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On the topological Degree

by

Joseph Weier

Let n be a positive integer > 1 ; E_i , for $i = n, n+1$, the i -dimensional Euclidean space; P_i^* , for $i = n, n+1$, an orientation of the i -dimensional finite Euclidean manifold P_i ; U an open set in E_{n+1} and V an open set in P_{n+1} ; A a simplicial 1-sphere in U and B such a one in V ; A^* an orientation of A and B^* an orientation of B .

By g, g' denoting continuous maps of P_{n+1} in P_n , we call the set consisting of all points p of P_{n+1} with $g(p) = g'(p)$, the set of the coincidences of (g, g') , the “singular form” of (g, g') . The pair (g, g') be named “normal”, if the singular form of (g, g') is either empty or composed of a finite number of pairwise disjoint simplicial 1-spheres.

Suppose φ_1, φ_2 are continuous maps of \bar{U} in E_n ; the set of the coincidences of (φ_1, φ_2) , the “singular form” of (φ_1, φ_2) , equal to A ; moreover B_1, \dots, B_m mutually disjoint simplicial 1-spheres of P_{n+1} , $B_1 = B$, $B_i \cdot \bar{V} = 0$ for $i > 1$; and (γ_1, γ_2) a normal pair of maps $\gamma_i: P_{n+1} \rightarrow P_n$; ΣB_i the singular form of (γ_1, γ_2) . Then we designate the B_i as the “singularities” of (γ_1, γ_2) and A as the “singularity” of (φ_1, φ_2) .

The significance of $n, E_n, E_{n+1}, P_n, P_{n+1}, U, V, A, B, A^*, B^*, P_n^*, P_{n+1}^*, \varphi_1, \varphi_2, \gamma_1, \gamma_2$ thus defined remain till the end of this paper.

By the way, I shall prove the following approximation theorem elsewhere. If γ denotes a continuous map of P_{n+1} in P_n and ε a positive number, then there are simplicial maps γ^1 and γ^2 of P_{n+1} in P_n homotopic to γ and having the further properties: the set of the coincidences of (γ^1, γ^2) is either empty or the union of a finite number of mutually disjoint 1-spheres, $d(\gamma, \gamma^1) < \varepsilon$ and $d(\gamma, \gamma^2) < \varepsilon$. More shortly: *one can normally approximate* (γ, γ) .

In Section 1, we associate with the orientated singularity A^* and just so with B^* an integer as its “degree” in such a way that degree of orientated singularities and classical degree are

corresponding concepts. Some simpler theorems in Section 2 enumerate properties which both these degrees have in common: topological invariance, invariance at deformations, and a decomposition property.

A known property of coincidences, relative to which we will compare singularities and coincidences, is pronounced in the next paragraph; whereby P signifies an $(n+1)$ -dimensional finite Euclidean manifold which possesses an orientation and lies in an Euclidean space.

Let c be a point of V and h_1, h_2 continuous maps of P_{n+1} in P ; c the only coincidence of (h_1, h_2) on V ; the degree of c at (h_1, h_2) equal to zero. Then there exists a pair (h'_1, h'_2) homotopic to (h_1, h_2) , consisting of maps $h'_i : P_{n+1} \rightarrow P$, and having the property: for $p \notin V$ hold the equations $h'_1(p) = h_1(p)$ and $h'_2(p) = h_2(p)$, on V there is no coincidence of (h'_1, h'_2) .

Is there any property of singularities being apt to stand comparison with this property of coincidences? In this problem Section 3 engages. First the following theorem. The singularity B of (γ_1, γ_2) having the degree zero, there exists a point b in V and a pair (g_1, g_2) homotopic to (γ_1, γ_2) , composed of maps $g_i : P_{n+1} \rightarrow P_n$, and of the fashion: $g_1(p) = \gamma_1(p)$ and $g_2(p) = \gamma_2(p)$ for $p \notin V$, the point b is the only coincidence of (g_1, g_2) on V . Perhaps you may say in brief: *singularities of the degree zero can be contracted on a single point*. Yet, an example in Section 3 shows that the resting point cannot always be removed.

