_r^*$

$$f_r(x) = \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, \omega \,.\, T_r^{-1}(x) \qquad [x \in T_r(\mathring{K}-G)] \tag{13.16}$$

Making use of Lemma 9.3 we see that in particular

$$f_r(y) = \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, \mathscr{T}_r^{-1} \,.\, \omega(y) \qquad [y \in T_r(H-G)] \tag{13.17}$$

Definition of $\not{f}_r$. For $z$ in the second component of $Y_r$ set

$$\not{f}_r(z) = \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, T_r^{-1}(z) \qquad [z \in \mathscr{T}_r(\mathscr{H})] \tag{13.18}$$

noting that

$$T_r I^{-1}(\mathscr{H}) \subset M^* \qquad (r > 0). \tag{13.19}$$

Since $\pi_1^*$ is defined on $M^*$, $\not{f}_r$ is defined on $\mathscr{T}_r\,(\mathscr{H})$.

We now verify the conditions (i), (ii), (iii) on $f$ and $\not{f}$ in Lemma 2.3.

Condition (i). The $W$ which appears in the general condition (i) of Lemma 2.3 is here to be replaced by

$$W_r = W \cap T_r(\mathring{K}-G) = T_r(H-G), \quad [W \text{ from } (10.2)] \tag{13.20}$$

where the second equality in (13.20) follows from Lemma 9.2. Making use of the definition of $f_r$ in (13.17) and of $\not{f}_r$ in (13.18), the condition (i), $\not{f}_r \,.\, \mu(p) = f_r(p)$, $p \in W_r$ reduces to the form

$$\mu | T_r(H-G) = \omega | T_r(H-G),$$

and is satisfied by virtue of the definition of $\mu$.

Condition (ii). The Condition (ii) has the general form

$$f(M-W) \cap \not{f}(\mathscr{M}) = \emptyset$$

in Lemma 2.3. Here $M-W$ is to be replaced by

$$T_r(\mathring{K}-G)-W_r = T_r(\mathring{K}-H)$$

in accord with (13.20). Thus Condition (ii) takes the equivalent forms

$$f_r[T_r(\mathring{K}-H)] \cap \not{f}_r[\mathscr{T}_r(\mathscr{H})] = \emptyset$$

$$\omega \,.\, T_r^{-1}[T_r(\mathring{K}-H)]\mathscr{T} \cup {}_r^{-1}[\mathscr{T}_r(\mathscr{H})] = \emptyset$$

$$\omega(\mathring{K}-H) \cap \mathscr{H} = \emptyset \tag{13.21}$$

and is satisfied in accord with (7.10).

Condition (iii). According to this condition $Y^*_r$, taken as $\Sigma'$ in Lemma 2.3, should equal

$$\begin{aligned}[f_r \,.\, T_r(\mathring{K}-G)] &\cup [f_r \,.\, \mathscr{T}_r(\mathscr{H})] \\ &= [\pi^*_1 \,.\, T_r \,.\, I^{-1} \,.\, \omega(\mathring{K}-G)] \cup [\pi^*_1 \,.\, T_r \,.\, I^{-1}(\mathscr{H})] \\ &= \pi^*_1 \,.\, T_r \,.\, I^{-1}(\mathring{\mathscr{K}}) = \pi^*_1 \,.\, T_r(\mathring{K})\end{aligned}$$

using (7.10). The last member is $\mathbf{Y}^*$, so that Condition (iii) is satisfied.

*If $s_r$ is defined by* (13.13) *then Lemma* 2.3 *implies that the mapping*

$$s_r : Y_r \to Y^*_r = \pi^*_1 \,.\, T_r(\mathring{K}) \qquad (r > 0) \tag{13.22}$$

*is a $C^\infty$-diffeomorphism onto $Y^*_r$.*

Definition of $s_{-1}$. Recall that

$$Y_{-1} = [L, \emptyset, \mathbf{X}] \quad Y^*_{-1} = [L, \emptyset, \mathbf{X}^*] \tag{13.23}$$

where $L$ is given in (10.28). An arbitrary point in $Y_{-1}$ is of the form $\pi_1(p)$ with $p \in L$. The point $\pi^*_1(p)$ is then a point in $Y^*_{-1}$. We define $s_{-1}$ by setting

$$s_{-1} \,.\, \pi_1(p) = \pi^*_1(p) \qquad (p \in L) \tag{13.24}$$

So defined $s_{-1}$ is a homeomorphism of $Y_{-1}$ onto $Y^*_{-1}$.

The definition of $s_+$. Recall that $Y_+ = \pi_1(L_+)$ where

$$L_+ = M - \bigcup_{r=1}^{\infty} T_r(B-G) \supset L \qquad \text{[Cf. (10.29)]} \tag{13.25}$$

We define a $C^\infty$-diffeomorphism of $Y_+$ into $\mathbf{X}^*$ by setting

$$s_+ \,.\, \pi_1(p) = \pi^*_1(p) \qquad (p \in L_+). \tag{13.26}$$

We must show that the relations

$$s_+|(Y_+ \cap Y_i) = s_i(Y_+ \cap Y_i) \quad (i = -1, 0, 1, \ldots) \tag{13.27}$$

are satisfied.

Verification of (13.27). This verification is immediate in in case $i = -1$, as one sees on comparing (13.26) and (13.24) for $p \in L$.

*The case $i > 0$ in* (13.27). To verify (13.27) in this case note the following. If for a point $p \in M$, $\pi_1(p)$ is in $Y_r$, then $p$ must be in the first component of $Y_r$, since $Y_r$ is $\mu$ represented. Hence

$$Y_+ \cap Y_r = \pi_1(L_+) \cap \pi_1 \,.\, T_r(\mathring{K}-G) = \pi_1 \,.\, T_r(\mathring{K}-B) \qquad (r > 0) \tag{13.28}$$

Any point in $Y_+ \cap Y_r$ is thus of the form $\pi_1(x)$ with $x$ in $T_r(\mathring{K}-B)$. For such an $x$ (13.13) gives

(13.29) $$s_r \,.\, \pi_1(x) = f_r(x) = \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, \omega \,.\, T_r^{-1}(x)$$

by (13.16). But $B$ has been so chosen that

$$\omega|(\mathring{K}-B) = I(\mathring{K}-B),$$

so that $\omega$ can be replaced by $I$ in (13.29). It follows then from (13.29), so reduced, that

(13.30) $$s_r \,.\, \pi_1(x) = \pi_1^*(x) \qquad (\pi_1(x) \in Y_+ \cap Y_r)$$

Now (13.30) is in agreement with (13.26) so that (13.27) holds for $i > 0$.

*The case* $i = 0$ *in* (13.27). As in the case $i > 0$, if for $p \in M$, the point $\pi_1(p)$ is in $Y_0$, then $p$ must be in the first component of $Y_0$ since $Y_0$ is $\mu$-represented. Hence

(13.31) $$Y_+ \cap Y_0 = \pi_1(L_+) \cap \pi_1(H'-G') = \pi_1(H'-G')$$

For such a $\pi_1(p)$

(13.32) $$s_0 \,.\, \pi_1(p) = \pi_1^*(p) \qquad (\pi_1(p) \in Y_+ \cap Y_0)$$

in accord with the definition (13.8) of $s_0$. On comparing (13.32) with (13.26) we see that (13.27) holds for $i = 0$.

