Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

CONGRUENCES OF MODULAR FORMS AND THE IWASAWA λ-INVARIANTS

Yuichi Hirano

Tome 146
Fascicule 1

2018

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du Centre national de la recherche scientifique
pages 1-79

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel de la Société Mathématique de France.

Fascicule 1, tome 146, mars 2018

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`http://www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

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Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél. : (33) 01 44 27 67 99
`bulletin@smf.emath.fr` • `http://smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Stéphane SEURET

Bull. Soc. Math. France
146 (1), 2018, p. 1-79

CONGRUENCES OF MODULAR FORMS AND THE IWASAWA λ-INVARIANTS

BY YUICHI HIRANO

ABSTRACT. — In this paper, we show how congruences between cusp forms and Eisenstein series of weight $k \geq 2$ give rise to corresponding congruences between the algebraic parts of the critical values of the associated L-functions. This is a generalization of results of Mazur, Stevens, and Vatsal in the case where $k = 2$. As an application, by proving congruences between the p-adic L-function of a certain cusp form and the product of two Kubota-Leopoldt p-adic L-functions, we prove the Iwasawa main conjecture (up to p-power) for cusp forms at ordinary primes p when the associated residual Galois representations are reducible. This is a generalization of Greenberg and Vatsal in the case where $k = 2$.

RÉSUMÉ (*Congruences de formes modulaires et λ-invariants d'Iwasawa*). — Dans cet article, nous montrons comment les congruences entre formes paraboliques et séries d'Eisenstein de poids $k \geq 2$ donnent lieu à des congruences entre les parties algébriques des valeurs critiques des fonctions L associées. C'est une généralisation des travaux de Mazur, Stevens et Vatsal dans le cas où $k = 2$. Comme application, en prouvant des congruences entre la fonction p-adique L d'une certaine forme parabolique et le produit de deux fonctions de Kubota-Leopoldt p-adiques L, nous prouvons la conjecture principale d'Iwasawa (à puissance p près) pour les formes paraboliques à nombres premiers ordinaires p lorsque les représentations de Galois résiduelles associées sont réductibles. C'est une généralisation des travaux de Greenberg et Vatsal dans le cas où $k = 2$.

Texte reçu le 14 janvier 2015, modifié le 4 juillet 2016, accepté le 2 juin 2016.

YUICHI HIRANO, Graduate School of Mathematical Sciences, The University of Tokyo, 8-1 Komaba 3-chome, Meguro-ku, Tokyo, 153-8914, Japan • *E-mail :* yhirano@ms.u-tokyo.ac.jp

Mathematical subject classification (2010). — 11F33, 11F67, 11F75, 11R23, 14F30.

Key words and phrases. — Congruences of modular forms, special values of L-functions, Iwasawa theory, Eisenstein series, Mellin transform, p-adic Hodge theory.

BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE 0037-9484/2018/1/$ 5.00

doi:10.24033/bsmf.2752

0. Introduction

0.1. Introduction. — The purpose of this paper is to show how congruences between the Fourier coefficients of Hecke eigenforms give rise to corresponding congruences between the special values of the associated L-functions. The study of this topic was initiated by Mazur [25] using the arithmetic of the modular curve $X_0(l)$, where l is a prime number, in order to investigate a weak analog of the Birch and Swinnerton-Dyer conjecture. Mazur's congruence formula was generalized to other congruence subgroups by Stevens [33]. Furthermore, by the theory of higher weight modular symbols, Ash and Stevens [2] have examined congruences between special values of the L-functions of cusp forms of higher weight over $\mathrm{SL}_2(\mathbb{Z})$ and those of the L-functions of cusp forms of weight 2 over $\Gamma_0(l)$. Moreover, Vatsal [39] has proved congruences between special values of the L-functions of two cusp forms of higher weight over $\Gamma_0(N)$, where N is a more general positive integer. Also, he obtained congruences between special values of the L-functions of cusp forms of weight 2 and those of the L-functions of Eisenstein series of weight 2. Moreover, Greenberg and Vatsal [16] used Vatsal's congruences [39] to study the Iwasawa invariants of elliptic curves in towers of cyclotomic fields. In particular, they provided evidence for the Iwasawa main conjecture for elliptic curves. Their work was motivated by Kato's results on the Iwasawa main conjecture for modular forms [21].

In this paper, we present a way to obtain congruences of the special values of the L-functions from congruences between cusp forms and Eisenstein series of weight $k \geq 2$. This is a generalization of the works explained above by Mazur [25], Stevens [33], and Vatsal [39].

Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and $\varpi \in \mathcal{O}$ a uniformizer.

THEOREM 0.1 (= Theorem 2.10). — *Let p be an odd prime number, r a positive integer, and k an integer with $2 \leq k \leq p-1$. Let $f = \sum_{n=1}^{\infty} a(n,f)\ e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. Assume that the residual Galois representation $\bar{\rho}_f$ associated to f is reducible and of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix},$$

and either ξ_1 or ξ_2 is unramified at p. Assume also that there exists an Eisenstein series $G = E_k(\psi_1, \psi_2) \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ (for the definition, see Theorem 3.18) such that G satisfies the assumptions of Theorem 1.9 and $f \equiv G \pmod{\varpi^r}$ (for the definition, see before Theorem 2.10). Then there exist a parity $\alpha \in \{\pm 1\}$ (explicitly given by (A.27)*), a complex number $\Omega_f^\alpha \in \mathbb{C}^\times$, and a p-adic unit $u \in \mathcal{O}^\times$ such that, for every primitive Dirichlet character χ whose conductor m_χ is prime to N, the following congruence holds:*

(1) *if* $(m_\chi, p) = 1$, *then, for each* j *with* $0 \leq j \leq k-2$ *and* $\alpha = \chi(-1)(-1)^j$,

$$\tau(\bar{\chi})\frac{L(f,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}} \pmod{\varpi^r}.$$

(2) *if* $p|m_\chi$, *we assume that* $m_\chi \in \varpi^r\mathcal{O}$, χ *is non-exceptional (see Definition 2.11), and* $\alpha = \chi(-1)$. *Then*

$$\tau(\bar{\chi})\frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^\alpha} \equiv u\tau(\bar{\chi})\frac{L(G,\chi,1)}{2\pi\sqrt{-1}} \pmod{\varpi^r}.$$

The organization of this paper is as follows.

In §1, we generalize Stevens's results [33, 34]. We construct a desired 1-cocycle π_g associated to a modular form g of weight $k \geq 2$ (Definition 1.2) and prove that π_g is integral, that is, π_g takes values in $L_{k-2}(\mathcal{O})$ under some assumption (Theorem 1.9). In terms of Schoenberg's cocycle, Stevens gave a generalization of the Mazur's congruence formula [25] to general congruence subgroups [33]. Also, he expected that these methods would be generalized to higher weight modular forms and to Hilbert modular forms [33]. The construction of such cocycles π_g associated to modular forms g of weight k has been accomplished so far only in the case of weight $k = 2$ mainly because of certain combinatorial problem arising in the higher weight case $k > 2$. Indeed, a discrete subgroup Γ acts on $L_{k-2}(\mathcal{O})$ trivially only in the case $k = 2$.

In §2, we generalize Vatsal's results [39].

If a Hecke eigenform $f = \sum_{n=1}^\infty a(n,f)e(nz)$ of weight $k \geq 2$ and an Eisenstein series $G = \sum_{n=0}^\infty a(n,G)e(nz)$ of weight $k \geq 2$ are related by a congruence of the Fourier coefficients $a(n,f) \equiv a(n,G) (\bmod\ \varpi^r)$ for all $n \geq 0$, we derive congruences between the special values of the associated L-functions (Theorem 2.10). One of the key ingredients in Vatsal's proof [39] is to describe the special values of the L-functions attached to the modular form G as a linear combination of 1-cocycles π_G due to the work of Stevens [33], which allows us to prove congruences between the special values by using cohomological arguments.

In Appendix A, we give a relation between p-adic modular forms and p-adic parabolic cohomologies of Hecke modules in the case the residual Galois representations $\bar{\rho}_f (= \rho_f (\bmod\ \varpi))$ associated to a cusp forms f is reducible by using integral p-adic Hodge theory. Our problem on the special values of the L-functions is closely related to a multiplicity-one theorem, which is introduced by Mazur. In the case $\bar{\rho}_f$ is irreducible, $k < p$, and a level N is prime to p, a multiplicity-one theorem is known to be valid by p-adic Hodge theory for open varieties with non-constant coefficients [10]. In particular, Theorem A.12, which may be regarded as p-adic Eichler-Shimura isomorphism, is crucial to define the canonical periods Ω_f^α associated to f and prove congruences between $\pi_f^\alpha/\Omega_f^\alpha$ and π_G^α modulo ϖ^r.

In §3, we generalize Greenberg-Vatsal's results [16]. Using Vatsal's congruences, it is devoted to an application to the Iwasawa main conjecture for elliptic curves under certain assumptions. In the same manner, Theorem 0.1 is used to establish a congruence between a p-adic L-function attached to f and the product of two Kubota-Leopoldt p-adic L-functions (Theorem 3.19). Then, following the work of Kato [21], we will prove the following theorem, which has not been treated by Skinner and Urban [32]:

THEOREM 0.2. — *Let p be an odd prime number and k an integer such that $2 \leq k \leq p-1$. Let $f \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. We assume that the residual Galois representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}/\varpi)$ associated to f is reducible and of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}$$

and that

$$\text{(Assumption)} \qquad \begin{aligned} &\psi \text{ is unramified at } p \text{ and odd, and} \\ &\varphi \text{ is ramified at } p \text{ and even.} \end{aligned}$$

Then $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$. In particular, the Iwasawa main conjecture for such f is true up to ϖ-power.

The work of §1, §2, and §3 is based on the author's master thesis at the University of Tokyo in 2010. After I had finished writing this paper, I found a result obtained by Heumann and Vatsal [17], which is almost the same one as Theorem 0.1 (1) (in the case $(m_\chi, p) = 1$) in this paper. We also treat the case $p | m_\chi$ (Theorem 0.1 (2)) and apply Theorem 0.1 (2) to the Iwasawa main conjecture.

0.2. Notation. — In this paper, p and l always denote distinct prime numbers.

We denote by $\mathbb{N}$ the set of natural numbers (that is, positive integers), denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_p$) the ring of rational integers (resp. p-adic integers), and also denote by $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) the rational number field (resp. the p-adic number field). We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and fix embeddings

$$\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C},$$

where $\mathbb{C}$ denotes the complex number field.

We assume that every ring is commutative with unity. For a ring R and $n \in \mathbb{N}$, we use the following notation:

$$\begin{aligned} \mathrm{M}_n(R) &= \{(n \times n)\text{-matrices with entries in } R\}, \\ \mathrm{GL}_n(R) &= \{M \in \mathrm{M}_n(R) | M \text{ is an invertible matrix}\}, \\ \mathrm{SL}_n(R) &= \{M \in \mathrm{GL}_n(R) | \det(M) = 1\}. \end{aligned}$$

Moreover, if R is a subring of $\mathbb{R}$, we put

$$\mathrm{GL}_n^+(R) = \{M \in \mathrm{GL}_n(R) | \det(M) > 0\}.$$

Let $\mathfrak{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ be the upper half plane and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper half plane obtained by adding the cusps. Then $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $\mathfrak{H}$ by

$$\alpha z = \frac{az+b}{cz+d}$$

for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $z \in \mathfrak{H}$. Let $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

The principal congruence subgroups are the subgroup $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ defined by

$$\Gamma(N) = \left\{ \alpha \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

where N is a positive integer. A congruence subgroup is a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ containing a principal congruence group. The smallest integer $N > 0$ for which

$$\Gamma(N) \subset \Gamma$$

is called the level of Γ.

We will be mostly interested in the following special congruence subgroups:

$$\Gamma_0(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\},$$
$$\Gamma_1(N) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \;\middle|\; a \equiv d \equiv 1 \bmod N \right\}.$$

Let k be a positive integer ≥ 2. For any function f on $\mathfrak{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$, we define the function $f|_k\gamma$ on $\mathfrak{H}$ by

$$f|_k\gamma(z) = \det(\gamma)^{k-1} f(\gamma z)(cz+d)^{-k}.$$

We simply write $f|_k\gamma$ for $f|\gamma$ if there is no risk of confusion. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and N a positive integer such that $\Gamma(N) \subset \Gamma$. Any holomorphic function on $\mathfrak{H}$ satisfying $f|_k\gamma = f$ for all $\gamma \in \Gamma(N)$ has the Fourier expansion of the form:

$$\sum_{n=0}^{\infty} a(n,f)\, e\left(\frac{nz}{N}\right),$$

where $e(z) = \exp(2\pi\sqrt{-1}z)$.

We define the space $M_k(\Gamma, \mathbb{C})$ of modular forms of weight k with respect to Γ to be the space of holomorphic functions f on $\mathfrak{H}$ satisfying the following conditions:

(a) $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

(b) $a\,(n, f|_k\alpha) = 0$ if $n < 0$ for each $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Here we note that the function $f|_k\alpha$ is invariant under the action of $\alpha^{-1}\Gamma\alpha$ and hence $f|_k\alpha$ has the Fourier expansion. We define the space $S_k(\Gamma, \mathbb{C})$ of cusp forms to be the subspace of $M_k(\Gamma, \mathbb{C})$ consisting of $f \in M_k(\Gamma, \mathbb{C})$ satisfying the following condition:

(c) $a\,(0, f|_k\alpha) = 0$ for any $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

Let $\varepsilon\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character modulo N. We put

$$M_k(\Gamma_0(N), \varepsilon, \mathbb{C}) = \Big\{ f \in M_k(\Gamma_1(N), \mathbb{C}) \;\Big|\; f|_k\gamma = \varepsilon(d) f \text{ for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \Big\},$$

$$S_k(\Gamma_0(N), \varepsilon, \mathbb{C}) = M_k(\Gamma_0(N), \varepsilon, \mathbb{C}) \cap S_k(\Gamma_1(N), \mathbb{C}).$$

We remark that $M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ is trivial if $\varepsilon(-1) \neq (-1)^k$.

For a ring R and a non-negative integer n, we denote by $L_n(R)$ the degree n part $\mathrm{Sym}^n_R(RX \oplus RY)$ of the polynomial algebra $R[X, Y]$. Thus, $L_n(R)$ consists of the homogeneous polynomials of degree n in two-variables X and Y, with coefficients in R. The semigroup $\Sigma = \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})$ acts on $L_n(R)$ by

$$\gamma \cdot P(X, Y) = P((X, Y)\det(\gamma)^t\gamma^{-1}).$$

If R is a $\mathbb{Q}$-algebra, we also define the action of Σ on $L_n(R)$ by

$$\gamma \star P(X, Y) = P((X, Y)^t\gamma^{-1}).$$

In the similarly way, $\mathrm{GL}_2^+(\mathbb{Q})$ acts on $L_n(R)$ for $\mathbb{Q}$-algebra R and it is denoted by $\star$. We simply denote $\det(\alpha)\alpha^{-1}$ by α^ι for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We remark that these three actions coincide if they are restricted to $\mathrm{SL}_2(\mathbb{Z})$.