Some theorems used in the following easily result from known¹⁾ properties of the Brouwer degree.

1. The degree of a singularity.

If m is a positive integer and $q = (\alpha_1, \dots, \alpha_m)$, $r = (\beta_1, \dots, \beta_m)$ are points of the Euclidean m -space E_m , $q+r$ means the point $(\alpha_1+\beta_1, \dots, \alpha_m+\beta_m)$ and $d(q, r)$ the Euclidean distance from q to r . "Simplexes" are Euclidean and open. If C signifies a 2-simplex in E_m and D the topological (topological and simplicial) image of $\bar{C}-C$, then D is said to be a "1-sphere" ("simplicial 1-sphere"). If just one point of the set M is attached to each point p of the set N by the map f , we denote the first point by $f(p)$. The pair (φ_1, φ_2) is said to be a pair of \bar{U} in E_n . Let, for

¹⁾ See for instance: P. J. Hilton, "An introduction to homotopy theory", Cambridge Univ. Press, vol. 43 (1953).

$i = 1, 2$, g_i be a map of P_{n+1} in P_n homotopic to γ_i , then (g_1, g_2) is called a pair "homotopic" to (γ_1, γ_2) .

Let a be a point of A , then we will define an "index" of a under (φ_1, φ_2) relative to A^* as follows.

Be denoted by S an n -simplex in U with $a \in S$ and $A \cdot \bar{S} = a$, by E_n^* and E_{n+1}^* the natural orientations of E_n and E_{n+1} . Let a_1, \dots, a_{n+1} points of E_{n+1} with the properties: the points a, a_1, \dots, a_{n+1} are linearly independent; the orientation induced by (aa_1, \dots, aa_{n+1}) into E_{n+1} concurs with the orientation E_{n+1}^* ; the 1-simplex with the vertexes a and a_1 lies in A ; the orientation induced by aa_1 into A and the orientation A^* agree; the points a_2, \dots, a_{n+1} lie in S . Let S^* be the orientation induced by (aa_2, \dots, aa_{n+1}) into S . Furthermore let T be an n -simplex in E_n , T^* the orientation which E_n^* induces into T , t an affine map of \bar{T} on \bar{S} with $t(T^*) = S^*$, b the point in T determined by $t(b) = a$. Let f be defined by

$$f(p) = \varphi_1 t(p) - \varphi_2 t(p), \quad p \in \bar{T},$$

as map of \bar{T} in E_n . Then b is the only fixed point of f , the index of b at f is said to be the index of a at (φ_1, φ_2) with respect to A^* .

You instantly verify that the last definition is unique and has the further property: if A^{**} means the orientation opposite to A^* , α^* and α^{**} are the indexes of a at (φ_1, φ_2) relative to A^* and A^{**} respectively, then $\alpha^* = -\alpha^{**}$. One easily sees:

There is an integer α such that, for each point p of A , the index of p at (φ_1, φ_2) referring to A^ is equal to α .* Then we will define α to be the "degree" of A^* under (φ_1, φ_2) , more exactly the degree of A under (φ_1, φ_2) with respect to (E_{n+1}^*, E_n^*) . Correspondingly one may declare the "degree" of B^* under (γ_1, γ_2) with respect to (P_{n+1}^*, P_n^*) .

2. Elementary properties of a singularity.

From the topological invariance of the fixed point index insues:

THEOREM 1. *The degree of A^* is topologically invariant, more precisely: Let t be a topological map of E_{n+1} onto itself such that $t(E_{n+1}^*) = E_{n+1}^*$, $t(A)$ a simplicial 1-sphere, $f_1 = t\varphi_1 t^{-1}$, and $f_2 = t\varphi_2 t^{-1}$. Then (f_1, f_2) represents a normal pair of mappings $f_i : t(U) \rightarrow E_n$, $t(A)$ is the only singularity of (f_1, f_2) , and the degree of $t(A^*)$ at (f_1, f_2) is equal to the degree of A^* at (φ_1, φ_2) .*

We will show:

THEOREM 2. Let $(\varphi_1^\tau, \varphi_2^\tau)$, $0 \leq \tau \leq 1$, be normal pairs of maps $\varphi_i^\tau : \bar{U} \rightarrow E_n$ which continuously depend on τ and A , for $0 \leq \tau \leq 1$, the only singularity of $(\varphi_1^\tau, \varphi_2^\tau)$. Then the degree of A^* at $(\varphi_1^0, \varphi_2^0)$ is equal to the degree of A^* at $(\varphi_1^1, \varphi_2^1)$.

