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Note on Fourier Series

by

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Suppose $f(t)$ is integrable L in $(-\pi, \pi)$ and periodic outside, and suppose that its Fourier series is

$$\tfrac{1}{2}a_0+\sum_{n=0}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=0}^{\infty}A_n(t)\,. \tag{1}$$

Then the allied series is

$$\sum_{n=1}^{\infty}(b_n\cos nt-a_n\sin nt)=\sum_{n=1}^{\infty}B_n(t). \tag{2}$$

Let us write

$$\begin{aligned}\varphi(t)&=\tfrac{1}{2}\{f(x+t)+f(x-t)\}\\ \psi(t)&=\tfrac{1}{2}\{f(x+t)-f(x-t)\}\end{aligned} \tag{3}$$

and

$$\begin{aligned}s_n&=\sum_{m=0}^{n}A_m(x)=\sum_{m=0}^{n}A_m\\ \bar{s}_n&=\sum_{m=1}^{n}B_m(x)=\sum_{m=1}^{n}B_m.\end{aligned} \tag{4}$$

The following theorem was recently given by Hardy [1].

Theorem A. If

$$|\varphi(t)|=o\left(\log\frac{1}{t}\right)\quad (C,1)\ [2] \tag{5}$$

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[1] Hardy 5, 108.

[2] We suppose that $t>0$, and say that $\chi(t)=o\{L(1/t)\}\,(C,\alpha)$, $\alpha>0$, as $t\to 0$ if $\frac{1}{t^{\alpha}}\int_0^t(t-u)^{\alpha-1}\chi(u)du=o\{L(1/t)\}$ as $t\to 0$. We also say that $s_n=o\{L(n)\}\ (C,\alpha)$, $\alpha>-1$, as $n\to\infty$ if

$$s_n^{\alpha}=\frac{1}{A_n^{\alpha}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}s_{\nu}=o\{L(n)\}$$

as $n\to\infty$, where $A_n^{\alpha}=\dfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)}$; s_n^{α} is the Cesàro mean of order α of s_n.

as $t \to 0$, *then a necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o(\log n) \tag{6}$$

as $n \to \infty$ *is that*

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\left(\log \frac{1}{t}\right) \tag{7}$$

as $t \to 0$.

The problem arises of relaxing conditions (5) and (7). We do this in theorem 1, and at the same time obtain a sharper conclusion than (6).

Theorem 1. If

$$|\varphi(t)| = O\left(\log \frac{1}{t}\right) \qquad (C, 1) \tag{8}$$

as $t \to 0$, *then a necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o(\log n) \qquad (C, -1+\delta) \tag{9}$$

as $n \to \infty$, *for any* $\delta > 0$, *is that*

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\left(\log \frac{1}{t}\right) \qquad (C, k) \tag{10}$$

as $t \to 0$, *for some* k.

This theorem can be further generalised by replacing the functions $\log \frac{1}{t}$ and $\log n$ by $L\left(\frac{1}{t}\right)$ and $L(n)$ respectively, where $L(x)$ is a logarithmico-exponential function such that $1 \prec L(x) \preceq x$ as $x \to \infty$ [2a]). We obtain then

Theorem 2. If

$$|\varphi(t)| = O\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C, 1) \tag{11}$$

as $t \to 0$, *then a necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o\{L(n)\} \qquad (C, -1+\delta) \tag{12}$$

as $n \to \infty$, *for any* $\delta > 0$, *is that*

$$\int_t^\pi \frac{\varphi(u)}{u} du = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C, k) \tag{13}$$

as $t \to 0$, *for some* k.

[2a]) See HARDY 3. We shall suppose throughout the paper that $L(x)$ satisfies these conditions unless the contrary is explicitly stated.

The theorem becomes trivial when $L(x) = x$, since $A_n = o(1)$ as $n \to \infty$. When $L(x) = 1$ it remains true if restated as follows.

Theorem 3. If

$$|\varphi(t)| = O(1) \qquad (C, 1) \tag{14}$$

as $t \to 0$, *then a necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} \tag{15}$$

should be summable $(C, -1+\delta)$, *for any* $\delta > 0$, *is that*

$$\int_0^\pi \frac{\varphi(u)}{u} du \tag{16}$$

should exist as a Cesàro integral of some order.

We shall only give the proof of theorem 2. Theorem 1 is included in theorem 2, and the proof of theorem 3 can readily be constructed from that of theorem 2. We employ the following lemmas.

Lemma 1 [3]). *If* $x^{\beta-\delta} \leqq L(x) \leqq x^{\beta+\delta}$ *as* $x \to \infty$, *for every* $\delta > 0$, *and if* $\alpha + \beta > 1$, *then, as* $t \to 0$,

$$\int_t^\eta u^{-\alpha} L\left(\frac{1}{u}\right) du \sim \frac{t^{1-\alpha}}{\alpha+\beta-1} L\left(\frac{1}{t}\right). \tag{17}$$

Lemma 2. If (11) *holds, then* $s_n = O\{L(n)\}$ (C, δ), *for every* $\delta > 0$.

