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The logic of Negationless Mathematics

by

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Arnhem

Griss stated a new method of treating intuitionistic mathematics without using negation [1]). The intention of this paper is to give a corresponding logical system.

1. We first list the syntactical rules and afterwards give their interpretation.

The following signs are used:

1. atomic formulas $F(x)$, $F(x, y)$, $F(x, y, z)$, . . ., $G(x)$, . . ., $=(x, y)$, $\#(x, y)$; x, y, z, . . . are called variables; they are supposed to be different,

2. undefined signs $\wedge$, $\vee$, (Ev), (v), $\rightarrow_{vs}$ (v stands for a variable, vs for a sequence of different variables).

Definition by induction of a *well-formed formula* (wff):

a. every atomic formula is a wff,
b. if p and q are wff, then $p \wedge q$ and $p \vee q$ are wff,
c. if p and q are wff, then $p \rightarrow_{vs} q$ is a wff,
d. if p is a wff, then $(Ev)p$ and $(v)p$ are wff.

Definition by induction of *free* and *bound variables*:

a. any variable occurring in an atomic formula is free in that formula,
b. any variable that is free (bound) in p, is free (bound) in $p \wedge q$ and in $p \vee q$,
any variable that is free (bound) in q, is free (bound) in $p \wedge q$ and in $p \vee q$,

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[1]) G. F. C. Griss, Negatieloze intuïtionistische wiskunde, Versl. Ned. Akad. v. Wetensch., afd. Natuurk., LIII (1944), p. 261—268,

Negationless intuitionistic mathematics, Verh. Kon. Ned. Akad. v. Wetensch., XLIX (1946), p. 1127—1133, LIII (1950), p. 456—463,

Logique des mathématiques intuitionistes sans négation, Comptes rendus des séances de l'Acad. des Sc., t. 227 (1948), p. 946—948.

c. any variable belonging to vs that is free in p (or q), is bound in $p \rightarrow_{vs} q$, and is said to be bound by $\rightarrow_{vs}$,

d. if the variable v is free in p, it is bound in $(Ev)p$ and in $(v)p$, and is said to be bound by (Ev) and (v), respectively,

e. any variable that is free in p (or q) and not bound by $\rightarrow_{vs}$, by (Ev) or by (v), is free in $p \rightarrow_{vs} q$, $(Ev)p$, $(v)p$, respectively.

An arbitrary wff will be written $p, q, r, \ldots$. If we want to express, that the wff p contains the free variable x, we write $p(x)$. This only means that p contains the free variable x, but not that x is the only variable that is free in p.

There is no difference between p and $p(x)$ occurring in the same derivation. $p(x)$ is written at those places where it is essential to remember that x occurs free in p; at other places of the same derivation p may be written.

If in a derivation first, e.g., $p(x)$ and afterwards $p(y)$ occurs, then with $p(y)$ is meant the wff, that is generated from $p(x)$ by replacing every x, that is free in $p(x)$ by y.

If $x, y, z, \ldots$ are all the free variables of p, then $\exists p$ stands for $(Ex)(Ey)(Ez) \ldots p$. If no variable is free in p, then $\exists p$ stands for p. The order of the (Ex), (Ey), (Ez), ... is indifferent. This will be shown afterwards (7.21).

If vs is the sequence of variables $x, y, z, \ldots$, then $\exists_{vs} p$ stands for $(Ex)(Ey)(Ez) \ldots p$, and $(vs)p$ for $(x)(y)(z) \ldots p$.

$p, q, r, \ldots$ are not signs belonging to the system. They are merely names for arbitrary wff. So they belong to the metasystem. The signs vs, $\exists$, $\exists_{vs}$, (vs), $\rightarrow_{vs}$ belong also to the metasystem.

Interpretation. The sign $\wedge$ is used for conjunction, $\vee$ for disjunction, (Ev) is the existentional operator, (v) the all-operator.

Wff without free variables are to be interpreted as propositions, wff with free variables as propositional functions.

$p(x) \rightarrow_x q(x)$ means that the class determined by the propositional function $p(x)$ (short: the class $p(x)$) is included in the class $q(x)$. [1]) $p(x, y) \rightarrow_{xy} q(x, y)$ means that the class (of pairs x, y) $p(x, y)$ is included in the class $q(x, y)$. $p(x, y) \rightarrow_x q(x, y)$ is to be interpreted as the class of those y for which $p(x, y)$ is included in $q(x, y)$, etc.

$(Ex)p(x)$ is a proposition; $(Ex)p(x, y)$ is the class of those y for which an x exists that satisfies $p(x, y)$. The same holds for the all-operator.

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[1]) It is supposed in this and the next two paragraphs, that p and q contain no free variables different from those mentioned between brackets.

There is still some difficulty with the interpretation of e.g. $p(x) \rightarrow_{xy} q(x, y)$. If $p(x)$ was a class of x's, it could not be seen in what respect this class might be included in $q(x, y)$. But we may also interpret $p(x)$ as the class of pairs x, y that satisfy $p(x)$. So if $p(x)$ as a propositional function of x, is satisfied by a, it is satisfied by any pair a, y. This situation is analogous to solving the equations $x + 1 = 0$ and $3x + 5y = 2$. The solution is $x = -1$, $y = 1$, as $x = -1$ and y arbitrary will satisfy $x + 1 = 0$. So there is the same kind of ambiguity in the interpretation of propositional functions as in the meaning of an equation.

In the same way it is possible to interpret a wff without free variables as a propositional function. A wff without free variables that is true may be interpreted as an all-class, an all-class of pairs, etc., and is to be compared with an identical equation. This kind of interpretation enables us in formal respect not to discern any more between propositions and propositional functions. We can restrict ourselves to theorems about arbitrary wff that may or may not contain free variables.

It is clear from the foregoing that $p \wedge q$ is to be interpreted as the product of the classes p and q, $p \vee q$ as their sum.

The negationless method. Before continuing it will be necessary to explain in brief the fundamental ideas of Griss' method.

Griss accepts that in constructing a mathematical system we progress from true propositions to other propositions that are also true. Perhaps we may, when making a rough calculation, find the impossibility that some theorem will ever be a part of our system. That result may be very instructive for the investigator, but it is not a part of the system itself. When I am building a house it may be of great importance to decide that I shall not use a certain kind of bricks, but this decision does not make those bricks part of the house. So in the mathematical system only those propositions will occur that are true. And as these propositions are all affirmative (the contradictory propositions being only possible in „rough"), a sign for negation is useless in his system.

Another fundamental feature of Griss' method is that he accepts that in constructing one is always constructing something and so never will construct nothing. In accordance with this view he declares that the null-class does not exist. Every propositional function has the property that it can be satisfied. So the product of two classes is not always a class. If the classes have no lement in common their product is not the null-class, but merely senseless.

The propositional calculus. In a mathematical Griss-system only true propositions will occur. So there is no reason for linking them by a sign for disjunction or implication. „In rough" we may find that a certain proposition can be proved as soon as A has been proved and also as soon as B has been proved. And then we might say that $A \vee B$ implies C. But in the system itself we shall never progress from A (or from B) to another proposition before A (or B) has been proved. So in the system itself the disjunction of propositions is useless. The same holds for the implication. In rough we may convince ourselves that B can be proved as soon as A has been proved, but this consideration is not a part of the system itself. Formally linking propositions by $\vee$ or $\rightarrow$ is possible, but the interpretation of the result is the same as the interpretation of their conjunction (this remark is of importance, as we shall formally treat propositions and propositional functions in the same way).

Propositions may be linked by conjunction. As the kind of linking obeys the same laws as the linking of propositional functions, there is no reason for a separate propositional calculus.

Axioms and derivations.

There are two kinds of *axioms*:

a. axioms of the form $\dfrac{p_0\, p_1 \cdots p_n}{p}$,

b. axioms of the form: if $\dfrac{p_0\, p_1 \cdots p_n}{p}$, then $\dfrac{q_0\, q_1 \cdots q_m}{q}$.

Definition by induction of $\dfrac{P}{p}$ (*p is derivable from P*).

In this definition P, Q, Q_1, Q_2 stand for arbitrary finite sequences of wff.

1. If every q, that belongs to Q, belongs to P, and if $\dfrac{Q}{p}$ is an axiom, then $\dfrac{P}{p}$.

2. If $\dfrac{Q}{p}$ is an axiom and every q, that belongs to Q, is derivable from P, then $\dfrac{P}{p}$.

3. If

$a.\ \frac{Q_1}{q}$,

$b.$ „if $\frac{Q_1}{q}$, then $\frac{Q_2}{p}$" is an axiom,

$c.$ every q, that belongs to Q_2, is derivable from P,

then $\frac{P}{p}$.

Semantical remark.

The meaning of a derivation of the form $\frac{p(x, y, \ldots)}{q(x, y, \ldots)}$ is, that for arbitrary values of $x, y, \ldots\ q(x, y, \ldots)$ can be derived from $p(x, y, \ldots)$. So there is a close connection between the derivation of one propositional function from another and the inclusion of the classes determined by the two functions.

The use of dots.

We discern left and right dots. Left dots stand to the left of a letter or of $\sim$, right dots to the right. The scope of a left (right) complex of dots is extended to the left (right) until a right (left) complex of dots is reached, that consists of an equal or a larger number of dots, or, if this is not the case, to the end of the formula.

$\wedge$ and $\vee$ bind stronger than $\rightarrow_{vs}$.

Final remarks.

In principle logical theorems can be dispensed with. Their purpose is merely to enable abbreviations in the mathematical process. Instead of a large quantity of applications of the logical axioms one application of a logical theorem may be used.

The mathematician will perhaps say that he is not reasoning in detail according to the logical axioms. But the logician only says that it is possible to rebuild the mathematical system by using the logical axioms. As soon as it turns out that his logical system is unable to describe the mathematical system, the logical system should be altered. On the other hand the considerations of the logician may be of some influence on mathematical thought.

Investigating the logical system it will appear that it obeys its own rules and axioms. But as its structure is very simple, only few of its axioms are sufficient for its own foundation. This last remark has a metalogical character and will not be analyzed further.

We now start building the logical system. Axioms will be marked A, definitions D and theorems without a letter. At the end of the bar the numbers of the (main) axioms, definitions and theorems are mentioned that are used.

Definitions are merely used as abbreviations.

I. The functional calculus without considering the inner structure of the wff

2. *The axioms of conjunction.*

A2.0 $$\frac{p \quad q}{p \wedge q}$$

A2.1 $$\frac{p \wedge q}{p},\ \frac{p \wedge q}{q}$$

A2.0 does not mean that any two propositional functions (classes) have a product. We must not forget that an axiom can only then be applied, when the premisses are derived formulas. So the meaning will be: any x (or any pair x, y, etc.) that satisfies p and q, will also satisfy $p \wedge q$.

In case p and q are propositions A2.0 simply says that two derived propositions may be conjuncted.

2.0 $$\frac{p \wedge q}{q \wedge p}$$

Proof. $$\cfrac{\cfrac{p \wedge q}{q}\,\mathrm{A2.1} \qquad \cfrac{p \wedge q}{p}\,\mathrm{A2.1}}{q \wedge p}\,\mathrm{A2.0}$$

2.1 $$\frac{p \wedge q \,.\wedge r}{p \wedge .\, q \wedge r}$$

Proof. $$\cfrac{\cfrac{\cfrac{p \wedge q \,.\wedge r}{p \wedge q}\,\mathrm{A2.1}}{p}\,\mathrm{A2.1} \qquad \cfrac{\cfrac{\cfrac{p \wedge q \,.\wedge r}{p \wedge q}\,\mathrm{A2.1}}{q}\,\mathrm{A2.1} \qquad \cfrac{p \wedge q \,.\wedge r}{r}\,\mathrm{A2.1}}{q \wedge r}\,\mathrm{A2.0}}{p \wedge .\, q \wedge r}\,\mathrm{A2.0}$$

2.2 $$\frac{p}{p}$$

Proof. $$\cfrac{\cfrac{p \quad p}{p \wedge p}\,\mathrm{A2.0}}{p}\,\mathrm{A2.1}$$

3. *Axioms about* $\rightarrow_{vs}$ *avd* $\exists_{vs}$.

A3.0 $\dfrac{p \quad p \rightarrow_{vs} q}{q}$

A3.2 $\dfrac{p}{(Ex)p}$

A3.3 $\dfrac{p \rightarrow_{vs} q}{\exists_{vs} p}$, $\dfrac{p \rightarrow_{vs} q}{\exists_{vs} q}$

A3.4 If $\dfrac{p \quad r}{q}$, then $\dfrac{\exists_{vs} p \quad r}{p \rightarrow_{vs} q}$.

No variable of vs must be free in r. The premiss r may be dropped.

The $\exists_{vs}$-operator is of extreme importance in negationless logic. E.g., in ordinary logic no one would hesitate to accept $\dfrac{p \rightarrow_{vs} q}{p \wedge r \rightarrow_{vs} q}$. But in negationless logic this derivation is only possible, if it is known that $p \wedge r$ exists. Therefore the premiss $\exists_{vs} . p \wedge r$ has to be added. From this example it is seen that in many cases additional premisses of the form $\exists_{vs} p$ will distinguish the present calculus from the usual logical calculi.

A3.2 states that any wff (class) that previously occurs as a conclusion, exists. For repeated application of this axiom leads to $\exists p$ and to $\exists_{vs} p$.

A3.3 states that, if previously it has been proved that $p(x, y, \ldots)$ is included in $q(x, y, \ldots)$, then there is a sequence $x, y, \ldots$ that satisfies $p(x, y, \ldots)$ and also a sequence that satisfies $q(x, y, \ldots)$.

It is not clear that in A3.4 the premiss $\exists_{vs} p$ must be added. For if in a mathematical system this axiom is applied, p is the conclusion of a preceding derivation and so the condition $\exists_{vs} p$ will always be fulfilled. Still we are not in accordance with the intention of our system, if $\exists_{vs} p$ is cancelled. For according to 2.2 $\dfrac{p \quad r}{p}$. Canceling $\exists_{vs} p$ we would find $\dfrac{r}{p \rightarrow_{vs} p}$. And then A3.3 would give the conclusion $\exists_{vs} p$. The derivation of the existence of an arbitrary p from an arbitrary premiss r is certainly not in accordance with our aim.

3.00 $\dfrac{\exists_{vs} p}{p \rightarrow_{vs} p}$

Proof. 2.0, A3.4.

3.01 $$\frac{\exists_{vs} pq}{p \wedge q \to_{vs} p}, \quad \frac{\exists_{vs} pq}{p \wedge q \to_{vs} q}$$ [1]

Proof. **A2.1, A3.4.**

3.10 $$\frac{p \to_{vs} q \quad \exists_{vs} pr}{p \wedge r \to_{vs} q}$$

Proof. $$\cfrac{\cfrac{\cfrac{p \wedge r^*}{p}\,\text{A2.1} \qquad p \to_{vs} q}{q}\,\text{A3.0} \qquad \exists_{vs} pr}{p \wedge r \to_{vs} q}\,\text{A3.4}$$

Remark. The full proof is:

$$\cfrac{\cfrac{p \wedge r}{p}\,\text{A2.1} \qquad p \to_{vs} q}{q}\,\text{A3.0} \text{ and so } \frac{p \to_{vs} q \quad \exists_{vs} pr}{p \wedge r \to_{vs} q}\,\text{A3.4}$$

So $p \wedge r$ turns out not to be a premiss of the derivation of $p \wedge r \to_{vs} q$ from $p \to_{vs} q$ and $\exists_{vs} pr$. This is the meaning of the asteric in the above proof.

3.11 $$\frac{p \to_{vs} q \wedge r}{p \to_{vs} q}$$

Proof. $$\cfrac{\cfrac{\cfrac{p^* \quad p \to_{vs} q \wedge r}{q \wedge r}\,\text{A3.0}}{q}\,\text{A2.1} \qquad \cfrac{p \to_{vs} q \wedge r}{\exists_{vs} p}\,\text{A3.3}}{p \to_{vs} q}\,\text{A3.4}$$

3.12 $$\frac{p \to_{vs} q \quad \exists_{vs} pr}{p \wedge r \to_{vs} q \wedge r}$$

Proof. $$\cfrac{\cfrac{\cfrac{\cfrac{p \wedge r^*}{p}\,\text{A2.1} \qquad p \to_{vs} q}{q}\,\text{A3.0} \qquad \cfrac{p \wedge r^*}{r}\,\text{A2.1}}{q \wedge r}\,\text{A2.0} \qquad \exists_{vs} pr}{p \wedge r \to_{vs} q \wedge r}\,\text{A3.4}$$

3.20 $$\frac{p \to_{vs} q \quad q \to_{vs} r}{p \to_{vs} r}$$

Proof. $$\cfrac{\cfrac{\cfrac{p^* \quad p \to_{vs} q}{q}\,\text{A3.0} \qquad q \to_{vs} r}{r}\,\text{A3.0} \qquad \cfrac{p \to_{vs} q}{\exists_{vs} p}\,\text{A3.3}}{p \to_{vs} r}\,\text{A3.4}$$

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[1] $\exists_{vs} pq$ is short for $\exists_{vs} . p \wedge q$.

3.21 $$\frac{p \rightarrow_{vs} q \quad p \rightarrow_{vs} r}{p \rightarrow_{vs} q \wedge r}$$

Proof. $$\cfrac{\cfrac{p^* \quad p \rightarrow_{vs} q}{q}\ \text{A3.0} \quad \cfrac{p^* \quad p \rightarrow_{vs} r}{r}\ \text{A3.0} \quad \cfrac{p \rightarrow_{vs} q}{\exists_{vs} p}\ \text{A3.3}}{p \rightarrow_{vs} q \wedge r}\ \text{A2.0, A3.4}$$

3.22 $$\frac{p \rightarrow_{vs} q \quad r \rightarrow_{vs} s \quad \exists_{vs} pr}{p \wedge r \rightarrow_{vs} q \wedge s}$$

Proof. $$\cfrac{\cfrac{p \rightarrow_{vs} q \quad \cfrac{\exists_{vs} pr}{p \wedge r \rightarrow_{vs} p}\ 3.01}{p \wedge r \rightarrow_{vs} q}\ 3.20 \quad \cfrac{r \rightarrow_{vs} s \quad \cfrac{\exists_{vs} pr}{p \wedge r \rightarrow_{vs} r}\ 3.01}{p \wedge r \rightarrow_{vs} s}\ 3.20}{p \wedge r \rightarrow_{vs} q \wedge s}\ 3.21$$

3.30 If $\dfrac{p}{q}$, then $\dfrac{\exists_{vs} p}{\exists_{vs} q}$ and $\dfrac{\exists p}{\exists q}$.

Proof. 1. $$\cfrac{\cfrac{\exists_{vs} p}{p \rightarrow_{vs} q}\ \text{A3.4}}{\exists_{vs} q}\ \text{A3.3}$$ 2. $$\cfrac{\cfrac{\exists p}{p \rightarrow_{vs'} q}\ \text{A3.4}}{\exists_{vs'} q}\ \text{A3.3}$$

In the 2nd derivation vs' is the sequence of free variables of p. We have still to derive $\exists q$ from $\exists_{vs'} q$. In case q contains a free variable, that does not belong to vs', we apply A3.2. In case a variable of vs' is not free in q, we apply a theorem that will be proved afterwards (7.00).

3.300 If $\dfrac{p \quad r}{q}$ and no free variable of r is free in p, then $\dfrac{\exists_{vs} p \quad r}{\exists_{vs} q}$ and $\dfrac{\exists p \quad r}{\exists q}$.

Proof. Similar.

3.301. $$\frac{\exists p \quad \exists q}{\exists\, .\, p \wedge \exists q}$$

Proof. $$\frac{p \quad \exists q}{p \wedge \exists q}\ \text{A2.0}$$

Further 3.300.