PROOF. It suffices to show that, given a point a of A , the index of a at $(\varphi_1^0, \varphi_2^0)$ relative to A^* and the index of a at $(\varphi_1^1, \varphi_2^1)$ relative to A^* are equal.

To prove this, let the significance of S , T , t , and b be the one defined in the first section; moreover f^τ , for $0 \leq \tau \leq 1$, determined by

$$f^\tau(p) = \varphi_1^\tau t(p) - \varphi_2^\tau t(p), \quad p \in \bar{T},$$

as map of \bar{T} in E_n . Then, for $0 \leq \tau \leq 1$, the point b is the only fixed point of f^τ , so the index of b under f^0 equal to the index of b under f^1 . This already yields the assertion.

If A' is a 1-sphere in U , we denote A and A' as “neighbouring”, provided the statements I and II are true. I. There is a homotopy $(t^\tau, 0 \leq \tau \leq 1)$ of topological maps $t^\tau : A \rightarrow U$ such that t^0 is the identity, $t^1(A) = A'$, and

$$d(p, t^\tau(p)) < 2d(p, t^1(p))$$

for all (p, τ) with $p \in A$ and $0 \leq \tau \leq 1$. II. The homotopies $(t_i^\tau, 0 \leq \tau \leq 1)$, $i = 1, 2$, being conditioned like $(t^\tau, 0 \leq \tau \leq 1)$, then the orientations $t_1^1(A^*)$, $t_2^1(A^*)$ of A' agree. The orientation $t^1(A^*)$ of A' we name the orientation “induced” by A^* into A' .

In the last paragraph replacing A , A' , A^* , U by B , B' , B^* , V respectively, you obtain the definition of a 1-sphere B' such that B and B' are “neighbouring” and the definition of the orientation which B^* “induces” into B' .

Now let us establish:

THEOREM 3. If α denotes the degree of A^* under (φ_1, φ_2) and $\alpha_1, \dots, \alpha_m$ are integers with $\sum \alpha_i = \alpha$, then there are simplicial 1-spheres A_1, \dots, A_m in U by pairs disjoint and a normal pair (f_1, f_2) of maps $f_i : \bar{U} \rightarrow E_i$ with the properties: $f_1(p) = \varphi_1(p)$ and $f_2(p) = \varphi_2(p)$ for $p \in \bar{U} - U$; for $i = 1, \dots, m$, the spheres A_i and A are neighbouring; the A_i are the singularities of (f_1, f_2) ; A_i^* being the orientation which A^* induces into A_i , the number α_i represents the degree of A_i^* at (f_1, f_2) .

PROOF. Let T be an n -simplex in E_n . Then you easily see that there are points a^τ , $0 \leq \tau \leq 1$, of A continuously dependent on τ and n -simplexes S^τ , $0 \leq \tau \leq 1$, continuously dependent on τ , too, and a homotopy $(t^\tau, 0 \leq \tau \leq 1)$ of affine maps $t^\tau : \bar{T} \rightarrow \bar{S}^\tau$

with the properties: $a^0 = a^1$, $S^0 = S^1$, and $t^0 = t^1$; for $0 < |\tau_1 - \tau_2| < 1$ there hold $a^{\tau_1} \neq a^{\tau_2}$ and $\bar{S}^{\tau_1} \cdot \bar{S}^{\tau_2} = 0$; $a^\tau \in S^\tau$ for all τ ; if, for all τ , f^τ denotes the map defined by

$$f^\tau(p) = \varphi_1 t^\tau(p) - \varphi_2 t^\tau(p), \quad p \in \bar{T},$$

and b^τ the point of T where $t^\tau(b^\tau) = a^\tau$, then the index of b^τ at f^τ is equal to α . Thus, Theorem 3 easily follows from