Moreover, for $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$ and a function G on $\mathfrak{H}$, we have the pull-back formula

$$\alpha^*(G(z)(X - zY)^{k-2}dz) = (G|\alpha)(z)\alpha \star (X - zY)^{k-2}dz.$$

Furthermore, for $\alpha, \beta \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\begin{aligned} \alpha^*(G(z)\beta \star (X - zY)^{k-2}dz) &= \beta \star (\alpha^*(G(z)(X - zY)^{k-2}dz) \\ &= (G|\alpha)(z)(\beta\alpha) \star (X - zY)^{k-2}dz. \end{aligned} \tag{0.1}$$

0.3. Acknowledgement. — I would like to express my gratitude to Professor Takeshi Tsuji for providing helpful comments and suggestions, and pointing out mathematical mistakes during the course of my study. In particular, the work in Appendix A would have been impossible without his insight and guidance. I heartily thank the referee for providing helpful comments for improvement.

1. Integrality of 1-cocycles

1.1. Preliminary. — Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$. Then $G \in M_k(\Gamma, \mathbb{C})$ has the Fourier expansion of the form

$$G(z) = \sum_{n=0}^{\infty} a(n, G)\, e(nz).$$

Thus, we may regard $M_k(\Gamma_1(N), \mathbb{C}) = \bigoplus_{\varepsilon:(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}} M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ as a subspace of $\mathbb{C}[[e(z)]]$. For a subring A of $\mathbb{C}$, we put

$$\begin{aligned}
M_k(\Gamma_1(N), A) &= M_k(\Gamma_1(N), \mathbb{C}) \cap A[[e(z)]],\\
S_k(\Gamma_1(N), A) &= S_k(\Gamma_1(N), \mathbb{C}) \cap A[[e(z)]],\\
M_k(\Gamma_0(N), \varepsilon, A) &= M_k(\Gamma_0(N), \varepsilon, \mathbb{C}) \cap A[[e(z)]],\\
S_k(\Gamma_0(N), \varepsilon, A) &= S_k(\Gamma_0(N), \varepsilon, \mathbb{C}) \cap A[[e(z)]].
\end{aligned}$$

Let χ be a Dirichlet character whose conductor m_χ is prime to N. We put

$$(G \otimes \chi)(z) = \sum_{n=0}^{\infty} a(n, G)\, \chi(n) e(nz).$$

We note that, if $G \in M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$, then $G \otimes \chi \in M_k(\Gamma_0(M), \varepsilon\chi^2, \mathbb{C})$ where $M = \operatorname{lcm}\{N, m_\chi^2, m_\chi m_\varepsilon\}$. The Dirichlet series

$$\sum_{n=1}^{\infty} a(n, G)\chi(n) n^{-s}$$

converges absolutely for $\mathrm{Re}(s) > k$ and extends to a meromorphic function on the complex plane with a possible simple pole at $s = k$. For each $G \in M_k(\Gamma, \mathbb{C})$, let $L(G, \chi, s)$ denote this analytic continuation. If χ is the trivial character, we simply write $L(G, s)$. We define $D(G, \chi, s)$ as

(1.1)

$$\begin{aligned}
D(G, \chi, s) &= \int_0^{\sqrt{-1}\infty} (\widetilde{G \otimes \chi})(z)(X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz\\
&= \sum_{j=0}^{k-2} \binom{k-2}{j} \sqrt{-1}^{j+1} \Gamma(s+j) \left(\frac{1}{2\pi}\right)^{s+j} L(G, \chi, s+j) X^{k-2-j} (-Y)^j,
\end{aligned}$$

where $\widetilde{G}(z) = G(z) - a(0, G)$ (see, for example, [33, Proposition 2.1.2] and the proof of [28, Theorem 4.3.5]). We call $D(G, \chi, s)$ the Mellin transform of G twisted by χ. The integral $D(G, \chi, s)$ converges absolutely for $\mathrm{Re}(s) > k$ and extends to a meromorphic function on the complex plane with simple poles at $s = -(k-2), \dots, -1, 0$ and $2, 3, \dots, k$ (see Proposition 1.1 (2)). We are interested in the special values of $L(G,\chi,s)$ at $s=1,\dots,k-1$, that is, in the special value of $D(G,\chi,s)$ at $s=1$.

Proposition 1.1. — *Let $G\in M_k(\Gamma,\mathbb{C})$, and χ a Dirichlet character whose conductor m_χ is prime to N.*

(1) *If $\alpha=\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}\in \mathrm{GL}_2^+(\mathbb{Q})$, then we have*

$$a(0,G|\alpha)=\frac{a^{k-1}}{d}a(0,G),$$
$$\widetilde{G|\alpha}=\widetilde{G}\big|\alpha.$$

(2) *The integral $D(G,\chi,s)$ converges absolutely for $\mathrm{Re}(s)>k$ and extends to a meromorphic function on the complex plane with simple poles at $s=-(k-2),\dots,-1,0$ and $2,3,\dots,k$.*

Proof. — (1) By definition, $(G|\alpha)(z)=\frac{a^{k-1}}{d}G(\alpha z)$. Then we have $a(0,G|\alpha)=\frac{a^{k-1}}{d}a(0,G)$. Moreover, by definition,

$$\begin{aligned}(\widetilde{G|\alpha})(z)&=(G|\alpha)(z)-a(0,G|\alpha)\\&=\frac{a^{k-1}}{d}(G(\alpha z)-a(0,G))\\&=\frac{a^{k-1}}{d}\widetilde{G}(\alpha z)\\&=(\widetilde{G}\big|\alpha)(z).\end{aligned}$$

(2) For $\mathrm{Re}(s)>k$,

$$\begin{aligned}D(G,\chi,s)&=\int_0^{\sqrt{-1}\infty}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz\\&=\int_{\sqrt{-1}}^{\sqrt{-1}\infty}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz\\&\quad+\int_0^{\sqrt{-1}}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz.\end{aligned}$$

Now we calculate the second term. We put $y=\mathrm{Im}(z)$. Then we get

$$\begin{aligned}&\int_0^{\sqrt{-1}}(\widetilde{G\otimes\chi})(z)(X-zY)^{k-2}\mathrm{Im}(z)^{s-1}dz\\&\quad=\int_0^{\sqrt{-1}}(G\otimes\chi)(z)(X-zY)^{k-2}y^{s-1}dz\\&\qquad-\int_0^{\sqrt{-1}}a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz\end{aligned}$$

$$
\begin{aligned}
&= \int_{\sigma^{-1}0}^{\sigma^{-1}\sqrt{-1}} (G\otimes\chi)(\sigma z)(X-\sigma zY)^{k-2}y^{1-s}d\sigma z \\
&\quad - \int_0^{\sqrt{-1}} a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz \\
&= -\int_{\sqrt{-1}}^{\sqrt{-1}\infty} ((G\otimes\chi)|\sigma)(z)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - \int_0^{\sqrt{-1}} a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz \\
&= -\int_{\sqrt{-1}}^{\sqrt{-1}\infty} ((\widetilde{G\otimes\chi})|\sigma)(z)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - \int_{\sqrt{-1}}^{\sqrt{-1}\infty} a(0,(G\otimes\chi)|\sigma)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - \int_0^{\sqrt{-1}} a(0,G)\chi(0)(X-zY)^{k-2}y^{s-1}dz \\
&= -\int_{\sqrt{-1}}^{\sqrt{-1}\infty} ((\widetilde{G\otimes\chi})|\sigma)(z)\sigma\cdot(X-zY)^{k-2}y^{1-s}dz \\
&\quad - a(0,(G\otimes\chi)|\sigma)\sqrt{-1}\sum_{j=0}^{k-2}\binom{k-2}{j}(\sqrt{-1}X)^jY^{k-2-j}\frac{-1}{j+2-s} \\
&\quad - a(0,G)\chi(0)\sqrt{-1}\sum_{j=0}^{k-2}\binom{k-2}{j}X^{k-2-j}(-\sqrt{-1}Y)^j\frac{1}{s+j}.
\end{aligned}
$$

Here the third equality follows from (0.1). By setting $s=1$, the second term is equal to

$$
a(0,(G\otimes\chi)|\sigma)\int_0^{\sqrt{-1}} \sigma\cdot(X-zY)^{k-2}dz.
$$

This proves (2). □

1.2. Construction of 1-cocycles. — In order to define a desired cocycle with good arithmetic and p-adic properties, we need to choose some special coboundary element as in [33].

Definition 1.2. — For a congruence subgroup Γ, let $G\in M_k(\Gamma,\mathbb{C})$. For $\alpha,\beta\in \mathrm{GL}_2^+(\mathbb{Q})$ and $z_0\in\mathfrak{H}$, we define the map

$$
\pi_{G,\beta}(z_0)\colon \mathrm{GL}_2^+(\mathbb{Q})\longrightarrow L_{k-2}(\mathbb{C})
$$

by

$$\begin{aligned}\pi_{G,\beta}(z_0)(\alpha) = &\int_{z_0}^{\alpha z_0} (G|\beta)(z)\beta \star (X - zY)^{k-2} dz \\ &+ \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\beta\alpha})(z)\beta\alpha \star (X - zY)^{k-2} dz \\ &- a(0, G|\beta\alpha) \int_0^{z_0} \beta\alpha \star (X - zY)^{k-2} dz \\ &- \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\beta})(z)\beta \star (X - zY)^{k-2} dz \\ &+ a(0, G|\beta) \int_0^{z_0} \beta \star (X - zY)^{k-2} dz.\end{aligned}$$

We remark that the second and fourth integrals converge absolutely by using Proposition 1.1 (2). If $\beta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we simply write π_G instead of $\pi_{G,\beta}$. Then we have

$$\pi_{G,\beta}(z_0)(\alpha) = \beta \star \pi_{G|\beta}(z_0)(\alpha). \tag{1.2}$$

Remark 1.3. — If G is a cusp form, then, for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, by using (0.1),

$$\pi_G(z_0)(\alpha) = \int_{\sqrt{-1}\infty}^{\alpha\sqrt{-1}\infty} G(z)(X - zY)^{k-2} dz$$

is the usual Eichler-Shimura cocycle.

Proposition 1.4. — *For each $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, the value $\pi_{G,\beta}(z_0)(\alpha)$ is independent of $z_0 \in \mathfrak{H}$.*

Proof. — For any $z_0, z_0' \in \mathfrak{H}$ and $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, we have

$$\begin{aligned}\pi_{G,\beta}(z_0)(\alpha) - \pi_{G,\beta}(z_0')(\alpha) = &\int_{z_0}^{\alpha z_0} (G|\beta)(z)\beta \star (X - zY)^{k-2} dz \\ &- \int_{z_0'}^{\alpha z_0'} (G|\beta)(z)\beta \star (X - zY)^{k-2} dz \\ &+ \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\beta\alpha})(z)\beta\alpha \star (X - zY)^{k-2} dz \\ &- \int_{z_0'}^{\sqrt{-1}\infty} (\widetilde{G|\beta\alpha})(z)\beta\alpha \star (X - zY)^{k-2} dz \\ &- \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\beta})(z)\beta \star (X - zY)^{k-2} dz\end{aligned}$$

$$+\int_{z_0'}^{\sqrt{-1}\infty}(\widetilde{G|\beta})(z)\beta\star(X-zY)^{k-2}dz$$
$$-a(0,G|\beta\alpha)\int_0^{z_0}\beta\alpha\star(X-zY)^{k-2}dz$$
$$+a(0,G|\beta\alpha)\int_0^{z_0'}\beta\alpha\star(X-zY)^{k-2}dz$$
$$+a(0,G|\beta)\int_0^{z_0}\beta\star(X-zY)^{k-2}dz$$
$$-a(0,G|\beta)\int_0^{z_0'}\beta\star(X-zY)^{k-2}dz.$$

By using the pullback formula (0.1), we get

$$\begin{aligned}
\pi_{G,\beta}(z_0)(\alpha)-\pi_{G,\beta}(z_0')(\alpha) &= \int_{\alpha z_0'}^{\alpha z_0}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\
&\quad+\int_{z_0}^{z_0'}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz\\
&\quad+\int_{z_0}^{z_0'}(\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\
&\quad-\int_{z_0}^{z_0'}(\widetilde{G|\beta})(z)\beta\star(X-zY)^{k-2}dz\\
&\quad-a(0,G|\beta\alpha)\int_{z_0'}^{z_0}\beta\alpha\star(X-zY)^{k-2}dz\\
&\quad+a(0,G|\beta)\int_{z_0'}^{z_0}\beta\star(X-zY)^{k-2}dz\\
&=\int_{z_0'}^{z_0}(G|\beta\alpha)(z)\beta\alpha\star(X-zY)^{k-2}dz\\
&\quad-\int_{z_0'}^{z_0}(\widetilde{G|\beta\alpha})(z)\beta\alpha\star(X-zY)^{k-2}dz\\
&\quad-a(0,G|\beta\alpha)\int_{z_0'}^{z_0}\beta\alpha\star(X-zY)^{k-2}dz\\
&\quad-a(0,G|\beta)\int_{z_0}^{z_0'}\beta\star(X-zY)^{k-2}dz\\
&\quad+\int_{z_0}^{z_0'}(G|\beta)(z)\beta\star(X-zY)^{k-2}dz
\end{aligned}$$

$$
\begin{aligned}
&- \int_{z_0}^{z_0'} (\widetilde{G|\beta})(z)\beta \star (X - zY)^{k-2} dz \\
&= 0. \qquad \square
\end{aligned}
$$

By Proposition 1.4, we simply write $\pi_{G,\beta}$ instead of $\pi_{G,\beta}(z_0)$. This map π_G is important for a cohomological treatment of $D(G,\chi,s)$, which we state in the next section. As a preparation for it, the rest of this section is devoted to the proof of some properties of π_G. The proof is based on the method of Stevens [33]. We put

$$
D_\alpha(G,s) = \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha})(z)\alpha \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz
$$

for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. If $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we simply write $D(G,s)$ instead of $D_\alpha(G,s)$. We remark that $D_\alpha(G,s) = \alpha \star D(G|\alpha, s)$ and hence this integral converges absolutely for $\mathrm{Re}(s) > k$ by using Proposition 1.1 (2).

Proposition 1.5. — (1) *For any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$, we have*

$$
D_\beta(G,1) = -\pi_{G,\beta}(\sigma),
$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(2) *If* $\tau = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, *then we have*

$$
\pi_{G,\alpha}(\sigma) = \pi_{G,\alpha\tau}(\sigma).
$$

Proof. — (1) It follows from the proof of Proposition 1.1 (2).