3.31 If $\dfrac{p}{q}$, then $\dfrac{p \wedge r}{q \wedge r}$.

Proof.
$$\dfrac{\dfrac{\dfrac{p \wedge r}{p}\,\text{A2.1}}{q} \qquad \dfrac{p \wedge r}{r}\,\text{A2.1}}{q \wedge r}\,\text{A2.0}$$

3.32 If $\dfrac{p}{q}$, then $\dfrac{r \rightarrow_{vs} p}{r \rightarrow_{vs} q}$.

Proof.
$$\dfrac{\dfrac{\dfrac{r \rightarrow_{vs} p}{\exists_{vs}\, p}\,\text{A3.3}}{p \rightarrow_{vs} q}\,\text{A3.4} \qquad r \rightarrow_{vs} p}{r \rightarrow_{vs} q}\,3.20$$

3.33 If $\dfrac{q}{p}$ and $\dfrac{\exists_{vs}\, p}{\exists_{vs}\, q}$, then $\dfrac{p \rightarrow_{vs} r}{q \rightarrow_{vs} r}$.

Proof.
$$\dfrac{\dfrac{\dfrac{\dfrac{p \rightarrow_{vs} r}{\exists_{vs}\, p}\,\text{A3.3}}{\exists_{vs}\, q}}{q \rightarrow_{vs} p}\,\text{A3.4} \qquad p \rightarrow_{vs} r}{q \rightarrow_{vs} r}\,3.20$$

3.40 $\dfrac{p \wedge q}{\exists_{vs}\, p}$

Proof. A2.1, A3.2.

3.41 $\dfrac{p \rightarrow_{vs} q}{\exists_{vs}\, pq}$

Proof.
$$\dfrac{\dfrac{\dfrac{\dfrac{p \rightarrow_{vs} q}{\exists_{vs}\, p}\,\text{A3.3}}{p \rightarrow_{vs} p}\,3.00 \qquad p \rightarrow_{vs} q}{p \rightarrow_{vs} p \wedge q}\,3.21}{\exists_{vs}\, pq}\,\text{A3.3}$$

3.42 $\dfrac{\exists_{vs}\, pq}{\exists_{vs}\, p}$, $\dfrac{\exists pq}{\exists p}$

Proof. A2.1, 3.30.

4. *Disjunction.*

A4.0 $\dfrac{p \quad \exists q}{p \vee q}$

A4.1 $$\frac{p \vee q}{q \vee p}$$

A4.2 $$\frac{\exists . p \vee q}{\exists p}$$

A4.3 $$\frac{p \rightarrow_{vs} r \qquad q \rightarrow_{vs} r}{p \vee q \rightarrow_{vs} r}$$

A4.4 $$\frac{p \vee q . \wedge r \qquad \exists pr \qquad \exists qr}{p \wedge r . \vee . q \wedge r}$$

In A4.0 $p \vee q$ can only be derived from p, if q is a propositional function. So the premiss $\exists q$ has to be added.

In A4.4 $p \wedge r . \vee . q \wedge r$ can only be derived, if the products $p \wedge r$ and $q \wedge r$ exist.

4.00 $$\frac{\exists . p \vee q}{\exists q}$$

Proof. A4.1, 3.30, A4.2.

4.01 $$\frac{\exists p \quad \exists q}{\exists . p \vee q}, \quad \frac{\exists_{vs} p \quad \exists_{vs} q}{\exists_{vs} . p \vee q}$$

Proof. A4.0, 3.300 and, if necessary, A3.2 and 7.00.

4.02 $$\frac{p \vee q}{\exists p}, \quad \frac{p \vee q}{\exists q}$$

Proof. A3.2, A4.2 or 4.00.

4.10 If $\frac{p}{r}$ and $\frac{q}{r}$, then $\frac{p \vee q}{r}$.

Proof. vs is the sequence of free variables of p and q.

$$\cfrac{\cfrac{\cfrac{\cfrac{p \vee q}{\exists_{vs} p}\ \text{4.02, A3.2}^{1)}}{p \rightarrow_{vs} r}\ \text{A3.4} \qquad \cfrac{\cfrac{p \vee q}{\exists_{vs} q}\ \text{4.02, A3.2}^{1)}}{q \rightarrow_{vs} r}\ \text{A3.4}}{p \vee q \rightarrow_{vs} r}\ \text{A4.3} \qquad p \vee q}{r}\ \text{A3.0}$$

4.100 If $\frac{p \quad s}{r}$ and $\frac{q \quad s}{r}$, then $\frac{p \vee q \quad s \quad \exists ps \quad \exists qs}{r}$.

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1) A3.2 should be applied, in case p and q do not contain the same free variables.

Proof.

$$\cfrac{\cfrac{\cfrac{p \vee q \qquad s}{p \vee q \,.\wedge s}\,\text{A2.0} \qquad \exists ps \ \exists qs}{p \wedge s \,.\vee.\, q \wedge s}\,\text{A4.4}}{r}\,\text{A2.1, 4.10}$$

4.11 If $\frac{p \quad p \vee q}{r}$ and $\frac{q \quad p \vee q}{r}$, then $\frac{p \vee q}{r}$.

Proof. vs is the sequence of free variables of p.

$$\cfrac{\cfrac{\cfrac{p \vee q}{p \vee q.\wedge.p \vee q}\,\text{A2.0} \qquad \cfrac{\cfrac{\cfrac{p \vee q}{\exists p}\,\text{4.02} \quad \cfrac{p \vee q}{\exists q}\,\text{4.02}}{p \rightarrow_{vs} p \vee q}\,\text{A4.0, A3.4}}{\exists.p \wedge.p \vee q}\,\text{3.41, A3.2}\ ^{1)} \qquad \cfrac{p \vee q}{\exists.q \wedge.p \vee p}\,\text{analogous}}{p \wedge.p \vee q:\vee:q \wedge.p \vee q}\,\text{A4.4}}{r}\,\text{A2.1, 4.10}$$

4.2 If $\frac{p}{q}$ and $\frac{r}{s}$, then $\frac{p \vee r}{q \vee s}$.

Proof. (1)

$$\cfrac{\cfrac{r}{s} \quad \cfrac{\cfrac{p \vee r}{\exists p}\,\text{A3.2, A4.2}}{\exists q}\,\text{3.30}}{q \vee s}\,\text{A4.0}$$

(2)

$$\cfrac{\cfrac{p}{q} \quad \cfrac{\cfrac{p \vee r}{\exists r}\,\text{A3.2, A4.2}}{\exists s}\,\text{3.30}}{q \vee s}\,\text{A4.0}$$

Further (1), (2), 4.11.

4.30 $$\frac{\exists_{vs} p \qquad \exists_{vs} q\ (\text{or } \exists q)}{p \rightarrow_{vs} p \vee q}$$

Proof. We first prove $\frac{\exists_{vs} q}{\exists q}$, by applying (if necessary) A3.2 and 7.00 (cf. the proof of 3.30). Further A4.0, A3.4.

4.31 $$\frac{p \vee q \rightarrow_{vs} r}{p \rightarrow_{vs} r}$$

vs contains all free variables of p.

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{p \vee q \rightarrow_{vs} r}{\exists_{vs}.p \vee q}\,\text{A3.3}}{\exists.p \vee q}\,\text{A3.2, 7.00}}{\exists p}\,\text{A4.2}}{\exists_{vs} p}\,\text{A3.2} \qquad \cfrac{p \vee q \rightarrow_{vs} r}{\exists q}\,\text{analogous}}{p \rightarrow_{vs} p \vee q}\,\text{4.30} \qquad p \vee q \rightarrow_{vs} r}{p \rightarrow_{vs} r}\,\text{3.20}$$

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[1] A3.2 should be applied, if $p \wedge . p \vee q$ contains a free variable that does not belong to vs.

4.32 $$\frac{p \rightarrow_{vs} q \quad \exists_{vs} r}{p \rightarrow_{vs} q \vee r}$$

Proof. $$\dfrac{\dfrac{\dfrac{p \rightarrow_{vs} q}{\exists_{vs} q}\,\text{A3.3} \quad \exists_{vs} r}{q \rightarrow_{vs} q \vee r}\,\text{4.30} \qquad p \rightarrow_{vs} q}{p \rightarrow_{vs} q \vee r}\,\text{3.20}$$

4.33 $$\frac{p \rightarrow_{vs} q \quad \exists_{vs} r}{p \vee r \rightarrow_{vs} q \vee r}$$

Proof. $$\dfrac{\dfrac{p \rightarrow_{vs} q \qquad \dfrac{\dfrac{p \rightarrow_{vs} q}{\exists_{vs} q}\,\text{A3.3} \quad \exists_{vs} r}{q \rightarrow_{vs} q \vee r}\,\text{4.30}}{p \rightarrow_{vs} q \vee r}\,\text{3.20} \qquad \dfrac{\dfrac{p \rightarrow_{vs} q}{\exists_{vs} q}\,\text{A3.3} \quad \exists_{vs} r}{r \rightarrow_{vs} q \vee r}\,\text{4.30}}{p \vee r \rightarrow_{vs} q \vee r}\,\text{A4.3}$$

4.34 $$\frac{p \wedge r \quad \exists q}{p \vee q \,.\wedge r}$$

Proof. $$\dfrac{\dfrac{\dfrac{p \wedge r}{p}\,\text{A2.1} \quad \exists q}{p \vee q}\,\text{A4.0} \qquad \dfrac{p \wedge r}{r}\,\text{A2.1}}{p \vee q \,.\wedge r}\,\text{A2.0}$$

4.35 $$\frac{p \wedge r \,.\vee.\, q \wedge r}{p \vee q \,.\wedge r}$$

Proof. (1) $$\dfrac{\dfrac{\dfrac{p \wedge r}{p}\,\text{A2.1} \quad \dfrac{p \wedge r \,.\vee.\, q \wedge r}{\exists q}\,\text{4.02, 3.42}}{p \vee q}\,\text{A4.0} \qquad \dfrac{p \wedge r}{r}\,\text{A2.1}}{p \vee q \,.\wedge r}\,\text{A2.0}$$

(2) $$\frac{q \wedge r \quad p \wedge r \,.\vee.\, q \wedge r}{p \vee q \,.\wedge r}\ \text{analogous}$$

Further (1), (2), 4.11.

4.36 $$\frac{p \wedge q \,.\vee r}{p \vee r \,.\wedge.\, q \vee r}$$

Proof.

(1) $$\dfrac{\dfrac{\dfrac{p \wedge q}{p}\,\text{A2.1} \quad \dfrac{p \wedge q \,.\vee r}{\exists r}\,\text{4.02}}{p \vee r}\,\text{A4.0} \qquad \dfrac{p \wedge q \quad p \wedge q \,.\vee r}{q \vee r}\ \text{analogous}}{p \vee r \,.\wedge.\, q \vee r}\,\text{A2.0}$$

(2)
$$\dfrac{\dfrac{r \qquad \dfrac{p \wedge q \,.\, \vee r}{\exists p}\;\textbf{4.02, 3.42}}{p \vee r}\;\textbf{A4.0} \qquad \dfrac{r \quad p \wedge q \,.\, \vee r}{q \vee r}\;\text{analogous}}{p \vee r \,.\, \wedge \,.\, q \vee r}\;\textbf{A2.0}$$

Further (1), (2), 4.11.

4.360
$$\dfrac{p \wedge q \,.\, \vee \,.\, r \wedge s}{p \vee r \,.\, \wedge \,.\, p \vee s \,.\, \wedge \,.\, q \vee r \,.\, \wedge \,.\, q \vee s}$$

Proof. 4.36.

4.37
$$\dfrac{p \vee q \,.\, \vee r}{p \vee \,.\, q \vee r}$$

Proof. vs is the sequence of free variables of p, q and r.

$$\dfrac{p \vee q \,.\, \vee r \qquad \dfrac{\dfrac{\dfrac{\dfrac{p \vee q \,.\, \vee r}{\exists_{vs} q \quad \exists_{vs} r}\;\textbf{4.02, 4.00, A3.2}}{q \rightarrow_{vs} q \vee r}\;\textbf{4.30} \qquad \dfrac{p \vee q \,.\, \vee r}{\exists_{vs} p}\;\textbf{4.02, A4.2, A3.2}}{p \vee q \rightarrow_{vs} p \vee \,.\, q \vee r}\;\textbf{4.33} \qquad \dfrac{\exists_{vs} r \quad \dfrac{\dfrac{r^* \quad \exists_{vs} q}{q \vee r}\;\textbf{A4.0} \qquad \exists_{vs} p}{p \vee \,.\, q \vee r}\;\textbf{A4}}{r \rightarrow_{vs} p \vee \,.\, q \vee r}\;\textbf{A3.4}}{p \vee q \,.\, \vee r \,.\, \rightarrow_{vs} p \vee \,.\, q \vee r}\;\textbf{A4.3}}{p \vee \,.\, q \vee r}\;\textbf{A3.0}$$

4.38
$$\dfrac{p \vee q \,.\, \wedge \,.\, r \vee s \quad \exists pr \quad \exists ps \quad \exists qr \quad \exists qs}{p \wedge r \,.\, \vee \,.\, p \wedge s \,.\, \vee \,.\, q \wedge r \,.\, \vee \,.\, q \wedge s}$$

Proof. (1)
$$\dfrac{p \wedge r \quad \exists s}{p \wedge \,.\, r \vee s}\;\textbf{A2.1, A4.0, A2.0}$$

(2)
$$\dfrac{p \vee q \,.\, \wedge \,.\, r \vee s \qquad \dfrac{\exists pr \qquad \dfrac{\exists qs}{\exists s}\;\textbf{3.42}}{\exists \,.\, p \wedge \,.\, r \vee s}\;\text{(1), }\textbf{3.300} \qquad \dfrac{\exists qr \quad \exists qs}{\exists \,.\, q \wedge \,.\, r \vee s}\;\text{analogous}}{p \wedge \,.\, r \vee s : \vee : q \wedge \,.\, r \vee s}\;\textbf{A4.4}$$

(3)
$$\dfrac{\dfrac{p \wedge \,.\, r \vee s \quad \exists pr \quad \exists ps}{p \wedge r \,.\, \vee \,.\, p \wedge s}\;\textbf{A4.4} \qquad \dfrac{\exists qr \quad \exists qs}{\exists : q \wedge r \,.\, \vee \,.\, q \wedge s}\;\textbf{4.01}}{p \wedge r \,.\, \vee \,.\, p \wedge s \,.\, \vee \,.\, q \wedge r \,.\, \vee \,.\, q \wedge s}\;\textbf{A4.0}$$

(4)
$$\dfrac{q \wedge \,.\, r \vee s \quad \exists pr \quad \exists ps \quad \exists qr \quad \exists qs}{p \wedge r \,.\, \vee \,.\, p \wedge s \,.\, \vee \,.\, q \wedge r \,.\, \vee \,.\, q \wedge s}\;\text{analogous}$$

(5)
$$\dfrac{\dfrac{\exists pr \quad \exists qs}{\exists : p \wedge \,.\, r \vee s}\;\text{see (2)} \qquad \exists pr \quad \exists ps \quad \exists qr \quad \exists qs}{\exists : p \wedge \,.\, r \vee s \,.\, \wedge \exists pr \wedge \exists ps \wedge \exists qr \wedge \exists qs}\;\textbf{3.301}$$

(6)
$$\dfrac{\exists pr \quad \exists ps \quad \exists qr \quad \exists qs}{\exists : q \wedge \,.\, r \vee s \,.\, \wedge \exists pr \wedge \exists ps \wedge \exists qr \wedge \exists qs}\;\text{analogous}$$

Further (2), (3), (4), (5), (6), 4.100.

4.39 $$\frac{p \vee r \,.\wedge .\, q \vee r \quad \exists pq \quad \exists pr \quad \exists qr}{p \wedge q \,.\vee r}$$

Proof.

(1) $$\frac{\dfrac{p \quad q}{p\wedge q}\,\text{A2.0} \quad \dfrac{\exists qr}{\exists r}\,3.42 \quad [\exists pq]}{p\wedge q\,.\vee r}\,\text{A4.0}$$

(2) $$\frac{[p]\ r\ [\exists qr]\ \exists pq}{p\wedge q\,.\vee r}\,\text{A4.0}$$

(3) $$\frac{p \quad q\vee r \quad \exists pq \quad \exists pr \quad \exists qr}{p\wedge q\,.\vee r}\ \text{(1), (2), 3.301, 4.100}$$

(4) $$\frac{r\ [q\vee r]\ \exists pq\ [\exists pr]\ [\exists qr]}{p\wedge q\,.\vee r}\,\text{A4.0}$$

(5) $$\frac{\exists pr \quad \dfrac{\exists pq}{\exists q}\,3.42}{\exists\,.\,p\wedge .\,q\vee r}\,4.34,\ 3.300$$

(6) $$\frac{\dfrac{\dfrac{\exists pr}{\exists r}\,3.42}{\exists rr}\,3.30 \quad \dfrac{\exists pq}{\exists q}\,3.42}{\exists\,.\,r\wedge .\,q\vee r}\,4.34,\ 3.300$$

Further (3), (4), (5), (6), 3.301, 4.100.

Remark. With the square brackets in the 2nd and the 6th line of the proof is meant, that the addition of the premisses $\exists pq$, p, $\exists qr$ and $q \vee r$, $\exists pr$, $\exists qr$ is not wanted for the derivation there but afterwards for using 4.100.

II. The general functional calculus

5. We shall now introduce all- and existentional-operators.

A5.0 $$\frac{(x)p}{p}$$

A5.1 If $\dfrac{p \quad q}{r}$, then $\dfrac{p \quad (x)q}{(x)r}$.

p must not contain x as a free variable; p may be dropped.

5.0 If $\dfrac{p \quad q}{r}$, then $\dfrac{p \quad (Ex)q}{(Ex)r}$.

p must not contain x as a free variable; p may be dropped.

Proof. $$\cfrac{\dfrac{p \quad (Ex)q}{q \rightarrow_x r}\,\text{A3.4}}{(Ex)r}\,\text{A3.3}$$

5.1 $$\frac{(x)p}{(Ex)p}$$

Proof. A5.0, A3.2.

5.20 $$\frac{(x)\,.\,p \wedge q}{(x)p}, \quad \frac{(x)\,.\,p \wedge q}{(x)q}$$

Proof. A2.1, A5.1.

5.21 $$\frac{(x)p \quad (x)q}{(x)\,.\,p \wedge q}$$

Proof. (1) $$\dfrac{p \quad \dfrac{(x)q}{q}\,\text{A5.0}}{p \wedge q}\,\text{A2.0}$$

Further (1), A5.1.

5.22 $$\frac{(x)p \quad (Ex)q}{(Ex)\,.\,p \wedge q}$$

Proof. (1) $$\dfrac{q \quad \dfrac{(x)p}{p}\,\text{A5.0}}{p \wedge q}\,\text{A2.0}$$

Further (1), 5.0.

5.3 $$\frac{(x)p \quad (Ex)q}{q \rightarrow_x p}$$

Proof. (1) $$\frac{(x)p\ [q]}{p}\,\text{A5.0}$$

Further (1), A3.4.

6. *Rules of substitution.*

A6.0 If $\dfrac{p(x)}{q(x)}$, then $\dfrac{p(y)}{q(y)}$,

if $\dfrac{p}{q(x)}$ and p does not contain x as a free variable, then $\dfrac{p}{q(y)}$, if $\dfrac{p(x)}{q}$ and q does not contain x as a free variable, then $\dfrac{p(y)}{q}$.

$p(x)$ and $q(x)$ must not contain y as a bound variable. $p(y)$ and $q(y)$ are formed from $p(x)$ and $q(x)$ by substituting y for x at every place where x is free.

A6.1 If x is bound in p, y does not occur in p and p is transformed into p' by substituting y for the bound variable x at every place where it occurs (including in the binding operators), then $\frac{p}{p'}$.