LEMMA 1. *Let S be an n -simplex in E_n ; and $(a_i^\tau, 0 \leq \tau \leq 1)$, $i = 1, \dots, m$, curves of points a_i^τ of S ; for $0 \leq \tau \leq 1$, the points $a_1^\tau, \dots, a_m^\tau$ mutually disjoint; $(f^\tau, 0 \leq \tau \leq 1)$ a homotopy of maps $f^\tau: \bar{S} \rightarrow E_n$; $p \neq f^\tau(p)$ for all (p, τ) with $p \in \bar{S} - S$ and $0 \leq \tau \leq 1$; further $a_1^\tau, \dots, a_m^\tau$, for $i = 0, 1$, the fixed points of f^i ; and, for $k = 1, \dots, m$, the index of a_k^0 at f^0 equal to the index of a_k^1 at f^1 . Then there exists a homotopy $(g^\tau, 0 \leq \tau \leq 1)$ of maps $g^\tau: \bar{S} \rightarrow E_n$ such that: $f^\tau(p) = g^\tau(p)$ for all (p, τ) where either $p \in \bar{S}$ and $\tau = 0, 1$ or $p \in \bar{S} - S$ and $0 \leq \tau \leq 1$; for $0 \leq \tau \leq 1$, the points $a_1^\tau, \dots, a_m^\tau$ are the fixed points of g^τ .*

PROOF. It suffices to show the following simpler proposition.

Let a_1, a_2 be different points of S and f^0, f^1 continuous maps of \bar{S} in E_n with the properties: $f^0(p) = f^1(p)$ for $p \in \bar{S} - S$; for $i = 0, 1$, the points a_1, a_2 are the fixed points of f^i ; for $k = 1, 2$, the index of a_k at f^0 is equal to the index of a_k at f^1 . Then there is a homotopy $(g^\tau, 0 \leq \tau \leq 1)$ of maps $g^\tau: \bar{S} \rightarrow E_n$ such that the following holds: $g^\tau(p) = f^0(p)$ for all (p, τ) with $p \in \bar{S} - S$ and $0 \leq \tau \leq 1$; $g^0 = f^0$ and $g^1 = f^1$; for $0 \leq \tau \leq 1$, the points a_1, a_2 are the only fixed points of g^τ .

To establish this, first let T denote an $(n-1)$ -simplex with $T \subset S$, $\bar{T} - T \subset \bar{S} - S$, and the property: if S_1, S_2 are both the components of the set $S - T$, we have $a_1 \in S_1$ and $a_2 \in S_2$. Following a known theorem on the fixed point index, there is a homotopy $(g^\tau, 0 \leq \tau \leq 1/2)$ of maps $g^\tau: \bar{S} \rightarrow E_n$ which disposes of the properties: $g^\tau(p) = f^0(p)$ for all (p, τ) with $p \in \bar{S} - S$ and $0 \leq \tau \leq 1/2$; $g^0 = f^0$;

$$g^{\tau/2}(p) = f^1(p) \text{ for } p \in \bar{T};$$

for $0 \leq \tau \leq 1/2$, the points a_1 and a_2 are the only fixed points of g^τ .

So it remains to show:

Let a be a point of S and f, f' continuous maps of \bar{S} in E_n : $f(p) = f'(p)$ for $p \in \bar{S} - S$, a the only fixed point of f and just so the only fixed point of f' . Then there is a homotopy $(h^\tau, 0 \leq \tau \leq 1)$ of maps $h^\tau: \bar{S} \rightarrow E_n$ such that: $h^\tau(p) = f(p)$ for all (p, τ) with

$p \in \bar{S} - S$ and $0 \leq \tau \leq 1$; $h^0 = f$ and $h^1 = f'$; a represents the only fixed point of h^{τ} for $0 \leq \tau \leq 1$.

The last proposition, however, is true, as you may easily verify.

