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The Baire category of independent sets

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Let us denote the linear continuum by C . Suppose that to every $x \in C$ there corresponds a set $P(x) \subset C$ such that $x \notin P(x)$ and x is not a limit point of $P(x)$. Two points x and y of C are said to be independent, provided that $x \notin P(y)$ and $y \notin P(x)$. A subset of C is said to be independent, provided that every pair of points of this subset is independent.

Fodor [1], [2] has obtained results concerning the Lebesgue measure of independent sets. The present note contains several theorems regarding the Baire category of independent sets, a few of which are somewhat analogous to Fodor's results. In the proof of Theorem 1 we make use of an idea due to Lázár [4]. Our theorems are valid for more general sets than C , as will be seen from the proofs.

THEOREM 1. *There always exists an independent set of second category.*

Proof: We have assumed that no point x of C is a limit point of $P(x)$, and therefore we can associate with every $x \in C$ an interval $J(x)$ with rational endpoints, such that $J(x) \cap P(x)$ is empty and $x \in J(x)$. The set of all intervals with rational endpoints is enumerable; denote these intervals by $J_1, J_2, \dots, J_n, \dots$. For every natural number n , let C_n be the set of points $x \in C$ with the property that $J(x) = J_n$. Then $C = \bigcup_{n=1}^{\infty} C_n$. If C_n were of first category for every n , C would also be of first category [3, p. 130], which is impossible [3, p. 136]. Hence, there exists a natural number k such that C_k is of second category. If x and y are any two points of C_k , then $P(x) \cap C_k$ and $P(y) \cap C_k$ are both empty, so that x and y are independent, and consequently C_k is an independent set. This completes the proof.

A consequence of Theorem 1 and a theorem [3, p. 134] on category, is that there always exists an independent set which is of second category in every subinterval of some interval of C . It is not true, however, that there always exists an independent

set which is of second category in every interval of C . This follows immediately from

THEOREM 2. *There does not always exist an independent set which is everywhere dense in C .*

Proof: If $x \in C$, define $P(x)$ to be the set of all real numbers y satisfying the relation $[x] + 2 \leq y < [x] + 3$, where $[x]$ denotes the greatest integer in x . Now suppose that D , a subset of C , is everywhere dense in C . Then D must contain a point x such that $0 \leq x < 1$, and a point y such that $2 \leq y < 3$; since $y \in P(x)$, x and y are not independent, and hence D cannot be an independent set.

A fortiori [3, p. 135] there does not always exist an independent set which is a residual subset of C . A sufficient condition for the nonexistence of a residual independent set is furnished by

THEOREM 3. *Let M , a subset of C , be of second category, and suppose that $P(x)$ is of second category for every $x \in M$. Then there does not exist a residual independent set.*

Proof: If R is a residual set, then [3, p. 134] $R \cap M$ is not empty; let $x \in R \cap M$. Since $P(x)$ is of second category, it again follows that $R \cap P(x)$ is not empty; let $y \in R \cap P(x)$. Now x and y are not independent, and hence R cannot be an independent set.

THEOREM 4. *There does not always exist an independent set which is residual in some interval of C .*

PROOF: It is possible (see, e.g., [5, p. 208]) to express C as the union of enumerably many mutually exclusive sets $E_1, E_2, \dots, E_n, \dots$, each of which is of second category in every interval of C . If $x \in C$, let n be that natural number for which $x \in E_n$, and define $P(x)$ to be the set of all elements of E_n lying outside the interval of length $1/n$ with x as midpoint. Now suppose that S , a subset of C , is residual in some interval, K , of C . Since each set E_n ($n = 1, 2, 3, \dots$) is of second category in every subinterval of K , it follows [3, pp. 130, 134] that $S \cap K \cap E_n$ ($n = 1, 2, 3, \dots$) is everywhere dense in K . Hence, if n is sufficiently large, there exists an $x \in S \cap K \cap E_n$ such that the interval of length $1/n$ with x as midpoint has both endpoints in the interior of K . This implies the existence of a subinterval, L , of K such that $L \cap E_n \subset P(x)$, and since $S \cap K \cap E_n$ is everywhere dense in K , there exists a $y \in S \cap L \cap E_n$. Thus S contains two elements x and y which are not independent, and therefore S cannot be an independent set.