Thus (13.27) holds without exception, so that if **s** is defined by (13.5), it follows from Lemma 2.4 that **s** is a $C^\infty$-diffeomorphism of **X** onto **X***.

The definition of **s** is explicit in Lemma 13.1.

LEMMA 13.1. *There exists a $C^\infty$-diffeomorphism* **s** *of* **X** *onto* **X*** *defined by the following conditions*:

(13.33) $$\mathbf{s} \,.\, \pi_1(p) = \pi_1^*(p) \qquad [p \in H'-G']$$

(13.34) $$\mathbf{s} \,.\, \pi_2(q) = \pi_2^*(q) \qquad [q \in \mathscr{H}']$$

(13.35) $$\mathbf{s} \,.\, \pi_1(x) = \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, \omega \,.\, T_r^{-1}(x) \qquad [x \in T_r(\mathring{K}-G),\ r>0]$$

(13.36) $$\mathbf{s} \,.\, \pi_2(z) = \pi_1^* \,.\, T_r \,.\, I^{-1} \,.\, \mathscr{T}_r^{-1}(z) \qquad [z \in \mathscr{T}_r(\mathscr{H}),\ r > 0]$$

(13.37) $$\mathbf{s} \,.\, \pi_1(p) = \pi_1^*(p) \qquad [p \in L]$$

## § 14. A special $C^\infty$-diffeomorphism D.

The mapping **D** which we shall define in this section, taken with the mappings **t** and **s** already defined in § 11 and § 13 respectively, is essential for our derivation of a solution of a problem of type $K$.

*The mapping* **a**. To define **D** we shall need a $C^\infty$-diffeomorphism

**a** of $D$ into $E$, where $D$ is an open $n$-interval of $E$ of the form

$$D: (a < x_i < b_i) \qquad (i = 1, \ldots, n) \tag{14.1}$$

Let $c$ and $\rho$ be constants on the open interval (0, 1). We shall refer to the $n$-subinterval of $D$

$$D_c: (a < x_i < a+c(b_i-a)) \qquad (i = 1, \ldots, n) \tag{14.2}$$

Bearing in mind the fact that $\rho c$ is a constant on (0, 1), $D_{\rho c}$ is well-defined and $D \supset D_c \supset D_{\rho c}$. Observe that $D$, $D_c$, and $D_{\rho c}$ are geometrically similar $n$-intervals with common vertex $(a, a, \ldots, a) \in E$. We shall define **a** over $D$ in such a fashion that

$$\mathbf{a}(D) = D_c, \ \mathbf{a}|D_{\rho c} = \mathbf{u}|D_{\rho c} \tag{14.3}$$

where **u** is the identity on $E$.

*The mapping* ${}_c\lambda_\rho$. Let ${}_c\lambda_\rho$ map the interval (0, 1) onto the interval (0, $c$) in such a way that

$$\begin{aligned} {}_c\lambda_\rho(t) &= t \quad (0 < t \leqq \rho c) \\ {}_c\lambda_\rho'(t) &> 0 \quad (0 < t < 1) \end{aligned}$$

and ${}_c\lambda_\rho$ is of class $C^\infty$. The existence of ${}_c\lambda_\rho$ is readily established as an indefinite integral of a suitably chosen function.

Definition of **a**. The required mapping **a** may be defined as one in which $(x) \to (x')$ in $E$ with

$$x'_i - a = (b_i - a)\,{}_c\lambda_\rho\left(\frac{x_i - a}{b_i - a}\right) \qquad (i = 1, \ldots, n) \tag{14.4}$$

It is readily verified that **a** has the desired properties. It will be noted that **a** depends upon $c$ and $\rho$ as well as upon $D$. However $c$ and $\rho$ will be chosen and fixed.

*The choice of $D$ and $c$.* Recall that $K$, $H'$ and $H''$ are defined by the respective conditions

$$\begin{aligned} &K: (-1 \leqq x_i \leqq 1) && (i = 1, \ldots, n) \\ &H': (-1 < x_j < 1),\ (-1 < x_n < -d), && (j = 1, \ldots, n-1) \\ &H'': (-1 < x_j < 1),\ (d < x_n < 1), && (j = 1, \ldots, n-1) \end{aligned}$$

Recall also that $d > 0$ was chosen in § 7 so small that

$$H' \supset G', \ H'' \supset G''$$

We shall take $D$ as an $n$-interval of the form

$$(-1 < x_i < b_i) \quad (b_i > 1)(i = 1, \ldots, n) \tag{14.5}$$

such that $D$ contains the fixed point $P$ of the radial transformation $R$, and is geometrically similar to $H'$. These conditions do not uniquely determine $D$. It is however sufficient that some choice of $D$ be made. Once a choice of $D$ is made it is clear that for suitable choice of $c$, with $0 < c < 1$

$$D_c = H'. \tag{14.6}$$

We suppose $D$ and $c$ so chosen and fixed.

*The choice of* $\rho$. Since $H' \supset G'$ it is possible to choose $\rho$ such that $0 < \rho < 1$ and $|\rho - 1|$ is so small that

$$D_{\rho c} \supset G' \tag{14.7}$$

We suppose $\rho$ so chosen and fixed. We note that

$$D_{\rho c} - G' \subset H' - G' = W^* \tag{14.8}$$

*The point* $\mathbf{a}(P)$. Since $P$ is in $D$, $\mathbf{a}(P)$ is in $H'$. Since $\mathbf{a}(P) \neq P$, $\mathbf{a}(P)$ is not in the set $D_{\rho c}$, because $D_{\rho c}$ is pointwise invariant under $\mathbf{a}$. Hence

$$\mathbf{a}(P) \in H' - D_{\rho c} \subset H' - G' = W^* \tag{14.9}$$

*The set* $M^* \cap D$. Note that

$$M^* \cap D = D - P - G'; \quad \mathbf{a}(M^* \cap D) = H' - G' - \mathbf{a}(P) \tag{14.10}$$

Definition of $X^*_D$. The composite manifold $\mathbf{X}^*$ was defined in § 12. We here introduce the subset

$$X^*_D = [M^* \cap D, \mathscr{H}', \mathbf{X}^*] \tag{14.10$'$}$$

of $\mathbf{X}^*$. This set is $\mu^*$-represented. In fact the Condition (1.14) here takes the form

$$\mu^*(M^* \cap D \cap W^*) = \mathscr{H}' \cap \mathscr{W}^* \tag{14.11}$$

Since $W^*$ is included in $M^*$ and in $D$, and since $\mathscr{H}' \supset \mathscr{W}^*$, Condition (14.11) reduces to the form $\mu^*(W^*) = \mathscr{W}^*$ and is satisfied.

*The point* $\mathbf{P}^{**} \in Y^*_0$. In (12.8) we have introduced the set

$$Y^*_0 = [H' - G', \mathscr{H}', \mathbf{X}^*] = \pi^*_2(\mathscr{H}') \quad \text{[Cf. (13.11)]} \tag{14.12}$$

Observe that $X^*_D \supset Y^*_0$ since $M^* \cap D \supset H' - G'$. Set

$$\pi^*_1 \,.\, \mathbf{a}(P) = \mathbf{P}^{**} \in Y^*_0 \qquad \text{[Cf. (14.9)]} \tag{14.12$'$}$$

We shall prove the following lemma.