(2) We have

$$
\begin{aligned}
D_{\alpha\tau}(G,s) &= \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha\tau})(z)\alpha\tau \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz \\
&= \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha}\Big|\tau)(z)\alpha\tau \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz \quad \text{(by Prop. 1.1 (1))} \\
&= \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha})(z)\alpha \star (X - zY)^{k-2} \mathrm{Im}(\tau^{-1}z)^{s-1} dz \quad \text{(by (0.1))} \\
&= \left(\frac{v}{u}\right)^{s-1} \int_0^{\sqrt{-1}\infty} (\widetilde{G\big|\alpha})(z)\alpha \star (X - zY)^{k-2} \mathrm{Im}(z)^{s-1} dz \\
&= \left(\frac{v}{u}\right)^{s-1} D_\alpha(G,s).
\end{aligned}
$$

This proves (2) by setting $s = 1$. $\square$

Proposition 1.6. — *The map π_G has the following properties:*

(1) (cocycle condition) For any $\alpha_1, \alpha_2 \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\pi_G(\alpha_1\alpha_2) = \pi_{G,\alpha_1}(\alpha_2) + \pi_G(\alpha_1).$$

More generally, for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\pi_{G,\beta}(\alpha_1\alpha_2) = \pi_{G,\beta\alpha_1}(\alpha_2) + \pi_{G,\beta}(\alpha_1).$$

(2) For any $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ with $c \geq 0$,

$$\pi_G(\alpha) = \begin{cases} a(0,G)\displaystyle\int_0^{\frac{a}{c}} (X - zY)^{k-2}dz \\ \quad + a(0,G|\alpha)\displaystyle\int_{-\frac{d}{c}}^{0} \alpha \star (X - zY)^{k-2}dz + \pi_{G,\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(\sigma) & \text{if } c > 0, \\ \\ a(0,G)\displaystyle\int_0^{\frac{b}{d}} (X - zY)^{k-2}dz & \text{if } c = 0, \end{cases}$$

where $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\delta = \det(\alpha)$.

(3) $\pi_G((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})) = a(0,G)\displaystyle\int_0^1 (X - zY)^{k-2}dz.$

Proof. — (1) For any $\alpha_1, \alpha_2 \in \mathrm{GL}_2^+(\mathbb{Q})$,

$$\begin{aligned} &\pi_{G,\alpha_1}(\alpha_2) + \pi_G(\alpha_1) \\ &= \int_{z_0}^{\alpha_2 z_0} (G|\alpha_1)(z)\alpha_1 \star (X - zY)^{k-2}dz + \int_{z_0}^{\alpha_1 z_0} G(z)(X - zY)^{k-2}dz \\ &\quad + \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha_1\alpha_2})(z)\alpha_1\alpha_2 \star (X - zY)^{k-2}dz \\ &\quad - \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha_1})(z)\alpha_1 \star (X - zY)^{k-2}dz + \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha_1})(z)\alpha_1 \star (X - zY)^{k-2}dz \\ &\quad - a(0,G|\alpha_1\alpha_2)\int_0^{z_0} \alpha_1\alpha_2 \star (X - zY)^{k-2}dz - \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G}(z)(X - zY)^{k-2}dz \\ &\quad + a(0,G|\alpha_1)\int_0^{z_0} \alpha_1 \star (X - zY)^{k-2}dz - a(0,G|\alpha_1)\int_0^{z_0} \alpha_1 \star (X - zY)^{k-2}dz \\ &\quad + a(0,G)\int_0^{z_0} (X - zY)^{k-2}dz \\ &= \pi_G(\alpha_1\alpha_2). \end{aligned}$$

(2) Let $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q}) \right\}$. We use the Bruhat decomposition

$$\mathrm{GL}_2^+(\mathbb{Q}) = B\sigma B \sqcup B.$$

First suppose that $c = 0$. By definition,

$$\begin{aligned}\pi_G(z_0)(\alpha) = &\int_{z_0}^{\alpha z_0} \widetilde{G}(z)(X - zY)^{k-2}dz + \int_{z_0}^{\alpha z_0} a(0, G)(X - zY)^{k-2}dz \\ &+ \int_{z_0}^{\sqrt{-1}\infty} (\widetilde{G|\alpha})(z)\alpha \star (X - zY)^{k-2}dz - a(0, G|\alpha) \int_0^{z_0} \alpha \star (X - zY)^{k-2}dz \\ &- \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G}(z)(X - zY)^{k-2}dz + a(0, G) \int_0^{z_0} (X - zY)^{k-2}dz.\end{aligned}$$

When z_0 tends to $\sqrt{-1}\infty$, so does αz_0. Then, the first, third and fifth terms converge to 0. Thus we obtain

$$\begin{aligned}\pi_G(\alpha) &= \lim_{z_0 \to \sqrt{-1}\infty} \left(a(0, G)\int_0^{\alpha z_0} (X - zY)^{k-2}dz - a(0, G|\alpha)\int_0^{z_0} \alpha \star (X - zY)^{k-2}dz \right) \\ &= \lim_{z_0 \to \sqrt{-1}\infty} \left(a(0, G)\int_0^{\alpha z_0} (X - zY)^{k-2}dz - a(0, G)\int_{\alpha 0}^{\alpha z_0} (X - zY)^{k-2}dz \right) \\ &= a(0, G) \int_0^{\frac{b}{d}} (X - zY)^{k-2}dz.\end{aligned}$$

Next we consider the case $c > 0$. By Proposition 1.6 (1) and the decomposition

$$\alpha = \begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix} \sigma \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix},$$

where $\delta = \det(\alpha)$, we get

$$\pi_G(\alpha) = \pi_G(\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}) + \pi_{G,\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}\sigma}(\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}) + \pi_{G,\begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}}(\sigma).$$

Here we note that $\pi_{G,\begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix}} = \pi_G$ and $\pi_G(\begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix}) = 0$ by the above case. We have already obtained formulas about the first and second terms by the case $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ considered above. Indeed, we have

$$\begin{aligned}
&\pi_{G,\left(\begin{smallmatrix} \delta & a \\ 0 & c \end{smallmatrix}\right)\sigma}\left(\left(\begin{smallmatrix} c & d \\ 0 & 1 \end{smallmatrix}\right)\right) \\
&= a\left(0, G\middle|\left(\begin{smallmatrix} \delta & a \\ 0 & c \end{smallmatrix}\right)\sigma\right)\int_0^d \begin{pmatrix} \delta & a \\ 0 & c \end{pmatrix}\sigma \star (X - zY)^{k-2}dz \\
&= a\left(0, G\middle|\alpha\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} c^{-1} & -c^{-1}d \\ 0 & 1 \end{smallmatrix}\right)\right)\int_0^d \alpha\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)\left(\begin{smallmatrix} c^{-1} & -c^{-1}d \\ 0 & 1 \end{smallmatrix}\right)\star (X - zY)^{k-2}dz \\
&= a(0, G|\alpha)\int_{-\frac{d}{c}}^0 \alpha \star (X - zY)^{k-2}dz.
\end{aligned}$$

Here the final equality follows from that, for $\beta = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})^+$ and $\alpha \in \mathrm{GL}_2(\mathbb{Q})^+$, by Proposition 1.1 and (0.1),

$$a\left(0, (G|\alpha)\middle|\beta\right)\alpha\beta \star (X - zY)^{k-2}dz = \beta \star (a\,(0, G|\alpha)\,\alpha \star (X - zY)^{k-2}dz),$$

and hence

$$a\left(0, (G|\alpha)\middle|\beta\right)\int_0^d \alpha\beta \star (X - zY)^{k-2}dz = a\,(0, G|\alpha)\int_{\beta(0)}^{\beta(d)} \alpha \star (X - zY)^{k-2}dz.$$

Thus, we obtain the formula as claimed.

(3) It follows immediately from (2). □

Remark 1.7. — (1) If $\Gamma = \Gamma_1(N)$, then, for any $\beta \in \Gamma$, we have $\pi_{G,\beta} = \beta \star \pi_{G|\beta} = \beta \cdot \pi_G$ by (1.2).

(2) The restriction of π_G to $\Gamma_1(N)$ is a 1-cocycle on $\Gamma_1(N)$, that is, $\pi_G \in Z^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ (for the definition, see § 2.1).

1.3. Integrality. — The value of $D(G, \chi, s)$ at $s = 1$ is described in terms of π_G as follows.

Lemma 1.8. — *Let $G \in M_k(\Gamma_1(N), \mathbb{C})$ and χ a Dirichlet character whose conductor m_χ is prime to N. Fix $b_1, \dots, b_{\varphi(m_\chi)} \in \mathbb{Z}$ such that $\{\bar{b}_1, \dots, \bar{b}_{\varphi(m_\chi)}\} = (\mathbb{Z}/m_\chi\mathbb{Z})^\times$, where $\bar{b}_i$ is the image of b_i under the natural map $\mathbb{Z} \to \mathbb{Z}/m_\chi\mathbb{Z}$ and φ is the Euler function. Then,*

$$\tau(\bar{\chi})D(G, \chi, 1) = -\sum_{i=1}^{\varphi(m_\chi)} \bar{\chi}(b_i)\begin{pmatrix} 1 & -\frac{b_i}{m_\chi} \\ 0 & 1 \end{pmatrix} \star \pi_{G,\left(\begin{smallmatrix} 1 & b_i \\ 0 & m_\chi \end{smallmatrix}\right)}(\sigma),$$

where $\tau(\bar{\chi}) = \sum_{i=1}^{\varphi(m_\chi)} \bar{\chi}(b_i)e(\frac{b_i}{m_\chi})$ is the Gauss sum of $\bar{\chi}$.

Proof. — We put $m = m_\chi$. We have

$$\tau(\bar\chi)(G\otimes\chi)(z) = \sum_{i=1}^{\varphi(m)} \bar\chi(b_i)G(z+\frac{b_i}{m}).$$

Thus,

$$\begin{aligned}
&\tau(\bar\chi)D(G,\chi,s)\\
&= \sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\int_0^{\sqrt{-1}\infty} \widetilde{G(z+\frac{b_i}{m})}(X-zY)^{k-2}y^{s-1}dz \qquad (y=\mathrm{Im}(z))\\
&= \sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}^{-1}\star\int_0^{\sqrt{-1}\infty} \widetilde{G(z+\frac{b_i}{m})}\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star(X-zY)^{k-2}y^{s-1}dz\\
&= \sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\int_0^{\sqrt{-1}\infty} \left(\widetilde{G\Big|\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}}\right)(z)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star(X-zY)^{k-2}y^{s-1}dz.
\end{aligned}$$

We remark that

$$D_{\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}}(G,s) = \int_0^{\sqrt{-1}\infty} \left(\widetilde{G\Big|\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}}\right)(z)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star(X-zY)^{k-2}y^{s-1}dz.$$

By Proposition 1.5 (1), we have

$$D_{\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}}(G,1) = -\pi_{G,\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}}(\sigma). \tag{1.3}$$

Therefore, we obtain

$$\tau(\bar\chi)D(G,\chi,1) = -\sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\pi_{G,\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}}(\sigma).$$

In addition, by using Proposition 1.5 (2), we have

$$\begin{aligned}
\tau(\bar\chi)D(G,\chi,1) &= -\sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\pi_{G,\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & m\end{pmatrix}}(\sigma)\\
&= -\sum_{i=1}^{\varphi(m)} \bar\chi(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\pi_{G,\begin{pmatrix}1 & b_i\\ 0 & m\end{pmatrix}}(\sigma).
\end{aligned}$$

We have proved the lemma. □

We fix a rational odd prime number p such that $(p,N)=1$. Let S be a set of rational prime numbers satisfying the following properties:

(1) both $(m,pN)=1$ and $(\varphi(m),p)=1$ hold for all $m\in S$;

(2) S has non-empty intersection with every arithmetic progression of the form $\{d + cpNe | e \in \mathbb{Z}\}$ for all pair $(c, d) \in \mathbb{Z}^2$ such that $c > 0$, $(p, cd) = 1$, $d \not\equiv 1 \pmod{p}$, and $(d, cN) = 1$

(for example, $S = \{m | m$ is a prime number such that $(m, pN) = 1$ and $m \not\equiv 1 \pmod{p}\}$).

We remark that $p \notin S$. For such a set S, let $\mathfrak{X}_S$ denote the set of Dirichlet characters χ whose conductor m_χ belongs to S. For $m \in S$, we fix $b_1, \dots, b_{\varphi(m)} \in \mathbb{Z}$ such that $\{b_1, \dots, b_{\varphi(m)}\} = (\mathbb{Z}/m\mathbb{Z})^\times$.

Theorem 1.9 (Integrality). — *Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$. Suppose that $G \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ and that the following conditions hold:*

(1) $k < p + 1$;
(2) $a(0, G|\alpha) \in \mathcal{O}$ *for each* $\alpha \in \mathrm{SL}_2(\mathbb{Z})$;
(3) $D_{\begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(G, 1) \in L_{k-2}(\mathcal{O})$ *for each* $m \in S$ *and* i;
(4) $\pi_G(\sigma) \in L_{k-2}(\mathcal{O})$.

Then π_G is integral, that is, $\pi_G(\Gamma_0(N)) \subset L_{k-2}(\mathcal{O})$.

Proof. — We put $\Gamma = \Gamma_0(N)$ and

$$\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma.$$

In the case where $c = 0$, we have $\pi_G(\gamma) \in L_{k-2}(\mathcal{O})$ by Proposition 1.6 (2) and the assumptions (1) and (2). In the case where $c \neq 0$, we may assume that $c > 0$. Indeed, we have $\pi_G(-\gamma) = \pi_G(\gamma)$ by using Proposition 1.6 (1), (2), and (1.2). Then, by Proposition 1.6 (2), we have

$$\begin{aligned}\pi_G(\gamma) = a(0, G) \int_0^{\frac{a}{cN}} (X - zY)^{k-2} dz \\ + a(0, G|\gamma) \int_{-\frac{d}{cN}}^0 \gamma \cdot (X - zY)^{k-2} dz + \pi_{G, \begin{pmatrix} 1 & \frac{a}{cN} \\ 0 & 1 \end{pmatrix}}(\sigma).\end{aligned}$$

We prove that $\pi_G(\gamma)$ is integral in two cases.

Case 1. — Assume that $(p, c) = 1$.

It is enough to prove that $\pi_G(\gamma)$ is integral in the case where $(p, d) = 1$ and $d \not\equiv 1 \pmod{p}$. Indeed, if $p | d$ or $d \equiv 1 \pmod{p}$, then we put

$$\gamma' := \gamma \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ab' + b \\ cN & cNb' + d \end{pmatrix} \in \Gamma.$$

Since $(p, cN) = 1$, note that $cNb' + d \not\equiv 0, 1 \pmod{p}$ for some $b' \in \mathbb{Z}$. Then, by applying the cocycle condition (Proposition 1.6 (1)) for π_G to the element γ', we get

$$\pi_G(\gamma') = \pi_G(\gamma) + \varepsilon(d)\gamma \cdot \pi_G(\left(\begin{smallmatrix} 1 & b' \\ 0 & 1 \end{smallmatrix}\right)).$$

Now the integrality of $\pi_G(\gamma)$ follows from the integrality of $\pi_G(\gamma')$.