Like Theorem 1, 2, and 3 one can prove:

The degree of B is topologically invariant. If $(\gamma_1^{\tau}, \gamma_2^{\tau})$, $0 \leq \tau \leq 1$, are normal pairs of maps $\gamma_i^{\tau}: P_{n+1} \rightarrow P_n$ which continuously depend on τ and if B , for $0 \leq \tau \leq 1$, represents a singularity of $(\gamma_1^{\tau}, \gamma_2^{\tau})$, then the degree of B^ at (γ_1^0, γ_2^0) and the degree of B^* at (γ_1^1, γ_2^1) are equal. Let β be the degree of B^* at (γ_1, γ_2) and β_1, \dots, β_m integers with $\sum \beta_i = \beta$; then there are mutually disjoint simplicial 1-spheres B_1, \dots, B_m in V and a normal pair (g_1, g_2) homotopic to (γ_1, γ_2) , composed of maps $g_i: P_{n+1} \rightarrow P_n$, and provided with the following properties: $g_1(p) = \gamma_1(p)$ and $g_2(p) = \gamma_2(p)$ for $p \notin V$; for $i = 1, \dots, m$, the spheres B_i and B are neighbouring; the B_i are the singularities of (g_1, g_2) on V ; by B_i^* denoting the orientation which B^* induces into B_i , one obtains β_i to be the degree of B_i^* under (g_1, g_2) .*

3. Singularities of the degree zero.

The singularity A of (φ_1, φ_2) be called “unessential” if, for every open set U_1 of E_{n+1} with $A \subset U_1 \subset U$, there are continuous maps $f_i: \bar{U} \rightarrow E_n$ such that: $f_1(p) = \varphi_1(p)$ and $f_2(p) = \varphi_2(p)$ for $p \notin U_1$, $f_1(p) \neq f_2(p)$ for $p \in \bar{U}_1$. We designate A as “essential” singularity if it is not unessential. Correspondingly one defines the “essentiality” and “unessentiality” of B . Hereupon holds:

THEOREM 4. *The singularity A of (φ_1, φ_2) being unessential, its degree is equal to zero.*

PROOF. Let a be a point of A , S and n -simplex of U with $a \in S$ and $A \cdot \bar{S} = a$, T an n -simplex in E_n , and t an affine map of \bar{T} onto \bar{S} . Let f be defined by $f(p) = \varphi_1 t(p) - \varphi_2 t(p)$, $p \in \bar{T}$, as map of \bar{T} in E_n . To establish that the index of a at (φ_1, φ_2) and thus the degree of A at (φ_1, φ_2) is equal to zero, it is sufficient to show: there exists a continuous map $f': \bar{T} \rightarrow E_n$ which has no fixed point and agrees with f on $\bar{T} - T$.

May U_1 denote an open set in E_{n+1} with $A \subset U_1 \subset U$ and $(\bar{S} - S) \cdot \bar{U}_1 = 0$. Then the unessentiality of A yields continuous maps $\varphi'_i: \bar{U} \rightarrow E_n$, $i = 1, 2$, such that $\varphi'_1(p) = \varphi_1(p)$ and $\varphi'_2(p) = \varphi_2(p)$ for $p \notin U_1$, $\varphi'_1(p) \neq \varphi'_2(p)$ for $p \in \bar{U}_1$. Now setting $f'(p) = \varphi'_1 t(p) - \varphi'_2 t(p)$ for $p \in \bar{T}$, we obtain a map $f': \bar{T} \rightarrow E_n$ of the desired kind.