Lemma 14.1. *There exists a* $C^\infty$*-diffeomorphism*

$$\mathbf{D} : X^*_D \to Y^*_0 - \mathbf{P}^{**} \tag{14.13}$$

*onto* $Y^*_0 - \mathbf{P}^{**}$.

In defining $\mathbf{D}$ the procedure of Lemma 2.3 will be followed.

We identify $\Sigma'$ of Lemma 2.3 with $Y_0^* - \mathbf{P}^{**}$, define two $C^\infty$-diffeomorphisms into $Y_0^* - \mathbf{P}^{**}$,

$$(14.14) \qquad f : M^* \cap D \to Y_0^* - \mathbf{P}^{**}; \; \ell : \mathscr{H}' \to Y_0^* - \mathbf{P}^{**},$$

and show that Conditions (i), (ii), (iii) of Lemma 2.3 are satisfied. The set which should replace $W$ in Lemma 2.3 is

$$(14.15) \qquad (M^* \cap D) \cap W^* = W^*$$

so that Conditions (i), (ii), (iii) of Lemma 2.3 will take the form

$$(14.16)(\text{i}) \qquad (\ell \mu^*)|W^* = f|W^*$$

$$(14.16)(\text{ii}) \qquad \ell[(M^* \cap D) - W^*] \cap \ell(\mathscr{H}') = \emptyset$$

$$(14.16)(\text{iii}) \qquad f(M^* \cap D) \cup \ell(\mathscr{H}') = Y_0^* - \mathbf{P}^{**}$$

Once $f$ and $\ell$ have been defined and shown to satisfy the relations (14.16), we shall define $\mathbf{D}$ in accord with Lemma 2.3 by setting

$$(14.17) \qquad \mathbf{D} \,.\, \pi_1^*(p) = f(p) \qquad (p \in M^* \cap D)$$

$$(14.18) \qquad \mathbf{D} \,.\, \pi_2^*(q) = \ell(q) \qquad (q \in \mathscr{H}')$$

Definition of $f$. Since (14.10) holds we can define $f$ by setting

$$(14.19) \qquad f(p) = \pi_1^* \,.\, \mathbf{a}(p) \qquad (p \in M^* \cap D)$$

and conclude that

$$(14.20) \qquad f(M^* \cap D) = \pi_1^*(H' - G') - \mathbf{P}^{**} = \pi_1^*(W^*) - \mathbf{P}^{**}$$

($\alpha$) *Thus $f$ is a $C^\infty$-diffeomorphism of $M^* \cap D$ into $Y_0^* - \mathbf{P}^{**}$.*

Definition of $\ell$. Since $\mathscr{H}'$ is the union of the disjoint sets $\mathscr{W}^*$ and $\mathscr{G}'$, $\ell$ can be defined over $\mathscr{H}'$ by setting

$$(14.21) \qquad (\ell \mu^*)|W^* = f|W^*$$

in accord with (14.16)(i), and by setting

$$(14.22) \qquad \ell|\mathscr{G}' = \pi_2^*|\mathscr{G}'$$

Definition (14.22) taken with (14.18), implies that $\mathbf{D}$ reduces to the identity over $\pi_2^*(\mathscr{G}')$. We continue by proving ($\beta$).

($\beta$) *The mapping $\ell$ is a $C^\infty$-diffeomorphism of $\mathscr{H}'$ into $Y_0^* - \mathbf{P}^{**}$.*

Note first that $\ell|\mathscr{W}^*$ is a $C^\infty$-diffeomorphism of $\mathscr{W}^*$ into $\pi_1^*(H' - G')$ in accord with (14.21) and (14.20). To show that $\ell$ is a $C^\infty$-diffeomorphism into $Y_0^*$ it will be sufficient to recall that $\mathscr{H}' = \mathscr{W}^* \cup \mathscr{G}'$, and to exhibit an open neighborhood $\mathscr{N}$ of $\mathscr{G}'$ relative to $\mathscr{E}$ such that $\mathscr{N} \subset \mathscr{H}'$ and $\ell|\mathscr{N}$ is a $C^\infty$-diffeomorphism of $\mathscr{N}$ into $Y_0^*$.

To that end recall that $D_{\rho c} - G' \subset W^*$ by (14.8). It follows from (14.21) and (14.19) that

$$(14.23) \qquad \not{f} \cdot \mu^*(p) = \pi_1^* \cdot \mathbf{a}(p) \qquad [p \in D_{\rho c} - G']$$

Since $\mathbf{a}(p) = p$ for $p \in D_{\rho c}$, and since $D_{\rho c} - G' \subset W^*$ we conclude from (14.23) that

$$(14.24) \qquad \not{f} \cdot \mu^*(p) = \pi_1^*(p) = \pi_2^* \cdot \mu^*(p) \qquad [p \in D_{\rho c} - G']$$

We introduce an open neighborhood $\mathscr{N}$ of $\mathscr{G}'$ relative to $\mathscr{E}$ by setting

$$\begin{aligned}\mathscr{N} &= \mathscr{H}' - \omega'(Cl_E[H' - D_{\rho c}]) = \mathscr{H}' - \omega'(H' - D_{\rho c}) \text{ (Cf. Lemma 7.2)}\\ &= [\mathscr{H}' - \omega'(H' - G')] \cup \omega'(D_{\rho c} - G') = \mathscr{G}' \cap \mu^*(D_{\rho c} - G') \subset \mathscr{H}'\end{aligned}$$

Relations (14.24) and (14.22) imply that

$$\not{f} | \mathscr{N} = \pi_2^* | \mathscr{N}$$

Thus $\not{f}$ is a $C^\infty$-diffeomorphism of $\mathscr{N}$ into $\pi_2^*(\mathscr{H}') = Y_0^*$.

Finally $\not{f}$ maps $\mathscr{H}'$ into $Y_0^* - \mathbf{P}^{**}$. In fact $\not{f}(\mathscr{W}^*)$, as defined by (14.21) and (14.20), does *not* contain $\mathbf{P}^{**}$. Nor does $\not{f}(\mathscr{G}') = \pi_2^*(\mathscr{G}')$ as defined by (14.22), since

$$\mathbf{P}^{**} = \pi_1^* \cdot \mathbf{a}(P) \in \pi_1^*(W^*) = \pi_2^*(\mathscr{W}^*) \qquad \text{[Cf. (14.9)]}$$

This completes the proof of $(\beta)$.

Now that $f$ and $\not{f}$ are admissibly defined we shall verify Conditions (14.17). Of these conditions (14.16)(i) is implied by (14.21).