We remark that $(cN)^{-1} \in \mathcal{O}$ by assumption. Then, for the proof of the integrality of $\pi_G(\gamma)$, by the formula above and the assumptions (1) and (2), it is enough to show that $\pi_{G,\begin{pmatrix} 1 & a \\ 0 & cN \end{pmatrix}}(\sigma) \in L_{k-2}(\mathcal{O})$. Therefore, for the proof in this case, it suffices to show that $\pi_G(\gamma')$ is integral by choosing $b', d' \in \mathbb{Z}$ such that $\gamma' = \begin{pmatrix} a & b' \\ cN & d' \end{pmatrix} \in \Gamma$. Indeed, we have $\pi_G(\gamma') \equiv \pi_{G,\begin{pmatrix} 1 & a \\ 0 & cN \end{pmatrix}}(\sigma) \pmod{L_{k-2}(\mathcal{O})}$. Since $(p, cd) = 1$, $d \not\equiv 1 \pmod{p}$, and $(d, cN) = 1$, there exists $e \in \mathbb{Z}$ such that $d + cpNe \in S$. We put $m = d + cpNe$ and

$$\gamma' = \gamma \begin{pmatrix} 1 & ep \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b' \\ cN & m \end{pmatrix} \in \Gamma,$$

where $b' = aep + b$. By applying the cocycle condition (Proposition 1.6 (1)) for π_G to the element

$$\gamma'\sigma = \begin{pmatrix} a & b' \\ cN & m \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b' & -a \\ m & -cN \end{pmatrix},$$

we get

$$\pi_G(\gamma'\sigma) = \pi_G(\gamma') + \varepsilon(m)\gamma' \cdot \pi_G(\sigma).$$

Since $\gamma' \cdot \pi_G(\sigma)$ is integral by the assumption (4), for the proof of the integrality of $\pi_G(\gamma')$, it suffices to show that $\pi_G(\gamma'\sigma)$ is integral. Using Proposition 1.6 (2), we have

$$\begin{aligned} \pi_G(\gamma'\sigma) = a(0, G) \int_0^{\frac{b'}{m}} (X - zY)^{k-2} dz \\ + a(0, G|\gamma'\sigma) \int_{\frac{cN}{m}}^{0} \gamma'\sigma \cdot (X - zY)^{k-2} dz + \pi_{G,\begin{pmatrix} 1 & b' \\ 0 & m \end{pmatrix}}(\sigma). \end{aligned}$$

Therefore, by the assumptions (1) and (2), it is enough to show that the final term is integral. It follows from (1.3) and the assumption (3).

Case 2. — Assume that $p|c$.

We put

$$\gamma' := \gamma \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} a + bN & b \\ (c+d)N & d \end{pmatrix} \in \Gamma.$$

Then, by applying the cocycle condition (Proposition 1.6 (1)) for π_G to the element γ', we get

$$\pi_G(\gamma') = \pi_G(\gamma) + \varepsilon(d)\gamma \cdot \pi_G(\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)).$$

Since $(p, c+d) = 1$ and $(p, N) = 1$, we see that both $\pi_G(\gamma')$ and $\pi_G((\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}))$ are integral by Case 1, and therefore so is $\pi_G(\gamma)$. Now this completes the proof of the theorem. ☐

REMARK 1.10. — Theorem 1.9 is a partial generalization of [34, Theorem 1.3].

2. Congruences for L-functions

2.1. Group cohomology. — To state our theorem, we need to recall some properties about group cohomology. We define an action of $\mathrm{GL}_2(\mathbb{Q})$ on the upper half complex plane $\mathfrak{H}$ as follows. For $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ with $\det(\alpha) > 0$, α act on $\mathfrak{H}$ by the usual linear fractional transformation. For $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, τ act on $\mathfrak{H}$ by $\tau z = -\bar{z}$. If $\det(\alpha) < 0$, then we define $\alpha(z) = (\alpha\tau)(\tau(z))$. This action is associative and so is well-defined. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

DEFINITION 2.1 (The standard $R[\Gamma]$-free resolution of R). — Let R be a commutative ring and M a left $R[\Gamma]$-module. We define $F_q = (R[\Gamma])^{\otimes(q+1)}$ and regard it as an $R[\Gamma]$-module via the multiplication of $R[\Gamma]$ on the first factor. Then F_q is a free $R[\Gamma]$-module with a basis $\{[\gamma_1, \dots, \gamma_q] = 1 \otimes \gamma_1 \otimes \cdots \otimes \gamma_q | \gamma_i \in \Gamma\}$. We define the $R[\Gamma]$-linear boundary map $\partial_q \colon F_q \to F_{q-1}$ by $\partial_1[\gamma] = \gamma - 1$ and

$$\begin{aligned}\partial_q[\gamma_1, \dots, \gamma_q] &= \gamma_1[\gamma_2, \dots, \gamma_q] \\ &\quad + \sum_{j=1}^{q-1} (-1)^j [\gamma_1, \dots, \gamma_j\gamma_{j+1}, \dots, \gamma_q] + (-1)^q [\gamma_1, \dots, \gamma_{q-1}]\end{aligned}$$

for $q > 1$. It is well known that (F_*, ∂_*) is a $R[\Gamma]$-free resolution of R. Let $C^i = C^i(\Gamma, M)$ be the space of functions on Γ^i with values in M for $i \geq 1$, and M for $i = 0$. Note that $\mathrm{Hom}_{R[\Gamma]}(F_q, M) \cong C^q$. Then the differential map $d^i \colon C^i \to C^{i+1}$ induced by ∂_* on F_* is given by $d^0 u(\gamma) = (\gamma - 1)u$ for $u \in M$ if $i = 0$, and if $i > 0$,

$$\begin{aligned}d^i u(\gamma_1, \dots, \gamma_{i+1}) &= \gamma_1 u(\gamma_2, \dots, \gamma_{i+1}) \\ &\quad + \sum_{j=1}^{i} (-1)^j u(\gamma_1, \dots, \gamma_j\gamma_{j+1}, \dots, \gamma_{i+1}) + (-1)^{i+1} u(\gamma_1, \dots, \gamma_i).\end{aligned}$$

The associated i-th cohomology group of Γ with coefficients in M is given by

$$H^i(\Gamma, M) = Z^i(\Gamma, M)/B^i(\Gamma, M),$$

where

$$Z^i(\Gamma, M) = \ker(d^i \colon C^i \to C^{i+1}) \quad \text{and} \quad B^i(\Gamma, M) = \mathrm{im}(d^{i-1} \colon C^{i-1} \to C^i).$$

We fix a base point $z_0 \in \mathfrak{H}$. For $G \in M_k(\Gamma_1(N), \mathbb{C})$ and $\gamma \in \Gamma_1(N)$, we define $\omega_G(z_0) \in C^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ by

$$\omega_G(z_0)(\gamma) = \int_{z_0}^{\gamma z_0} G(z)(X - zY)^{k-2} dz.$$

Then we have $\omega_G(z_0) \in Z^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$.

Also we have $\pi_G \in Z^1(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ by Proposition 1.6 (1) and (1.2). Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to R^\times$ be a character and

$$\Delta = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \;\middle|\; \det(\alpha) \neq 0, c \equiv 0 \pmod N, (a, N) = 1 \right\}.$$

We define an $R[\Delta]$-module $L_{k-2}(\varepsilon, R)$ as follows: let $L_{k-2}(\varepsilon, R)$ be the R-module $L_{k-2}(R)$ with left $R[\Delta]$-action by

$$\gamma \bullet P(X, Y) = \varepsilon(d) \gamma \cdot P(X, Y)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$. For $G \in M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ and $\gamma \in \Gamma_0(N)$, we define $\omega_G(z_0) \in C^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$ by

$$\omega_G(z_0)(\gamma) = \int_{z_0}^{\gamma z_0} G(z)(X - zY)^{k-2} dz.$$

Then we have $\omega_G(z_0) \in Z^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$.

Also we have $\pi_G \in Z^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$ by Proposition 1.6 (1) and (1.2).

For each cusp $s \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$, let Γ_s denote the stabilizer of s in Γ, and let π_s be a generator of Γ_s:

$$\Gamma_s = \{\alpha \in \Gamma | \alpha s = s\} = \{\pm \pi_s^m \in \Gamma | m \in \mathbb{Z}\}.$$

Let $Z(\Gamma)$ be a representative set for Γ-equivalence classes of cusps, which is a finite set. Then we note that for each cusp $s \in \mathbb{P}^1(\mathbb{Q})$, we can find $\gamma \in \Gamma$ and $s_0 \in Z(\Gamma)$ such that $\gamma s = s_0$. We consider the set of all conjugates of π_s in Γ for all $s \in Z(\Gamma)$, which is denoted by P. The parabolic cohomology group of Γ with coefficients in M is given by

$$H^1_{\mathrm{par}}(\Gamma, M) = Z^1_{\mathrm{par}}(\Gamma, M)/B^1(\Gamma, M),$$

where

$$Z^1_{\mathrm{par}}(\Gamma, M) = \{u \in Z^1(\Gamma, M) | u(\pi) \in (\pi - 1)M \text{ for all } \pi \in P\}.$$

If $f \in S_k(\Gamma_1(N), \mathbb{C})$ (resp. $f \in S_k(\Gamma_0(N), \varepsilon, \mathbb{C})$), we have $\omega_f(z_0), \pi_f \in Z^1_{\mathrm{par}}(\Gamma_1(N), L_{k-2}(\mathbb{C}))$ (resp. $\omega_f(z_0), \pi_f \in Z^1_{\mathrm{par}}(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$).

2.2. The Hecke eigenvalues. — We recall the definitions of the Hecke operators on group cohomology and the space of modular forms. Let $\Gamma, \Gamma' < \mathrm{SL}_2(\mathbb{Z})$ be congruence subgroups. For any $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, we have a decomposition $\Gamma\alpha\Gamma' = \coprod_i \Gamma\alpha_i$ as a disjoint union. We denote $\det(\alpha)\alpha^{-1}$ simply by α^ι for any $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. Let $\langle \Gamma, \Gamma', \alpha^\iota \rangle$ be the semi-group in $\mathrm{GL}_2(\mathbb{Q})$ generated by α^ι for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ and two congruences subgroups Γ and Γ'. For any $\langle \Gamma, \Gamma', \alpha^\iota \rangle$-module M, we define the Hecke operator $[\Gamma\alpha\Gamma']$ as follows. For each $\gamma \in \Gamma'$, we can write $\alpha_i\gamma = \gamma_i\alpha_j$ for a unique j with $\gamma_i \in \Gamma$. For each cocycle $u : \Gamma \to M \in Z^1(\Gamma, M)$, we define $v = u\big|[\Gamma\alpha\Gamma']$ by $v(\gamma) = \sum_i \alpha_i^\iota u(\gamma_i)$. The operator $[\Gamma\alpha\Gamma']$ is a well-defined linear operator from $H^1(\Gamma, M)$ into $H^1(\Gamma', M)$. Also $[\Gamma\alpha\Gamma']$ sends $H^1_{\mathrm{par}}(\Gamma, M)$ into $H^1_{\mathrm{par}}(\Gamma', M)$.

We consider the case $\Gamma = \Gamma' = \Gamma_0(N)$ or $\Gamma_1(N)$. If $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$ for a prime number l, we abbreviate $[\Gamma\alpha\Gamma]$ to $T(l)$. We have the following lemma ([28, Lemma 4.5.6 (1)]):

Lemma 2.2. — *An explicit left coset decomposition is given by*

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & l^e \end{pmatrix} \Gamma_0(N) = \begin{cases} \displaystyle\coprod_{\substack{0 \le f \le e, \\ 0 \le r < l^f \text{ with } (r,\, l^f,\, l^{e-f})=1}} \Gamma_0(N) \begin{pmatrix} l^{e-f} & r \\ 0 & l^f \end{pmatrix} & \text{if } (l, N) = 1, \\[2em] \displaystyle\coprod_{0 \le r < l^e} \Gamma_0(N) \begin{pmatrix} 1 & r \\ 0 & l^e \end{pmatrix} & \text{if } l|N, \end{cases}$$

as a disjoint union.

Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$. We define the Hecke operator $T(l)$ on $M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ for a prime number l. We put the disjoint decompositions $\Gamma\alpha\Gamma = \coprod_i \Gamma\alpha_i$, where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$. Then we define

$$f|T(l) = \sum_i \varepsilon(\alpha_i) f|\alpha_i,$$

where $\varepsilon(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \varepsilon(a)$. Here we note that it is independent of the choice of $\{\alpha_i\}_i$ and $f|T(l) \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$. Moreover, we define the Hecke operator $T(l^e)$ for a prime number l and an integer $e \geq 1$ inductively by

$$T(l^{e+1}) = \begin{cases} T(l)T(l^e) - \varepsilon(l)l^{k-1}T(l^{e-1}) & \text{if } (l, N) = 1, \\ T(l)^{e+1} & \text{if } l|N, \end{cases}$$

where we define $T(1)$ to be the identity map. More generally, we can define the Hecke operator $T(m)$ by

$$T(l)T(l') = T(l')T(l),$$
$$T(m) = \prod_l T(l^{e_l})$$

for different primes l and l' and each positive integer $m = \prod_l l^{e_l}$ for primes l.

Using Lemma 2.2, for $A = \mathcal{O}$ or $\mathbb{C}$, the Hecke operators on $H^1(\Gamma_0(N), L_{k-2}(\varepsilon, A))$ and $M_k(\Gamma_0(N), \varepsilon, A)$ can be described explicitly. We prove that the map from $M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$ to $H^1(\Gamma_0(N), L_{k-2}(\varepsilon, \mathbb{C}))$ sending G to the class of π_G is Hecke equivariant (see (2.2) below). In order to do it, we make the following calculations. We abbreviate $\Gamma_0(N)$ to Γ. We fix $G \in M_k(\Gamma_0(N), \varepsilon, \mathbb{C})$. For a prime number l, we put $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix}$, and $G' = G|[\Gamma\alpha\Gamma] \in M_k(\Gamma, \varepsilon, \mathbb{C})$. By the pull-back formula (0.1), for any $\gamma \in \Gamma$,

$$\begin{aligned}
\omega_{G'}(z_0)(\gamma) &= \int_{z_0}^{\gamma z_0} G'(z)(X - zY)^{k-2} dz \\
&= \sum_i \varepsilon(\alpha_i) \int_{z_0}^{\gamma z_0} (G|\alpha_i)(z)(X - zY)^{k-2} dz \\
&= \sum_i \alpha_i^{\iota} \bullet \int_{\alpha_i z_0}^{\alpha_i \gamma z_0} G(z)(X - zY)^{k-2} dz.
\end{aligned}$$

For any $\gamma \in \Gamma$, by the definition of π_G,

$$\pi_G(\gamma) = \int_{z_0}^{\gamma z_0} G(z)(X - zY)^{k-2} dz + (\gamma - 1) \bullet I_G(X, Y), \tag{2.1}$$

where

$$I_G(X, Y) = \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G}(z)(X - zY)^{k-2} dz - a(0, G) \int_0^{z_0} (X - zY)^{k-2} dz.$$

We simply write the above equation for

$$\pi_G(\gamma) = \omega_G(z_0)(\gamma) + (\gamma - 1) \bullet I_G(X, Y).$$

Further, for any $w \in \mathfrak{H}$, we define $F(z_0)(w) = \int_{z_0}^{w} G(z)(X - zY)^{k-2} dz - I_G(X, Y)$. For any $\gamma \in \Gamma$, we put $u(z_0)(w)(\gamma) = F(z_0)(\gamma w) - \gamma \bullet F(z_0)(w)$.