In the proof of Theorem 4, $P(x)$ was chosen to be of second

category for every $x \in C$. Does Theorem 4 remain valid if $P(x)$ is required to be a "thinner" set for every $x \in C$? Of course if $P(x)$ is required to be empty for every $x \in C$, then Theorem 4 is trivially false. The next theorem indicates, however, that the "thinness" of $P(x)$ has very little effect on the truth of Theorem 4. The proof of Theorem 5 obviously constitutes an alternative proof of Theorem 4.

THEOREM 5. *The assumption that $P(x)$ contains at most one point for every $x \in C$ does not imply the existence of an independent set which is residual in some interval of C .*

PROOF: There are enumerably many closed (nondegenerate) intervals of C with rational endpoints. There are [3, p. 344] 2^{\aleph_0} G_δ -subsets of C that are everywhere dense in C , and likewise, for every closed interval, H , of C with rational endpoints, there are 2^{\aleph_0} G_δ -subsets of H that are everywhere dense in H ; all together, then, this makes $2^{\aleph_0} \cdot \aleph_0 = 2^{\aleph_0}$ subsets, which may be arranged in a transfinite sequence,

$$(1) \quad G_0, G_1, \dots, G_\xi, \dots \quad (\xi < \omega_\gamma),$$

where ω_γ is the initial number [3, p. 43] of $Z(2^{\aleph_0})$. Every $G_\xi (\xi < \omega_\gamma)$ contains [3, pp. 135, 128] 2^{\aleph_0} points.

Now we define, by means of transfinite induction, a sequence of distinct points

$$x_0, x_1, \dots, x_\xi, \dots \quad (\xi < \omega_\gamma)$$

and a sequence of points

$$y_0, y_1, \dots, y_\xi, \dots \quad (\xi < \omega_\gamma)$$

as follows. Let x_0 be an arbitrary point of G_0 , and let y_0 be any other point of G_0 . Suppose that $0 < \alpha < \omega_\gamma$, and that we have defined x_β and y_β for every $\beta < \alpha$. There are fewer than 2^{\aleph_0} points x_β with $\beta < \alpha$, whereas G_α contains 2^{\aleph_0} points. Let x_α be an arbitrary point of G_α such that $x_\alpha \neq x_\beta$ ($\beta < \alpha$), and let y_α be any other point of G_α . This completes the induction.

For every $\xi < \omega_\gamma$, let $P(x_\xi) = \{y_\xi\}$; for every $x \in C$ such that $x \neq x_\xi$ ($\xi < \omega_\gamma$), let $P(x)$ be the empty set.

Suppose that T , a subset of C , is residual in some interval of C ; then T is residual in some closed subinterval, Q , of this interval, with rational endpoints. Hence [3, p. 135], there exists a $\xi < \omega_\gamma$ such that $G_\xi \subseteq T \cap Q$, which implies that x_ξ and y_ξ belong to T . Since x_ξ and y_ξ are not independent, T cannot be independent.

THEOREM 6. *Let d be a positive number, and suppose that, for a residual set, R , of elements $x \in C$, the distance between x and $P(x)$ is not less than d . Then, if D is any interval of C of length d , there exists an independent set which is residual in D .*

PROOF: The set $R \cap D$ is residual in D , and if x and y are distinct points of $R \cap D$, both different from the possible end-points of D , then x and y are independent, because $P(x)$ and $P(y)$ cannot contain any points in the interior of D .

THEOREM 7. *Let d be a positive number, and suppose that, for a set, S , of elements $x \in C$, which is of second category in every interval of C , the distance between x and $P(x)$ is not less than d . Then, if D is any interval of C of length d , there exists an independent set which is of second category in every subinterval of D ; but there does not always exist an independent set which is everywhere dense in C , even if $S = C$.*

PROOF: The first and second parts of the conclusion follow from arguments analogous to those used in proving Theorems 6 and 2, respectively.

THEOREM 8. *Let d be a positive number. Suppose that, for every $x \in C$, $P(x)$ consists of at most one point, and the distance between x and $P(x)$ is not less than d . Then there does not always exist a residual independent set.*

PROOF: It is only necessary to modify the proof of Theorem 5 in two essential respects: let the terms of the sequence (1) be the G_δ -subsets of C that are everywhere dense in C , and subject each y_ξ ($\xi < \omega_\gamma$) to the additional condition that the distance between x_ξ and y_ξ be not less than d .

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