VERIFICATION OF (14.16)(ii). Since

$$\not{f}(\mathscr{H}') = \not{f}(\mathscr{W}^*) \cup \not{f}(\mathscr{G}') = f(W^*) \cup \pi_2^*(\mathscr{G}')$$

by (14.21) and (14.22), Condition (14.16)(ii) is equivalent to the pair of conditions

$$(14.25) \qquad f[(M^* \cap D) - W^*] \cap f(W^*) = \emptyset$$

$$(14.26) \qquad f[(M^* \cap D) - W^*] \cap \pi_2^*(\mathscr{G}') = \emptyset$$

Condition (14.25) is satisfied since $f$ is a homeomorphism. With the aid of (14.20) we see that the left member of (14.26) is included in

$$\pi_1^*(W^*) \cap \pi_2^*(\mathscr{G}') = \pi_2^*(\mathscr{W}^* \cap \mathscr{G}') = \emptyset$$

so that (14.16)(ii) holds.

VERIFICATION OF (14.16)(iii). The left member of (14.16)(iii) is included in $Y_0^* - \mathbf{P}^{**}$ in accord with $(\alpha)$ and $(\beta)$, and in turn includes

$$[\pi_1^*(W^*) - \mathbf{P}^{**}] \cup \pi_2^*(\mathscr{G}') = \pi_2^*(\mathscr{W}^* \cup \mathscr{G}') - \mathbf{P}^{**} = Y_0^* - \mathbf{P}^{**}.$$

[Cf. (14.20), (14.22)

Thus (14.16)(iii) holds.

It follows from Lemma 2.3 that **D** is a $C^\infty$-diffeomorphism of $X_D^*$ onto $Y_0^*$—$\mathbf{P}^{**}$.

This completes the proof of Lemma 14.1.

We shall be explicit in describing **D**.

LEMMA 14.2. *The* $C^\infty$*-diffeomorphism* **D** *of Lemma* 14.1 *is such that*

$$\mathbf{D} \,.\, \pi_1^*(p) = \pi_1^* \,.\, \mathrm{a}(p) \qquad [p \in M^* \cap D] \tag{14.27$'$}$$

$$\mathbf{D} \,.\, \pi_2^*(q) = \pi_2^*(q) \qquad [q \in \mathscr{G}'] \tag{14.27$''$}$$

The relations (14.27) define **D** at *each* point of $X_D^*$. For $X_D^*$ equals

$$\pi_1^*(M^* \cap D) \cup \pi_2^*(\mathscr{H}') = \pi_1^*(M^* \cap D) \cup \pi_2^*(G')$$

by virtue of the relation $\mathscr{H}' = \mathscr{W}^* \cup \mathscr{G}'$ and the inclusion in $\pi_1^*(M \cap D)$ of $\pi_1^*(W^*) = \pi_2^*(\mathscr{W}^*)$.

## § 15. Neighborhoods of P ∊ Ext X and P* ∊ Ext X*.

The point $P \in E$ and the point $\mathscr{P} = I(P) \in \mathscr{E}$ have their ordinary euclidean neighborhoods. We shall extend **X** and **X*** by adding ideal points **P** and **P*** respectively. These extensions of **X** and **X***, topologized as below, will be denoted by

$$\text{Ext } \mathbf{X}, \quad \text{Ext } \mathbf{X}^* \tag{15.0}$$

The $C^\infty$-diffeomorphism **t** of **X** onto $\mathscr{E}-\mathscr{P}$ was defined in § 11. Let $\mathbf{t}^e$ be an *extension* of **t** to Ext **X** such that $\mathbf{t}^e(\mathbf{P}) = \mathscr{P}$. After having defined a base $(N_m)$ for neighborhoods of **P** relative to Ext **X**, we shall show that $\mathbf{t}^e$ is a homeomorphism of Ext **X** onto $\mathscr{E}$.

*The neighborhoods* $N_m$ *of* **P**, $m = 0, 1, \ldots$. Set

$$N'_m = [R^m(M \cap D), \mathscr{R}^m(\mathscr{M}), \mathbf{X}] \qquad (m = 0, 1, \ldots) \tag{15.1}$$

$$N_m = N'_m \cup \mathbf{P} \tag{15.2}$$

We first show that

$$R^m(M \cap D) \subset M, \ \mathscr{R}^m(\mathscr{M}) \subset \mathscr{M} \qquad \text{[Cf. (10.1)]} \tag{15.3}$$

so that (15.1) defines a subset of **X**. Set $D-P = D'$ and note that

$$R^m(D') \supset R^r(\mathring{K}); \ R^m(D) \cap R^p(\mathring{K}) = \emptyset \tag{15.4}$$

for $m \leqq r$ and $0 \leqq p < m$ respectively. We have

$$\begin{aligned} R^m(M \cap D) &= R^m(M \cap D') \\ &= R^m[D' - \bigcup_{r=0}^{\infty} R^r(G)] \qquad (15.5) \\ &= [R^m(D') - \bigcup_{p=m}^{\infty} R^p(G)] \subset M. \end{aligned}$$

Moreover

$$\mathscr{R}^m(\mathscr{M}) = \overset{\infty}{\underset{r=m}{\cup}} \mathscr{R}_r(\mathscr{H}) \subset \mathscr{M} \tag{15.6}$$

This establishes (15.3).

It is readily shown that $N'_m$ is $\mu$-represented in (15.1). We leave the proof to the reader. We shall set $I(D) = \mathscr{D}$, $I(D') = \mathscr{D}'$ and prove the following.

LEMMA 15.1. $\mathbf{t}(N'_m) = \mathscr{R}^m(\mathscr{D}')$ $(m \geqq 0)$.

To prove this lemma we need the relations, $(r \geqq 0)$

$$X_r \cap N'_m = X_r \qquad (m \leqq r) \tag{15.7}$$

$$X_r \cap N'_m = \emptyset \qquad (m > r) \tag{15.8}$$

$$\mathbf{t}(X_r \cap N'_m) = \mathscr{R}^r(\mathring{\mathscr{K}}) \qquad (m \leqq r) \tag{15.9}$$

$$\mathbf{t}(X_r \cap N'_m) = \emptyset \qquad (m > r) \tag{15.10}$$

$$\mathbf{t}(X_{-1} \cap N'_m) = \mathscr{R}^m(\mathscr{D}') - \overset{\infty}{\underset{r=m}{\cup}} \mathscr{R}^r(\mathring{\mathscr{K}}) \qquad (m \geqq 0) \tag{15.11}$$

where the subsets $X_i$ of $\mathbf{X}$ are defined in § 10.

Equations (15.7) and (15.8) are valid since the first and second components of $X_r$ in its $\mu$-representation as a subset of $\mathbf{X}$ are included in the corresponding components of $N'_m$ when $m \leqq r$, while corresponding components do not meet when $m > r$. This follows readily from (15.4), and (15.5) in the case of first components, and is immediate in the case of second components. Relations (15.9) and (15.10) follow from (15.7) and (15.8) respectively, recalling that $\mathbf{t}(X_r) = \mathscr{R}^r(\mathring{\mathscr{K}})$. § 11.

To verify (15.11) recall that $X_{-1} = \pi_1(A)$ where

$$A = E - P - \overset{\infty}{\underset{r=0}{\cup}} R^r(\mathring{K}) \qquad \text{[Cf. (10.16)]} \tag{15.12}$$

Now $A \cap W = \emptyset$, so that whenever a point $\pi_1(p)$, $(p \in A)$ is in $N'_m$ $p$ must be in the first component of $N'_m$. Hence

$$\begin{aligned} X_{-1} \cap N'_m &= \pi_1(A \cap [R^m(M \cap D)]) \\ &= \pi_1(R^m(D') - \overset{\infty}{\underset{r=m}{\cup}} R(\mathring{K})) \end{aligned} \tag{15.13}$$

using (15.5) and (15.12). Relation (15.11) follows from (15.13) since $\mathbf{t} \,.\, \pi_1(p) = I(p)$ for $p \in A$. Cf. (11.11).