Then, for any $\gamma \in \Gamma$,

$$
\begin{aligned}
&u(z_0)(w)(\gamma)\\
&= \int_{z_0}^{\gamma w} G(z)(X-zY)^{k-2}dz - I_G(X,Y)\\
&\quad - \gamma \bullet \int_{z_0}^{w} G(z)(X-zY)^{k-2}dz + \gamma \bullet I_G(X,Y)\\
&= \int_{z_0}^{\gamma w} G(z)(X-zY)^{k-2}dz - \int_{\gamma z_0}^{\gamma w} G(z)(X-zY)^{k-2}dz + (\gamma-1)\bullet I_G(X,Y)\\
&= \int_{z_0}^{\gamma z_0} G(z)(X-zY)^{k-2}dz + (\gamma-1)\bullet I_G(X,Y)\\
&= \omega_G(z_0)(\gamma) + (\gamma-1)\bullet I_G(X,Y)\\
&= \pi_G(z_0)(\gamma).
\end{aligned}
$$

This value is independent of the choice of $w \in \mathfrak{H}$ and hence we simply write $u(z_0)(\gamma)$ instead of $u(z_0)(w)(\gamma)$. By the definition of $F(z_0)(w)$ and the above calculations, for any $\gamma \in \Gamma$, we get

$$
\begin{aligned}
\omega_{G'}(z_0)(\gamma) &= \sum_i \alpha_i^{\iota} \bullet (F(z_0)(\alpha_i\gamma z_0) - F(z_0)(\alpha_i z_0))\\
&= \sum_i \alpha_i^{\iota} \bullet (F(z_0)(\gamma_i\alpha_j z_0) - F(z_0)(\alpha_i z_0))\\
&= \sum_i \alpha_i^{\iota} \bullet (u(z_0)(\gamma_i) + \gamma_i \bullet F(z_0)(\alpha_j z_0)) - \sum_i \alpha_i^{\iota} \bullet F(z_0)(\alpha_i z_0)\\
&= u(z_0)\big|[\Gamma\alpha\Gamma](\gamma) + (\gamma-1)\bullet\left(\sum_i \alpha_i^{\iota}\bullet F(z_0)(\alpha_i z_0)\right).
\end{aligned}
$$

Then, by the above calculations, for any $\gamma \in \Gamma$, we obtain

$$
\begin{aligned}
(2.2)\quad \pi_{G'}(\gamma) &= \omega_{G'}(z_0)(\gamma) + (\gamma-1)\bullet I_{G'}(X,Y)\\
&= \pi_G\big|[\Gamma\alpha\Gamma](\gamma) + (\gamma-1)\bullet\left(I_{G'}(X,Y) + \sum_i \alpha_i^{\iota}\bullet F(z_0)(\alpha_i z_0)\right).
\end{aligned}
$$

We now prove the following proposition.

Proposition 2.3. — (1) *Suppose $G' = G|[\Gamma\alpha\Gamma] = \lambda(l,G)G$ for $\lambda(l,G) \in \mathbb{C}$. Then we have $\pi_{G'} = \lambda(l,G)\pi_G$.*

(2) *$I_{G'}(X,Y) + \sum_i \alpha_i^{\iota} \bullet F(z_0)(\alpha_i z_0) = \sum_i \alpha_i^{\iota} \bullet \pi_G(\alpha_i)$.*

(3) *Let $G \in M_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ be a Hecke eigenform, $\lambda(m,G)$ the eigenvalue of $T(m)$, $\varpi \in \mathcal{O}$ a uniformizer, and r a non-negative integer. Assume that $k < p$, $a(0,G) \equiv 0 \pmod{\varpi^r\mathcal{O}}$, and π_G is integral, that is, $\pi_G(\Gamma_0(N)) \subset L_{k-2}(\varepsilon,\mathcal{O})$. Then $[\pi_G]\big|T(m) = \lambda(m,G)[\pi_G]$ in $H^1(\Gamma_0(N), L_{k-2}(\varepsilon,\mathcal{O}/\varpi^r))$ for any positive integer m.*

Proof. — (1) It follows from (2.1).

(2) First we calculate $I_{G'}(X,Y)$. By the definition of $I_{G'}(X,Y)$, we have

$$
\begin{aligned}
I_{G'}(X,Y) &= \int_{z_0}^{\sqrt{-1}\infty} \widetilde{G'}(z)(X-zY)^{k-2}dz - a(0,G')\int_0^{z_0}(X-zY)^{k-2}dz \\
&= \sum_i \varepsilon(\alpha_i)\int_{z_0}^{\sqrt{-1}\infty} \widetilde{(G|\alpha_i)}(z)(X-zY)^{k-2}dz \\
&\quad - \sum_i \varepsilon(\alpha_i)a(0,G|\alpha_i)\int_0^{z_0}(X-zY)^{k-2}dz \\
&= \sum_i \alpha_i^{\iota}\bullet\int_{z_0}^{\sqrt{-1}\infty} \widetilde{(G|\alpha_i)}(z)\alpha_i\star(X-zY)^{k-2}dz \\
&\quad - \sum_i \alpha_i^{\iota}\bullet a(0,G|\alpha_i)\int_0^{z_0}\alpha_i\star(X-zY)^{k-2}dz.
\end{aligned}
$$

Therefore, by the definition of $\pi_G(\alpha_i)$, we have

$$
\begin{aligned}
&I_{G'}(X,Y) + \sum_i \alpha_i^{\iota}\bullet F(z_0)(\alpha_i z_0) \\
&\quad = I_{G'}(X,Y) + \sum_i \alpha_i^{\iota}\bullet\Bigg[\int_{z_0}^{\alpha_i z_0} G(z)(X-zY)^{k-2}dz \\
&\qquad - \int_{z_0}^{\sqrt{-1}\infty}\widetilde{G}(z)(X-zY)^{k-2}dz + a(0,G)\int_0^{z_0}(X-zY)^{k-2}dz\Bigg] \\
&\quad = \sum_i \alpha_i^{\iota}\bullet\pi_G(\alpha_i),
\end{aligned}
$$

as required.

(3) We fix a prime number l. Using Lemma 2.2, (2.2) and (2), we obtain

$$
\begin{aligned}
\pi_{G'}(\gamma) - \pi_G\big|[\Gamma\alpha\Gamma](\gamma) &= (\gamma-1)\bullet\left(I_{G'}(X,Y) + \sum_i \alpha_i^{\iota}\bullet F(z_0)(\alpha_i z_0)\right) \quad \text{(by (2.2))} \\
&= (\gamma-1)\bullet\left(\sum_i \alpha_i^{\iota}\bullet\pi_G(\alpha_i)\right) \qquad \text{(by (2))} \\
&= (\gamma-1)\bullet\left(\sum_i \alpha_i^{\iota}\bullet a(0,G)\int_0^{\frac{r_i}{l^{f_i}}}(X-zY)^{k-2}dz\right) \\
&\equiv 0 \pmod{\varpi^r\mathcal{O}}
\end{aligned}
$$

for any $\gamma\in\Gamma$. Here the third equality follows from Proposition 1.6 (2) and the last congruence follows from an explicit calculation with $\sum_{r=1}^{p-1} r^{j+1}\equiv 0 \pmod p$

for any non-negative integer j such that $j+1 < p-1$ if $l = p$. Therefore, by using (1), we prove (3). $\square$

2.3. Canonical periods. — We put $\Gamma_0 = \Gamma_0(N)$ and $\Gamma_1 = \Gamma_1(N)$. Let $f \in S_k(\Gamma_0, \varepsilon, \mathcal{O})$ be a normalized Hecke eigenform. We assume that if $k > 2$, then $(p, N) = 1$ and $k-2 < p$. Let Y denote the modular curve $\Gamma_1\backslash\mathfrak{H}$ with Γ_1-structure. Let $\mathcal{L}_{k-2}(\mathcal{O})$ be the local system on Y corresponding to the Γ_1-module $L_{k-2}(\mathcal{O})$. For a prime number l, we simply write $T_l = T(l)$ if $(l, N) = 1$ and $U_l = T(l)$ if $l|N$. We denote by $\mathfrak{M}_f$ a maximal ideal of the Hecke algebra generated by ϖ, $T_l - a(l, f)$ (for $(l, N) = 1$), $U_l - a(l, f)$ (for $l|N$), and $\langle d\rangle - \varepsilon(d)$. For $\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we consider the complex conjugation $[\Gamma_1\tau\Gamma_1]$ on $H^1_{\mathrm{par}}(\Gamma_1, L_{k-2}(\mathcal{O}))$ defined in §2.2. We note that the complex conjugation $[\Gamma_1\tau\Gamma_1]$ commutes with T_l and U_l for any prime number l because $\Gamma_1\tau\Gamma_1 = \Gamma_1\tau$ and $\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \Gamma_1 = \coprod_i \Gamma_1\alpha_i = \coprod_i \Gamma_1\tau^{-1}\alpha_i\tau$.

Proposition 2.4. — *For each parity $\alpha \in \{\pm 1\}$,*

the α-eigenspace $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^{\alpha}_{\mathfrak{M}_f}$ is free of rank 1 over $\mathcal{O}$.

Proof. — The Eichler-Shimura isomorphism and the q-expansion principle over $\mathbb{C}$ imply that

$$H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathbb{C}))^{\alpha}_{\mathfrak{M}_f} \simeq H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathbb{C}))^{\alpha}_{\mathfrak{M}_f}, \tag{2.3}$$

whose dimension over $\mathbb{C}$ is equal to 1. Then it suffices to show that $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. First suppose that $k = 2$. By considering the exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi} \mathcal{O} \to \mathcal{O}/\varpi \to 0$ and taking its cohomology, we see that $H^1_{\mathrm{c}}(Y, \mathcal{O})$ is torsion-free. Next suppose that $k > 2$. We note that, if $(p, N) = 1$ and $k-2 < p$, then

$$H^0(Y, \mathcal{L}_{k-2}(A)) \simeq H^0(\Gamma_1, L_{k-2}(A)) = 0 \text{ for } A = \mathcal{O},\ \mathcal{O}/\varpi \tag{2.4}$$

because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \in \Gamma_1$. Thus, by considering the exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi} \mathcal{O} \to \mathcal{O}/\varpi \to 0$ and taking its cohomology, we see that $H^1(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. In particular, $H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. Thus it suffices to show that the kernel of $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O})) \twoheadrightarrow H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ is torsion-free. The Gysin sequence with the help of (2.4) implies that the kernel is identified with the boundary cohomology of degree 0 and hence it is torsion-free as desired. $\square$

Proposition 2.5. — *For each parity $\alpha \in \{\pm 1\}$, the canonical morphism induces an isomorphism*

$$H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^{\alpha}_{\mathfrak{M}_f} \simeq H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^{\alpha}_{\mathfrak{M}_f}.$$

Proof. — It suffices to show the injectivity of this morphism. As mentioned in the proof of Proposition 2.4, both $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ and $H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$ are torsion-free. Hence the injectivity follows from the isomorphism (2.3). □

For each parity $\alpha \in \{\pm 1\}$, we define the canonical period Ω^α_f. We choose a generator $[\delta_f]^\alpha_{\mathrm{c}}$ (resp. $[\delta_f]^\alpha$) of $H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^\alpha_{\mathfrak{M}_f}$ (resp. $H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))^\alpha_{\mathfrak{M}_f}$).

Let $\Omega^\bullet(Y, \mathbb{C})$ denote the complex of $\mathbb{C}$-valued C^∞-differential Γ_1-invariant forms in $\mathfrak{H}$. Moreover, let $\Omega^\bullet_{\mathrm{c}}(Y, \mathbb{C})$ denote the complex of forms in $\Omega^\bullet(Y, \mathbb{C})$ which, together with their exterior differentials, are fast decreasing at each cusp $s \in Z(\Gamma_1)$. By [4, Theorem 5.2], we have

$$H^1_{\mathrm{dR}}(Y, \Omega^\bullet_{\mathrm{c}}(Y, \mathbb{C}) \otimes_{\mathbb{C}} L_{k-2}(\mathbb{C})) \simeq H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathbb{C})).$$

Let $[\omega_f]_{\mathrm{dR}} \in H^1_{\mathrm{dR}}(Y, \Omega^\bullet_{\mathrm{c}}(Y, \mathbb{C}) \otimes_{\mathbb{C}} L_{k-2}(\mathbb{C}))$ be the de Rham cohomology class attached to f. Let $[\omega_f]_{\mathrm{c}} \in H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathbb{C}))$ (resp. $[\omega_f] \in H^1_{\mathrm{par}}(Y, \mathcal{L}_{k-2}(\mathbb{C}))$) be the image of $[\omega_f]_{\mathrm{dR}}$. We note that, by (2.1), the cocycle π_f defines the same cohomology class as $\omega_f(z_0)$ and also $[\omega_f] = [\omega_f(z_0)]$ via the comparison theorem between Betti cohomology and group cohomology (cf. [3, Proposition 2.5]). By using the proof of Proposition 2.3, the Hecke eigenvalues of the cohomology classes $[\omega_f]_{\mathrm{c}}$ and $[\omega_f]$ are the same as those of f. We write $[\omega_f]^\alpha_{\mathrm{c}}$ and $[\omega_f]^\alpha$ for the projections to the α-parts. Thus, by Proposition 2.4 and Proposition 2.5, there exist complex numbers $\Omega^\alpha_{f,\mathrm{c}}$, $\Omega^\alpha_f \in \mathbb{C}^\times$ such that

$$\begin{aligned} [\omega_f]^\alpha_{\mathrm{c}} &= \Omega^\alpha_{f,\mathrm{c}} [\delta_f]^\alpha_{\mathrm{c}}, \\ [\omega_f]^\alpha &= \Omega^\alpha_f [\delta_f]^\alpha. \end{aligned} \tag{2.5}$$

We note that, by the definition, $\Omega^\alpha_{f,\mathrm{c}}$ is equal to Ω^α_f up to $\mathcal{O}^\times$.

Proposition 2.6. — *For each parity $\alpha \in \{\pm 1\}$, let*

$$\pi^\alpha_f = \frac{1}{2} \left(\pi_f + \alpha \pi_f | [\Gamma_1 \tau \Gamma_1] \right).$$

Then the image of Γ_0 under the map $\pi^\alpha_f / \Omega^\alpha_f$ and $\pi^\alpha_f(\sigma)/\Omega^\alpha_f \in L_{k-2}(\mathbb{C})$ are contained in $L_{k-2}(\mathcal{O})$.

Proof. — By the proof of Theorem 1.9, it suffices to show the integrality of the coefficients of $X^{k-2-j}Y^j$ in $D(f,1)/\Omega^\alpha_f \in L_{k-2}(\mathbb{C})$ for each j with $\alpha = (-1)^j$ and $D_{\left(\begin{smallmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{smallmatrix}\right)}(f,1)/\Omega^\alpha_f \in L_{k-2}(\mathbb{C})$ for each $m \in S$, i, and j with $\alpha = \chi(-1)(-1)^j$. Here we note that $\pi_f(\sigma) = -D(f,1)$ by Proposition 1.5 (1). In order to prove this integrality, we give a cohomological treatment of the special values of the L-functions.

