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On a conjecture of Nelder in Mathematical Statistics

by

Bengt Ulin

1. The following problem has previously been attacked in this journal [1]):

Let C denote the class of distribution functions $F(x)$ of one variable ($F(x) = 0$, $x<0$; $F(x)$ non-decreasing, continuous to the right and $\lim\limits_{x=\infty} F(x) = 1$). Is it then true that

$$\sup_{F \in C} (I_0 I_2 - I_1^2)$$

where

$$I_\nu = \int_0^\infty x^\nu e^{-x} dF(x), \qquad \nu = 0, 1, 2,$$

is attained for

$$F(x) \equiv F_0(x) = \begin{cases} \frac{1}{2}, & 0 \leqq x < 2 \\ 1, & x \geqq 2, \end{cases}$$

and is $F_0(x)$ a unique extremal function?

In the following it will be shown that this conjecture is true.

2. In order to establish the existence of an extremal function we select from an extremal sequence a subsequence converging to a limit distribution $F(x) \in C$. This function is extremal, because we may proceed to the limit under the integral sign in the integrals I_ν corresponding to the subsequence.

3. Put $S = I_0 I_2 - I_1^2$ and let $F(x) \in C$ be arbitrary. We consider this distribution as a set function $\alpha(e)$. Let I_h: $t-h \leqq x < t+h$ be a (small) interval, on which there is a mass $\varepsilon(h) > 0$. According to whether the point $x = t$ carries a mass $\varepsilon_1 > 0$ or not, we now vary $\alpha(e)$ by removing a mass ε, $0 < \varepsilon \leqq \varepsilon_1$, from $x = t$, or the mass $\varepsilon(h)$ from I_h, to a point $x = u$ and obtain the respective functions

$$\alpha^*(e) = \alpha(e) - \varepsilon\delta_t(e) + \varepsilon\delta_u(e)$$
$$\alpha^*(e) = \alpha(eI_h') + \varepsilon(h)\delta_u(e),$$

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1) M. Hammersley: On a conjecture of Nelder. Vol. 10, p. 241—244, (1952).

where I_h' is the complement of I_h and $\delta_x(e)$ the Dirac function of the point x. α^* will determine a function $F^* \varepsilon C$. With obvious notations,

$$I_\nu^* - I_\nu = \left(u^\nu e^{-u} - t^\nu e^{-t} + \eta_\nu(h)\right)\varepsilon, \qquad \nu = 0, 1, 2,$$

where $\eta_\nu = 0$ if $x = t$ carries a mass, and otherwise $|\eta_\nu| < Ah$, $A =$ const. and $\varepsilon(h) \to 0$ as $h \to 0$. Hence we obtain

$$S^* - S = \{\varphi(u) - \varphi(t)\}\varepsilon + 0(\varepsilon) \tag{1}$$

where

$$\varphi(x) = e^{-x}(I_0 x^2 - 2I_1 x + I_2).$$

The curve $y = \varphi(x)$ is sketched in the figure below, where ξ denotes the greater root of $\varphi'(x) = 0$.

If the mass of $\alpha(e)$ were not concentrated to the points $x=0$ and $x = \xi$, it will be seen from (1) that we may make $S^* > S$ by choosing $u = \xi$ in a variation of the type described. We conclude that only a step function

$$G(x;\, m,\, \xi) = \begin{cases} m & ,\ 0 \leqq x < \xi,\ 0 < m < 1 \\ 1-m, & x \geqq \xi, \end{cases} \tag{2}$$

moreover such that $I_2 = \varphi(0) = \varphi(\xi)$, may satisfy the inequality

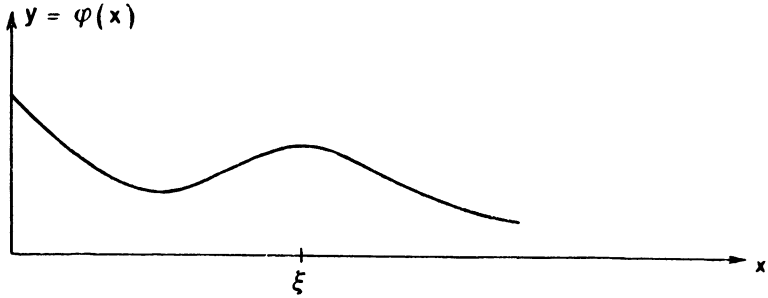
$$S^* - S \leqq 0 \tag{3}$$

for every permitted variation. As $F(x)$ is not extremal, unless (3) is always valid, we must seek the extremal functions among the functions $G(x)$. From (2) it follows that

$$S = m(1-m)\xi^2 e^{-\xi}.$$

This expression obtains its maximum for $m = \frac{1}{2}$, $\xi = 2$ only.

Thus $F_0(x)$ is the unique extremal function.



(Oblatum 16-12-53).