We may suppose without loss of generality that $0 < \delta < 1$. We have to show that

$$I(n) = \int_0^\eta \varphi(t) \varkappa_n^\delta(t) dt = O\{L(n)\},$$

as $n \to \infty$, where $\varkappa_n^\delta(t)$ is the n-th Fejér kernel of order δ, and $0 < \eta \leqq \pi$. M. Riesz [4]) has shown that

$$|\varkappa_n^\delta(t)| \begin{cases} \leqq An \\ \leqq An^{-\delta} t^{-1-\delta} \end{cases}$$

for $n > 0$, $0 < t < \pi$, $0 < \delta < 1$. Write

$$I(n) = \int_0^{1/n} + \int_{1/n}^\eta = I_1 + I_2 .$$

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[3]) Hardy 3, 37.

[4]) Riesz 10.

Then

$$|I_1| \leqq An\int_0^{1/n}|\varphi(u)|du = O\{L(n)\}$$

by hypothesis, and, if $\Phi(t)=\int_0^t|\varphi(u)|du$,

$$\begin{aligned}|I_2| &\leqq An^{-\delta}\int_{1/n}^{\eta}|\varphi(u)|\,u^{-1-\delta}\,du\\ &\leqq An^{-\delta}\left|\Phi\left(\frac{1}{n}\right)\right|n^{1+\delta}+An^{-\delta}\int_{1/n}^{\eta}\Phi(u)u^{-2-\delta}\,du\\ &= O\{L(n)\}+n^{-\delta}\int_{1/n}^{\eta}O\left\{L\left(\frac{1}{u}\right)\right\}u^{-1-\delta}\,du\\ &= O\{L(n)\},\end{aligned}$$

by lemma 1 [5]).

Lemma 3. Necessary and sufficient conditions that (12) *should hold, for a given* $\delta=\delta_0>0$, *are that it should hold for some* $\delta>0$ *and that* $s_n=o\{L(n)\}$ (C,δ_0).

Let $d_n=\sum_{\nu=1}^{n}\frac{s_\nu}{\nu}$, and let d_n^α be the n-th Cesàro mean of order α or d_n. Then it is easily verified [6]) that, for $\alpha>0$, $n>0$,

$$\alpha(d_n^{\alpha-1}-d_n^\alpha)=s_n^\alpha-s_0. \tag{18}$$

Also if $d_n^\alpha=o\{L(n)\}$ then $d_n^\beta=o\{L(n)\}$ for $\beta>\alpha>-1$. From (18) it then follows by induction that necessary and sufficient conditions that $d_n^{\delta-1}=o\{L(n)\}$ for $\delta=\delta_0$ are that this should hold for some δ and that $s_n^{\delta_0}=o\{L(n)\}$.

Lemma 4. If $s_n=O\{L(n)\}(C,\delta)$, *for a given* $\delta>0$, *and* $s_n=o\{L(n)\}$ (C,k), *for some* k, *then* $s_n=o\{L(n)\}$ (C,δ'), *for every* $\delta'>\delta$.

This is a particular case of a theorem of Dixon and Ferrar [7]).

Lemma 5. A necessary and sufficient condition that

$$s_n=o\{L(n)\}\quad(C) \tag{19}$$

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[5]) Here A denotes some constant, not necessarily the same at each occurrence.

[6]) Cf. Kogbetliantz 8, 30.

[7]) Dixon and Ferrar 2, theorem II. See also Riesz 11.

as $n \to \infty$ *is that*

$$\varphi(t) = o\left\{L\left(\frac{1}{t}\right)\right\} \quad (C) \tag{20}$$

as $t \to 0$.

The corresponding result with $L(x) = 1$ is due to Hardy and Littlewood [8]). The proof of lemma 5 is on the same lines. The properties of $L(x)$ required have been given by Hardy [9]).

Lemma 6 [10]). *A necessary and sufficient condition that*

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} = o\{L(n)\} \quad (C) \tag{21}$$

as $n \to \infty$ *is that*

$$\int_t^\pi \frac{\varphi(u)}{u}\,du = o\left\{L\left(\frac{1}{t}\right)\right\} \quad (C) \tag{22}$$

as $t \to 0$.