Let χ be the trivial character or a Dirichlet character with conductor $m \in S$. We note that m is prime to p. Fix a representative set $\{b_i\}_i$ of $(\mathbb{Z}/m\mathbb{Z})^\times$ in $\mathbb{Z}$.

For each b_i, we consider the following subset H_{b_i} of $\mathfrak{H}$:

$$H_{b_i} = \frac{b_i}{m} + \sqrt{-1}\mathbb{R}_+^\times = \left\{ \frac{b_i}{m} + \sqrt{-1}y \;\middle|\; y \in \mathbb{R} \text{ with } y > 0 \right\}.$$

Then we have $H_{b_i} \to Y$ and it induces

$$H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(A)) \to H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \tag{2.6}$$

for $A = \mathcal{O}$, K, $\mathbb{C}$. Then, for each j with $0 \le j \le k-2$, we define the evaluation map

$$\mathrm{ev}^j_{b_i,A} : H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(A)) \to A \tag{2.7}$$

by the composition of

$$H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(A)) \xrightarrow{(2.6)} H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \xrightarrow{\begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix}} H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \tag{2.8}$$

and

$$H^1_{\mathrm{c}}(H_{b_i}, \mathcal{L}_{k-2}(A)) \xrightarrow{\text{coeff. of } X^{k-2-j}Y^j} H^1_{\mathrm{c}}(H_{b_i}, A) \xrightarrow{\text{trace}} A. \tag{2.9}$$

Here the second arrow of (2.8) is induced by

$$L_{k-2}(A) \to L_{k-2}(A)\,;\; P(X,Y) \mapsto \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star P(X,Y)$$

because m is prime to p, the first arrow of (2.9) is induced by

$$L_{k-2}(A) \to A\,;\; \sum_{j=0}^{k-2} a_j X^{k-2-j} Y^j \mapsto a_j,$$

and the second arrow of (2.9) is the trace map:

$$\omega \mapsto \int_{\sqrt{-1}\infty}^{\frac{b_i}{m}} \omega.$$

.

Proposition 2.7. — *Let χ be the trivial character or a character with conductor $m \in S$. Then*

$$-\tau(\bar{\chi}) D(f, \chi, 1) = \sum_{j=0}^{k-2} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}).$$

Proof. — Direct calculation shows that

$$\begin{aligned}\tau(\bar{\chi})D(f,\chi,1) &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star \int_0^{\sqrt{-1}\infty} (f\Big|\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix})(z)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star (X-zY)^{k-2}dz\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix} \star \int_{\frac{b_i}{m}}^{\sqrt{-1}\infty} f(z)(X-zY)^{k-2}dz\\ &= -\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}).\end{aligned}$$

Here the first equality follows from the proof of Lemma 1.8, the second equality follows from the pull-back formula (0.1), and the last equality follows from the definition of the evaluation map $\mathrm{ev}^j_{b_i,\mathbb{C}}$. □

We also treat the anti-holomorphic case.

Proposition 2.8. — *Under the same notation of Proposition 2.7,*

$$-\chi(-1)\tau^{\iota}\bullet\tau(\bar{\chi})D(f,\chi,1) = \sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1]).$$

Proof. — We note that $[\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1]$ corresponds to the de Rham cohomology class

$$\tau^{\iota}\bullet f(-\bar{z})(X-(-\bar{z})Y)^{k-2}d(-\bar{z}) = -f(-\bar{z})(X-\bar{z}Y)^{k-2}d\bar{z}$$

via the de Rham theorem (cf. [18, §6.4]). Thus,

$$\begin{aligned}&\sum_{j=0}^{k-2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]_{\mathrm{c}}|[\Gamma_1\tau\Gamma_1])\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\tau^{\iota}\bullet\int_{\sqrt{-1}\infty}^{\frac{b_i}{m}} f(-\bar{z})(X-(-\bar{z})Y)^{k-2}d(-\bar{z})\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\begin{pmatrix}1 & -\frac{b_i}{m}\\ 0 & 1\end{pmatrix}\tau^{\iota}\bullet\int_{\sqrt{-1}\infty}^{-\frac{b_i}{m}} f(z)(X-zY)^{k-2}dz\\ &= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\tau^{\iota}\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\bullet\int_{\sqrt{-1}\infty}^{-\frac{b_i}{m}} f(z)(X-zY)^{k-2}dz\\ &= \chi(-1)\tau^{\iota}\bullet\sum_{i=1}^{\varphi(m)} \bar{\chi}(-b_i)\begin{pmatrix}1 & \frac{b_i}{m}\\ 0 & 1\end{pmatrix}\star\int_{\sqrt{-1}\infty}^{-\frac{b_i}{m}} f(z)(X-zY)^{k-2}dz\\ &= -\chi(-1)\tau^{\iota}\bullet\tau(\bar{\chi})D(f,\chi,1).\end{aligned}$$

Here the third equality follows from

$$\begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \tau^{\iota} = \tau^{\iota} \begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}$$

and the last equality follows from the proof of Lemma 1.8. □

PROPOSITION 2.9. — *For each $\alpha \in \{\pm 1\}$, under the same notation of Proposition 2.7,*

$$\sum_{j=0}^{k-2} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]^{\alpha}_{\mathrm{c}}) = \tau(\bar{\chi}) \sum_{j=0}^{k-2} \left(\frac{1+\alpha\chi(-1)(-1)^j}{2} \right) \binom{k-2}{j} \cdot j! \left(\frac{1}{2\pi\sqrt{-1}} \right)^{j+1} L(f,\chi,j+1) X^{k-2-j} Y^j.$$

Proof. — Direct calculation shows that

$$\begin{aligned} \sum_{j=0}^{k-2} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \mathrm{ev}^j_{b_i,\mathbb{C}}([\omega_f]^{\alpha}_{\mathrm{c}}) &= \frac{1}{2} \sum_{j=0}^{k-2} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \mathrm{ev}^j_{b_i,\mathbb{C}} \left([\omega_f]_{\mathrm{c}} + \alpha [\omega_f]_{\mathrm{c}} | [\Gamma_1 \tau \Gamma_1] \right) \\ &= -\frac{1}{2} \tau(\bar{\chi}) D(f,\chi,1) - \frac{1}{2} \alpha \chi(-1) \tau^{\iota} \bullet \tau(\bar{\chi}) D(f,\chi,1). \end{aligned}$$

Here the last equality follows from Proposition 2.7 and Proposition 2.8. Hence, our proposition follows from (1.1):

$$D(f,\chi,1) = -\sum_{j=0}^{k-2} \binom{k-2}{j} j! \left(\frac{1}{2\pi\sqrt{-1}} \right)^{j+1} L(f,\chi,j+1) X^{k-2-j} Y^j. \quad □$$

Therefore, Proposition 2.6 follows from Proposition 2.9, the functoriality of the evaluation map $\mathrm{ev}^j_{b_i,A}$ for A, the integrality of $[\omega_f]^{\alpha}_{\mathrm{c}} / \Omega^{\alpha}_f \in H^1_{\mathrm{c}}(Y, \mathcal{L}_{k-2}(\mathcal{O}))$, Lemma 1.8, and (1.3). □

2.4. Congruences of special values. — For modular forms $f, g \in M_k(\Gamma, \mathcal{O})$ and a positive integer $r \in \mathbb{Z}$, we define a congruence of modular forms $f \equiv g \pmod{\varpi^r}$ by $a(m,f) \equiv a(m,g) \pmod{\varpi^r}$ for any integer $m \in \mathbb{Z}$.

THEOREM 2.10. — *Let p be an odd prime number, r a positive integer, and k an integer with $2 \le k \le p-1$. Let $f = \sum_{n=1}^{\infty} a(n,f)\, e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a p-ordinary normalized Hecke eigenform. Assume that the residual Galois representation $\bar{\rho}_f$ associated to f is reducible of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix},$$

and either ξ_1 or ξ_2 is unramified at p. Assume also that there exists an Eisenstein series $G = E_k(\psi_1, \psi_2) \in M_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ (for the definition, see Theorem 3.18) such that G satisfies the assumptions of Theorem 1.9 and $f \equiv$

$G \pmod{\varpi^r}$. Then there exist a parity $\alpha \in \{\pm 1\}$ (explicitly given by (A.27)), a complex number $\Omega_f^\alpha \in \mathbb{C}^\times$, and a p-adic unit $u \in \mathcal{O}^\times$ such that, for every primitive Dirichlet character χ whose conductor m_χ is prime to N, the following congruence holds:

(1) *if $(m_\chi, p) = 1$, then, for each j with $0 \leq j \leq k-2$ and $\alpha = \chi(-1)(-1)^j$,*

$$\tau(\bar{\chi}) \frac{L(f,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}\Omega_f^\alpha} \equiv u\tau(\bar{\chi}) \frac{L(G,\chi,1+j)}{(2\pi\sqrt{-1})^{1+j}} \pmod{\varpi^r};$$

(2) *if $p|m_\chi$, we assume that $m_\chi \in \varpi^r\mathcal{O}$, χ is non-exceptional (see Definition 2.11) and $\alpha = \chi(-1)$. Then*

$$\tau(\bar{\chi}) \frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^\alpha} \equiv u\tau(\bar{\chi}) \frac{L(G,\chi,1)}{2\pi\sqrt{-1}} \pmod{\varpi^r}.$$

Proof. — We put $\Gamma = \Gamma_0(N)$. By Proposition 2.3 (3), we get $[\pi_G]^\alpha \big| T(m) \equiv a(m,G)[\pi_G]^\alpha \pmod{\varpi^r}$ and $[\delta_f]^\alpha \big| T(m) \equiv a(m,f)[\delta_f]^\alpha \pmod{\varpi^r}$ for any positive integer m. We will see that $[\pi_G]^\alpha$ is non-trivial in $H^1_{\mathrm{par}}(\Gamma, L_{k-2}(\varepsilon, \mathcal{O}/\varpi))$ by a mod p non-vanishing theorem ([14, Lemma 3, page 430 (cf. the remark at the end of the proof, page 432)]) and (2.10) represented as below. Therefore, by Theorem A.12, there exists a p-adic unit $u \in \mathcal{O}^\times$ such that $[\delta_f]^\alpha = u[\pi_G]^\alpha$ in $H^1_{\mathrm{par}}(\Gamma_1(N), L_{k-2}(\mathcal{O}/\varpi^r))^\alpha[\mathfrak{M}_f] \simeq \mathcal{O}/\varpi^r$. Let $\delta_f^\alpha = \pi_f^\alpha/\Omega_f^\alpha$ which is integral by Proposition 2.6 and represents $[\delta_f]^\alpha$. Hence, for some $Q(X,Y) \in L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$, we obtain $\delta_f^\alpha - u\pi_G^\alpha = \partial Q(X,Y)$ in $Z^1(\Gamma, L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r))$.

Let χ be a non-trivial primitive Dirichlet character, whose conductor is denoted by m_χ. We fix $b_1, \ldots, b_{\varphi(m_\chi)} \in \mathbb{Z}$ such that $\{\bar{b}_1, \ldots, \bar{b}_{\varphi(m_\chi)}\} = (\mathbb{Z}/m_\chi\mathbb{Z})^\times$.

We consider in two cases.

(i) We treat the case $(p, m_\chi) = 1$.

We put $m = m_\chi$. For each b_i, we put

$$\gamma_{b_i} = \begin{pmatrix} a_i & b_i p^h \\ c_i p^h & m \end{pmatrix} \in \Gamma$$

for some choice of $a_i, c_i, h \in \mathbb{Z}$ with $p^h \in \varpi^r\mathcal{O}$. An explicit calculation with the cocycle condition (Proposition 1.6 (1)) and Theorem 1.9 shows that

$$\pi_G(\gamma_{b_i}\sigma) = \pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma) \in L_{k-2}(\varepsilon, \mathcal{O}).$$

Here we recall that $\pi_G(\sigma) = -D(G,1) \in L_{k-2}(\varepsilon, \mathcal{O})$ by Proposition 1.5 (1). By the choice of h, the action of γ_{b_i} on $L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$ is given by

$$\gamma_{b_i} \bullet P(X,Y) \equiv \varepsilon(m)P(mX, m^{-1}Y) \pmod{\varpi^r}.$$

We remark that the action of γ_{b_i} on $L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$ is independent of b_i. On the other hand, by using Proposition 1.6 (2) with $\pi_G(\gamma_{b_i}\sigma) \in L_{k-2}(\varepsilon, \mathcal{O})$ and our assumption, we get

$$\pi_G(\gamma_{b_i}\sigma) \equiv \pi_{G,\left(\begin{smallmatrix}1 & b_ip^h\\ 0 & m\end{smallmatrix}\right)}(\sigma) \pmod{\varpi^r}.$$

Here we remark that $a(0, G|\gamma_{b_i}\sigma) \in \mathcal{O}$. Therefore, by Lemma 1.8, computing modulo ϖ^r, we obtain

$$\begin{aligned}
\tau(\bar{\chi})D(G,\chi,1) &= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\begin{pmatrix}1 & -\frac{b_ip^h}{m}\\ 0 & 1\end{pmatrix} \star \pi_{G,\left(\begin{smallmatrix}1 & b_ip^h\\ 0 & m\end{smallmatrix}\right)}(\sigma)\\
&\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\begin{pmatrix}1 & -\frac{b_ip^h}{m}\\ 0 & 1\end{pmatrix} \star \pi_G(\gamma_{b_i}\sigma)\\
&\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G(\gamma_{b_i}\sigma)\\
&= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\{\pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma)\}\\
&\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G(\gamma_{b_i}) \pmod{\varpi^r}.
\end{aligned}$$

By definition, we recall that

$$\pi_G|[\Gamma\tau\Gamma](\gamma_{b_i}) = \tau^{\iota} \bullet \pi_G(\gamma'_{b_i}),$$

where

$$\gamma'_{b_i} = \tau\gamma_{b_i}\tau^{-1} = \begin{pmatrix}a_i & -b_ip^h\\ -c_ip^h & m\end{pmatrix} \in \Gamma.$$

In a similar way as above, we get

$$\chi(-1)\tau(\bar{\chi})D(G,\chi,1) \equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G(\gamma'_{b_i}) \pmod{\varpi^r}.$$

Therefore, computing modulo ϖ^r, we obtain

$$\begin{aligned}
\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\pi_G^{\alpha}(\gamma_{b_i}) &= \frac{1}{2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_ip^h)\left(\pi_G(\gamma_{b_i}) + \alpha\tau^{\iota} \bullet \pi_G(\gamma'_{b_i})\right)\\
&\equiv -\frac{1}{2}\left(1 + \alpha\chi(-1)\tau^{\iota}\right) \bullet \tau(\bar{\chi})D(G,\chi,1)\\
&= \tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\left(\frac{1+\alpha\chi(-1)(-1)^j}{2}\right)\\
&\quad \cdot j!\left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1} L(G,\chi,j+1)X^{k-2-j}Y^j.
\end{aligned} \tag{2.10}$$

Here the last equality follows from (1.1). We put

$$\tau(\bar{\chi})D(G,\chi,1)^{\alpha}=-\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})D(G,\chi,1).$$