Now the union of the sets $X_i$, for $i = -1, 0, 1, \ldots$, is $\mathbf{X}$, so that

$$\overset{\infty}{\underset{i=-1}{\cup}} \mathbf{t}(X_i \cap N'_m) = \mathbf{t}(N'_m) = \mathscr{R}^m(\mathscr{D}')$$

using (15.9), (15.10) and (15.11). This establishes the lemma.

The space Ext $\mathbf{X}$ is topologized giving the points of Ext $\mathbf{X} - \mathbf{P}$

$=$ **X** their neighborhoods in **X** and taking the ensemble $(N_m)$ as a base for the neighborhoods of **P**. The reader will verify that Ext **X** is a Hausdorff space.

COROLLARY 15.1. *The mapping* $\mathbf{t}^e$ *is a homeomorphism of* Ext **X** *onto* $\mathscr{E}$.

Since **t** is a homeomorphism of **X** onto $\mathscr{E}-\mathscr{P}$ it is sufficient to note that

$$\mathbf{t}^e(N_m) = \mathscr{R}^m(\mathscr{D}) \qquad (m = 0, 1, \ldots),$$

and that $R^m(\mathscr{D})$ is a base for neighborhoods of $\mathscr{P}$ relative to $\mathscr{E}$.

*The neighborhoods* $N_m^*$ *of* **P***. As a base for the neighborhoods of **P*** relative to Ext **X*** we shall take the subsets $\mathbf{N}_m^*$ of Ext **X*** of the form $\mathbf{N}^{*\prime} \cup \mathbf{P}^*$ where

$$N_m^{*\prime} = [R^m(D'), \emptyset, \mathbf{X}^*] \qquad (m = 1, 2, \ldots) \tag{15.14}$$

Before coming to the fundamental Lemma 15.2, we establish two relations.

$$\begin{cases} T_r(\mathring{K}) \cap R^m(D') = T_r(\mathring{K}). & (m \leqq r-1) \quad (15.15)' \\ T_r(\mathring{K}) \cap R^m(D') = \emptyset. & (m \geqq r+1) \quad (15.15)'' \end{cases}$$

PROOF OF (15.15). From (9.6)

$$T(\mathring{K}) \subset \text{Int}\,[\mathring{K} \cup R(\mathring{K})] \subset D'$$

By definition $T_{r+1} = R^r T$, so that for $r > 0$

$$T_r(\mathring{K}) \subset R^{r-1}(D') \subset R^m(D') \qquad (m \leqq r-1)$$

establishing (15.15)′. Further

$$T_r(\mathring{K}) \subset \text{Int}[R^{r-1}(\mathring{K}) \cup R^r(\mathring{K})]$$

The last set does not intersect $R^{r+1}(D')$ and hence does not intersect $R^m(D')$ for $m \geqq r+1$. Thus (15.15)″ holds.

By definition (10.21)

$$Y_r = [T_r(\mathring{K}-G), \mathscr{T}_r(\mathscr{H}), \mathbf{X}] \quad (r = 1, 2, \ldots) \tag{15.16}$$

Concerning $Y_r$ we shall prove the following

$$Y_r \cap N_m' = Y_r \qquad (m \leqq r-1) \tag{15.17)'$$

$$Y_r \cap N_m' = \emptyset \qquad (m \geqq r+1) \tag{15.17)''$$

PROOF OF (15.17)′. To establish (15.17)′ we show that the two components of $Y_r$ relative to **X** are included in the corresponding components of $N_1^m$. Lemma 9.2 implies that

$$T_r(\mathring{K}-G) \cap \bigcup_{\nu=0}^{\infty} R^\nu(G) = \emptyset \tag{15.18}$$

It follows then from (15.15)′ that

$$T_r(\mathring{K}-G) \subset R^m(D') - \overset{\infty}{\underset{p=m}{\cup}} R^p(G) \qquad (m \leqq r-1) \tag{15.19}$$

The right member of (15.19) is the first component of $N'_m$ as given by (15.5). The second component of $Y_r$ is

$$\mathscr{T}_r(\mathscr{H}) = \mathscr{R}^{r-1}(\mathscr{H}'') \cup \mathscr{R}^r(\mathscr{H}') \subset \overset{\infty}{\underset{p=m}{\cup}} \mathscr{R}^p(\mathscr{H}) \tag{15.20}$$

provided $m \leqq r-1$. Since the right member of (15.20) is the second component of $N'_m$, (15.17)′ follows.

Proof of (15.17)″. If $m \geqq r+1$ it follows from (15.15)″ that the members of (15.19) do not intersect, nor do the extreme members of (15.20). Since $Y_r$ is $\mu$-represented (15.17)″ follows.

The mapping $\mathbf{s}$ of $\mathbf{X}$ onto $\mathbf{X}^*$ defined in § 13 will be given an *extension* $\mathbf{s}^e$ over Ext $\mathbf{X}$ by setting $\mathbf{s}^e(\mathbf{P}) = \mathbf{P}^*$. With this understood the basic lemma on $(N^*_m)$ follows.

Lemma 15.2. $N^*_{m+1} \subset \mathbf{s}^e(N_m) \subset N^*_{m-1}$ $(m = 2, 3, \ldots)$

*The second inclusion.* We shall introduce the set

$$L_m = R^m(D') - \overset{\infty}{\underset{r=m}{\cup}} T_r(\mathring{K}) \qquad (m = 1, 2, \ldots) \tag{15.21}$$

$$\subset [E-P-\overset{\infty}{\underset{r=1}{\cup}} T_r(\mathring{K})-H'] = L \qquad \text{[Cf. (10.28)]}$$

That $L_m \subset L$ is implied by the relations

$$R^m(D') \subset E-P;\ H' \cap R^m(D') = \emptyset \qquad (m > 0)$$

$$T_p(\mathring{K}) \cap R^m(D') = \emptyset \quad (p = 1, \ldots, m-1)(m > 1)$$

of which the last follows from (15.15)″. Noting that for $m = 0, 1, \ldots$

$$N'_m \supset \overset{\infty}{\underset{p=m+1}{\cup}} Y_p \qquad \text{[by (15.17)′]} \tag{15.22}$$

we shall show that in accord with Lemma 1.4

$$N'_m - \overset{\infty}{\underset{p=m}{\cup}} Y_p = [L_m, \emptyset, \mathbf{X}] \qquad (m > 0) \tag{15.23}$$

Proof of (15.23). One verifies the fact that the second component of the left member of (15.23) is $\emptyset$ by showing that

$$[\text{2nd comp } N'_m] \subset [\text{2nd comp } \overset{\infty}{\underset{p=m}{\cup}} Y_p], \qquad (m > 0)$$

recalling that the second component of $Y_p$ is $\mathscr{T}_p(\mathscr{H})$.