6.00 $$\frac{p(x, x)}{(Ey)p(x, y)}$$

Proof. (1) $\frac{p(x, y)}{(Ey)p(x, y)}$ A3.2

Further (1), A6.0.

6.01 $$\frac{(Ex)p(x, x)}{(Ex)(Ey)p(x, y)}$$

Proof. 6.00, 5.0.

6.02 $$\frac{(x)p(x, x)}{(x)(Ey)p(x, y)}$$

Proof. 6.00, A5.1.

7. A7 If x does not occur as a free variable in p, then $\frac{(Ex)p}{(x)p}$.

If x does not occur as a free variable in p, the theorems 7.00—7.03 hold.

7.00 $\frac{(Ex)p}{p}$ *Proof.* A7, A5.0.

7.01 $\frac{(x)p}{p}$ *Proof.* A5.0.

7.02 $\frac{p}{(Ex)p}$ *Proof.* A3.2.

7.03 $\frac{p}{(x)p}$ *Proof.* 7.02, A7.

7.10 If q and r do not contain x as a free variable and $\frac{p\ q}{r}$, then $\frac{(Ex)p\ q}{r}$.

Proof. 5.0, 7.00.

7.11 If p does not contain x as a free variable and $\frac{p}{q}$, then $\frac{p}{(x)q}$.

Proof. 7.03, A5.1.

7.20 $$\frac{(x)(y)p}{(y)(x)p}$$

Proof. $$\frac{\dfrac{(x)(y)p}{(y)p}\ \text{A5.0}}{p}\ \text{A5.0}$$

Further twice 7.11.

7.21 $$\frac{(Ex)(Ey)p}{(Ey)(Ex)p}$$

Proof. $$\frac{\dfrac{p}{(Ex)p}\ \text{A3.2}}{(Ey)\,(Ex)p}\ \text{A3.2}$$

Further twice 7.10.

7.22 $$\frac{(Ex)(y)p}{(y)(Ex)p}$$

Proof. $$\frac{p}{(Ex)p}\ \text{A3.2}$$

Further A5.1, 7.10.

7.3 $$\frac{\exists_{vs}p}{\exists p}$$

Proof. If *vs* contains variables, that are not free in *p*, they can be dropped (7.00). Further A3.2.

8. *Implication.*

There is some difference in meaning between

$$p \rightarrow_{xy} q \text{ and } (x)\,.\,p \rightarrow_{y} q.$$

In both cases *p* is a part of *q*. But in the second case, *p* is a part of *q for any x*. That is only possible, if for any *x p* exists, i.e., $(x)(Ey)p$.

8.0 $$\frac{(x)\,.\,p \rightarrow_{y} q}{p \rightarrow_{xy} q}$$

Proof. $$\frac{\dfrac{\dfrac{(x)\,.\,p \rightarrow_{y} q}{p \rightarrow_{y} q}\ \text{A5.0} \qquad p^{*}}{q}\ \text{A3.0} \qquad \dfrac{\dfrac{(x)\,.\,p \rightarrow_{y} q}{(Ex)\,.\,p \rightarrow_{y} q}\ \text{5.1}}{(Ex)(Ey)p}\ \text{A3.3, 5.0}}{p \rightarrow_{xy} q}\ \text{A3.4}$$

8.1 $$\frac{p \rightarrow_{xy} q \quad (x)(Ey)p}{(x)\,.\,p \rightarrow_{y} q}$$

Proof.

$$\cfrac{\cfrac{p \rightarrow_{xy} q \quad p^*}{q}\,\text{A3.0} \qquad (Ey)p}{p \rightarrow_y q}\,\text{A3.4}$$

Further A5.1.

8.2 $$\frac{(x) \,.\, p \rightarrow_y q}{(x)(Ey)p}$$

Proof. A3.3, A5.1.

8.3 $$\frac{p(x) \rightarrow_x q(x) \quad p(y)}{q(y)}\;\text{1)}$$

y must not be bound in p or q.

Proof. A3.0, A6.0.

The theorem is proved in the same way for propositional functions containing more than one variable.

8.30 $$\frac{p(x) \vee q(x) \rightarrow_x r(x) \quad p(y)}{r(y)}$$

y must not be bound in $p(x)$ or $r(x)$.

Proof. If y is bound in $q(x)$, by A6.1 $q(x)$ can be transformed into $q'(x)$ not containing y as a bound variable.

$$\cfrac{p(x) \vee q(x) \rightarrow_x r(x) \qquad \cfrac{p(y) \quad \cfrac{\cfrac{p(x) \vee q(x) \rightarrow_x r(x)}{\exists q(x)}\,\text{A3.3, A3.2, A4.2}}{\exists q(y)}\,\text{A6.1 }^{2)}}{p(y) \vee q(y)}\,\text{A4.0}}{r(y)}\,\text{8.3}$$

8.31 $$\frac{p \rightarrow_{xy} q \quad (Ey)p}{p \rightarrow_y q}$$

Proof.

$$\cfrac{\cfrac{p \rightarrow_{xy} q \quad p^*}{q}\,\text{A3.0} \qquad (Ey)p}{p \rightarrow_y q}\,\text{A3.4}$$

8.32 If p does not contain y as a free variable, then

$$\frac{p \rightarrow_{xy} q}{p \rightarrow_x (y)q}.$$

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1) Or, if x is not free in q, $\dfrac{p(x) \rightarrow_x q \quad p(y)}{q}$.

2) It is allowed that y is free in $q(x)$; this will become clear in section 10.

Proof. (1)
$$\dfrac{\dfrac{(y)\,.\,p \rightarrow_{x} q}{p \rightarrow_{xy} q}\,8.0 \qquad p}{q}\,\text{A3.0}$$

$$\dfrac{\dfrac{\dfrac{p \rightarrow_{xy} q}{(y)(Ex)p}\,\text{A3.3, A7}}{(Ex)p}\,\text{A5.0} \qquad \dfrac{\dfrac{p \rightarrow_{xy} q \quad \dfrac{\dfrac{p \rightarrow_{xy} q}{(Ey)(Ex)p}\,\text{A3.3}}{(y)(Ex)p}\,\text{A7}}{(y)\,.\,p \rightarrow_{x} q}\,8.1 \qquad \dfrac{p^{*}}{(y)p}\,7.03}{(y)q}\,\text{(1), A5.1}}{p \rightarrow_{x} (y)q}\,\text{A3.4}$$

8.33 If q does not contain y as a free variable, then

$$\frac{p \rightarrow_{xy} q}{(Ey)p \rightarrow_{x} q}.$$

Proof. (1)
$$\frac{p \rightarrow_{xy} q \quad p}{q}\,\text{A3.0}$$

$$\dfrac{\dfrac{p \rightarrow_{xy} q \quad (Ey)p^{*}}{q}\,\text{(1), 5.0, 7.00} \qquad \dfrac{p \rightarrow_{xy} q}{(Ex)(Ey)q}\,\text{A3.3}}{(Ey)p \rightarrow_{x} q}\,\text{A3.4}$$

8.34 If x is not free in p and q, then $\dfrac{p \rightarrow_{vs} q}{p \rightarrow_{vs'} q}$.

vs does not contain x; vs' consists of vs and x.

Proof.
$$\dfrac{\dfrac{\dfrac{p \rightarrow_{vs} q}{\exists_{vs} p}\,\text{A3.3}}{\exists_{vs'} p}\,7.02 \qquad \dfrac{p \rightarrow_{vs} q \quad p^{*}}{q}\,\text{A3.3}}{p \rightarrow_{vs'} q}\,\text{A3.4}$$

8.35
$$\frac{p(x, y) \rightarrow_{xy} q(x, y) \qquad (Ex)p(x, x)}{p(x, x) \rightarrow_{x} q(x, x)}$$

Proof. 8.3, A3.4.

9. *The basic relations* $=$ *and* $\#$.

It is possible to apply the logical theory to a field of individuals. We presuppose that the individuals are discernable. In case we want to express that two individuals are discernable, we write $x \# y$, in case they are identical $x = y$. The relations $=$ and $\#$ are introduced as basic relations of our logical system by means of the axioms A9.0—3.

$x = y$ and $x \# y$ are atomic formulas (cf. D9.0—1).

The use of the propositional function $x \# y$ renders it impossible that there is only one individual, or, more precise, makes it necessary that there are at least two discernable individuals. So after adjunction of the sign $\#$, this theory cannot be applied to a field that consists of only one individual.

Formally this circumstance might be expressed by the axiom $(Ex)(Ey)x \# y$. But this axiom is not an axiom similar to the others, but, one might say, a material axiom (as it supposes a special property of the scope of the field of individuals). Adding a material axiom implies adding material theorems. Instead of splitting the theorems in two different kinds, we prefer writing the theorems that presuppose the „axiom" $(Ex)(Ey)x \# y$, in the usual way, $\dfrac{(Ex)(Ey)x \# y}{p}$. But we shall omit the premiss $(Ex)(Ey)x \# y$ in the formulation of theorems, except in case it is the only premiss.

D9.0 $\quad x = y =_{df} = (x, y)$

D9.1 $\quad x \# y =_{df} \# (x, y)$

Following Griss[1]) we choose as axioms:

A9.0 $$\frac{x = y \quad p(x)}{p(y)}$$

p must not contain y as a bound variable.

A9.1 $$\frac{x \# z \rightarrow_z y \# z}{y = x}\ ^{2})$$

A9.2 $$\frac{x \# y}{z \# x \vee z \# y}$$

A9.3 $$\frac{x \# y}{y \# x}$$

9.0 $$\frac{\exists \#}{(x)(Ey)x \# y}, \quad \frac{\exists \#}{(y)(Ex)x \# y}$$

Proof. (1) $\dfrac{x \# u}{(Ey)x \# y}$ A3.2, A6.1 $\quad$ (2) $\dfrac{x \# v}{(Ey)x \# y}$ A3.2, A6.1

(3) $$\cfrac{\cfrac{u \# v}{x \# u \vee x \# v}\ \text{A9.2}}{(Ey)x \# y}\ \text{(1), (2), 4.10}$$

(4) $$\frac{u \# v}{(x)(Ey)x \# y}\ \text{(3), 7.11}$$

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[1]) Versl. Ned. Akad. v. Wetensch., afd. Natuurk., LIII (1944), p. 262 and 266.

[2]) From an intuitionistic point of view this axiom is suspect; cf. section **15**. Griss proves that it is valid for real numbers.

$$\begin{array}{c}\exists\# \\ \hline (x)(Ey)x \# y \\ \hline (y)(Ex)y \# x \\ \hline (y)(Ex)x \# y\end{array}\quad\begin{array}{l}(4),\ 7.10 \\ \text{A6.1} \\ \text{A9.3, 5.0, A5.1}\end{array}$$

9.1 $\begin{array}{c}\exists\# \\ \hline (x)x = x\end{array}$

Proof. (1) $\begin{array}{c}(Ey)x \# y \\ \hline x \# y \rightarrow_y x \# y \\ \hline x = x\end{array}\begin{array}{l}3.00 \\ \text{A9.1}\end{array}\qquad\begin{array}{c}\exists\# \\ \hline (x)(Ey)x \# y \\ \hline (x)x = x\end{array}\begin{array}{l}9.0 \\ (1),\ \text{A5.1}\end{array}$

9.10 $\begin{array}{c}\exists\# \\ \hline \exists=\end{array}$

Proof. $\begin{array}{c}\exists\# \\ \hline (x)x = x \\ \hline (Ex)x = x \\ \hline (Ex)(Ey)x = y\end{array}\begin{array}{l}9.1 \\ 5.1 \\ 6.01\end{array}$

9.11 $\begin{array}{c}\exists\# \\ \hline (x)(Ey)x = y\end{array}$

Proof. 9.1, 6.02.

9.2 $\begin{array}{c}x = y \quad y = z \\ \hline x = z\end{array}$

Proof. A9.0.

9.20 $\begin{array}{c}x \# z \quad x = y \\ \hline y \# z\end{array}$

Proof. A9.0.

9.21 $\begin{array}{c}x = y \\ \hline x \# z \rightarrow_z y \# z\end{array}$

Proof. $\begin{array}{c}\begin{array}{cc} & \begin{array}{c}\exists\# \\ \hline (x)(Ez)x \# z \\ \hline (Ez)x \# z\end{array} \\ x = y & \end{array} \\ \hline x \# z \rightarrow_z y \# z\end{array}\begin{array}{l}9.0 \\ \text{A5.0} \\ 9.20,\ \text{A3.4}\end{array}$

9.3 $\begin{array}{c}x = y \\ \hline y = x\end{array}$

Proof. $\begin{array}{c}x = y \\ \hline x \# z \rightarrow_z y \# z \\ \hline y = x\end{array}\begin{array}{l}9.21 \\ \text{A9.1}\end{array}$

10. *Disjunction.*

A10 $$\frac{p(x) \vee q(x) \qquad p(y) \rightarrow_{y} y \# x}{q(y)}$$

This axiom says: if x belongs to the sumclass of p and q, but is different (discernable) from all the members of the class p, then x belongs to q. Or, in ordinary language, if x belongs to the sumclass of p and q, but not to p, it belongs to q. But in the last sentence it is negated that x belongs to p, perhaps because $p(x)$ turns out to be contradictory. The former sentence is free from negation, because it only says that all the members of the class p are different from x.

Perhaps the following example makes the difference clearer. I am looking for my fountain-pen. I ask: "Is it on my writing-table?" I find it in my pocket. And now I say: "It is not on my writing-table, for it is in my pocket." That is a negated sentence. But I can also investigate every object on my writing-table and always find: this object is different from my fountain-pen. Then all the objects on the table are different from my fountain-pen. And if I know in some way that my fountain-pen belongs to the sumclass of the objects on my table and in my pocket, I am able to conclude (A10): my fountain-pen is in my pocket.

D10 If x is the only free variable of $p(x)$, then
$\sim p(x) =_{df} p(y) \rightarrow_{y} y \# x$.
If x and y are the only free variables of $p(x, y)$, then
$\sim p(x, y) =_{df} p(u, v) \rightarrow_{uv} u \# x \vee v \# y$, etc.

By this definition a kind of negation is introduced. But this operation, $\sim$, is based upon the relation of difference. So it is not a negation in the proper sense, as it has nothing to do with refutation or contradiction. Still, formally, it has many properties in common with the usual negation.

We are now able to formulate A10 in a simpler form:

A10.0 $$\frac{p \vee q \quad \sim p}{q}$$

This axiom is more general, as the number of variables is arbitrary.

Remark. There is still some ambiguity with respect to the "negation" of propositional functions with variables that have been identified.

E.g., if in $\sim p(x, y)$ the variables are identified, we get according to D10

$$\sim p(x, x) =_{df} p(u, v) \rightarrow_{uv} u \# x \vee v \# x.$$

But if we consider $p(x, x)$ as a propositional function with one variable, then D10 says

$$\sim p(x, x) =_{df} p(y, y) \rightarrow_{y} y \# x.$$

We choose the first definition. This is done by the following decision: if a variable in a propositional function is repeated, the function should formally be treated as a function of two (or more) identified variables and not as a function of one variable. So identification of variables does not reduce the number of variables.

$\sim x \# x$, considered as a function of one variable, would be nonsense as the class $x \# x$ is empty. So $x \# x$ would be senseless and cannot be negated.

$\sim x \# x$, considered as a function of two identified variables means $u \# v \rightarrow_{uv} u \# x \vee v \# x$, and this is significant. It can be derived from $\exists\#$ by A9.2 and A3.4. (Cf. 10.12.)

Under certain existentional conditions there is no harm in negating a function of two variables in the same way as a function of one variable. This will be shown in 10.10—11.

We define:

$$\exists p(x, x, z) =_{df} (Ex)(Ey)(Ez)p(x, y, z), \text{ etc.}$$

So if the $\exists$-operator is applied to a wff with identified free variables, the variables should first be changed into different variables and then they all should be bound by (*Evs*).

The definitions of $\sim$ and $\exists$ applied to wff with identified free variables have the following consequence. If a theorem has been proved for wff without identified free variables, the corresponding theorem for wff with identified free variables is an immediate consequence of it (by means of A6.0). Mind that the $\exists$-premisses remain unchanged, when free variables are identified in the premisses and the conclusion of a derivation. So in proofs we are always allowed to suppose that all free variables are different.

Remark. It seems that by the following derivation we are able to construct a disjunction of two wff of which one represents an empty class.

$$\dfrac{f(x) \qquad \dfrac{\dfrac{\exists\#}{\exists y \# z}\ \text{def } \exists}{\exists y \# y}\ \text{def } \exists}{f(x) \vee y \# y}\ \text{A4.0}$$

But this wff should not be interpreted as the sum of the class $f(x)$ and the empty class $y \# y$. We first form the class of triples (x, y, z), that satisfy $f(x) \vee y \# z$ and from this class we form the subclass of those triples of which the second and third element are identical. So we find as interpretation the class of triples (x, y, y) of which x satisfies $f(x)$ and y is arbitrary.

$$10.10 \qquad \dfrac{\sim p(x, y) \qquad (Ex)p(x, y)}{p(u, y) \rightarrow_u u \# x}$$

Proof. Suppose that p contains just two free variables.

$$\dfrac{(Ex)p(x, y) \qquad \dfrac{\dfrac{\dfrac{\sim p(x, y)}{p(u,v)\rightarrow_{uv} u\#x \vee v\#y}\,\text{D10, 2.2} \quad p(u, y)^*}{u\#x\vee y\#y}\,8.3 \quad \dfrac{\dfrac{\exists\#}{z\#u\rightarrow_{zu} z\#y\vee u\#y}\,\text{A9.2, A3.4}}{\sim y\#y}\,\text{D10}}{u\#x}\,\text{A10.0}}{p(u,y)\rightarrow_u u\#x}\,\text{A6.1, A3.4}$$

In case p contains more than two free variables, the proof is similar.

More generally we prove in the same way:

10.100 If $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m$ is the sequence of free variables of p, then

$$\dfrac{\sim p \qquad (Ex_0)(Ex_1)\ldots(Ex_n)p}{p(u_0, u_1, \ldots, u_n, y_0, y_1, \ldots, y_m)\rightarrow_{u_0 u_1 \ldots u_n} u_0\#x_0\vee u_1\#x_1\vee\ldots\vee u_n\#x_n}$$

$$10.11 \qquad \dfrac{p(u, y) \rightarrow_u u \# x}{\sim p(x, y)}$$

Proof. Suppose that p contains just two free variables.

$$\dfrac{p(u, y) \rightarrow_u u \# x \qquad \dfrac{\exists\#}{u \# x \rightarrow_u u \# x \vee v \# y}\,\text{A4.0, 9.0, A3.4}}{p(u, y) \rightarrow_u u \# x \vee v \# y}\,3.20$$

Further 7.11, 8.0, D10.

In case p contains more than two free variables, the proof is similar.

More generally we prove in the same way:

10.110 If $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m$ is the sequence of free variables of p, then

$$\dfrac{p(u_0, u_1, \ldots, u_n, y_0, y_1, \ldots, y_m)\rightarrow_{u_0 u_1 \ldots u_n} u_0\#x_0\vee u_1\#x_1\vee\ldots\vee u_n\#x_n}{\sim p}$$

10.12 $$\frac{\sim p(x, x) \quad (Ex)p(x, x)}{p(y, y) \rightarrow_y y \# x}$$

Proof. Suppose that p contains just two free variables.

$$\dfrac{\dfrac{\dfrac{\dfrac{\sim p(x, x)}{p(u, v) \rightarrow_{uv} u \# x \vee v \# x}\,\text{D10} \qquad p(y, y)^*}{y \# x \vee y \# x}\,8.3}{y \# x}\,4.10 \qquad \dfrac{(Ex)p(x, x)}{(Ey)p(y, y)}\,\text{A6.1}}{p(y, y) \rightarrow_y y \# x}\,\text{A3.4}$$

The proof is analogous in case p contains more than two free variables.