By the cocycle condition (Proposition 1.6 (1)), we have

$$\begin{aligned}\pi_f(\gamma_{b_i}\sigma)&=\pi_f(\gamma_{b_i})+\gamma_{b_i}\bullet\pi_f(\sigma),\\(\pi_f|[\Gamma\tau\Gamma])(\gamma_{b_i}\sigma)&=\tau^{\iota}\bullet\pi_f(\gamma'_{b_i}(-\sigma))\\&=\tau^{\iota}\bullet\bigl(\pi_f(\gamma'_{b_i})+\gamma'_{b_i}\bullet\pi_f(-\sigma)\bigr)\\&=(\pi_f|[\Gamma\tau\Gamma])(\gamma_{b_i})+\gamma_{b_i}\bullet(\pi_f|[\Gamma\tau\Gamma])(\sigma).\end{aligned}$$

Thus we get

$$\delta_f^{\alpha}(\gamma_{b_i}\sigma)=\delta_f^{\alpha}(\gamma_{b_i})+\gamma_{b_i}\bullet\delta_f^{\alpha}(\sigma)\in L_{k-2}(\varepsilon,\mathcal{O}),$$

where the integrality follows from Proposition 2.6. On the other hand, we have

$$\begin{aligned}\delta_f^{\alpha}(\gamma_{b_i}\sigma)&=\frac{1}{2\Omega_f^{\alpha}}(\pi_f(\gamma_{b_i}\sigma)+\alpha(\pi_f|[\Gamma\tau\Gamma])(\gamma_{b_i}\sigma))\\&=\frac{1}{2\Omega_f^{\alpha}}\left(\pi_f(\gamma_{b_i}\sigma)+\alpha\tau^{\iota}\bullet\pi_f(\gamma'_{b_i}(-\sigma))\right)\\&=\frac{1}{2\Omega_f^{\alpha}}\left(\pi_{f,\left(\begin{smallmatrix}1&b_ip^h\\0&m\end{smallmatrix}\right)}(\sigma)+\alpha\tau^{\iota}\bullet\pi_{f,\left(\begin{smallmatrix}1&-b_ip^h\\0&m\end{smallmatrix}\right)}(\sigma)\right).\end{aligned}$$

Here the last equality follows from that $\pi_f(\gamma'_{b_i}(-\sigma))=\pi_f(\gamma'_{b_i}\sigma)$ and Proposition 1.6 (2). Therefore, by Lemma 1.8, computing modulo ϖ^r, we obtain

$$\begin{aligned}\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_ip^h)\delta_f^{\alpha}(\gamma_{b_i})&=\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_ip^h)\left(\delta_f^{\alpha}(\gamma_{b_i}\sigma)-\gamma_{b_i}\bullet\delta_f^{\alpha}(\sigma)\right)\\&\equiv\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_ip^h)\delta_f^{\alpha}(\gamma_{b_i}\sigma)\equiv\sum_{i=1}^{\varphi(m)}\bar{\chi}(b_ip^h)\begin{pmatrix}1&-\frac{b_ip^h}{m}\\0&1\end{pmatrix}\star\delta_f^{\alpha}(\gamma_{b_i}\sigma)\\&=-\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}\\&=\tau(\bar{\chi})\sum_{j=0}^{k-2}\binom{k-2}{j}\left(\frac{1+\alpha\chi(-1)(-1)^j}{2}\right)\\&\quad\cdot j!\left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1}\frac{L(f,\chi,j+1)}{\Omega_f^{\alpha}}X^{k-2-j}Y^j.\end{aligned}$$

We put

$$\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}^{\alpha}=-\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}.$$

Since $\delta_f^\alpha - u\pi_G^\alpha = \partial Q(X,Y)$ for some $Q(X,Y) \in L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$, we have

$$\begin{aligned}
\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^\alpha}^\alpha &- u\tau(\bar{\chi})D(G,\chi,1)^\alpha \pmod{\varpi^r} \\
&= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i p^h)(\gamma_{b_i} - 1) \bullet Q(X,Y) \\
&= \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i p^h)\{\varepsilon(m)Q(mX, m^{-1}Y) - Q(X,Y)\} \\
&= 0
\end{aligned}$$

if χ is non-trivial, as required.

(ii) We consider the case $p|m_\chi$.

This case is more difficult because the relation $\pi_G(\gamma_b\sigma) \equiv \pi_{G,\begin{pmatrix}1 & b\\ 0 & m_\chi\end{pmatrix}}(\sigma) \pmod{\varpi^r}$ does not hold, since m_χ is not invertible in $\mathcal{O}$. Thus this case is more delicate. In order to obtain the congruence for special values of L-functions, we will make the substitution $Y = 0$.

We put $m = m_\chi$. Let $\mathfrak{p}$ be the maximal ideal of $\mathcal{O}[\chi]$.

Definition 2.11. — We say that a Dirichlet character χ is non-exceptional at $\mathfrak{p}$ if χ satisfies the following three conditions:

(a) $p|m$;

(b) for each $j \in \{k-2, k-1\}$, $\bar{\chi}(x) \not\equiv x^j \pmod{\mathfrak{p}}$ for some $x \in \mathbb{Z}$;

(c) $\chi(x) \not\equiv x \pmod{\mathfrak{p}}$ for some $x \in \mathbb{Z}$.

Sublemma. — *Assume that χ is non-exceptional at $\mathfrak{p}$. Then,*

$$A(\chi) = \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\frac{b_i}{m}, \quad A_j(\chi) = \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i}{m}\right)^j \ (j \in \{k-2, k-1\})$$

are $\mathfrak{p}$-integral.

Proof. — We treat the case $A_j(\chi)$ (the case $A(\chi)$ is similar). Let $x \in \mathbb{Z}$ such that $\bar{\chi}(x) - x^j$ is a $\mathfrak{p}$-unit of $\mathcal{O}[\chi]$. If $(m,x) = 1$, then $\{b_i x\}_i$ is a set of representatives of $(\mathbb{Z}/m\mathbb{Z})^\times$ and hence we get

$$\begin{aligned}
(\bar{\chi}(x) - x^j)A_j(\chi) &= \sum_{i=1}^{\varphi(m)} \chi(b_i)\bar{\chi}(x)\left(\frac{b_i}{\varphi(m)}\right)^j - \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i x}{m}\right)^j \\
&\equiv \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i x}{m}\right)^j - \sum_{i=1}^{\varphi(m)} \chi(b_i)\left(\frac{b_i x}{m}\right)^j \mod \mathcal{O}[\chi] \\
&\equiv 0 \mod \mathcal{O}[\chi].
\end{aligned}$$

Suppose that $d = (m,x) \neq 1$. We put $m = dm'$ and $x = dx'$. Since χ is primitive, we have

$$\begin{aligned} x^j A_j(\chi) &= \sum_{i=1}^{\varphi(m)} \chi(b_i) \left(\frac{b_i x'}{m'}\right)^j \\ &\equiv \sum_{s=1}^{\varphi(m')} \left\{ \sum_{b_i \equiv s(\bmod m')} \chi(b_i) \right\} \left(\frac{sx'}{m'}\right)^j \mod \mathcal{O}[\chi] \\ &\equiv 0 \mod \mathcal{O}[\chi]. \end{aligned}$$

□

We define

$$\gamma_{b_i} = \begin{pmatrix} a_i p^h & b_i \\ c_i & m \end{pmatrix} \in \Gamma$$

for some choice of $a_i, c_i, h \in \mathbb{Z}$ with $p^h \in \varpi^r \mathcal{O}$. By the choice of h and our assumption that $m \in \varpi^r \mathcal{O}$, the action of γ_{b_i} on $L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$ is given by

$$\gamma_{b_i} \bullet P(X,Y) \equiv \varepsilon(m) P(-b_i Y, -c_i X) \pmod{\varpi^r}.$$

By the definition of γ_{b_i},

$$\gamma_{b_i}\sigma = \begin{pmatrix} a_i p^h & b_i \\ c_i & m \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b_i & -a_i p^h \\ m & -c_i \end{pmatrix}.$$

By the cocycle condition (Proposition 1.6 (1)) and Theorem 1.9, we have

$$\pi_G(\gamma_{b_i}\sigma) = \pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma) \in L_{k-2}(\varepsilon, \mathcal{O}).$$

On the other hand, by using Proposition 1.6 (2), we get

$$\begin{aligned} \pi_G(\gamma_{b_i}\sigma) = a(0,G) &\int_0^{\frac{b_i}{m}} (X - zY)^{k-2} dz \\ &+ \varepsilon(m) a(0, G|\sigma) \int_{\frac{c_i}{m}}^{0} \gamma_{b_i}\sigma \star (X - zY)^{k-2} dz + \pi_{G, \begin{pmatrix} 1 & b_i \\ 0 & m \end{pmatrix}}(\sigma). \end{aligned}$$

Therefore, by Lemma 1.8, we obtain

$$\begin{aligned} \tau(\bar{\chi}) D(G,\chi,1) &= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_{G, \begin{pmatrix} 1 & b_i \\ 0 & m \end{pmatrix}}(\sigma) \\ &= -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_G(\gamma_{b_i}\sigma) \\ &\quad - \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \left\{ -a(0,G) \int_0^{\frac{b_i}{m}} (X - zY)^{k-2} dz \right\} \end{aligned}$$

$$-\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \left\{ -\varepsilon(m) a(0, G|\sigma) \int_{\frac{c_i}{m}}^{0} \gamma_{b_i}\sigma \star (X - zY)^{k-2} dz \right\}.$$

Then an explicit calculation shows that the coefficients of X^{k-2} in the second and final terms are

$$a(0,G) \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \frac{b_i}{m}, \quad \varepsilon(m) a(0, G|\sigma) \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(-1)^{k-j-1}}{j+1} \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \frac{c_i^{k-1}}{m},$$

respectively. Thus they are integral and congruent to 0 modulo ϖ^r, since both $A(\chi)$ and $A_{k-1}(\chi)$ are integral by Sublemma and both $a(0, G)$ and $a(0, G|\sigma)$ belongs to $\varpi^r \mathcal{O}$ by our assumptions. Therefore, in the same way as the case (i), computing modulo ϖ^r, we obtain

$$\begin{aligned} \tau(\bar{\chi}) D(G, \chi, 1)\Big|_{Y=0} &\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star \pi_G(\gamma_{b_i}\sigma)\Big|_{Y=0} \\ &\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \pi_G(\gamma_{b_i}\sigma)\Big|_{Y=0} \\ &\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \{\pi_G(\gamma_{b_i}) + \gamma_{b_i} \bullet \pi_G(\sigma)\}\Big|_{Y=0} \\ &\equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \pi_G(\gamma_{b_i})\Big|_{Y=0} \pmod{\varpi^r}. \end{aligned}$$

Here the last equality follows from that, for any $P(X, Y) \in L_{k-2}(\mathcal{O})$,

$$\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \gamma_{b_i} \bullet P(X, Y)\Big|_{Y=0} \equiv 0 \pmod{\varpi^r}, \tag{2.11}$$

which is obtained by $m \in \varpi^r \mathcal{O}$ and $A_{k-2}(\chi)$ is integral by Sublemma. Similarly as above by substituting $-b_i$ and γ'_{b_i} for b_i and γ_{b_i} respectively, we have

$$\chi(-1)\tau(\bar{\chi}) D(G, \chi, 1)\Big|_{Y=0} \equiv -\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i) \pi_G(\gamma'_{b_i})\Big|_{Y=0} \pmod{\varpi^r},$$

where

$$\gamma'_{b_i} = \tau \gamma_{b_i} \tau^{-1} = \begin{pmatrix} a_i p^h & -b_i \\ -c_i & m \end{pmatrix} \in \Gamma.$$

Therefore, computing modulo ϖ^r, we obtain

$$\begin{aligned}(2.12)\quad \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\pi_G^{\alpha}(\gamma_{b_i})\Big|_{Y=0} &= \frac{1}{2}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\left(\pi_G(\gamma_{b_i}) + \alpha\tau^{\iota}\bullet \pi_G(\gamma'_{b_i})\right)\Big|_{Y=0} \\ &\equiv -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet \tau(\bar{\chi})D(G,\chi,1)\Big|_{Y=0} \\ &= \tau(\bar{\chi})\left(\frac{1+\alpha\chi(-1)}{2}\right)\frac{L(G,\chi,1)}{2\pi\sqrt{-1}}X^{k-2}.\end{aligned}$$

Here the last equality follows from (1.1). We put

$$\tau(\bar{\chi})D(G,\chi,1)^{\alpha} = -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet \tau(\bar{\chi})D(G,\chi,1)\Big|_{Y=0}.$$

In the same way as the case (i) with the help of (2.11), computing modulo ϖ^r, we obtain

$$\begin{aligned}\sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\delta_f^{\alpha}(\gamma_{b_i})\Big|_{Y=0} &\equiv -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet \tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}\Big|_{Y=0} \\ &= \tau(\bar{\chi})\left(\frac{1+\alpha\chi(-1)}{2}\right)\frac{L(f,\chi,1)}{(2\pi\sqrt{-1})\Omega_f^{\alpha}}X^{k-2}\end{aligned}$$

and put

$$\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}^{\alpha} = -\frac{1}{2}\left(1+\alpha\chi(-1)\tau^{\iota}\right)\bullet \tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}\Big|_{Y=0}.$$

Since $\delta_f^{\alpha} - u\pi_G^{\alpha} = \partial Q(X,Y)$ for some $Q(X,Y) \in L_{k-2}(\varepsilon, \mathcal{O}/\varpi^r)$, we have

$$\begin{aligned}\tau(\bar{\chi})\frac{D(f,\chi,1)}{\Omega_f^{\alpha}}^{\alpha} - u\tau(\bar{\chi})D(G,\chi,1)^{\alpha} &\equiv \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)(\gamma_{b_i}-1)\bullet Q(X,Y)\Big|_{Y=0} \\ &\equiv \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)\{\varepsilon(m)Q(-b_iY,-c_iX) - Q(X,Y)\}\Big|_{Y=0} \\ &\equiv 0 \pmod{\varpi^r} \quad \text{(by (2.11))}\end{aligned}$$

if χ is non-trivial and non-exceptional. We have completed the proof of our theorem. □

3. Application to the Iwasawa invariants

In this section, we first compare the Iwasawa invariant of non-primitive Selmer groups associated to modular forms with that of Selmer groups associated to Dirichlet characters. Next, in order to provide evidence for the Iwasawa main conjecture, we prove congruences between the p-adic L-function of a certain cusp form and a product of two Kubota-Leopoldt p-adic L-functions.

3.1. Iwasawa modules. — In this subsection, we summarize basic results on Iwasawa modules to define the Iwasawa invariants. We refer the reader to [42] for proofs. Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$, ϖ a uniformizer, and $\Lambda = \mathcal{O}[[T]]$ the power series ring in one variable T over $\mathcal{O}$.

Definition 3.1. — A polynomial $P(T) \in \mathcal{O}[T]$ is said to be distinguished if $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$ with $a_i \in \varpi\mathcal{O}$ for $0 \le i \le n-1$.