It follows that (15.23) holds if

$$[\text{1st comp } N'_m] - [\text{1st comp } \overset{\infty}{\underset{r=m}{\cup}} Y_r] = L_m \qquad (m > 0) \tag{15.24}$$

or equivalently, using (15.5) and (15.16), if

$$(15.25) \qquad [R^m(D') - \bigcup_{r=m}^{\infty} R^r(G)] - [\bigcup_{r=m}^{\infty} [T_r(\mathring{K} - G)] = L_m$$

That (15.25) holds is verified with the aid of (9.20). Hence (15.23) is valid.

It follows from (15.23) that

$$(15.26) \qquad N'_m \subset [\bigcup_{p=m}^{\infty} Y_p] \cup [L_m, \emptyset, \mathbf{X}] \qquad (m > 0)$$

We now apply $\mathbf{s}$ to the members of (15.26). Recall that $L_m \subset L$, so that $\mathbf{s} \,.\, \pi_1(L_m) = \pi_1^*(L_m)$ in accord with (13.24), and that for $p > 0$

$$\mathbf{s}(Y_p) = \pi_1^* \,.\, T_p(\mathring{K}) \qquad \text{[by (13.22)].}$$

Hence (15.26) and (15.15)′ imply that

$$\mathbf{s}(N'_m) \subset \pi_1^*[\bigcup_{p=m}^{\infty} T_p(\mathring{K}) \cup L_m] \subset \pi_1^* \,.\, R^{m-1}(D') = N^{*\prime}_{m-1}$$

for $m = 2, 3, \ldots$.

This establishes the second inclusion in the lemma.

*The first inclusion.* It follows from (15.22) and (15.23) that

$$N'_m \supset [\bigcup_{p=m+1}^{\infty} Y_p] \cup [L_m, \emptyset, \mathbf{X}]$$

$$\mathbf{s}(N'_m) \supset \pi_1^*[\bigcup_{p=m+1}^{\infty} T_p(\mathring{K}) \cup L_m] \qquad \text{[by (13.22), (13.37)]}$$

$$= \pi_1^*[R^m(D') - T_m(\mathring{K})] \supset \pi_1^* \,.\, R^{m+1}(D') = N^{*\prime}_{m+1} \qquad \text{[by (15.15′), (15.21)]}$$

since

$$R^m(D') \supset R^{m+1}(D'); \; T_m(\mathring{K}) \cap R^{m+1}(D') = \emptyset \quad \text{[by (15.15)″]}$$

The first inclusion in the lemma is thereby established.

COROLLARY 15.2. *The mapping* $\mathbf{s}^e$ *is a homeomorphism of* Ext $\mathbf{X}$ *onto* Ext $\mathbf{X}^*$.

*An extension* $\mathbf{D}^e$ *of* $\mathbf{D}$. Observe that

$$X_D^* = (M^* \cap D, \mathscr{M}^*, \mathbf{X}^*] \qquad \text{[Cf. (14.10)]}$$

is a subset of $\mathbf{X}^*$ which includes $N^{*\prime}_m$, $m > 0$, since

$$(15.27) \qquad M^* \cap D = D' - G' \supset R^m(D') \qquad (m > 0)$$

It is therefore appropriate to extend $X_D^*$ by adding the ideal point $\mathbf{P}^* \in$ Ext $\mathbf{X}$ to $X_D^*$. We denote this extension by Ext $\mathbf{X}_D^*$, and extend the topology of $X_D^*$ by regarding Ext $\mathbf{X}_D^*$ as a subset of Ext $\mathbf{X}^*$. Let the $C^\infty$-diffeomorphism $\mathbf{D} : X_D^* \to Y_0^*$ of Lemma 14.1 be given an extension over Ext $X_D^*$ by setting

$$(15.28) \qquad \mathbf{D}^e(\mathbf{P}^*) = \mathbf{P}^{**} = \pi_1^* \,.\, \mathrm{a}(P) \in Y_0^* \qquad \text{[Cf. (14.12)′]}$$

We complete Lemma 14.1 by the following.

LEMMA 15.3. *The extension* $\mathbf{D}^e$ *of* $\mathbf{D}$ *is a homeomorphism*

$$\mathbf{D}^e : \text{Ext } X_D^* \to Y_0^* \quad [\text{onto } Y_0^*].$$

Let $p$ be given in $R^m(D')$, $m > 0$. Since $p$ is then in $M^* \cap D$, $\pi_1^*(p)$ is in $X_D^*$, and in accord with Lemma 14.2

$$\mathbf{D} \,.\, \pi_1^*(p) = \pi_1^* \,.\, \mathbf{a}(p) \in Y_0^* \tag{15.29}$$

Since $\pi_1^*(p)$ represents an arbitrary point in $N_m^{*\prime}$

$$D'(N_m^{*\prime}) = \pi_1^* \,.\, \mathbf{a} \,.\, R^m(D').$$

Taken with (15.28) this gives

$$\mathbf{D}^e(N_m^*) = \pi_1^* \,.\, \mathbf{a} \,.\, R^m(D) \qquad (m = 1, 2, \ldots) \tag{15.30}$$

The base $(N_m^*)$ for neighborhoods of $\mathbf{P}^*$ relative to $X_D^*$ thus has for image the ensemble (15.30). This ensemble is clearly a base for neighborhoods of $\mathbf{P}^{**}$, relative to $Y_0^*$.

The lemma follows.

## § 16. Proof of Theorem 0.1.

It follows from Corollary 4.2 and Lemmas 6.1 and 8.1 that the first class of problems can be "effectively" mapped into the class of problems of type $K$. To establish the existence of a solution of Theorem 0.1, it is therefore sufficient to establish a solution of a problem.

$$[\omega, H', \mathscr{H}']_K \tag{16.1}$$

of type $K$. Such a problem is defined by means of Lemma 7.1. As observed in a Note in § 7 no generality is lost if the $n$-cube $K$ on which the problem (16.1) is defined is the special $n$-cube

$$(-1 \leqq x_i \leqq 1) \qquad (i = 1, \ldots, n) \tag{16.2}$$

introduced in (9.1). We continue with a lemma.

LEMMA 16.1. *In order that the problem* (16.1) *admits a solution it is sufficient that there exists a* $C_0^\infty$-*diffeomorphism*

$$\mathbf{g} : H' \to Y_0^* \qquad [\text{Cf. (12.8)}]$$

*onto the subset* $Y_0^*$ *of* $\mathbf{X}^*$ *such that for some compact subset* $\Omega$ *of* $H'$ *which includes* $G'$

$$\pi_1^*(p) = \mathbf{g}(p) \qquad (p \in H' - \Omega) \tag{16.3}$$

Assuming that $\mathbf{g}$ exists we shall define a solution $\lambda_\omega$ of problem (16.1). Recall that

$$Y_0^* = [H'-G', \mathscr{H}', \mathbf{X}^*] = \pi_2^*(\mathscr{H}') \quad \text{[Cf. (13.11)]}$$

We can then define $\lambda_\omega$ by the condition

$$\text{(16.4)} \qquad \pi_2^* \cdot \lambda_\omega(p) = \mathbf{g}(p) \qquad (p \in H')$$

So defined $\lambda_\omega$ is a $C_0^\infty$-diffeomorphism of $H'$ onto $\mathscr{H}'$.