10.12 shows that from the negation of $p(x, x)$, considered as a function of *two* variables, can be derived the negation of $p(x, x)$, considered as a function of *one* variable, but only if the premiss $(Ex)p(x, x)$ is valid. If this premiss were not valid, the conclusion would be senseless.

More generally we prove in the same way:

10.120 If $x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m$ is the sequence of free variables of p, then

$$\frac{\sim p(x, x, \ldots, x, y_0, y_1, \ldots, y_m) \quad (Ex)p(x, x, \ldots, x, y_0, y_1, \ldots, y_m)}{p(z, z, \ldots, z, y_0, y_1, \ldots, y_m) \rightarrow_z z \# x}$$

10.20 $$\frac{\sim x = y}{x \# y}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\sim x = y \qquad \dfrac{\exists \#}{(Eu)\, u = y}\,9.11,\ \text{A5.0}}{u = y \rightarrow_u u \# x}\,10.10 \qquad \dfrac{\exists \#}{y = y}\,9.1,\ \text{A5.0}}{y \# x}\,8.3}{x \# y}\,\text{A9.3}$$

10.21 $$\frac{\sim x \# y}{x = y}$$

Proof.

$$\dfrac{\dfrac{\sim x \# y \qquad \dfrac{\exists \#}{(Eu)\, u \# y}\,9.0,\ \text{A5.0}}{u \# y \rightarrow_u u \# x}\,10.10}{x = y}\,\text{A9.1}$$

10.22 $$\frac{x = y}{\sim x \# y}$$

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{u \# v^*}{x \# u \vee x \# v}\,\text{A9.2} \qquad x = y}{x \# u \vee y \# v}\,\text{A9.0, 4.2} \qquad \exists\#}{u \# v \rightarrow_{uv} u \# x \vee v \# y}\,\text{A3.4, A9.3}}{\sim x \# y}\,\text{D10}$$

10.23 $\dfrac{x \# y}{\sim x = y}$

Proof.

$$\cfrac{\cfrac{\cfrac{\cfrac{x \# y}{u \# x \vee u \# y}\,\text{A9.2} \qquad u = v^*}{u \# x \vee v \# y}\,\text{A9.0, 4.2} \qquad \cfrac{\exists\#}{\exists=}\,9.10}{u = v \rightarrow_{uv} u \# x \vee v \# y}\,\text{A3.4}}{\sim x = y}\,\text{D10}$$

10.40 $\dfrac{p(x) \quad \exists \sim p}{\sim \sim p(x)}$

Proof. Suppose that p contains only one free variable.

$$\cfrac{\cfrac{\cfrac{\cfrac{\sim p(y)^*}{p(z) \rightarrow_z z \# y}\,\text{D10} \qquad p(x)}{x \# y}\,8.3 \qquad \exists \sim p}{\sim p(y) \rightarrow_y x \# y}\,\text{A3.4}}{\sim \sim p(x)}\,\text{A9.3, D10}$$

If the number of free variables of p is more than one, the proof is similar.

10.41 $\dfrac{\exists \sim p}{\exists p}$ *Proof.* D10, A3.3, 7.10.

10.42 $\dfrac{\exists \sim p}{\exists \sim \sim p}$ *Proof.* 10.41, 10.40, 5.0.

10.43 $\dfrac{\exists_{vs} \sim p}{p \rightarrow_{vs} \sim \sim p}$ *Proof.* 7.3, 10.40, 10.41, A3.4.

10.50 If $\dfrac{p}{q}$, then $\dfrac{\sim q \quad \exists p}{\sim p}$.

Proof. Suppose x is the only free variable of p and of q.

$$\cfrac{\cfrac{\cfrac{\sim q(x)}{q(y) \rightarrow_y y \# x}\,\text{D10} \qquad \cfrac{\exists p}{p(y) \rightarrow_y q(y)}\,\text{A3.4, A6.1}}{p(y) \rightarrow_y y \# x}\,3.20}{\sim p(x)}\,\text{D10}$$

The proof is analogous, if p and q contain more than one, but the same free variables.

Suppose that p and q do not contain the same free variables. E.g., the free variables of p are x and y, of q x and z. Then we define

$$p' =_{df} p \vee z = z$$
$$q' =_{df} q \vee y = y.$$

Now first we prove $\dfrac{p'}{q'}$ (9.1). Therefore $\dfrac{\sim q' \;\; \exists p'}{\sim p'}$. And from this we prove $\dfrac{\sim q \;\; \exists p}{\sim p}$ (10.100, 10.110).

10.50 remains valid, if a premiss is added, that does not contain a free variable that is free in p or q.

10.51 $\dfrac{p \rightarrow_{vs} q \qquad \exists_{vs} \sim q}{\sim q \rightarrow_{vs} \sim p}$

Proof. Similar.

10.52 If $\dfrac{p}{\sim q}$, then $\dfrac{q \;\; \exists p}{\sim p}$.

Proof. Suppose x is the only free variable of p and of q.

$$\dfrac{\dfrac{\exists p \qquad \dfrac{q(y) \qquad \dfrac{p(x)^*}{q(y) \rightarrow_y y \# x}\,\text{D10}}{y \# x}\,\text{A3.0}}{p(x) \rightarrow_x y \# x}\,\text{A3.4}}{\sim p(y)}\,\text{D10}$$

Further similar to the proof of 10.50.

10.52 remains valid, if a premiss is added, that does not contain a free variable that is free in p or q.

10.53 $\dfrac{p \rightarrow_{vs} \sim q}{q \rightarrow_{vs} \sim p}$

Proof. A3.3, 10.42, 10.51, 10.43, 3.20.

10.60 $\dfrac{(vs)\,.\,p \vee q \qquad \exists_{vs} \sim p}{\sim p \rightarrow_{vs} q}$

Proof.

$$\dfrac{\dfrac{\dfrac{(vs)\,.\,p \vee q}{p \vee q}\,\text{A5.0} \qquad \sim p^*}{q}\,\text{A10.0} \qquad \exists_{vs} \sim p}{\sim p \rightarrow_{vs} q}\,\text{A3.4}$$

10.610 $$\frac{(vs) \,.\, p \vee \sim q}{\sim\sim q \rightarrow_{vs} p}$$

Proof. A3.2, 4.02, 10.42, 10.60.

10.611 $$\frac{(vs) \,.\, p \vee \sim q}{q \rightarrow_{vs} p}$$

Proof. A3.2, 4.02, 10.610, 10.43, 3.20.

10.70 $$\frac{\sim p \vee \sim q \qquad \exists \,.\, p \wedge q}{\sim .\, p \wedge q}$$

Proof. $$\frac{\sim p \qquad \exists \,.\, p \wedge q}{\sim .\, p \wedge q}\ \text{A2.1, 10.50}$$

Further 3.301, 4.100.

10.71 $$\frac{\sim .\, p \vee q}{\sim p \wedge \sim q}$$

Proof. Let p and q contain one free variable; the free variable of p is the same as the free variable of q.

$$\dfrac{\dfrac{\sim .\, p \vee q}{p(y) \vee q(y) \rightarrow_{v} y \# x}\,\text{D10} \qquad \dfrac{\dfrac{p(y)^{*} \quad \dfrac{\sim .p \vee q}{\exists q}\,\text{D10, A3.3, 4.00}}{p(y) \vee q(y)}\,\text{A4.0} \qquad \dfrac{\sim .p \vee q}{\exists p}\,\text{D10, A3.3, A4.2}}{p(y) \rightarrow_{v} p(y) \vee q(y)}\,\text{A3.4}}{\dfrac{p(y) \rightarrow_{v} y \# x}{\sim p(x)}\,\text{D10}}\,\text{3.20}$$

Further a similar derivation of $\sim q(x)$, and A2.0.
The proof is analogous in the other cases (cf. the proof of 10.50).

10.72 $$\frac{\sim p \wedge \sim q}{\sim \ .\, p \vee q}$$

Proof. (1) $$\dfrac{\dfrac{p \vee q \quad \exists \sim p \quad \exists \sim q}{\sim\sim p \vee \sim\sim q}\,\text{10.40, 4.2} \qquad \exists \,.\sim p \wedge \sim q}{\sim .\ \sim p \wedge \sim q}\,\text{10.70}$$

$$\dfrac{\sim p \wedge \sim q \quad \dfrac{\sim p \wedge \sim q}{\exists \sim p} \ \dfrac{\sim p \wedge \sim q}{\exists \sim q}\,\text{A3.2, 3.42} \quad \dfrac{\dfrac{\sim p \wedge \sim q}{\exists p} \quad \dfrac{\sim p \wedge \sim q}{\exists q}\,\begin{array}{l}\text{A3.2, 3.42,}\\ \text{10.41}\end{array}}{\exists \,.p \vee q}\,\text{4.01} \qquad \dfrac{\sim p \wedge \sim q}{\exists .\sim p \wedge \sim q}\,\text{A3.2}}{\sim .p \vee q}\,\text{(1), 1}$$

10.8 $$\frac{\sim .\, p \wedge q \quad p}{\sim q}$$

Proof. Let x be the only free variable of p and of q.

$$\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\sim .\, p(x)\wedge q(x)}{)\wedge q(z)\rightarrow_{yz} y\#x\vee z\#x}^{\text{D10}} \quad p(x) \quad q(z)^*}{x\#x\vee z\#x}\,8.3 \qquad \dfrac{\exists\#}{\sim x\#x}\,\text{A9.2, A3.4, D10}}{z\#x}\,\text{A10.0} \qquad \dfrac{\sim .\, p(x)\wedge q(x)}{(Ez)q(z)}\,\text{D10, A3.3, 3.42, A6.1}}{\dfrac{q(z)\rightarrow_{z} z\#x}{\sim q(x)}\,\text{D10}}\,\text{A3.4}$$

The proof is similar if p and q contain more or different free variables.

10.80 $$\frac{\sim .\, y = x \wedge z = x}{y \# x \vee z \# x}$$

Proof.

$$\dfrac{\genfrac{}{}{0pt}{}{\exists\#}{{}_{vx}t=x\wedge v=x}^{\text{9.1, 5.1, 6.01}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\sim .\, y=x\wedge z=x}{t=u\wedge v=w\rightarrow_{tuvw} t\#y\vee u\#x\vee v\#z\vee w\#x}\,\text{D10} \quad t=x\wedge v=x^*}{x\#y\vee x\#x\vee x\#z\vee x\#x}\,\text{8.3, A9.0} \quad \dfrac{\exists\#}{\sim x\#x}\,\text{A9.2}}{y\#x\vee z\#x}\,\text{A10.0}}{t=x\wedge v=x\rightarrow_{tvx} y\#x\vee z\#x}\,\text{A3.4} \quad \dfrac{\exists\#}{x=x\wedge x=x}\,9.1}{y\#x\vee z\#x}\,8.3$$

This theorem is not in conflict with intuitionism. It merely shows that in negationless intuitionistic mathematics $y = x \wedge z=x$ can only be "negated" in those cases in which $y \# x \vee z \# x$ can be proved. So *the possibilities of "negating" in this system are more restricted than in normal intuitionism.*

11. *Individual constants.*

In the application of the theory it may be possible, that individual constants are substituted for variables. For this reason and for other reasons, that will appear later, we enlarge the used signs with

3. individual constants a, b, ...

As a metasystematical symbol for an arbitrary individual constant, we shall use the letter c.

To the definition of a well-formed formula we add:

e. if $p(x)$ is a wff and c an individual constant, and $p(x)$ is changed into q by replacing every x, that is free in $p(x)$, by c, then q is a wff.

The new wff q is written $p(c)$.

The following axioms are added.

A11.0 $\dfrac{(x)p(x)}{p(c)}$

A11.1 $\dfrac{p(c)}{(Ex)p(x)}$

A11.2 If $\dfrac{p(x)}{q(x)}$, then $\dfrac{p(c)}{q(c)}$.

11.0 If x does not occur as a free variable in q, and $\dfrac{p(x)}{q}$, then $\dfrac{p(c)}{q}$.

Proof. A11.1, 7.10.

11.1 If x does not occur as a free variable in q, and $\dfrac{p}{q(x)}$, then $\dfrac{p}{q(c)}$.

Proof. 7.11, A11.0.

If a wff containing an individual constant is negated, the constant is treated in the same way as a free variable. So, e.g.,

$$\sim p(c) =_{df} p(x) \rightarrow_x x \# c,$$

$$\sim p(x, c, c) =_{df} p(y, z, u) \rightarrow_{yzu} y \# x \vee z \# c \vee u \# c.$$

A constant occurring twice is treated in the same way as two identified free variables.

If an $\exists$-operator is applied on a wff containing an individual constant, the constant has to be replaced by a free variable. So, e.g.,

$$\exists p(c) =_{df} \exists p(x),$$

$$\exists p(x, c, c) =_{df} \exists p(x, y, z).$$

This has the following consequence. If a theorem has been proved for wff not containing individual constants, the corresponding theorem for wff containing individual constants is an immediate consequence of it (by means of A11.2). Note that the $\exists$-premisses remain unchanged, when free variables are replaced by individual constants in the premisses and the conclusion of a derivation. So in proofs we are always allowed to suppose that the wff do not contain individual constants.

Remark. If $c_1 \# c_2$, then $f(x) \vee c_1 = c_2$ is to be interpreted as the class of triples (x, c_1, c_2), of which x satisfies $f(x)$.

Definition. If $\frac{p}{q}$ and $\frac{q}{p}$, we say that p and q are *equivalent*, and write $p \equiv q$.

The relation $\equiv$ is a metasystematical relation.

11.2 If $p \equiv q$ and $r \equiv s$, then
a. $p \wedge r \equiv q \wedge s$ (A2.1, A2.0)
b. $p \vee r \equiv q \vee s$ (4.2)
c. $p \rightarrow_{vs} r \equiv q \rightarrow_{vs} s$ (A3.0, A3.3, 3.30, A3.4)
d. $\exists p \equiv \exists q$ (3.30)
e. $(Ex)p \equiv (Ex)q$ (5.0)
f. $(x)p \equiv (x)q$ (A5.1)
g. $p(c) \equiv q(c)$ (A11.2).

From 11.2 *it is seen by induction that, if* $\frac{p}{q}$ *and* $\frac{q}{p}$, *p and q are interchangeable.*

Semantical remarks.

Suppose that x is the only free variable of $p(x)$ and of $q(x)$. We remember, that $p(x) \vee q(x)$ is a propositional function determining the sumclass of the classes determined by $p(x)$ and $q(x)$. So $p(c) \vee q(c)$ will mean, that c is a member of this class. Therefore $p(c) \vee q(c)$ is not a disjunction of the propositions $p(c)$ and $q(c)$. In case only one of $p(c)$ and $q(c)$ is true, $p(c)$ and $q(c)$ would not both be a proposition and the disjunction $p(c) \vee q(c)$, if understood as a disjunction of propositions, would be senseless. But $p(c) \vee q(c)$ understood as one proposition and not composed out of two propositions is not senseless and merely means, that c belongs to the sumclass $p(x) \vee q(x)$.

There is another difficulty. Suppose that x and y are the only free variables of $p(x, y)$ and $q(x, y)$. How is $p(x, c) \vee q(x, c)$ to be understood? Again it does not mean the disjunction of $p(x, c)$ and $q(x, c)$. For it is possible, that there exists no x for which, e.g., $p(x, c)$ holds. And then $p(x, c)$ is not a propositional function. So we should not be able to form $p(x, c) \vee q(x, c)$ as soon as one of the two does not represent a class that is not empty.

Therefore we choose a different interpretation, that is closely connected with the interpretation of $p(c) \vee q(c)$. The propositional function $p(x, y) \vee q(x, y)$ determines a class of pairs (x, y). Now we decide, that $p(x, c) \vee q(x, c)$ determines the subclass of those pairs of which y is the individual constant c. This interpretation is independent of the existence of the functions $p(x, c)$ and $q(x, c)$ separately.

E.g., $x \# 0$ in the theory of whole numbers is equivalent with $x < 0 \vee x > 0$. The proposition $1 \# 0$ is true. Therefore the proposition $1 < 0 \vee 1 > 0$ is true too, though $1 < 0$ separately is not a proposition.

And $|x| \# |y|$ is equivalent with $x^2 < y^2 \vee x^2 > y^2$. So $x^2 < 0 \vee x^2 > 0$ determines the class of those x that are $\# 0$. But $x^2 < 0$ separately is not a propositional function.

This causes some difficulties. The meaning of $x^2 < 0 \vee x^2 > 0$ depends on 0 being or not being obtained by substitution in a preceding formula. In the former case it is sensible, in the latter senseless. We decide, that every individual constant appearing explicitly in a formula is supposed to be introduced by substitution for a free variable. In the next section we shall see how it will be possible to construct propositional functions in which individual constants occur implicitly that are not supposed to be introduced by substitution.

12. *Note about definitions.*

We mentioned in section **10**, that after identifying two (or more) free variables of a wff, we would formally treat the wff as a wff with the original number of free variables. Under certain circumstances it is preferable to treat a wff with identified variables as a wff with a reduced number of variables. This is done by means of a definition. In case, e.g., we want to treat $p(x, x)$ as a wff with a reduced number of variables, we define:

$$q(x) =_{df} p(x, x).$$

The identified variables of p, that are to be treated as one variable of q, should be mentioned explicitly between the brackets after p and q in the definition.

We will allow a definition of this kind only in case

$$(Ex)p(x, x)$$

has been derived (to avoid the construction of empty classes).

A12.0 If $q(x) =_{df} p(x, x)$, then $\dfrac{\sim q(x)}{\sim p(x, x)}$.

12.0 If $q(x) =_{df} p(x, x)$, then $q(x)$ and $p(x, x)$ are interchangeable.

Proof. (1) $\dfrac{\sim p(x, x) \ (Ex)p(x, x)}{\sim q(x)}$ 10.12

(2) $\dfrac{\sim q(x)}{\sim p(x, x)}$ A12.0

$$(3)\quad \frac{[\exists p(x, x)]\quad (Ex)p(x, x)}{\exists q(x)}\ \text{def}\ \exists$$

$$(4)\quad \frac{[\exists q(x)]\quad (Ex)p(x, x)}{\exists p(x, x)}\ 6.01$$

As the only formal difference between $q(x)$ and $p(x, x)$ is, that they are to be treated in different ways when the $\sim$- or $\exists$-operator is applied and as it has been supposed that $(Ex)p(x, x)$ has been derived, they are interchangeable (11.2).

We mentioned in section **11**, that after replacing a free variable by an individual constant, we would formally treat the wff as a wff with a free variable instead of the constant. Under certain circumstances it is preferable to treat a wff, after replacing a free variable by a constant, formally as a wff with a reduced number of variables. This is done again by means of a definition. In case, e.g., $p(x, c)$ has been formed from $p(x, y)$ and we want to treat $p(x, c)$ formally as a wff with one free variable less than $p(x, y)$, we define:

$$q(x) =_{df} p(x, c).$$

c should be mentioned in the definition explicitly between the brackets after p and not between those after q.

We will allow a definition of this kind only in case

$$(Ex)p(x, c)$$

has been derived.

A12.1 If $q(x) =_{df} p(x, c)$, then $\dfrac{\sim q(x)}{\sim p(x, c)}$.