Like Lemma 1 you can prove:

LEMMA 2. *Let S be an n -simplex in E_n and a a point of S . Suppose $(f^\tau, 0 \leq \tau \leq 1)$ to be a homotopy of maps $f^\tau: \bar{S} \rightarrow E_n$ with the properties: $p \neq f^\tau(p)$ for all (p, τ) where $p \in \bar{S} - S$ and $0 \leq \tau \leq 1$; the point a is the only fixed point of f^0 and just so of f^1 , the index of a at f^0 is equal to zero. Then there exists a homotopy $(g^\tau, 0 \leq \tau \leq 1)$ of maps $g^\tau: \bar{S} \rightarrow E_n$ which have the properties: $g^\tau(p) = f^\tau(p)$ for all (p, τ) where either $p \in \bar{S}$ and $\tau = 0, 1$ or $p \in \bar{S} - S$ and $0 \leq \tau \leq 1$; for $0 < \tau < 1$, the map g^τ has no fixed point.*

A modified inversion of Theorem 4 is given by

THEOREM 5. *Let the degree of A at (φ_1, φ_2) be zero. Then there exists a point a in U and continuous maps $f_i: \bar{U} \rightarrow E_n$ with the properties: $f_1(p) = \varphi_1(p)$ and $f_2(p) = \varphi_2(p)$ for $p \in \bar{U} - U$, the point a is the only coincidence of (f_1, f_2) .*

PROOF. Let S^τ , $0 \leq \tau \leq 1$, be n -simplexes of U continuously dependent on τ such that: for all τ , the intersection $A \cdot S^\tau$ consists of a single point a^τ ; for $\tau_1 \neq \tau_2$, the intersection $\bar{S}^{\tau_1} \cdot \bar{S}^{\tau_2}$ is empty. The union of all S^τ with $0 < \tau < 1$ we denote by S .

Following Lemma 2, there are continuous maps $g_i: \bar{U} \rightarrow E_n$, $i = 1, 2$, of the condition: $g_1(p) = \varphi_1(p)$ and $g_2(p) = \varphi_2(p)$ for $p \notin S$, $g_1(p) \neq g_2(p)$ for $p \in S$.

The set $A - S$ is homeomorph to a closed segment. Thus there exists an open set U_1 in E_{n+1} such that $A - S \subset U_1 \subset U$ and \bar{U}_1 is homeomorph to the closure of a simplex. Let a be a point of U_1 . Then there exists a continuous map w of $\bar{U}_1 - a$ onto $\bar{U}_1 - U_1$ so that $w(p) = p$ for $p \in \bar{U}_1 - U_1$.

Hereupon we set $\lambda(p) = d(p, a)/(d(p, a) + d(p, \bar{U}_1 - U_1))$ for all points $p \in \bar{U}_1 - a$, and $f_1 = g_1$, moreover

$$f_2(p) = g_1(p) + \lambda(p)(g_2w(p) - g_1w(p)) \text{ for } p \in \bar{U}_1 - a,$$

further $f_2(p) = g_2(p)$ for $p \in \bar{U}_1 - U_1$, and $f_2(a) = g_1(a)$.

For the sake of finishing the argumentation it suffices to show that a represents the only coincidence of (f_1, f_2) on \bar{U}_1 : If p means a point of $\bar{U}_1 - a$, we have

$$f_2(p) - f_1(p) = \lambda(p)(g_2w(p) - g_1w(p)),$$

besides $\lambda(p) > 0$, and $g_2w(p) \neq g_1w(p)$, thus $f_2(p) \neq f_1(p)$.

Similarly as Theorem 4 and 5 one can prove:

The singularity B of (γ_1, γ_2) being unessential, its degree is equal to zero. The degree of B under (γ_1, γ_2) being zero, there is a point b in V and a pair (g_1, g_2) homotopic to (γ_1, γ_2) , consisting of maps $g_i: P_{n+1} \rightarrow P_n$, and of the further condition: $g_1(p) = \gamma_1(p)$ and $g_2(p) = \gamma_2(p)$ for $p \notin V$, the point b is the only coincidence of (g_1, g_2) on \bar{V} .

The precise inversion of Theorem 4 is not correct:

There exist singularities of the degree zero which are essential.

PROOF. Let S be a 4-simplex in E_4 , T a 3-simplex in E_3 , a a point of S , and b a point of T . Set $f_1(p) = b$ for $p \in \bar{S}$. Further, let f_2 be a continuous map of \bar{S} onto \bar{T} with the properties: $f_2(a) = b$,

$$f_2(p) \neq b \text{ for } p \neq a,$$

the map $f_2|_{\bar{S}-S}$ represents an essential map of the 3-sphere $\bar{S}-S$ on the 2-sphere $\bar{T}-T$. Following a known theorem ²⁾, such a map exists.