It remains to show that $\lambda_\omega$ satisfies the boundary condition (7.13) associated with the problem (16.1). It is thereby sufficient to show that

$$\text{(16.5)} \qquad \lambda_\omega(p) = \omega(p) \qquad (p \in H'-\Omega)$$

for the set $\Omega$ given in Lemma 16.1. Now

$$\text{(16.6)} \qquad \pi_2^* \cdot \omega(p) = \pi_2^* \cdot \mu^*(p) = \pi_1^*(p) \quad [p \in H'-G' = W^*]$$

by virtue of the definitions of $\pi_1^*$, $\pi_2^*$, and $\mu^*$. From (16.3) and (16.4) we find that

$$\text{(16.7)} \qquad \pi_2^* \cdot \lambda_\omega(p) = \pi_1^*(p) \qquad (p \in H'-\Omega)$$

A comparison of (16.6) and (16.7) shows that (16.5) holds.

This establishes the lemma.

In terms of the mappings $\mathbf{t}$ of § 11, $\mathbf{s}$ of § 13 and of $I$ we shall define a mapping $\mathbf{k}$.

*The mapping* $\mathbf{k}$. A $C^\infty$-diffeomorphism

$$\text{(16.8)} \qquad \mathbf{k} : E-P \to \mathbf{X}^* \qquad [\text{onto } \mathbf{X}^*]$$

results from the sequence of $C^\infty$-diffeomorphisms,

$$\text{(16.9)} \qquad E-P \to \mathscr{E}-\mathscr{P} \to \mathbf{X} \to \mathbf{X}^*,$$

defined by $I$, $\mathbf{t}^{-1}$, $\mathbf{s}$ respectively. The $C^\infty$-diffeomorphism

$$\text{(16.10)} \qquad \mathbf{k} = \mathbf{s}\,\mathbf{t}^{-1} I$$

thereby maps $E-P$ onto $\mathbf{X}^*$.

*The n-interval Z*. We have chosen $D$ in § 14. Referring to (14.2) let $\sigma$ be a constant with $0 < \sigma < 1$ and with $|\sigma-1|$ so small that $D_\sigma$, like $D$, includes $\mathring{K}$ and the point $P$, and $\mathbf{a}(D_\sigma) \supset G'$. Such a choice of $\sigma$ is possible since $\mathbf{a}(D) = H' \supset G'$. The interval $D_\sigma$ is of the form

$$(-1 < x_i < c_i) \qquad (i = 1, \ldots, n)$$

With $\eta$ a constant such that $-1 < \eta < 0$ we introduce the $n$-interval

$$\text{(16.11)} \qquad Z : (\eta \leqq x_i \leqq c_i) \qquad (i = 1, \ldots, n).$$

Setting $\zeta = D_\sigma - Z$ we suppose $\eta+1$ is so small a positive constant that

$$\omega(p) = I(p) \qquad [p \in \zeta \cap \mathring{K}] \tag{16.11a}$$

$$\mathbf{a}(Z) \supset G' \tag{16.11b}$$

$$\zeta \cap R^r(\mathring{K}) = \emptyset \qquad (r > 0). \tag{16.11c}$$

Let $\gamma$ be the complement of $\zeta \cap \mathring{K}$ in $D-Z$ so that

$$D-Z = (\zeta \cap \mathring{K}) \cup \gamma. \tag{16.12}$$

One sees that $\gamma$ is a subset of each of the sets

$$A = E-P-\bigcup_{r=0}^{\infty} R(\mathring{K}) \qquad \text{[Cf. (10.16)]}$$

$$L = E-P-\bigcup_{r=1}^{\infty} T_r(\mathring{K})-H' \qquad \text{[Cf. (10.28)]}$$

For our purposes the essential properties of $Z$ may be summarized as follows.

(α) *The n-interval $Z$, given by* (16.11) *is closed in $E$, contains $P$, is a subset of $D$, and is such that* (16.11a), (16.11b) *and* (16.12) *hold with $\gamma \subset A \cap L$.*

Lemma 16.2. *The mapping* $\mathbf{k}$ *is a $C^\infty$-diffeomorphism of $E-P$ onto* $\mathbf{X}^*$ such that

(a) $\mathbf{k}|(D-Z) = \pi_1^*|(D-Z)$

(b) $\mathbf{k}(D') = [M^* \cap D, \mathcal{M}^*, \mathbf{X}^*] = X_D^*$.

*Proof of* (a). It follows from Lemma 11.1 that for $p \in A$

$$\mathbf{t}^{-1} . I(p) = \pi_1(p). \tag{16.13)$'$}$$

This relation is also valid for $p \in \zeta \cap \mathring{K}$ in accord with Lemma 11.1, since for such $p$, $\omega(p) = I(p)$. It follows from Lemma 13.1 that

$$\mathbf{s} . \pi_1(p) = \pi_1^*(p) \qquad [p \in L \cup (\zeta \cap \mathring{K})]. \tag{16.13)$''$}$$

The relation (16.13)″ follows from (16.37) for $p \in L$. To establish (16.13)″ for $p \in \zeta \cap \mathring{K}$ observe that

$$\mathring{K} = (\mathring{K} \cap L) \cup (\mathring{K} \cap H') \cup (\mathring{K} \cap T(\mathring{K}))$$

in accord with the formula for $L$. For $p$ not in $L$ but in $\zeta \cap \mathring{K}$, $p$ cannot be in $G$, since $\omega(p)$ is defined. Cf. (16.11a). There thus remain two cases: Case I, $p \in H'-G'$; Case II, $p = T(q)$, $q \in \mathring{K}$. In Case I, (13.33) serves to establish (16.13)″. In case II, Lemma 9.4 implies that $q \in \zeta \cap \mathring{K}$, hence $\omega(q) = I(q)$, so that $q$ is not in $G$. With $q$ thus in $\mathring{K}-G$, (13.35) applies to $x = p$ with $\omega(q) = I(q)$, and shows that (16.13)″ holds for the given $p$.

Thus equations (16.13) both hold for $p \in \gamma$ since $\gamma \subset A \cap L$, and also hold for $p \in \zeta \cap \mathring{K}$. By virtue of (16.12) equations (16.13) both hold for $p \in D-Z$, so that for such $p$, $\mathbf{k}(p) = \pi_1^*(p)$. This establishes (a).

*Proof of* (b). Note that $E-D$ is included both in $A$ and in $L$, so that both equations (16.13) hold for

$p \in E-D$. Now $E-D = M^*-D'$ so that

$$\mathbf{k}(M^*-D') = \pi_1^*(M^*-D') \tag{16.14}$$

Since $M^*-D'$ does not meet $W^*$, the set

$$\pi_1^*(M^*-D') = [M^*-D', \emptyset, \mathbf{X}^*] = \mathbf{k}(M^*-D') \tag{16.15}$$

is a $\mu^*$-represented subset of $\mathbf{X}^*$. We have successively

$$D' \cup (M^*-D') = E-P,$$
$$\mathbf{k}(D') \cup \mathbf{k}(M^*-D') = \mathbf{k}(E-P) = \mathbf{X}^*,$$
$$\mathbf{k}(D') = \mathbf{X}^*-[M^*-D', \emptyset, \mathbf{X}^*] = \mathbf{X}_D^*,$$

using (16.15) and Cor. 1.4. This establishes (b) in the lemma.