12.1 If $q(x) =_{df} p(x, c)$, then $q(x)$ and $p(x, c)$ are interchangeable.

Proof. (1)

$$\dfrac{\dfrac{(Ex)p(x, c)\qquad \dfrac{\dfrac{\dfrac{\sim p(x, c)}{p(y, z)\rightarrow_{yz} y \# x \vee z \# c}\ \text{D10}\qquad p(y, c)^*}{y \# x \vee c \# c}\ \text{8.3, A11.2}\qquad \dfrac{\exists \#}{\sim c \# c}\ \text{A9.2, A11.2}}{y \# x}\ \text{A10.0}}{p(y, c)\rightarrow_{y} y \# x}\ \text{A3.4}}{\sim q(x)}\ \text{D10}$$

$$(2)\quad \frac{\sim q(x)}{\sim p(x, c)}\ \text{A12.1}$$

$$(3)\quad \frac{[\exists p(x, c)]\quad (Ex)p(x, c)}{\exists q(x)}\ \text{def}\ \exists$$

$$(4)\quad \frac{[\exists q(x)]\ (Ex)p(x, c)}{\exists p(x, c)}\ \text{A11.1, def}\ \exists$$

Further 11.2.

In the same way as an individual constant is formally suppressed, we can formally suppress a free variable. This is done by means of a definition of the kind:

$$p(x) =_{df} q(x, y).$$

We will allow a definition of this kind only in case

$$(Ex)p(x, y)$$

has been derived.

A12.2 If $q(x) =_{df} p(x, y)$, then $\dfrac{\sim q(x)}{\sim p(x, y)}$.

12.2 If $q(x) =_{df} p(x, y)$, then $q(x)$ and $p(x, y)$ are interchangeable.

Proof. Similar to that of 12.1.

Remark. The axioms A12.0—2 seem to be suspect from an intuitionistic point of view. In an informal way we can show this. Suppose that $q(x) =_{df} p(x, x)$ and that $\sim q(x)$ has been derived. Then $q(y)$ implies $y \# x$. From this it is seen that $p(y, z)$ will imply the impossibility of $y = x \wedge z = x$. But it seems intuitionistically not allowed to derive $y \# x \vee z \# x$ from this impossibility. So it seems not to be allowed to derive $\sim p(x, x)$ from $\sim q(x)$. That this argument is wrong is shown by the theorem 10.80.

III. Propositional functions of higher level

13. We will enlarge our formalism with functions of propositional functions. The variables introduced up to now we shall call variables of level 0, the wff we shall call wff of level 0. The new functions will be called wff of level 1. Besides the variables of level 0, that stand for an arbitrary individual constant, we shall introduce variables of level 1, that stand for an arbitrary wff of level 0. We shall first formulate these expositions more precisely

To the signs of our system we add:

4. variabels of level 1 $f, g, \ldots,$
5. $\vee_1$.

In the following definition v stand for a variable of level 0, v_1 and v_1' for a variable of level 1, vs for a sequence of variables of level 0, vs_1 for a sequence of variables of level 1 and (or) 0.

Definition by induction of a *well-formed formula of level* 1 (wff$_1$):

a. every wff is a wff_1,
b. v_1 and $v_1(vs)$ are wff_1,
c. if p_1 and q_1 are wff_1, then $p_1 \wedge q_1$ and $p_1 \vee q_1$ are wff_1,
d. if p_1 and q_1 are wff_1, then $p_1 \rightarrow_{vs_1} q_1$ is a wff_1,
e. if p_1 is a wff_1, then $(Ev)p_1$, $(v)p_1$, $(Ev_1)p_1$ and $(v_1)p_1$ are wff_1,
f. if p_1 is a wff_1 and p_1 contains a free v_1, then $\sim p_1$ is a wff_1. [1]

An arbitrary wff_1 will be written $p_1, q_1, \ldots$. If we want to express that, e.g., p_1 contains some free variables, we will write these variables between brackets behind p_1. Again this does not mean that these variables are the only free variables occurring in p_1.

$p_1, q_1, \ldots$ do not belong to the system itself, but to the metasystem.

From the preceding axioms concerning wff (excepted A9.0—3, A10, A10.0 *and* A11.0—2, *that will be considered later) axioms concerning* wff_1 *can be formed in the following way.*

The signs standing for arbitrary wff $(p, q, \ldots)$ *are replaced by signs standing for arbitrary* wff_1 $(p_1, q_1, \ldots)$. *A variable of level* 0 *may remain unchanged but may also be replaced by a variable of level* 1. *Equal signs are replaced by equal signs, different signs by different signs.*

In the axioms A4.0, A4.2 *and* A4.4 *the sign* $\exists$ *binds only the free variables of level* 0.

In the axioms A6.0 and A6.1 the variables x and y must both remain unchanged or must both be replaced by variables of level 1.

Constants of level 1 *and variables of level* 1.

On the level 1 the wff (of level 0) play the same rôle as the individual constants on the level 0. Therefore we call an individual constant a constant of level 0 and a wff a constant of level 1.

We discern two kinds of variables of level 1, variables standing alone, e.g., f, and variables with a sequence of variables of level 0 between brackets put behind it, e.g., $f(x, y, z)$. The intention of the use of these two different kinds is, that for the variable f may be substituted an arbitrary wff and for $f(x, y, z)$ only an arbitrary wff with free variables x, y and z and no other free variables.

A derivation of a wff_1 that contains a free variable v_1, should be valid if for v_1 an arbitrary wff is substituted. The wff_1 f does not contain a free variable of level 0. So by means of 7.00 we would be able to prove $\frac{(Ex)f}{f}$. This would not fulfil our purpose, as it is

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[1] As, e.g., $\sim f$ is a wff_1, the sign $\sim$ cannot be eliminated by D10 from every wff_1 any more.

not allowed here to substitute a wff containing the free variable x for f. To avoid this consequence, we state the following rule.

Rule. A part of a wff$_1$ consisting of a free variable v_1 (without variables of level 0 between brackets put behind) should not be treated as not containing some free variable of level 0.

For similar reasons we state the rule:

Rule. A part $v_1(vs)$ of a wff$_1$ should be treated as containing all variables of vs free and containing no variables of level 0 free that do not belong to vs.

In accordance with these rules A11.0—2 are transformed into:

A11.0 $$\frac{(x)p_1(x)}{p_1(c)}, \quad \frac{(f)p_1(f) \quad \exists p}{p_1(p)}$$

A11.1 $$\frac{p_1(c)}{(Ex)p_1(x)}, \quad \frac{p_1(p)}{(Ef)p_1(f)}, \quad \frac{p_1(p(vs))}{(Ef)p_1(f(vs))}$$

vs is the sequence of free variables of p.

A11.2 If $\dfrac{p_1(x)}{q_1(x)}$, then $\dfrac{p_1(c)}{q_1(c)}$,

if $\dfrac{p_1(f)}{q_1(f)}$, then $\dfrac{p_1(p)}{q_1(p)}$,

if $\dfrac{p_1(f(vs))}{q_1(f(vs))}$, then $\dfrac{p_1(p(vs))}{q_1(p(vs))}$.

vs is the sequence of free variables of p.

The following theorem is a special case of 8.3.

13.0 $$\frac{p_1(f(vs)) \rightarrow_f q_1(f(vs)) \quad p_1(p(vs))}{q_1(p(vs))}$$

vs is the sequence of free variables of p.

Proof. A3.0, A11.2.

Negation and existentional operator.

The existentional operator may bind all variables of level 0 or all variables of level 1. So we will have to discern between two kinds of existentional operators. The $\exists$-operator binding all variables of level 0 we shall write $\exists$, as we used to do, and the $\exists$-operator binding all variables of level 1 we shall write $\exists_1$. So, e.g., $\exists f(x)$ means $(Ex)f(x)$ and $\exists_1 f(x)$ means $(Ef)f(x)$.

In the same way we discern between $\sim$ operating on all variables of level 0, and $\sim_1$ operating on all variables of level 1. So, e.g., $\sim f(x)$ means $f(y) \rightarrow_y y \# x$, and $\sim_1 f(x)$ means $g(x) \rightarrow_g g \#_1 f$ (for the meaning of $\#_1$ see below).

The rules of section **11** concerning the application of the $\sim$- and the $\exists$-operator on wff containing constants of level 0 are transformed into rules concerning the application of the $\sim_1$- and the $\exists_1$-operator on wff_1 containing constants of level 1.

So, e.g., (for $\vee_1$ see below)

$$\sim_1 . p \rightarrow_x \sim q =_{df} f \rightarrow_x \sim g . \rightarrow_{fg} f \#_1 p \vee_1 g \#_1 q$$

$$\exists_1 . p \rightarrow_x \sim q =_{df} (Ef)(Eg) . f \rightarrow_x \sim g$$

There is some ambiguity in these conventions. E.g., $(x)(Ey)x=y$ is a wff. But it has two parts that are also wff, viz., $(Ey)x = y$ and $x = y$. And so $\exists_1(x)(Ey)x = y$ might mean $(Ef)f$, $(Ef)(x)f$ and $(Ef)(x)(Ey)f$. So if we apply the $\sim_1$- or the $\exists_1$-operator to a wff_1 containing constants of level 1, it should be known by what kind of substitution the wff_1 has been obtained from a wff_1 containing no constants of level 1. In the given example we should have to discern between:

f (subst. $(x)(Ey)x = y$ for f),
$(x)f$ (subst. $(Ey)x = y$ for f),
$(x)(Ey)f$ (subst. $x = y$ for f).

As these difficulties can be avoided in the present paper, we will, merely for the sake of simplicity, not complicate our symbolism in this way. The following practical rule will be sufficient.

If in p_1 occur $p, q, r, \ldots$, we can transform p_1 into p_1' by replacing $p, q, r, \ldots$ by $f, g, h, \ldots$, respectively ($f, g, h, \ldots$ must not occur in p_1). Then it will in this paper tacitly be assumed that p_1 has been obtained from p_1' by substitution of $p, q, r, \ldots$ for $f, g, h, \ldots$. So, e.g., $\exists_1(x)(Ey)p$ will mean $(Ef)(x)(Ey)f$.

Disjunction of propositional functions of level 1.

Suppose that $p_1(f)$ and $q_1(f)$ are two wff_1, for which $\exists_1 p_1(f)$ and $\exists_1 q_1(f)$ have been derived. Then they represent two classes of classes, that are not empty. So we can form the sumclass $p_1(f) \vee q_1(f)$. Suppose further that $p(x)$ satisfies $p_1(f)$ (so $p_1(p(x))$). Then $p(x)$ will also satify the sumclass, so $p_1(p(x)) \vee q_1(p(x))$. From this result we would be able to derive by means of A3.2 and 4.00: $\exists q_1(p(x))$. But that should not be a consequence of the two suppositions that have been made.

If we interpret $p_1(p(x)) \vee q_1(p(x))$ as: p belongs to the sumclass of $p_1(f)$ and $q_1(f)$, it cannot be followed that $\exists q_1(p(x))$.

But if we interpret $p_1(p(x)) \vee q_1(p(x))$ as: x belongs to the sumclass of $p_1(p(x))$ and $q_1(p(x))$, then the derivation of $\exists q_1(p(x))$ should be allowed.

From this example it is seen that we will have to discern between a disjunction of two classes and a disjunction of two classes of classes. For the first kind of disjunction was used the sign $\vee$. For the disjunction of two classes of classes we will use the sign $\vee_1$.

To the definition of a wff$_1$ is added:

g. if p_1 and q_1 are wff$_1$, then $p_1 \vee_1 q_1$ is a wff$_1$.

The axioms about $\vee_1$ can be formed from the axioms about $\vee$ (A4.0—4 and A10.0) by replacing $\vee$ by $\vee_1$, $\exists$ by $\exists_1$ and $\sim$ by $\sim_1$. (A10 is a special case of A10.0 and may be cancelled.) In the definition of $\sim_1$ in D10 $\vee$ should be replaced by $\vee_1$, $\#$ by $\#_1$.

In the same way the theorems of level 0 can be transformed into theorems of level 1. The transformed theorems are quoted by the same numbers as the original ones.

The relations $=_1$ *and* $\#_1$.

The theory of $=$ and $\#$ could be translated into the language of level 1, if we would accept the translation of the axioms A9.0—3. This translation is:

A9.0 $$\frac{p =_1 q \quad p_1(p)}{p_1(q)}$$

A9.1 $$\frac{p \#_1 f \rightarrow_f q \#_1 f}{q =_1 p}$$

A9.2 $$\frac{p \#_1 q}{r \#_1 p \vee r \#_1 q}$$

A9.3 $$\frac{p \#_1 q}{q \#_1 p}$$

We mentioned already that the former axiom A9.1 was suspect from an intuitionistic point of view. On the level 1 the axiom is still more suspect. We will not accept it. If A9.1 is rejected the theorems 9.1 and 9.3 and all theorems derived from these disappear also, and besides 10.21 disappears. That is too much. The theorems

9.1 $$\frac{\exists \#_1}{(f) f =_1 f}$$

9.3 $$\frac{p =_1 q}{q =_1 p}$$

cannot be dispensed with. So we will have to keep 9.1 and 9.3 instead of A9.1. Then the whole theory of $=$ and $\#$ would be saved with the exception of A9.1 itself and the theorem 10.21.

The meaning of $p =_1 q$ is: any object or sequence of objects that satisfies p will satisfy q, and vice versa. So we define:

D13.0 $p_1 =_1 q_1 =_{df} p_1 \rightarrow_{vs} q_1 \,.\, \wedge \,.\, q_1 \rightarrow_{vs} p_1$
vs is the sequence of free variables of level 0 of p_1 and q_1.

From this definition A9.0 (for $=_1$) can be derived by induction with respect to the structure of p_1.

Further from D13.0 and 3.00 is derived $\dfrac{\exists p}{p =_1 p}$, which plays the same rôle in deductions as 9.1. The theorem 9.3 is an immediate consequence of D13.0.

The meaning of $p \#_1 q$ is: there exists an object that satisfies p and does not satisfy q (i.e., is different from all objects that satisfy q), or satisfies q and does not satisfy p. So we define:

D13.1 $p_1 \#_1 q_1 =_{df} \exists \,.\, p_1 \wedge \sim q_1 \vee_1 \exists \,.\, \sim p_1 \wedge q_1$

Note that it is not allowed to replace in this definition $\vee_1$ by $\vee$. For if we write $\vee$ instead of $\vee_1$, we are able to derive $\exists \sim p$ from $p \#_1 q$ (4.02, 3.42). This consequence might be mistaken, e.g., if p is the class $x = x$. But if we write $\vee_1$ and apply 4.02, we do not find $\exists \exists \,.\, \sim p \wedge q$ (which is identical with $\exists \,.\, \sim p \wedge q$) but $\exists_1 \exists \,.\, \sim p \wedge q$. And as p and q are constants of level 1 substituted in a function of level 1, $\exists_1 \exists \,.\, \sim p \wedge q$ is the same as $\exists_1 \exists \,.\, \sim f \wedge g$.

A9.3 is a consequence of D13.1; A9.2 will be proved later (13.92).

13.10 $\dfrac{\exists \,.\, p_1 \wedge \sim q_1}{p_1 \#_1 q_1}, \quad \dfrac{\exists \,.\, \sim p_1 \wedge q_1}{p_1 \#_1 q_1}$

Proof. $A(x) =_{df} x = a,\ F(x) =_{df} x \# a$

(1)
$$\dfrac{\dfrac{\dfrac{\dfrac{\exists \#}{a = a}\ \text{9.1, A11.0} \qquad \dfrac{\exists \#}{\sim a \# a}\ \text{A9.2, A3.4, D10, 7.11, A11.0}}{(Ex) \,.\, x = a \wedge \sim x \# a}\ \text{A11.1}}{\exists \,.\, A(x) \wedge \sim F(x)}\ \text{def } A(x),\ F(x)}{\exists_1 \exists \,.\, f \wedge \sim g}\ \text{A11.1}$$

(2)
$$\dfrac{\exists \#}{\exists_1 \exists \,.\, \sim f \wedge g}\ \text{similar}$$

Further (1), (2), A4.0, D13.1.

13.11 $\dfrac{\exists \,.\, f(x) \wedge \sim g(x)}{f(x) \#_1 g(x)}, \quad \dfrac{\exists \,.\, \sim f(x) \wedge g(x)}{f(x) \#_1 g(x)}$

Proof. Similar.

13.20 $\dfrac{\exists p}{(Ef)\,.\,p =_1 f}$

Proof. 9.1, A11.1.

13.21 $\dfrac{\exists\,\#}{(Ef)\ \exists_{vs} \sim f(x)}$

Proof. $A(x) =_{df} x = a$ (x belongs to vs.)

$$\begin{array}{cl}
\exists\# & \\
\hline
(Ex)\ x \# a & 9.0,\ \mathrm{A11.0} \\
\hline
(Ex) \sim x = a & 10.23,\ 5.0,\ \mathrm{A11.2} \\
\hline
\exists_{vs} \sim A(x) & \text{def } A(x),\ \mathrm{A3.2} \\
\hline
(Ef)\,\exists_{vs} \sim f(x) & \mathrm{A11.1}
\end{array}$$

13.22 $\dfrac{\exists\#}{(Ef)\ (Eg)\,\exists_{vs}\,.\,f \wedge g}$

Proof. By means of $A_1(x) =_{df} x = a$ and $A_2(x) =_{df} x = a$.

13.23 $\dfrac{\exists\#}{(Ef)\ (Eg)\ \exists\,.\,f \wedge \sim g}$, $\dfrac{\exists\#}{(Ef)\ (Eg)\,\exists\,.\sim f \wedge g}$

Proof. By means of $A(x) =_{df} x = a$ and $B(x) =_{df} x \# a$.

13.230 $\dfrac{\exists_{vs}\,p}{(Ef)\ \exists_{vs}\,.\sim f \wedge p}$

Proof. $C(x) =_{df} x \# y$

(1) $\dfrac{p}{\sim C(y) \wedge p}$ D10, A9.2, A2.0

$$\begin{array}{cl}
\exists_{vs}\,p & \\
\hline
\exists_{vs}\,.\sim C(y) \wedge p & (1),\ 3.30 \\
\hline
(Ef)\ \exists_{vs}\,.\sim f \wedge p & \mathrm{A11.1}
\end{array}$$

13.231 $\dfrac{\exists\#}{\exists_1\#_1}$

Proof. 13.23, 4.01, D13.1.

13.24 $\dfrac{\exists\#}{(Ef)(vs)f}$, $\dfrac{\exists\#}{(Ef)(x)f(x)}$

Proof. By means of $D(x) =_{df} x = x$.

13.25 $\dfrac{\exists\#}{(Ef)\sim_1(x)f(x)}$

Proof. $F(x) =_{df} x \# a$

$$\dfrac{\dfrac{\dfrac{\exists\#}{(Ef)(x)f(x)}\;\textbf{13.24}\qquad\dfrac{\dfrac{\dfrac{(x)f(x)^*}{f(a)}\;\textbf{A11.0}\qquad\dfrac{\exists\#}{\sim F(a)}\;\textbf{D10}}{\exists\,.\,f(x)\wedge\sim F(x)}\;\textbf{A11.1}}{f(x)\#_1 F(x)}\;\textbf{13.11, A11.2}}{(x)f(x)\rightarrow_f f(x)\#_1 F(x)}\;\textbf{A3.4}}{\dfrac{\sim_1(x)F(x)}{(Ef)\sim_1(x)f(x)}\;\textbf{A11.1}}\;\textbf{D10}$$

13.26 $$\dfrac{\exists\#}{(Ef)(Eg)\exists_{vs}\sim.\,f\vee g}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists\#}{\sim x\#x}\;\textbf{A9.2, D10}}{\sim x\#x\wedge\sim x\#x}\;\textbf{A2.0}}{\sim\sim\sim x\#x\wedge\sim\sim\sim x\#x}\;\textbf{10.42, 10.40}}{\sim.\sim\sim x\#x\vee\sim\sim x\#x}\;\textbf{10.72}}{\exists_{vs}\sim.\sim\sim x\#x\vee\sim\sim x\#x}\;\textbf{A3.2}}{(Ef)(Eg)\;\exists_{vs}\sim.\,f\vee g}\;\textbf{A11.1}$$

13.27 $$\dfrac{\exists\#}{(Ef)\sim_1\exists_{vs}\sim f}$$

Proof. $D(x) =_{df} x = x$

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists\#}{(Eg)\exists_{vs}\sim g(x)}\;\textbf{13.21}\dfrac{\dfrac{\exists_{vs}\sim g(x)^*}{\exists_{vs}.\,D(x)\wedge\sim g(x)}\;\textbf{9.1, 5.22}}{g(x)\#_1 D(x)}\;\textbf{7.3, 13.11, A11.2}}{\exists_{vs}\sim g(x)\rightarrow_g g(x)\#_1 D(x)}\;\textbf{A3.4}}{\sim_1\exists_{vs}\sim D(x)}\;\textbf{D10}}{(Ef)\sim_1\exists_{vs}\sim f}\;\textbf{A11.1}$$

13.28 $$\dfrac{\exists\#}{(Ef)f(x)}$$

Proof. By means of $D(x) =_{df} x = x$.