Now denote by C a simplicial 1-sphere ϵa in S , by R a 3-simplex in S with $a \in R$ and $C \cdot \bar{R} = a$. Let S_1 be an open set in E_4 such that

$$C-a \subset S_1, \bar{R} \cdot \bar{S}_1 = a, \text{ and } \bar{S}_1 \subset S;$$

further $\zeta(p) = d(p, C)/(d(p, C) + d(p, \bar{S}_1 - S_1))$ for all points p of S_1 ; and $g_2(p) = f_2(p)$ for $p \in \bar{S}-S_1$,

$$g_2(p) = b + \zeta(p)(f(p) - b) \text{ for } p \in S_1.$$

The pair (f_1, g_2) thus defined is regular, and C represents its only singularity.

The assumption, C be an unessential singularity of (f_1, g_2) , leads to a contradiction as follows. Then there would exist continuous maps $f^i: \bar{S} \rightarrow E_3$, $i = 1, 2$, so conditioned that: $f^1(p) = f_1(p)$ and $f^2(p) = g_2(p)$ for $p \in \bar{S}-S$, $f^1(p) \neq f^2(p)$ for all points p of \bar{S} .

We define f by $f(p) = b + (f^2(p) - f^1(p))$, $p \in \bar{S}$, as map of \bar{S} in E_3 , that disposes of the following properties: 1) the sphere $\bar{S}-S$ is essentially mapped on $\bar{T}-T$ by $f|_{\bar{S}-S}$, 2) for all points p of \bar{S} holds $b \neq f(p)$. Assertion 1) is true, since $f^1(p) = f_1(p) = b$ for $p \in \bar{S}-S$ and $f^2|_{\bar{S}-S} = f_2|_{\bar{S}-S}$ is an essential map of $\bar{S}-S$ on $\bar{T}-T$. From $f^1(p) \neq f^2(p)$, $p \in \bar{S}$, ensues the correctness of the second assertion. The affirmations 1) and 2), however, contradict to one another.

In order to prove, the degree of C at (f_1, g_2) be zero, it suffices to show: the index of a at (f_1, g_2) is zero. This to establish, let t be an affine map of \bar{T} on \bar{R} . Determine h by $h(p) = f_1 t(p) - g_2 t(p)$, $p \in \bar{T}$, as map of \bar{T} in E_3 . The point b is the only fixed point of h . We will show that the index of b under h is equal to zero.

²⁾ H. Hopf, „Zur Algebra der Abbildungen von Mannigfaltigkeiten”, Journal f. reine und angewandte Math., vol. 163 (1930), pp. 71–88.

For this purpose let R^τ , $0 \leq \tau \leq 1$, be 3-simplexes of S continuously dependent on τ such that $R^0 = R$ and $a \notin R^\tau$ for $\tau > 0$; further $(t^\tau, 0 \leq \tau \leq 1)$ a homotopy of affine maps $t^\tau: \bar{T} \rightarrow \bar{R}^\tau$ with $t^0 = t$; besides h^τ , for $0 \leq \tau \leq 1$, defined by

$$h^\tau(p) = f_1 t^\tau(p) - f_2 t^\tau(p), \quad p \in \bar{T},$$

as map of \bar{T} in E_3 .

On account of $\bar{R} \cdot \bar{S}_1 = a$ and $g_2(p) = f_2(p)$, $p \notin \bar{S}_1$, holds $f_2(p) = g_2(p)$ for $p \in \bar{R}$, hence $h^0 = h$. For all (p, τ) with $p \in \bar{T}$ and $0 < \tau \leq 1$, one has $t^\tau(p) \neq a$, consequently $f_1 t^\tau(p) \neq f_2 t^\tau(p)$; from which it follows that, for $0 < \tau \leq 1$, the map h^τ has no fixed point. Thus, the index of b under $h^0 = h$ is equal to zero.

And the proof is complete.

(Oblatum 3-11-55).

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