Let

$$\chi(t) = \int_t^\pi \frac{\varphi(u)}{u}du, \qquad \chi^*(t) = \int_t^\pi \varphi(u)\,\tfrac{1}{2}\cot\tfrac{1}{2}u\,du\,.$$

Then, since $\frac{1}{2}\cot\frac{1}{2}u - \frac{1}{u}$ is bounded in $(0, \pi)$, it is easy to see that $\chi(t) - \chi^*(t)$ tends to a limit as $t \to 0$. Also [11])

$$\chi^*(t) = \tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos nt\,,$$

where, for $n > 0$,

$$c_n = \frac{2}{\pi}\int_0^\pi \chi^*(t)\cos nt\,dt = \frac{s_n}{n} - \frac{1}{2}\frac{A_n}{n}\,.$$

Hence

$$\sum_{\nu=1}^{n} \frac{s_\nu}{\nu} - \sum_{\nu=1}^{n} c_\nu = \tfrac{1}{2}\sum_{\nu=1}^{n} \frac{A_\nu}{\nu}\,,$$

and the lemma will follow by applying lemma 5 to $\chi^*(t)$, if we show that (21) and (22) each imply

$$\sum_{\nu=1}^{n} \frac{A_\nu}{\nu} = o\{L(n)\} \quad (C)$$

as $n \to \infty$.

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[8]) Hardy and Littlewood 6, 70. See also Bosanquet 1.

[9]) Hardy 3, 37.

[10]) The case corresponding to $L(x) = 1$, in the modified form of theorem 3, was conjectured by Hardy and Littlewood 7, 242.

[11]) Hardy 4.

Now, by (18), (21) implies (19), and the first result follows easily by partial summation. Again, (22) implies (20), for, writing $\Phi(t)=\int_0^t \varphi(u)\,du$, we have

$$\chi(t)=\int_t^\pi \frac{\varphi(u)}{u}du = C-\frac{\Phi(t)}{t}+\int_t^\pi \frac{\Phi(u)}{u^2}du = o\left(\frac{1}{t}\right)$$

as $t\to 0$. Hence

$$\Phi(t)=\int_0^t u\frac{\varphi(u)}{u}du=[-u\chi(u)]_0^t+\int_0^t \chi(u)\,du=-t\chi(t)+\int_0^t \chi(u)\,du\,.$$

Hence (22) implies

$$\frac{\Phi(t)}{t}=o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C)\,,$$

which is equivalent to (20). Lemma 5 now gives the second result.

Theorem 2 is an immediate consequence of lemmas 2, 3, 4 and 6.

ALLIED SERIES.

The following analogue of theorem 2 is also true.

Theorem 4. If

$$|\psi(t)|=O\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C,\ 1) \tag{23}$$

as $t\to 0$, *then a necessary and sufficient condition that*

$$\sum_{\nu=1}^{n}\frac{\bar{s}_\nu}{\nu}=o\{L(n)\} \qquad (C,\ -1+\delta) \tag{24}$$

as $n\to\infty$, *for any* $\delta>0$, *is that*

$$\int_t^\pi \cot\tfrac{1}{2}u\,du\int_u^\pi \cot\tfrac{1}{2}v\,\psi(v)\,dv = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C)$$

as $t\to 0$.

We require the following additional lemmas, the proofs of which are analogous to those already given.

Lemma 7. If (23) *holds, then* $nB_n=O\{L(n)\}$ $(C,\ 1+\delta)$, *for every* $\delta>0$.

Lemma 8. Necessary and sufficient conditions that

$$\bar{s}_n=O\{L(n)\} \qquad (C,\ \delta),$$

for a given δ, *are that this be true for some* δ *and*

$$nB_n=O\{L(n)\} \qquad (C,\ 1+\delta).$$

Both these lemmas depend on the identity

$$\tau_n^\alpha = \alpha(\bar{s}_n^{\alpha-1} - \bar{s}_n^\alpha), \tag{26}$$

where τ_n^α is the n-th Cesàro mean of order α of nB_n.

The Fejér kernel for the Allied series is $\bar{\varkappa}_n^\delta(t)$, where $|\bar{\varkappa}_n^\delta(t)| \leqq An$ and $|\bar{\varkappa}_n^\delta - \frac{1}{2}\cot\frac{1}{2}t| \leqq A\,n^{-\delta}\,t^{-1-\delta}$, for $n > 0$, $0 < t < \pi$.

Lemma 9. A necessary and sufficient condition that

$$\bar{s}_n = o\{L(n)\} \qquad (C) \tag{27}$$

as $n \to \infty$ *is that*

$$\int_t^\pi \cot\tfrac{1}{2}u\,\psi(u)\,du = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C) \tag{28}$$

as $t \to 0$.

The lemma remains true when $L(x)=1$, this case being due to Hardy and Littlewood [12]).

Lemma 10. A necessary and sufficient condition that

$$\sum_{\nu=1}^{n} \frac{\bar{s}_\nu}{\nu} = o\{L(n)\} \qquad (C) \tag{29}$$

as $n \to \infty$ *is that*

$$\int_t^\pi \cot\tfrac{1}{2}u\,du \int_u^\pi \cot\tfrac{1}{2}v\,\psi(v)\,dv = o\left\{L\left(\frac{1}{t}\right)\right\} \qquad (C) \tag{30}$$

as $t \to 0$.

Theorem 4 follows from lemmas 7, 8 and 10, and lemmas 3 and 4 with $\bar{s}_n$ in place of s_n.

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[12]) Hardy and Littlewood, 7. See also Paley, 9.

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