13.280 $$\dfrac{\exists\#}{(Ef)\sim f(x)}$$

Proof. By means of $G(y) =_{df} x \# y$.

13.281 $$\frac{\sim f(x)}{\sim_1 f(x)}$$

Proof.
$$\cfrac{\cfrac{\cfrac{\cfrac{\sim f(x) \quad g(x)^*}{\exists \,.\, \sim f(x) \wedge g(x)}\,\text{A3.2}}{g(x) \#_1 f(x)}\,\text{13.11} \qquad \cfrac{\exists \#}{(Eg)g(x)}\,\text{13.28}}{g(x) \rightarrow_g g(x) \#_1 f(x)}\,\text{A3.4}}{\sim_1 f(x)}\,\text{D10}$$

13.29 $$\frac{\exists \#}{(Ef)(Ex)\,.\, f(x) \wedge x \# y}$$

Proof. By means of $C(x) =_{df} x \# y$.

The theorems 13.20—13.29 can be proved for variables of level 1 with a prescribed number of variables put behind it between brackets (so for $f(vs)$).

13.30 $$\frac{(vs)p \quad \exists_{vs} \sim q}{p \#_1 q}$$

Proof. (1)
$$\cfrac{\cfrac{\cfrac{(vs)p}{p}\,\text{A5.0} \qquad \sim q}{p \wedge \sim q}\,\text{A2.0}}{\exists_{vs} \,.\, p \wedge \sim q}\,\text{A3.2} \qquad \cfrac{\cfrac{(vs)p \quad \exists_{vs} \sim q}{\exists \,.\, p \wedge \sim q}\,\text{(1), 7.10, 7.3}}{p \#_1 q}\,\text{13.10}$$

13.40 $$\frac{(vs)p}{\sim_1 \exists_{vs} \sim p}$$

Proof. **13.30, 13.21, A3.4, D10.**

13.41 $$\frac{\sim_1 \exists_{vs} \sim p}{(vs)p}$$

Proof. $H(x) =_{df} x = y$.

(1) $$\frac{H(x) \quad \sim f(x)}{\sim f(y)}\,\text{A9.0}$$

(2) $$\frac{\exists \,.\, H(x) \wedge \sim f(x)}{\sim f(y)}\,\text{(1), 7.10}$$

(3) $$\frac{\sim H(x) \quad f(x)}{(Ex)\,.\, f(x) \wedge x \# y}\,\text{10.20, A3.2}$$

(4) $$\frac{\exists \,.\, \sim H(x) \wedge f(x)}{(Ex)\,.\, f(x) \wedge x \# y}\,\text{(3), 7.10}$$

(5) $$\frac{\sim f(y)}{\exists \sim f(x)}\,\text{A3.2}$$

(6) $$\cfrac{f(x) =_1 H(x) \qquad \cfrac{\exists \#}{(Ex) \sim H(x)}\,\text{9.0}}{(Ex) \sim f(x)}\,\text{A9.0}$$

(7)

$$\dfrac{\dfrac{\begin{array}{c}\exists\#\\ )\sim f(y)\end{array}{}^{\text{13.280}}\quad \sim_1\exists\sim f(x)}{\sim_1\sim f(y)}\,\text{(5), 10.50}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\sim_1(Ex)\sim f(x)\quad \dfrac{\dfrac{\dfrac{\exists\#}{\exists H(x)}\,\text{9.11}}{H(x)=_1H(x)}\,\text{3.00, D13.0}}{(Ef).f(x)=_1H(x)}\,\text{A11.1}}{\sim_1.f(x)=_1H(x)\ ^{1)}}\,\text{(6), 10.50}}{g=_1H(x)\to_g g\#_1f(x)}\,\text{D10}\quad \dfrac{\exists\#}{H(x)=_1H(x)}\,\text{D13.0}}{H(x)\#_1f(x)}\,\text{8.3}}{\sim f(y)\vee_1(Ex).f(x)\wedge x\#y}\,\begin{array}{l}\text{(2), (4), 13,280, 13.29, A4.(}\\ \text{4.10, D13.}\end{array}}{(Ex)\,.\,f(x)\wedge x\#y}\,\text{A10.0}$$

(7) allows us to define $h(x)=_{df}f(x)\wedge x\#y$.

(8)

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\#\\ \#x\end{array}{}^{\text{D10, A9.2}}\quad \dfrac{\dfrac{\dfrac{\sim_1\exists\sim f(x)}{(Ex).f(x)\wedge x\#y}\,\text{(7)}\quad \dfrac{\sim h(x)}{\sim.f(x)\wedge x\#y}\,\text{10.11}}{f(z)\wedge z\#u\to_{zu}z\#x\vee u\#y}\,\text{10.120}\quad f(x)\quad x\#u^*}{x\#x\vee u\#y}\,\text{8.3}}{u\#y}\,\text{A10.0}\qquad \dfrac{\exists\#}{(Eu)x\#u}\,\text{9.0}}{\dfrac{x\#u\to_u u\#y}{f(x)\quad \dfrac{}{x=y}\,\text{A9.1}}}\,\text{A3.4}$$

$$\dfrac{f(x)\quad \dfrac{x\#u\to_u u\#y}{x=y}\,\text{A9.1}}{f(y)}\,\text{A9.0}$$

(9)

$$\dfrac{\dfrac{\dfrac{\sim_1(Ex)\sim f(x)}{\sim g\to_g g\#_1f(x)}\,\text{D10}\quad \dfrac{\dfrac{\dfrac{\sim_1(Ex)\sim f(x)}{\exists h(x)}\,\text{(7), def } h}{\sim h(y)}\,\text{def } h\text{, A3.4, D10}}{\exists\sim h(x)}\,\text{A3.2}}{h(x)\#_1f(x)}\,\text{8.3}\qquad \dfrac{\exists h(x)\quad \dfrac{\sim_1(Ex)\sim f(x)}{\sim_1\exists\,.\,g(x)\wedge\sim f(x)}\,\text{3.30, 13.23, 10.50}}{\sim_1\exists\,.\,h(x)\wedge\sim f(x)}\,\text{A11.0}}{\dfrac{\exists\,.\sim h(x)\wedge f(x)}{f(y)}\,\text{(8), 7.10}}\,\text{D13.1, A10.0}$$

Further (9), 7.11, A6.1.

In a similar way we prove for arbitrary sequences vs and vs':

$$\frac{\sim_1\exists_{vs}\sim f(vs')}{(vs)f(vs')}\,.$$

(In the definition of $H(x)$ for y a variable is chosen that does not belong to vs or to vs'.)

13.41 is then followed by means of A11.2. (If some of the free variables of p are equal, the equal variables are replaced by differ-

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[1] $\sim_1$ should here only be applied to f and not to the constant H.

ent variables that do not belong to vs and do not occur in p. In vs the same changes are made. p is then transformed into a formula p'. The constants of level 0 of p' are replaced by variables that do not belong to vs and do not occur in p'. Now A11.2 (level 1) is applied and then A11.2 (level 0), A6.0 and A6.1.)

13.42 $$\frac{p \to_{vs} q}{\sim_1 \exists_{vs} . p \wedge \sim q}$$

Proof. (1) $$\cfrac{\cfrac{p \quad p \to_{vs} q}{q}\ \text{A3.0} \qquad \sim f}{\exists . q \wedge \sim f}\ \text{A3.2}$$

$$\cfrac{\cfrac{\cfrac{\exists_{vs} . p \wedge \sim f^{*} \quad p \to_{vs} q}{\exists . q \wedge \sim f}\ (1),\ 7.10}{q \#_1 f}\ 13.10 \qquad \cfrac{\cfrac{p \to_{vs} q}{\exists_{vs} p}\ \text{A3.3}}{(Ef)\, \exists_{vs} . p \wedge \sim f}\ 13.230}{\cfrac{\exists_{vs} . p \wedge \sim f . \to_{f} q \#_1 f}{\sim_1 \exists_{vs} . p \wedge \sim q}\ 10.11}\ \text{A3.4}$$

13.43 $$\frac{p \to_{vs} \sim q}{\sim_1 \exists_{vs} . p \wedge q}$$

Proof. $$\cfrac{\cfrac{f \to_{vs} \sim g}{\sim_1 \exists_{vs} . f \wedge \sim\sim g}\ \text{13.42 (proved for } f \text{ and } g \text{ similarly)}}{\sim_1 \exists_{vs} . f \wedge g}\ \text{A3.3, 10.40, 3.30, 13.22, 10.50}$$

Further A11.2.

13.5 $$\frac{\sim p \to_{vs} q \qquad \exists_{vs} \sim q}{(vs) . p \vee q} \qquad \text{(cf. 13.80)}$$

Proof. $$\cfrac{\cfrac{\cfrac{\cfrac{\sim f \to_{vs} g \quad \cfrac{\exists_{vs} \sim g}{g \to_{vs} \sim\sim g}\ 10.43}{\sim f \to_{vs} \sim\sim g}\ 3.20}{\sim_1 \exists_{vs} . \sim f \wedge \sim g}\ \text{13.43 (proof)}}{\sim_1 \exists_{vs} \sim . f \vee g}\ \text{10.71, 3.30, 13.26, 10.50}}{(vs) . f \vee g}\ 13.41$$

Further A11.2.

13.50 $$\frac{\sim p \to_{vs} q \qquad \exists_{vs} \sim q}{\sim q \to_{vs} p}$$ *Proof.* 13.5, 10.60.

13.51 $$\frac{\exists_{vs} \sim p}{\sim\sim p \to_{vs} p}$$ *Proof.* 3.00, 10.42, 13.50.

13.52 $$\frac{\sim\sim p}{p}$$ *Proof.* A3.2, 10.41, 13.51, A3.0.

13.53 $\dfrac{\exists_{vs} \sim p}{p \vee \sim p}$ *Proof.* 13.51, 13.5, A5.0.

This is a striking result. It should be noticed that 13.53 is not exactly the law of excluded middle. It is not necessary to interpret $(x) \,.\, p \vee q$ as: for every x it is decidable whether x belongs to p or to q. Even in negationless logic this formulation would not be allowable, as we can only say, that "x belongs to p" or "x belongs to q" if both these propositions are right. For this reason Griss chooses a different interpretation of the disjunction. He says: "La disjonction n'affirme rien d'un cas déterminé (particulier), mais elle nous donne la possibilité de démontrer un théorème pour tous les éléments d'un ensemble V en démontrant ce théorème pour les éléments de deux espèces dont la réunion est identique à V." ¹) So 13.53 should be interpreted: if it can be shown that every element of p belongs to q and every element of $\sim p$ belongs to q, then for any x it can be proved that x belongs to q.

From a (normal) intuitionistic point of view there might be elements from which it is not decidable whether they belong to p or to $\sim p$. According to 13.53 such an element would still belong to $p \vee \sim p$. So there might be objects that belong to a sumclass and for which it is not decidable to which of the two components of the sumclass they belong. Even this result is unacceptable for intuitionistic disjunction. So, if 13.53 is maintained, the disjunction of the present system would be essentially different from the intuitionistic disjunction.

13.54 If $\dfrac{\sim p}{q}$, then $\dfrac{\sim q \quad \exists \sim p}{p}$.

Proof. 10.50, 13.52.

13.55 If $\dfrac{\sim p}{\sim q}$, then $\dfrac{q \quad \exists \sim p}{p}$.

Proof. 10.52, 13.52.

13.54—55 remain valid, if (twice) a premiss is added, that does not contain a free variable that is free in p or q.

13.6 $\dfrac{\sim . p \wedge q \quad \exists . \sim p \vee \sim q}{\sim p \vee \sim q}$

Proof. (1) $\dfrac{\dfrac{\sim . \sim p \vee \sim q}{\sim \sim p \wedge \sim \sim q}\ 10.71}{p \wedge q}\ 13.52$

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¹) G. F. C. Griss, Logique des mathématiques intuitionistes sans négation. Comptes rendus des séances de l'Ac. des Sc., t. 227 (1948), p. 947.

$$\dfrac{\sim.p\wedge q \qquad \dfrac{\dfrac{\dfrac{\sim.p\wedge q}{\exists.p\wedge q}\,\text{A3.2, 10.41}\quad \dfrac{\exists.\sim p\vee\sim q}{\exists\sim p}\quad \dfrac{\exists.\sim p\vee\sim q}{\exists\sim q}\,\text{A4.2, 4.00}}{\exists.\sim\sim p\wedge\sim\sim q}\,\text{10.40, 3.300}}{\exists\sim.\sim p\vee\sim q}\,\text{10.72, 3.30}}{\sim p\vee\sim q}\,\text{(1), 10.50, 13.52}$$

13.70 $\dfrac{\exists_{vs}\sim p}{\sim_1(vs)p}$ *Proof.* 13.40, 13.24, 10.52; further A11.2.

13.71 $\dfrac{\sim_1(vs)p}{\exists_{vs}\sim p}$ *Proof.* 13.41, 13.27, 13.54; further A11.2.

10.70—72 and 13.6 state a relation between conjunction and disjunction that reminds of the two-valued logic. The same can be said of 13.40—41 and 13.70—71 with respect to the all- and existence-operators.

13.8 $\dfrac{\exists p}{\exists_{vs}\sim p\vee_1(vs)p}$

Proof.

$$\dfrac{\dfrac{\dfrac{\exists\#}{f)\sim_1\exists_{vs}\sim f}\,\text{13.27}\quad \dfrac{\dfrac{\sim_1(vs)f^*}{\exists_{vs}\sim f}\,\text{13.71}\quad \dfrac{\dfrac{\exists\#}{(Ef)\sim_1(x)f}\,\text{13.25}}{(Ef)\sim_1(vs)f}\,\text{A5.0, 13.24, 10.50, 3.}}{\sim_1(vs)f\rightarrow_f\exists_{vs}\sim f}\,\text{A3.4}}{(f).(vs)f\vee_1\exists_{vs}\sim f}\,\text{13.5}\qquad \exists p}{(vs)p\vee_1\exists_{vs}\sim p}\,\text{A11.0}$$

13.80 $\dfrac{\sim p\rightarrow_{vs}q}{(vs).p\vee q}$

Proof. By means of 13.5 and 13.8 (applied to q).

13.90 $\dfrac{p\#_1q\qquad\exists\sim r}{p\#_1r\vee_1q\#_1r}$

Proof. (1) $$\dfrac{\dfrac{\exists\#}{\exists_1p\#_1r}\,\text{13.231}\qquad\dfrac{\sim q\quad r}{q\#_1r}\,\text{A3.2, 13.10}}{p\#_1r\vee_1q\#_1r}\,\text{A4.0}$$

(2) $$\dfrac{\dfrac{\exists\#}{\exists_1q\#_1r}\,\text{13.231}\qquad\dfrac{p\quad\sim r}{p\#_1r}\,\text{A3.2, 13.10}}{p\#_1r\vee_1q\#_1r}\,\text{A4.0}$$

(3) $$\dfrac{p\quad\sim q\quad\dfrac{\exists\sim r}{r\vee\sim r}\,\text{13.53}\quad\dfrac{\exists\#}{\exists_1.p\wedge\sim q\wedge r}\quad\dfrac{\exists\#}{\exists_1.p\wedge\sim q\wedge\sim r}\,\text{1)}}{p\#_1r\vee_1q\#_1r}\,\text{4.1}$$

---

1) Proof by means of an example.

Further (3), 7.10, 4.100, D13.1.

13.91 $$\frac{p \#_1 q \qquad (vs)r}{p \#_1 r \vee_1 q \#_1 r}$$

Proof. Similar, using 5.22.

13.92 $$\frac{p \#_1 q \quad \exists r}{p \#_1 r \vee_1 q \#_1 r}$$

Proof. 13.90, 13.91, 13.8, 4.100.

The theorem 13.92 replaces the axiom A9.2.

14. *Correspondence between the present system and the system of Hilbert and Ackermann.*

We shall show in this section, that the structural difference between the present system and the ordinary two-valued logic is caused only by the existentional conditions and not by the lack of real negation.

Therefore we construct a new system of axioms that are generated from our system by canceling the existence-conditions. So in the axioms A3.4, A4.0, A4.3, A4.4, A11.0 (in section 13) these conditions are dropped and are further canceled the axioms A3.3, A4.2. In the derived theorems the existentional conditions have to be dropped, too.

We now compose the following transformation between this system (the system N) and the system of Hilbert and Ackermann[1]) (the system HA).

system HA	system N
$p \vee q$	$p \vee q$
$\bar{p}$	$\sim p$
$p \,\&\, q$	$p \wedge q$
$p \rightarrow q$	$\sim p \vee q$
$(Ex)f(x)$	$(Ex)f(x)$
$(x)f(x)$	$(x)f(x)$
$(x) \,.\, f(x) \rightarrow g(x)$	$f(x) \rightarrow_x g(x)$

A similar correspondence transforms $p \rightarrow_{vs} q$. Furthermore "p is provable" in HA corresponds with $\frac{\exists}{p}$ in N. The sign $\exists$ indicates, that the premiss is empty. This correspondence is not unambiguous. It will be shown, that if two transformations of a formula can be made, it does not matter which of the two is chosen.

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[1]) D. Hilbert and W. Ackermann, Grundzüge der theoretischen Logik, New York 1946, p. 23, 56, 57.

THEOREM 14.0. *The system HA transformed according to the above rules becomes part of the system N.*

Proof. The axioms of HA are:

1. $p \vee p \rightarrow p$
2. $p \rightarrow p \vee q$
3. $p \vee q \rightarrow q \vee p$
4. $p \rightarrow q \,.\rightarrow.\, r \vee p \rightarrow r \vee q$
5. $(x)f(x) \rightarrow f(y)$
6. $f(y) \rightarrow (Ex)f(x)$

Proof of the corresponding theorems of N.

1. $$\frac{\exists}{\sim . p \vee p . \vee p}$$

Proof. (1) $\dfrac{p \vee p}{p}$ 2.2, 4.10

$$\begin{array}{cl} \exists & \\ \hline & (1),\ \mathrm{A}3.4 \\ p \vee p \rightarrow_{vs} p & \\ \hline & 13.51,\ 3.20 \\ \sim\sim . p \vee p . \rightarrow_{vs} p & \\ \hline & 13.5,\ \mathrm{A}5.0 \\ \sim . p \vee p . \vee p & \end{array}$$

2. $$\frac{\exists}{\sim p \vee . p \vee q}$$

Proof. Similar, starting from A4.0.

3. $$\frac{\exists}{\sim . p \vee q . \vee . q \vee p}$$

Proof. Similar, starting from 2.0.

4. $$\frac{\exists}{\sim . \sim p \vee q . \vee \sim . r \vee p . \vee . r \vee q}$$

Proof. s is short for the consequence.