If use is made of Corollaries 15.1 and 15.2, then on setting

$$\mathbf{k}^e - \mathbf{s}^e(\mathbf{t}^e)^{-1}I, \tag{16.16}$$

we have the following lemma.

LEMMA 16.3. *The $C^\infty$-diffeomorphism* $\mathbf{k}$ *of $E-P$ onto* $\mathbf{X}^*$ *admits an extension* $\mathbf{k}^e$ *which is a $C^\infty$-diffeomorphism of $E$ onto* Ext $\mathbf{X}^*$ *and in which*

$$\mathbf{k}^e(P) = \mathbf{s}^e(\mathbf{P}) = \mathbf{P}^*. \tag{16.17}$$

We refer to the $C^\infty$-diffeomorphism $\mathbf{a}$ of $D$ onto $H'$ defined in § 14, to the point $\mathbf{a}(P) \in H'-G'$ of (14.9), and to $\mathbf{P}^{**} = \pi_1^* \,.\, \mathbf{a}(P)$ of (14.12)′ and prove the following lemma.

LEMMA 16.4. *There exists a $C_0^\infty$-diffeomorphism $\mathfrak{g}_P$ of $H'$ onto $Y_0^*$, of the general nature of $\mathfrak{g}$ of Lemma* 16.1, *and in particular such that $\mathfrak{g}_P$ defines a $C^\infty$-diffeomorphism of $H'-\mathbf{a}(P)$ onto $Y_0^*-\mathbf{P}^{**}$.*

Recall that the inverse of $\mathbf{a}$ is a $C^\infty$-diffeomorphism of $H'$ onto $D$, that $\mathbf{k}^e|D$ is a $C_0^\infty$-diffeomorphism of $D$ onto Ext $X_D^*$ (Lemmas (16.2, 16.3), and $\mathbf{D}^e$ is a $C_0^\infty$-diffeomorphism of Ext $X_D^*$ onto $Y_0^*$ (Lemmas 14.1, 15.3) such that the mapping,

$$\mathfrak{g}_P = \mathbf{D}^e \,.\, \mathbf{k}^e \,.\, (\mathbf{a})^{-1} : H' \to Y_0^* \tag{16.18}$$

is a $C_0^\infty$-diffeomorphism of $H'$ onto $Y_0^*$. In particular

$$\mathbf{a}(P) \to P \to \mathbf{P}^* \to \mathbf{P}^{**} \quad \text{[Cf. (15.28), (16.17)]} \tag{16.19}$$

under $\mathfrak{g}_P$. We shall show that $\mathfrak{g}_P$ satisfies the lemma.

Restricted to $H'-\mathbf{a}(P)$, $\mathfrak{g}_P$ maps

$$[H'-\mathbf{a}(P)] \to D' \to X_D^* \to [Y_0^* - \mathbf{P}^{**}]$$

as a $C^\infty$-diffeomorphism, in accord with the definition of **a**, and with Lemmas 16.2 and 14.1. It remains to choose a compact subset $\Omega$ of $H'$ with $\Omega \supset G'$, such that the boundary condition (16.3) is satisfied.

*The choice of* $\Omega$. With the $n$-interval $Z$ given by (16.11) and conditioned as in ($\alpha$), set $\Omega = \mathbf{a}(Z)$. The set $\Omega$ is compact. Moreover $\Omega \supset G'$ in accord with (16.11b). Since $\mathbf{a}(D) = H'$, by (14.3) and (14.6), and $D \supset Z$,

$$H' \supset \Omega \supset G', \quad H'-\Omega = \mathbf{a}(D-Z).$$

By definition of $\mathfrak{g}_P$

$$(16.19)' \qquad \mathfrak{g}_P(p) = \mathbf{D} \,.\, \mathbf{k} \,.\, \mathbf{a}^{-1}(p) = \mathbf{D} \,.\, \pi_1^* \,.\, \mathbf{a}^{-1}(p) \quad [p \in H'-\Omega]$$

since $\mathbf{a}^{-1}(p)$ is in $D-Z$ and Lemma 16.2(a) applies. Since **a** reduces to the identity on $G'$ [Cf. (14.7)], it follows from (16.11b) that $Z \supset G'$, and hence $D-Z \subset M^* \cap D$. With $\mathbf{a}^{-1}(p) \in M^* \cap D$, $p$ in (14.27)' can be replaced by $\mathbf{a}^{-1}(p)$ so that the last member of (16.19)'
reduces to

$$\pi_1^* \,.\, \mathbf{a} \,.\, \mathbf{a}^{-1}(p) = \pi_1^*(p) \qquad (p \in H'-\Omega)$$

Thus the boundary condition (16.3) is satisfied and Lemma 16.4 is established.

Lemma 16.4 combined with Lemma 16.1 gives us the fundamental corollary.

COROLLARY 16.1. *The problem* (16.1) *admits a solution implying the existence of a solution of each problem of the first class.*

As defined in (16.4) the solution $\lambda_\omega$ of problem (16.1) is such that

$$(16.20) \qquad \pi_2^* \,.\, \lambda_\omega(p) = \mathfrak{g}_P(p) = \mathbf{D}^e \,.\, \mathbf{k}^e \,.\, \mathbf{a}^{-1}(p) \qquad (p \in H')$$

The sequence (16.19) shows that $\mathfrak{g}_P$, and hence $\lambda_\omega$, may fail to be of class $C^\infty$ at $\mathbf{a}(P) \in H'-G'$. Under $\lambda_\omega$, as represented in (16.20),

$$H' \to D \to \text{Ext } X_D^* \to Y_0^* \to \mathscr{H}' \quad [\text{Cf. } (13.11)]$$

and in particular the exceptional point $\mathbf{a}(P)$ is transformed as follows:

$$\mathbf{a}(P) \to P \to \mathbf{P}^* \to \mathbf{P}^{**} \to \omega \,.\, \mathbf{a}(P)$$

This is in agreement with (16.19) up to the final image. To verify this final image recall that

$$(\pi_2^*)^{-1} \,.\, \pi_1^*(p) = \omega(p) \qquad (p \in W^*)$$

since $\mu^* = \omega | W^*$. The final image of $\mathbf{a}(P)$ is

$$(\pi_2^*)^{-1}(\mathbf{P}^{**}) = (\pi_2^*)^{-1} \,.\, \pi_1^* \,.\, \mathbf{a}(P) = \omega \,.\, \mathbf{a}(P) \text{ [Cf. (14.12)']}$$

since $\mathbf{a}(P)$ is in $W^*$. [Cf. (14.9)].

It is not difficult to show that the mapping $\Lambda_\Phi$, affirmed to exist in Theorem 0.1, may be chosen so that the "exceptional point" at which $\Lambda_\Phi$ may fail to be differentiable may be chosen arbitrarily on the interior of $S_{n-1}$, and its image under $\Lambda_\Phi$ arbitrarily on the interior of $\mathcal{M}_{n-1}$.

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