(1) $\dfrac{q}{s}$ A4.0 (2) $\begin{array}{cl} p \qquad \sim q & \\ \hline \sim . \sim p \vee q & 10.40,\ 10.72 \\ \hline s & \mathrm{A}4.0 \end{array}$ (3) $\begin{array}{cl} \sim r \quad \sim p & \\ \hline \sim . r \vee p & 10.72 \\ \hline s & \mathrm{A}4.0 \end{array}$

(4) $\dfrac{r}{s}$ A4.0

Further (1), (2), (3), (4), 13.53, 4.100.

5. $$\frac{\exists}{\sim (x)f(x) \vee f(y)}$$

Proof. A5.0, A6.0. Further similar to the proof of 1.

6. $$\frac{\exists}{\sim f(y) \vee (Ex)f(x)}$$

Proof. A3.2, A6.1. Further similar to the proof of 1.

The HA-rules:
1. if p and $p \rightarrow q$, then q,
2. if $p \rightarrow f(x)$ and p does not contain x, then $p \rightarrow (x)f(x)$,
3. if $f(x) \rightarrow p$ and p does not contain x, then $(Ex)f(x) \rightarrow p$,

are transformed into the following theorems of N.

1. If $\frac{\exists}{p}$ and $\frac{\exists}{\sim p \vee q}$, then $\frac{\exists}{q}$.

Proof. 10.40, A10.0.

2. If $\frac{\exists}{\sim p \vee f(x)}$ and p does not contain x as a free variable,

then $\frac{\exists}{\sim p \vee (x)f(x)}$.

Proof. (1) $$\frac{p \quad \frac{\exists}{\sim p \vee f(x)}}{f(x)} \text{ A10.0}$$ (2) $$\frac{\frac{p}{(x)f(x)} \text{ (1), 7.11}}{\sim p \vee (x)f(x)} \text{ A4.0}$$

(3) $$\frac{\sim p}{\sim p \vee (x)f(x)} \text{ A4.0}$$

Further (2), (3), 13.53, 4.10.

3. If $\frac{\exists}{\sim f(x) \vee p}$ and p does not contain x as a free variable,

then $\frac{\exists}{\sim (Ex)f(x) \vee p}$.

Proof. (1) $$\frac{f(x) \quad \frac{\exists}{\sim f(x) \vee p}}{p} \text{ A10.0}$$ (2) $$\frac{(Ex)f(x)}{p} \text{ (1), 7.10}$$

Further A3.4, 13.51, 3.20, 13.5, A5.0 (similar to the proof of the theorem of N corresponding to the first axiom of HA).

The system HA is further based on the definitions

$$p \mathbin{\&} q =_{df} \overline{\overline{p} \vee \overline{q}} \text{ and } p \rightarrow q =_{df} \overline{p} \vee q.$$

In the system N holds:

$$\frac{p \wedge q}{\sim . \sim p \vee \sim q} \text{ 10.40, 10.72}$$

$$\frac{\sim . \sim p \vee \sim q}{p \wedge q} \quad 10.71,\ 13.52$$

$p \to q$ and $\bar{p} \vee q$ have the same transform.

And further it can be proved in N, that, if $\frac{p}{q}$ and $\frac{q}{p}$, p and q are interchangeable (11.2).

This interchangeability corresponds with a definition in HA.

The two transformations of $(x) . f(x) \to g(x)$ to $(x) . \sim f(x) \vee g(x)$ and to $f(x) \to_x g(x)$ give interchangeable results (10.611, 13.5).

We shall now compose a *transformation of the system N into the system HA*. The rules mentioned at the beginning of this section are not sufficient, as there are no transformation-rules for the signs $=$ and $\#$. We shall first try to find wff, that are interchangeable with $x = y$ and $x \# y$ within the system N.

1. $x = y$ is interchangeable with $f(x) \to_f f(y)$.

Proof.

$$\frac{x = y}{f(x) \to_f f(y)} \quad \text{A9.0, A3.4}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{f(x) \to_f f(y) \qquad x = z^*}{y = z} \;\; 8.3}{x = z \to_z y = z} \;\; \text{A3.4}}{y \# z \to_z x \# z} \;\; 10.51}{x = y} \;\; \text{A9.1}$$

2. $x \# y$ is interchangeable with $\sim x = y$, and so with $\sim . f(x) \to_f f(y)$.

Proof. 10.20, 10.23.

In according with this we transform

$$x = y \text{ into } (f) . f(x) \to f(y),$$

$$x \# y \text{ into } \overline{(f) . f(x) \to f(y)}.$$

We still have no transformation of $\frac{p \quad q \ldots}{r}$. We decide that we shall first link the premisses by $\wedge$. Then we alter the result to $\frac{\exists}{p \wedge q \wedge \ldots \to r}$. From A3.0 and A3.4 it is seen that, if one of the two is provable in N, the other is too.

Theorem 14.1. *The system N (section* 13 *excluded) transformed according to the rules mentioned at the beginning of this section and those added in the last two paragraphs, becomes part of the system HA.*

Proof. This is easily verifyable in HA.

15. *The relations between the present system and intuitionism.* We saw in the preceding section, that the present system differs from the classical system by the addition of existentional-premisses. Besides negation is defined in a different way, but this is of no influence on the structure of the system. Therefore we might call the system the negationless classical calculus (NC).

From an intuitionistic point of view the system NC is too large. The theorems, that are unacceptable in intuitionism (if negation, disjunction, implication, existence are interpreted in the intuitionistic way), are 10.21, 13.41, 13.5—13.55, 13.6, 13.71, 13.8, 13.80, 13.90—92. What axiom is responsible for the derivation of these theorems? It turns out to be the axiom A9.1. If this axiom is rejected, all the above theorems disappear. But we cannot reject this axiom without losing too much. If A9.1 is rejected the theorems 9.1, 9.3 disappear, too. So we replace A9.1 by

A9.10 $\dfrac{\exists \#}{x = x}$

A9.11 $\dfrac{x = y}{y = x}$

The new system we call the negationless intuitionistic calculus (NI). This calculus consists of all the theorems of the present system, with the exception of the above series of intuitionistically unacceptable theorems.

Comparing these results with Griss 1), we see that the "negation" in NI is something between the Griss relations $\neq$ and $\#$. Griss' characterization of $\neq$ is:

1. $x = x$,
2. $x = y \rightarrow y = x$,
3. $x = y \wedge y = z \rightarrow x = z$,
4. $x \neq y \rightarrow y \neq x$,
5. $x = y \wedge y \neq z \rightarrow x \neq z$.

His characterization of $\#$ consists of these 5, replacing $\neq$ by $\#$, and

6. $x \# y \rightarrow (z) . z \# x \vee z \# y$,
7. $(z) . z \# y \rightarrow x \# z . \rightarrow x = y$.

The negation in NI obeys 1—6, the negation in NC obeys 1—7.

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1) G. F. C. Griss, Negatieloze intuïtionistische wiskunde, Verslagen Ned. Akad. v. Wetensch., Afd. Natuurkunde, LIII (1944), p. 266.

I think there is no harm in adding 6, as 6 is in our notation equivalent with $\frac{\exists\#}{\sim x \# x}$.

It seems somewhat strange that a calculus has been constructed with the pretention of being a negationless intuitionistic calculus and that in that calculus A9.1 first has been accepted and afterwards rejected as being intuitionistically suspect. The reason is that Griss proved that A9.1 holds for real numbers [1]). So the consequences of this axiom should hold for real numbers too. And one of the consequences is 13.53, the formal equivalent of the law of excluded middle. I am very astonished about this result, but cannot solve the riddle.

16. *Correspondence between the present system and the system of Heyting.*

We construct N′ from NI in the same way as we constructed N from NC.

We state the following transformation between the system N′ and the system of Heyting (system H) [2]).

system H	system N′
$p \wedge q$	$p \wedge q$
$p \vee q$	$p \vee q$
$p \supset q$	$p \rightarrow_{vs} q$
$\neg p$	$\sim p$

As far as wff without free variables are concerned the axioms of the Heyting propositional calculus are transformed into theorems of N′.

Proof. The transforms of the axioms 2.1—4.11 of Heyting are:

2.1 $\frac{\exists}{p \rightarrow_{vs} p \wedge p}$ *Proof.* A2.0, A3.4.

2.11 $\frac{\exists}{p \wedge q \rightarrow_{vs} q \wedge p}$ *Proof.* 2.0, A3.4.

2.12 $\frac{\exists}{p \rightarrow_{vs} q \cdot \rightarrow_{vs} \cdot p \wedge r \rightarrow_{vs} q \wedge r}$

Proof. A2.1, A3.0, A2.0, A3.4.

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[1]) l.c. p. 265.

[2]) A. Heyting, Die formalen Regeln der intuitionistischen Logik, Sitzungsberichte der Preuszischen Akad. v. Wiss., 1930, math. phys. Klasse, p. 45 sqq.

2.13 $$\frac{\exists}{p \rightarrow_{vs} q \cdot \land \cdot q \rightarrow_{vs} r \cdot \rightarrow_{vs} \cdot p \rightarrow_{vs} r}$$ *Proof.* 3.20, A3.4.

2.14 $$\frac{\exists}{q \rightarrow_{vs} \cdot p \rightarrow_{vs} q}$$ *Proof.* A3.4.

2.15 $$\frac{\exists}{p \land \cdot p \rightarrow_{vs} q \cdot \rightarrow_{vs} q}$$ *Proof.* A3.0, A3.4.

3.1 $$\frac{\exists}{p \rightarrow_{vs} p \lor q}$$ *Proof.* A4.0, A3.4.

3.11 $$\frac{\exists}{p \lor q \rightarrow_{vs} q \lor p}$$ *Proof.* A4.1, A3.4.

3.12 $$\frac{\exists}{p \rightarrow_{vs} r \cdot \land \cdot q \rightarrow_{vs} r \cdot \rightarrow_{vs} \cdot p \lor q \rightarrow_{vs} r}$$ *Proof.* A4.3, A3.4.

4.1 $$\frac{\exists}{\sim p \rightarrow_{vs} \cdot p \rightarrow_{vs} q}$$

Proof. As $\sim p$ is senseless, if p is not obtained from a wff $p(x)$ containing a free variable by substituting an individual constant c for x, we may write $p(c)$ instead of p.

$$\dfrac{\dfrac{\dfrac{\dfrac{\sim p(c)}{p(x) \rightarrow_x x \# c}\,\text{10.10, A11.2} \qquad p(c)}{c \# c}\,\text{8.3, A11.2}}{q \lor c \# c}\,\text{A4.0} \qquad \dfrac{\exists}{\sim c \# c}\,\text{A9.2, A3.4, D10, A11.2}}{q}\,\text{A10.0}$$

Further A3.4.

4.11 $$\frac{\exists}{p \rightarrow_{vs} q \cdot \land \cdot p \rightarrow_{vs} \sim q \cdot \rightarrow_{vs} \sim p}$$

Proof. $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \rightarrow_{vs} q \qquad p \rightarrow_{vs} \sim q}{p \rightarrow_{vs} q \land \sim q}\,\text{3.21} \qquad p^*}{q(c) \land \sim q(c)}\,\text{8.3}}{q(c) \land \cdot q(x) \rightarrow_x x \# c}\,\text{10.10, A11.2}}{c \# c}\,\text{8.3, A11.2}}{p \rightarrow_{vs} c \# c}\,\text{A3.4} \qquad \dfrac{\exists}{\sim c \# c}\,\text{A9.2, A3.4, D10, A11.2}}{\sim p}\,\text{10.51, A3.0}$$

Further A3.4.

The proof is still the same, if p or q contain free variables.

The axioms of N′ concerning propositions, viz. the axioms of the sections **2**, **3**, **4**, **10**, are provable in H.

So the propositional calculi of the systems N′ and H are isomorphic. But this isomorphism does not hold any more, as soon as the propositions are changed into propositional functions. In two of the preceding proofs it was essential, that p and q do not contain a free variable, viz. the proofs of 2.14 and 4.1.

It is easily seen, that 2.14 is not provable any more for propositional functions. Let x be a free variable of p and q. Then from

$$\frac{\exists}{q \rightarrow_x \cdot p \rightarrow_x q}$$

would be provable

$$\frac{q}{p \rightarrow_x q} \text{ A3.0.}$$

As x is not free in $p \rightarrow_x q$, then

$$\frac{(Ex)q}{p \rightarrow_x q} \text{ 7.10.}$$

This would be a theorem of N′, and so of N. But the transformation of section **14** does not give a theorem of HA. And so it cannot be a theorem of N′.

In the same way 4.1 is rejected.

17. *The relations between the present system and the system of Griss.*

Griss constructed a set of axioms for negationless intuitionistic logic. ¹) His axioms are:

1. Axioms concerning propositions.

2.1 $p \rightarrow p \,\&\, p$
2.11 $p \,\&\, q \rightarrow q \,\&\, p$
2.12 $(p \rightarrow q) \rightarrow (p \,\&\, r \rightarrow q \,\&\, r)$
2.13 $(p \rightarrow q) \,\&\, (q \rightarrow r) \rightarrow (p \rightarrow r)$
2.16 $p \,\&\, q \rightarrow p$

2. Rules of substitution.

1.11 From the proved formulas P and Q follows $P \,\&\, Q$.
1.12 From P and $P \rightarrow Q$ follows Q.
1.13 From R and $P \rightarrow Q$ follows $P \rightarrow Q \,\&\, R$.

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¹) G. F. C. Griss, Logic of negationless intuitionistic mathematics, Proc. Kon. Ned. Akad. v. Wetensch. LIV (1951), p. 41—49.

3. Axioms concerning classes.

a. axioms for the intersection. 1)

12.1 $a \subset a \cap a$

12.11 $a \chi b \rightarrow a \cap b \subset b \cap a$

12.12 $a \chi c \,\&\, b \chi c \,\&\, a \subset b \rightarrow a \cap c \subset b \cap c$

12.13 $a \subset b \,\&\, b \subset c \rightarrow a \subset c$

12.16 $a \chi b \rightarrow a \cap b \subset a$

12.17 $a \subset u$

($a \chi b$ means: the product of a and b exists; the constant u represents the all-class.)

b. axioms for the union.

3.1 $a \subset a \cup b$

3.11 $a \cup b \subset b \cup a$

3.12 $a \subset c \,\&\, b \subset c \rightarrow a \cup b \subset c$

3.13 $a \chi c \,\&\, b \chi c \rightarrow (a \cup b) \cap c \subset (a \cap c) \cup (b \cap c)$

c. axioms for χ and $\neq$.

5.1 $a \chi a$

5.11 $a \chi b \rightarrow b \chi a$

5.12 $a \chi b \,\&\, a \subset c \rightarrow b \chi c$

6.1 $a \neq u \,\&\, b \subset a \rightarrow b \neq u$

d. axioms for the complement. 2)

4.12 $a \chi b \,\&\, a \chi c \,\&\, c \neq u \,\&\, a \cap b \subset \neg c \rightarrow a \cap c \subset \neg b$

4.13 $a \neq u \rightarrow (a \cup b) \cap (\neg a \cup b) \subset b$

As the system of Griss (system G) and the present system (NI) do not possess the same possibilities of expression, a proper translation between the two systems is impossible. But a partial translation is possible. To enable this translation we replace the axioms 2.12 and 2.13 by the following rules:

2.12′ From $P \rightarrow Q$ follows $P \,\&\, R \rightarrow Q \,\&\, R$.

2.13′ From $P \rightarrow Q$ and $Q \rightarrow R$ follows $P \rightarrow R$.

The theorems 2.221—2.28 are then changed in a corresponding way; the rest of the theorems remain unchanged. (Personally I prefer these rules, as double-implications between propositions are avoided by them. I have some doubt with respect to the fact that these double-implications are in accordance with Griss' meaning of implication. 3))

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1) Griss only mentions that in 12.11, 12.12 (?), 12.16 and 3.13 χ-conditions have to be added, without specializing them.

2) The χ- and $\neq$-conditions in 4.12 and 4.13 are not mentioned explicitly by Griss.

3) Cf. Griss, l.c., p. 41.

An essential formal difference between the two systems is that in G the letters a, b, c, . . . represent non-empty classes (cf. 5.1).[1] In NI the existence of a class has to be guaranteed by a special premiss. Further in G there is formally made a difference between propositions and classes and in NI not. Thirdly in G there exists only an inclusion-relation ($\rightarrow_{vs}$ in NI) that binds all free variables.

We now shall compare G with NI1—5 (and A10.0).

The rules of translation are:

system G	system NI
$p \rightarrow q$	$\dfrac{\exists \quad p}{q}$ (for $\exists$ see below)
$p \;\&\; q$	$p \wedge q$
$a \subset b$	$p \rightarrow q$ ($\rightarrow$ binds all free variables)
$a \cap b$	$p \wedge q$
$a \cup b$	$p \vee q$
$\neg\, a$	$\sim p$
$a \chi b$	$\exists pq$
$a \neq u$	$\exists \sim p$
p is provable (if p is not an implication)	$\dfrac{\exists}{p}$
$p \rightarrow q$ is provable	$\dfrac{\exists \quad p}{q}$

1. *Transformation of the axioms of G into NI* with the proofs of the transforms. The sign $\exists$ is introduced in NI, representing the premiss that every class exists. This is expressed by the supplementary axiom:

A17 $\quad \dfrac{\exists}{\exists p}$

G2.1 $\quad \dfrac{p}{p \wedge p}$ A2.0

(which means: the axiom 2.1 of G is transformed into $\dfrac{p}{p \wedge p}$ and this is proved in NI by A2.0; the premiss $\exists$ is often omitted.)

G2.11 $\quad \dfrac{p \wedge q}{q \wedge p}$ 2.0

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[1] Consequently a rule should be added to G forbidding substitution of $b \cap c$ for a, if $b \chi c$ has not been proved, and of $\neg b$ for a, if $b \neq u$ has not been proved.

G2.12′ If $\frac{p}{q}$, then $\frac{p \wedge r}{q \wedge r}$ (3.31).

G2.13′ If $\frac{p}{q}$ and $\frac{q}{r}$, then $\frac{p}{r}$ (definition of derivation).

G2.16 $\frac{p \wedge q}{p}$ A2.1

G1.11 If $\frac{\exists}{p}$ and $\frac{\exists}{q}$, then $\frac{\exists}{p \wedge q}$ (A2.0).

G1.12 If $\frac{\exists}{p}$ and $\frac{\exists \;\; p}{q}$, then $\frac{\exists}{q}$ (definition of derivation).

G1.13 If $\frac{\exists}{r}$ and $\frac{\exists \;\; p}{q}$, then $\frac{\exists \;\; p}{q \wedge r}$ (A2.0).

G12.1 $\frac{\exists}{p \rightarrow p \wedge p}$ A2.0, A3.4

G12.11 $\frac{\exists pq}{p \wedge q \rightarrow q \wedge p}$ 2.0, A3.4

G12.12 $\frac{p \rightarrow q \;\; \exists pr \;\; [\exists qr]}{p \wedge r \rightarrow q \wedge r}$ 3.12

G12.13 $\frac{p \rightarrow q \,.\, \wedge \,.\, q \rightarrow r}{p \rightarrow r}$ 3.20

G12.16 $\frac{\exists pq}{p \wedge q \rightarrow p}$ 3.01

G3.1 $\frac{\exists}{p \rightarrow p \vee q}$ A4.0, A3.4

G3.11 $\frac{\exists}{p \vee q \rightarrow q \vee p}$ A4.1, 4.01, A3.4

G3.12 $\frac{p \rightarrow r \,.\, \wedge \,.\, q \rightarrow r}{p \vee q \rightarrow r}$ A4.3

G3.13 $\frac{\exists pr \quad \exists qr}{p \vee q \,.\, \wedge r \,.\, \rightarrow : p \wedge r \,.\, \vee \,.\, q \wedge r}$ A4.4, A3.4(and 3.42, A4.0, 3.300 for deriving $\exists : p \vee q \,.\, \wedge r$)

G5.1 $\frac{\exists}{\exists pp}$ A17, 3.30

G5.11 $\frac{\exists pq}{\exists qp}$ 2.0, 3.30

G5.12 $\frac{\exists pq \quad p \rightarrow r}{\exists qr}$ A3.0, 3.300

G6.1 $$\frac{\exists \sim p \quad q \to p}{\exists \sim q}\ 10.51,\ \text{A3.3}$$

G4.12 $$\frac{p \wedge q \to \sim r \quad \exists pr \quad [\exists pq] \quad [\exists \sim r]}{p \wedge r \to \sim q}$$

Proof.

$$\dfrac{\dfrac{p^* \quad \dfrac{r^* \quad \dfrac{\dfrac{\dfrac{p \wedge q \to \sim r}{\exists \sim r}\ \text{A3.3}}{\exists \sim\sim r}\ 10.42 \qquad p \wedge q \to \sim r}{\sim\sim r \to \sim . p \wedge q}\ 10.51}{\sim . p \wedge q}\ 10.43,\ \text{A3.0}}{\sim q}\ 10.8 \qquad \exists pr}{p \wedge r \to \sim q}\ \text{A3.4}$$

G4.13 $$\frac{\exists \sim p}{p \vee q . \wedge . \sim p \vee q : \to q}$$

I did not succeed in proving this theorem in NI. By means of 4.100 are provable

$$\frac{\exists \sim p \quad \exists pq}{p \vee q . \wedge . \sim p \vee q : \to q} \text{ and } \frac{\exists p \quad \exists \sim pq}{p \vee q . \wedge . \sim p \vee q : \to q}.$$

This does not imply that the transforms of the consequences of G4.13 are not provable in NI. On the contrary, by far most of them are.

For the investigation of G12.17 we have to enlarge the means of our translation. If a proposition in G contains u, we first reduce it by replacing $a \cap u$ and $u \cap a$ by a, and $a \cup u$ and $u \cup a$ by u. The reduced of a provable proposition is again provable in G. Then we translate

$a \subset u$ (if a is not u) by $\exists p$,
$u \subset a$ (if a is not u) by $(vs)p$ (vs binds all free variables of p),
$u \subset u$ by $\exists$,
$a \chi u$ and $u \chi a$ (if a is not u) by $\exists p$,
$u \chi u$ by $\exists$.

Further we declare that it is forbidden to replace in a proposition a variable by u, if (after being reduced) the proposition will then contain as a part $\neg u$ or $u \neq u$ (*rule about restricted substitution of* u).

Then G12.17 is transformed into $\frac{\exists}{\exists p}$ A17. Further it is easily seen that the transforms of the axioms of G, after u being substituted for any of the variables, are all provable in NI. (A5.0 and A5.1 are used in the proof.)

From this transformation it is seen: *if a formula is provable in G* (with 2.12′ and 2.13′ instead of 2.12 and 2.13 and with the above rule about restricted substitution of u added) *its transform is provable in* NI + A17 + *an axiom that is the transform of* G4.13.

But the system NI + A17 is essentially larger than NI, as from $\exists$ can be derived the existence of, e.g., $p \wedge q$ and $\sim p$. This corresponds with the derivation in G of $a \cap b \; \chi \; a \cap b$ and $\neg\, a \; \chi \; \neg\, a$ from G5.1.

If we want to restore the original system NI, we have to replace A17 by

A17′ If p is an elementary class, i.e., a class that has not been formed by conjunction, disjunction or negation, then $\dfrac{\exists}{\exists p}$ [1].

But then the formal accordance with G is broken. To regain this accordance we add to the system G a rule of substitution.

In the following rule $P(a)$ stands for a proposition containing a. If every a in $P(a)$ is replaced by b, we write the result $P(b)$.

Rule of substitution (*rule GS*). If b and a are elementary, then:

if $P(a)$, then $P(b)$,
if $P(a)$, then $P(b \cup c)$,
if $P(a)$ is not an implication and $P(a)$, then $b \; \chi \; c \rightarrow P(b \cap c)$,
if $P(a)$ is not an implication and $P(a)$, then $b \neq u \rightarrow P(\neg\, b)$,
if $P(a) \rightarrow Q(a)$, then $b \; \chi \; c \;\&\; P(b \cap c) \rightarrow Q(b \cap c)$,
if $P(a) \rightarrow Q(a)$, then $b \neq u \;\&\; P(\neg\, b) \rightarrow Q(\neg\, b)$.
In the last two lines P or Q may not contain a.

Now again *the system G* (*changed as mentioned above*) + *rule GS can be transformed into a part of the system NI* + *A*17′ + *the transform of G*4.13.

2. *Transformation of NI into G.*

We first transform NI into a new system NI′ that has the same properties as the system G.

a. We discern between wff containing a free variable and wff

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[1] As in any special case $\exists$ can be replaced by a conjunction of the form $\exists p \wedge \exists q \wedge \ldots$, the adjunction of $\exists$ does not mean an enlargement of the formalism. In other words: $\exists$ may be interpreted as being a sign of the metasystem.

not containing a free variable. The former are represented by p, q, r, . . ., the latter by P, Q, R, ...

b. $\rightarrow_{vs}$ is replaced by $\rightarrow$, $\exists_{vs}$ by $\exists$.

c. In every premiss the parts p, q, . . . are conjuncted and also the parts P, Q, ...

d. It is forbidden to conjunct p and P. (If the conclusion of a theorem has the form $p \wedge P$, the theorem should be split into two theorems having p and P as conclusion.)

e. An axiom or theorem of the form $\frac{P\ p}{r}$ is replaced by $\frac{P\ \ \exists p}{p \rightarrow r}$.

A3.4 is cancelled (as being of no use after this transformation).

f. A3.2 and A5.0 are cancelled.

g. We add the following axioms:

3.20, 3.22, 3.42,

A17.0 $\frac{\exists p}{\exists pp}$ (provable in NI by A2.0, 3.30)

A17.1 If $\frac{P\ \ \exists p}{p \rightarrow q}$ and $\frac{R\ \ \exists r}{r \rightarrow s}$, then $\frac{P \wedge R\ \ \ \exists pr}{\exists qs}$.

P and (or) R may be dropped. (A17.1 is provable in NI by 3.42, 3.22, A3.3.)

Now every provable theorem of NI is transformed into a provable theorem of NI′. We will show by two examples in what way the derivations are transformed. Suppose that we have in NI the following derivation:

$$\cfrac{\cfrac{P\ p}{q}\,(a) \qquad \cfrac{R\ r}{s}\,(b)}{t}\,(c)$$

This derivation is transformed into:

$$\cfrac{\cfrac{\cfrac{P \wedge R}{P}\ \ \cfrac{\exists pr}{\exists p}\,3.42}{p \rightarrow q}\,(a) \quad \cfrac{\cfrac{P \wedge R}{R}\ \ \cfrac{\exists pr}{\exists r}\,3.42}{r \rightarrow s}\,(b) \quad \cfrac{\cfrac{P \wedge R\ \ \ \exists pr}{\exists qs}\,\text{A17.1}}{q \wedge s \rightarrow t}\,(c) \quad \exists pr}{p \wedge r \rightarrow t}\ 3.22,\ 3.20$$

The derivation in NI

$$\cfrac{\cfrac{P}{Q}\,(a) \qquad \cfrac{R\ r}{s}\,(b)}{t}\,(c)$$

is transformed into

$$(1)\ \dfrac{R\quad \exists r}{r \to s}\,(b) \qquad\qquad (2)\ \dfrac{R\quad \exists r}{r \to s}\,(b)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \land R}{R \land R}\quad \dfrac{\exists r}{\exists rr}\,\text{A17.0}}{\exists ss}\,(1),\,(2),\,\text{A17.1}}{\exists s}\,3.42 \qquad \dfrac{\dfrac{P \land R}{P}}{Q}\,(a)}{s \to t}\,(c) \qquad \dfrac{\dfrac{P \land R}{R}\qquad \exists r}{r \to s}\,(b)}{r \to t}\,3.20$$

We shall now prove that the axioms of NI′ are transformed into theorems of G by the rules of translation mentioned above. We shall translate $p \land q$ by $a \cap b$ and $P \land Q$ by $p \mathbin{\&} q$. If p is not a conjunction or a negation, $\exists p$ will be translated by $a\,\chi\,a$.

In G we can prove:

GR From Q follows $P \to Q$ (G2.22, G2.21, G1.13, G2.13).

We mention first the number of the axiom, then the transform into NI′ (if this transform is different from the original axiom) and finally the numbers of the theorems of G by which the translation into G is proved.

A2.0 $\dfrac{\exists pq}{p \land q \to p \land q}$ (G12.21), $\dfrac{\exists pq}{p \land q \to q \land p}$ (G12.11),

$\dfrac{P \land Q}{P \land Q}$ (G2.21), $\dfrac{P \land Q}{Q \land P}$ (G2.11)

A2.1 $\dfrac{\exists pq}{p \land q \to p}$ (G12.16), $\dfrac{P \land Q}{P}$ (G2.16)

A3.0 $\dfrac{\exists p \quad p \to q}{p \to q}$ (G2.21)

A3.3 If p is neither a conjunction nor a negation, the translation of the first part of A3.3 is $a \subset b \to a\,\chi\,a$ (G5.1, GR). If p is a conjunction, the translation is $a_1 \cap a_2 \subset b \to a_1\,\chi\,a_2$ (A). This is not provable in G.
If p is a negation, the translation is $\neg\, a \subset b \to a \neq u$ (B). This again is not provable.
The three translations of the second part are:
$a \subset b \to b\,\chi\,b$ (G5.1, GR),
$a \subset b_1 \cap b_2 \to b_1\,\chi\,b_2$ (C) (not provable),
$a \subset \neg\, b \to b \neq u$ (D) (not provable).

3.20 (G12.13)

3.22 (G12.23)

3.42 If p is neither a conjunction nor a negation the translation into G is $a \chi b \to a \chi a$ (G5.1, GR).
If p is a conjunction the translation is $a_1 \cap a_2 \chi b \to a_1 \chi a_2$ (E), and this is not provable in G.
If p is a negation the translation is $\neg a \chi b \to a \neq u$ (F), and this is not provable in G.

A17.0 The three translations are, respectively,
$a \chi a \to a \chi a$ (G2.21),
$a_1 \chi a_2 \to a_1 \cap a_2 \chi a_1 \cap a_2$ (G5.1, GR),
$a \neq u \to \neg a \chi \neg a$ (G5.1, GR).

A17.1 If both p and r are neither a conjunction nor a negation, the translation is provable by G5.1, G1.13, G12.23, G5.12. In the other cases the translation is not provable. We give two examples.
1. p is a conjunction, r is neither a conjunction nor a negation. The translation is:
if $p \,\&\, a_1 \chi a_2 \to a_1 \cap a_2 \subset b$ and $r \,\&\, c \chi c \to c \subset d$, then $p \,\&\, r \,\&\, a_1 \cap a_2 \chi c \to b \chi d$.
The reason this cannot be proved is that the formula $a_1 \cap a_2 \chi c \to a_1 \chi a_2$ is missing in G. This is formula (E).

2. p is a negation, r is neither a conjunction nor a negation. The translation is:
if $p \,\&\, a \neq u \to \neg a \subset b$ and $r \,\&\, c \chi c \to c \subset d$, then $p \,\&\, r \,\&\, \neg a \chi c \to b \chi d$.
The reason this cannot be proved is that the formula $\neg a \chi c \to a \neq u$ cannot be proved. This is formula (F).

A4.0 $\dfrac{\exists p \qquad \exists q}{p \to p \vee q}$ (G3.1, GR)

A4.1 $\dfrac{\exists \,.\, p \vee q}{p \vee q \to q \vee p}$ (G3.11, GR)

A4.2 (G5.1, GR)

A4.3 (G3.3, G3.22)

A4.4 $\dfrac{\exists : p \vee q \,.\, \wedge r \qquad \exists pr \qquad \exists qr}{p \vee q \,.\, \wedge r \,.\, \to : p \wedge r \,.\, \vee \,.\, q \wedge r}$ (G3.13, G2.28)

A5.1 If $\dfrac{P \quad \exists p}{p \to q}$, then $\dfrac{P \quad (vs)p}{(vs)q}$ (G12.13).

Remember that $(vs)p$ is translated by $u \subset a$.

A10.0 $\dfrac{\exists : p \vee q \,.\, \wedge \sim q}{p \vee q \,.\, \wedge \sim q \,.\, \rightarrow p}$

The translation is very similar to G4.42. But as G4.42 has been derived from G4.13 and G12.221, it turns out to be $a \cup b \;\chi \neg\, a \;\&\; a \neq u \rightarrow (a \cup b) \cap \neg\, a \subset b$.
This would be the translation, if the premiss $\exists \sim q$ was added in A10.0. As this is not the case, the translation would still be provable, if $a \cup b \;\chi \neg\, a \rightarrow a \neq u$ would be provable. But this is again the unprovable formula (F).

As D10 cannot be translated into the language of G, adjunction of (E) and (F) does not make it sure that the further theorems about negation in NI would have provable transforms in G. The transforms of 10.40 (G4.3, G2.28), 10.41 (G5.1, GR) and 10.43 (G4.3) are provable. But for the proof of the transforms of 10.42, 10.50 and 10.51 there is another formula missing in G, viz., $a \neq u \rightarrow \neg\, a \neq u$ (G).

It is clear that (E) and (F) are not provable in G, for if they were, $a_1 \;\chi\; a_2$ and $a \neq u$ would be provable formulas in G (these formulas are derived by substituting $a_1 \cap a_2$ and $\neg\, a$ for b in (E) and (F), respectively). One might oppose that the adjunction of the rule of substitution GS would render these substitutions illegal. But the addition of GS diminishes the possibilities of deriving formulas and so by the addition of this rule the derivation of a rule that was not provable before cannot be rendered possible.

There is still one difficulty left. The axioms A3.2 and A5.0 were cancelled. They are of the form $\dfrac{p}{P}$ and $\dfrac{P}{p}$, respectively. And so they cannot be transformed into NI′. Now in NI any theorem $\dfrac{p}{P}$ is equivalent to $\dfrac{\exists p}{P}$ (7.10, A3.2), and $\dfrac{P}{p}$ to $\dfrac{P}{(vs)p}$ (7.11, A5.0). And so we transform a theorem or axiom $\dfrac{p}{P}$ of NI into $\dfrac{\exists p}{P}$ of NI′, and $\dfrac{P}{p}$ of NI into $\dfrac{P}{(vs)p}$ of NI′. The transforms of A3.2 and A5.0 are then provable theorems of NI′, as their premiss and conclusion are identified by the transformation.

We now add in NI′ the axiom

A17.2 If $\dfrac{\exists pq}{p \wedge q \rightarrow r}$, then $\dfrac{(vs)p \;\; \exists q}{q \rightarrow r}$.

This is provable in NI by 5.22, A5.0, A3.4.

Then the derivations in NI are transformed into derivations in NI′. E.g.,

$$\dfrac{\dfrac{p}{P} \quad q}{Q} \text{ is transformed into } \dfrac{\dfrac{\exists p}{P}\,7.10 \quad q}{Q}$$

$$\dfrac{\dfrac{P}{p}}{q} \text{ is transformed into } \dfrac{\dfrac{P}{(vs)p}}{(vs)q}\,\text{A5.1}$$

$$\dfrac{\dfrac{P}{p} \quad q}{r} \text{ is transformed into } \dfrac{\dfrac{P}{(vs)p} \quad \exists q}{q \rightarrow r}\,\text{A17.2}$$

The transformation of A17.2 into G is:
if $a \chi b \rightarrow a \cap b \subset c$, then $u \subset a \rightarrow b \subset c$.
Proof.
$u \chi b$ (G5.1, G12.17, G5.12)
$u \subset a \rightarrow a \chi b$ (G12.17, $u \chi b$, G5.12)
Further G12.221, G12.61, G12.13.

The relation between NI and NI′.
We shall call the theorems T_1 and T_2 *equivalent*, if
a. if T_1, then T_2 is provable,
b. if T_2, then T_1 is provable.
Then the relation between NI and NI′ is:
a. every theorem of NI is in NI equivalent with a theorem, that is provable in NI′,
b. every theorem of NI′ is provable in NI.

Conclusion.
We have compared three systems:
1. the system NI, as far as variables of propositional functions are not mentioned explicitly in the descriptions of the formulas,
2. the system NI′,
3. the system G.
And we have found:
1. to every theorem of NI there exists an equivalent theorem of NI′,
2. every theorem of NI′ can be transformed into a theorem of a system G′, consisting of G, the rule GS, the rule about restricted substitution of u and the formulas (A)—(G) [1]),

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[1]) Besides G2.12 and G2.13 have been replaced by G2.12′ and G2.13′ and it has been forbidden to replace p by a proposition containing $\rightarrow$.

and so

3. every theorem of NI can be transformed into a theorem of G′.

Besides:

4. every theorem of G′ can be transformed into a theorem of NI + A17′, excepted G4.13; the transform belongs to NI′.

So NI′ + A17′ and G′ are isomorphic, if to NI′ an axiom would be added that is the transform of G4.13.¹)

It seems doubtful to me that this axiom is necessary for a logical description of the method of negationless intuitionistic mathematics.

18. *A possible revision of NI.*

It is possible to strengthen the system NI by the addition of:

A4.5 If $\dfrac{p\quad s}{r}$ and $\dfrac{q\quad s}{r}$, then $\dfrac{p \vee q\quad s}{r}$.

This axiom is very similar to the theorem 4.100. But in 4.100 in the premiss of the last derivation $\exists ps$ and $\exists qs$ had to be added. Omission of these premisses means practically the supposition that, if it is known that the sumclass $p \vee q$ and the class s have an element in common, it is known that this element belongs to p and to s (and then r is derived by means of $\dfrac{p\ s}{r}$) or it is known that this element belongs to q and to s (and then r is derived by $\dfrac{q\ s}{r}$).

This supposition can be based on the supposition that, if it is known that an element belongs to the sumclass $p \vee q$, it is known that it belongs to p or it is known that it belongs to q. This supposition is accepted in ordinary intuitionistic mathematics. ²)

If A4.5 is accepted, the former axioms A4.3 and A4.4 can be proved. But further it would also be possible to prove the transform of G4.13.

G4.13 $$\frac{\exists q \quad \exists \sim p}{p \vee q \,.\wedge.\, \sim p \vee q : \rightarrow q}$$

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¹) With the restriction that there is some uncertainty about the theorems derived in NI by means of D10.

²) The interpretation of $\vee$ in NI, however, is different from the intuitionistic interpretation. Cf. the remark after 13.53. Therefore I have preferred to reject A4.5 in the system NI.

Proof.

$$(1)\quad \dfrac{\sim p \quad p \vee q}{q}\,\text{A10.0} \qquad (2)\quad \dfrac{q \quad [p \vee q]}{q}\,2.2$$

$$(3)\quad \dfrac{\dfrac{\dfrac{\dfrac{\exists \sim p}{\exists p}\,10.41 \qquad \exists q}{q \rightarrow p \vee q}\,4.30 \qquad \dfrac{\exists \sim p \qquad \exists q}{q \rightarrow \sim p \vee q}\,4.30}{q \rightarrow : p \vee q \,.\, \wedge \,.\, \sim p \vee q}\,3.21}{\exists : p \vee q \,.\, \wedge \,.\, \sim p \vee q}\,\text{A3.3}$$

Further (1), (2), A4.5, (3), A3.4.

(The premiss $\exists q$ corresponds with an extra premiss $b\,\chi\,b$ in G4.13, which changes G4.13 into an equivalent axiom.)

But, as far as I see, the transform of A4.5 into G is not provable in G and not in G'. So NI + A4.5 seems to be stronger than G'.

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