Theorem 3.2 (Weierstrass Preparation Theorem). — *If $f(T) \in \Lambda$ is non-zero, then we may uniquely write*

$$f(T) = \varpi^{\mu} P(T) U(T),$$

where $U(T) \in \Lambda$ is a unit, $P(T)$ is a distinguished polynomial, and μ is a non-negative integer.

For a non-zero element $f(T) \in \Lambda$, we define the Iwasawa λ-invariant and the Iwasawa μ-invariant of $f(T)$ by

$$\lambda(f(T)) = \deg(P(T)),\ \ \mu(f(T)) = \mu,$$

respectively.

Definition 3.3. — Two Λ-modules M and M' are said to be pseudo-isomorphic and we write $M \sim M'$, if there is a homomorphism $M \to M'$ with finite kernel and cokernel.

Theorem 3.4. — *Let M be a finitely generated Λ-module. Then*

$$M \sim \Lambda^{\oplus r} \oplus \left(\bigoplus_{i=1}^{s} \Lambda/(\varpi^{m_i}) \right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(f_j(T)^{n_j}) \right)$$

for some non-negative integers r, s, t, m_i, n_j, and distinguished and irreducible polynomials $f_j(T)$ for $1 \le j \le t$.

We say that a Λ-module is a torsion Λ-module if every element is annihilated by some power of the maximal ideal (ϖ, T). If M is a finitely generated torsion Λ-module, then $r = 0$. We define the Iwasawa λ-invariant, the Iwasawa μ-invariant, and the characteristic ideal of M by

$$\lambda(M) = \sum_{j=1}^{t} \deg(f_j(T)^{n_j}),\ \ \mu(M) = \sum_{i=1}^{s} m_i,\ \ \mathrm{Char}_{\Lambda}(M) = \left(\prod_{i=1}^{s} \varpi^{m_i} \prod_{j=1}^{t} f_j(T)^{n_j} \right),$$

respectively.

For a number field K, let K_∞ denote the cyclotomic $\mathbb{Z}_p$-extension of K and $\Lambda = \mathbb{Z}_p[[T]] \simeq \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n]$. Then $\mathrm{Gal}(K_\infty/K)$ is a finitely generated torsion Λ-module.

THEOREM 3.5 (Ferrero-Washington [11]). — *Let K be a finite abelian extension of $\mathbb{Q}$ and p a prime number. Then $\mu(\mathrm{Gal}(K_\infty/K))$ for K is equal to zero.*

3.2. Selmer groups. — We will recall general results on Selmer groups. We omit details, which can be found in [15], [16]. Let Σ be a finite set of primes of $\mathbb{Q}$ containing p and ∞, and let $\mathbb{Q}_\Sigma$ be the maximal extension of $\mathbb{Q}$ which is unramified outside Σ. Let F_p be a finite extension of $\mathbb{Q}_p$ and V_p a finite dimensional F_p-vector space endowed with a continuous F_p-linear action of $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. We put $d = \dim_{F_p}(V_p)$. Let $\mathcal{O}$ denote the ring of integers of F_p. Choose a $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$-stable $\mathcal{O}$-lattice T_p in V_p. We put $A = V_p/T_p$. Then A is a discrete $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$-module which is isomorphic to $(F_p/\mathcal{O})^d$ as an $\mathcal{O}$-module. We denote by $d^\pm$ the dimension of the (± 1)-eigenspaces of complex conjugation acting on V_p, respectively. Then we have $d = d^+ + d^-$. Since we have fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we can identify $G_{\mathbb{Q}_p}$ with a decomposition group for some prime of $\overline{\mathbb{Q}}$ above p. We will assume that V_p is ordinary at p, that is, V_p contains an F_p-vector subspace F^+V_p of dimension d^+ which is stable under the action of $G_{\mathbb{Q}_p}$. Let F^+A denote the image of F^+V_p in A under the canonical map $V_p \to A$.

$$\begin{tikzcd}
0 \arrow[r] & F^+V_p \arrow[r] \arrow[d] & V_p \arrow[d] & & \\
0 \arrow[r] & F^+A \arrow[r] & A = V_p/T_p \arrow[r] & A/F^+A \arrow[r] & 0.
\end{tikzcd}$$

For a pair (A, F^+A), we define the Selmer group of A in the sense of Greenberg [15] by

$$S_A(\mathbb{Q}_\infty) = S(\mathbb{Q}_\infty; A, F^+A) = \ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to \prod_{l\in\Sigma} \mathcal{H}_l(\mathbb{Q}_\infty, A)\right),$$

where $\mathcal{H}_l(\mathbb{Q}_\infty, A)$ is defined as follows: if $l \neq p$, we let

$$\mathcal{H}_l(\mathbb{Q}_\infty, A) = \prod_{\eta | l} H^1((\mathbb{Q}_\infty)_\eta, A),$$

where the product is taken over the finite set of primes η of $\mathbb{Q}_\infty$ lying above l. There is a unique prime η_p of $\mathbb{Q}_\infty$ lying above p. Let I_{η_p} denote the inertia subgroup of $G_{(\mathbb{Q}_\infty)_{\eta_p}}$. We define

$$\mathcal{H}_p(\mathbb{Q}_\infty, A) = \mathcal{H}_p(\mathbb{Q}_\infty; A, F^+A) = \mathrm{im}\left(H^1((\mathbb{Q}_\infty)_{\eta_p}, A) \to H^1(I_{\eta_p}, A/F^+A)\right).$$

We define the Iwasawa algebra Λ by $\Lambda = \mathcal{O}[[\Gamma]]$, where $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We know that the groups $H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)$, $H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)$, $\mathcal{H}_l(\mathbb{Q}_\infty, A)$, and $S_A(\mathbb{Q}_\infty)$ are discrete $\mathcal{O}$-modules with a natural continuous action of Γ. Hence these groups are regarded as Λ-modules and are known to be cofinitely generated, that is, their Pontryagin duals are finitely generated Λ-modules. The following corank formulas follow from the results in [15, §3, §4]:

Proposition 3.6. — *The following statements hold:*
(1) $\mathrm{corank}_\Lambda(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A)) = d^- + \mathrm{corank}_\Lambda(H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A))$.
(2) $\mathrm{corank}_\Lambda(\mathcal{H}_p(\mathbb{Q}_\infty, A)) = d^-$.
(3) $\mathrm{corank}_\Lambda(\mathcal{H}_l(\mathbb{Q}_\infty, A)) = 0$ *if* $l \neq p$.

We always assume that $S_A(\mathbb{Q}_\infty)$ is Λ-cotorsion in the rest of §3.2. Put $A^* = \mathrm{Hom}(T_p, \mu_{p^\infty})$. This is also a discrete $\mathcal{O}$-module equipped with a continuous action of $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. The next proposition, which is proved in [16, Proposition 2.1], is important in this paper.

Proposition 3.7. — *Assume that $S_A(\mathbb{Q}_\infty)$ is Λ-cotorsion and $H^0(\mathbb{Q}_\infty, A^*)$ is finite. Then the following sequence is exact:*

$$0 \to S_A(\mathbb{Q}_\infty) \to H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to \prod_{l\in\Sigma} \mathcal{H}_l(\mathbb{Q}_\infty, A) \to 0.$$

Next we recall the non-primitive Selmer groups of A in the sense of Greenberg. Let Σ_0 be any finite subset of Σ which does not contain neither p nor ∞. The non-primitive Selmer groups for (A, F^+A) and Σ_0 is defined by

$$S_A^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; A, F^+A) = \ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A) \to \prod_{l\in\Sigma\setminus\Sigma_0} \mathcal{H}_l(\mathbb{Q}_\infty, A)\right).$$

We have $S_A(\mathbb{Q}_\infty) \subset S_A^{\Sigma_0}(\mathbb{Q}_\infty)$ by the definition. We denote by $M^\vee$ the Pontryagin dual of any locally compact $\mathbb{Z}_p$-module M. We obtain the following corollary of Proposition 3.6 (3), Proposition 3.7, and [15, Proposition 2], which is proved in [16, Corollary 2.3].

Corollary 3.8. — *Under the assumption as in Proposition 3.7, we have*

$$S_A^{\Sigma_0}(\mathbb{Q}_\infty)/S_A(\mathbb{Q}_\infty) \cong \prod_{l\in\Sigma_0} \mathcal{H}_l(\mathbb{Q}_\infty, A)$$

as Λ-modules. In particular, $S_A^{\Sigma_0}(\mathbb{Q}_\infty)$ is Λ-cotorsion, and the following equalities hold:

$$\mathrm{corank}_\mathcal{O}(S_A^{\Sigma_0}(\mathbb{Q}_\infty)) = \mathrm{corank}_\mathcal{O}(S_A(\mathbb{Q}_\infty)) + \sum_{l\in\Sigma_0} \mathrm{corank}_\mathcal{O}(\mathcal{H}_l(\mathbb{Q}_\infty, A)),$$

$$\mu(S_A^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \mu(S_A(\mathbb{Q}_\infty)^\vee).$$

Next, in order to compare $\mathrm{corank}_{\mathcal{O}}(S_A^{\Sigma_0}(\mathbb{Q}_\infty))$ with $\mathrm{corank}_{\mathcal{O}}(S_A(\mathbb{Q}_\infty))$, we would like to find a generator of $\mathcal{H}_l(\mathbb{Q}_\infty, A)^\vee$. The following proposition is the result in [16, Proposition 2.4].

PROPOSITION 3.9. — *Let l be a prime number with $l \neq p$. Put $P_l(X) = \det((1 - \mathrm{Frob}_l\, X)|_{(V_p)_{I_l}}) \in \mathcal{O}[X]$ and $\mathcal{P}_l = P_l(l^{-1}\gamma_l) \in \Lambda = \mathcal{O}[[\Gamma]]$, where γ_l denotes the Frobenius automorphism corresponding to the prime l in $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. The characteristic ideal of the Λ-module $\mathcal{H}_l(\mathbb{Q}_\infty, A)^\vee$ is generated by $\mathcal{P}_l$.*

Let ϖ be a uniformizer of $\mathcal{O}$. Let $A[\varpi]$ denote the ϖ-torsion of A. We now define a Selmer group of $A[\varpi]$. For any subset Σ_0 of $\Sigma - \{p, \infty\}$, we define

$$\begin{aligned} S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty) &= S^{\Sigma_0}(\mathbb{Q}_\infty; A[\varpi], F^+A[\varpi]) \\ &= \ker\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A[\varpi]) \to \prod_{l \in \Sigma \setminus \Sigma_0} \mathcal{H}_l(\mathbb{Q}_\infty, A[\varpi]) \right), \end{aligned}$$

where $\mathcal{H}_l(\mathbb{Q}_\infty, A[\varpi])$ is defined by

$$\mathcal{H}_l(\mathbb{Q}_\infty, A[\varpi]) = \begin{cases} \prod_{\eta | l} H^1(I_\eta, A[\varpi]) & \text{if } l \neq p, \\ H^1(I_{\eta_p}, A[\varpi]/F^+A[\varpi]) & \text{if } l = p. \end{cases}$$

Under certain hypotheses, the next proposition obtained by [16, Proposition 2.8] allows us to describe $\lambda(S_A^{\Sigma_0}(\mathbb{Q}_\infty))$ in terms of the Galois module $A[\varpi]$. We put $\mathrm{Ram}(A) = \{l | l \neq p, \infty$ and the action of $G_{\mathbb{Q}_l}$ on A is ramified$\}$.

PROPOSITION 3.10. — *Let p be an odd prime number and Σ_0 a subset of $\Sigma - \{p, \infty\}$ containing $\mathrm{Ram}(A)$. Assume that I_{η_p} acts trivially on A/F^+A and $H^0(\mathbb{Q}_\infty, A) = 0$. Then we have*

$$S_A^{\Sigma_0}(\mathbb{Q}_\infty)[\varpi] \cong S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty).$$

Consequently, $S_A(\mathbb{Q}_\infty)$ is Λ-cotorsion, and has μ-invariant is zero if and only if $S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)$ is finite. If this is the case,

$$\lambda(S_A^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \dim_{\mathcal{O}/\varpi\mathcal{O}}(S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)).$$

Now we apply the general theory recalled above to the Iwasawa main conjecture of modular forms. Let $f = \sum_{n=1}^\infty a(n, f)e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ be a normalized Hecke eigenform and

$$\rho_f \colon G_{\mathbb{Q}} \to \mathrm{GL}(T_f) \simeq \mathrm{GL}_2(\mathcal{O})$$

the associated Galois representation, which satisfies

(1) ρ_f is unramified at all primes $l \nmid Np$,
(2) $\mathrm{Tr}(\rho_f(\mathrm{Frob}_l)) = a(l, f)$ for $l \nmid Np$,
(3) $\det(\rho_f(\mathrm{Frob}_l)) = \varepsilon(l)l^{k-1}$ for $l \nmid Np$,

(4) ρ_f is odd.

We write κ for the residue field of $\mathcal{O}$. Let $A_f = T_f \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p)$ denote the cofree $\mathcal{O}$-module of corank 2 with $G_{\mathbb{Q}}$-action via ρ_f. We assume that

(RR) the residual representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\kappa)$ is reducible.

Then $\bar{\rho}_f$ is of the form

$$\bar{\rho}_f \sim \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix},$$

that is, there exists an exact sequence

$$0 \to \Phi \to A_f[\varpi] \to \Psi \to 0 \tag{3.1}$$

of $\kappa[G_{\mathbb{Q}}]$-modules, where $G_{\mathbb{Q}}$ acts on Ψ via the character $\psi\colon G_{\mathbb{Q}} \to \kappa^\times$, and on Φ via the character $\varphi\colon G_{\mathbb{Q}} \to \kappa^\times$.

Hereafter we assume that f is p-ordinary. From the result of [27] and [43, Theorem 2.1.4], the restriction of ρ_f to the decomposition group D_p is of the form

$$\rho_f\big|_{D_p} \sim \begin{pmatrix} \chi_p^{k-1}\rho_1 & * \\ 0 & \rho_2 \end{pmatrix},$$

where $\rho_1, \rho_2\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are unramified characters such that ρ_2 sends the arithmetic Frobenius to a unit-root of $X^2 - a(p,f)X + \varepsilon(p)p^{k-1} = 0$ and χ_p is the p-adic cyclotomic character. Then F^+A_f is defined by the following exact sequence of $\mathcal{O}[G_{\mathbb{Q}_p}]$-modules:

$$0 \to F^+A_f \to A_f \to A_f/F^+A_f \to 0, \tag{3.2}$$

where $G_{\mathbb{Q}_p}$ acts on F^+A_f via the character $\chi_p^{k-1}\rho_1\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$, and on A_f/F^+A_f via the character $\rho_2\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$. We can define the Selmer group of (A_f, F^+A_f) by

$$S_{A_f}(\mathbb{Q}_\infty) = S(\mathbb{Q}_\infty; A_f, F^+A_f).$$

Let $\Sigma_0 = \{l \in \mathbb{N} | l \text{ is a prime number such that } l|N\}$ and $\Sigma = \Sigma_0 \cup \{p, \infty\}$ a finite set of places of $\mathbb{Q}$. Then the non-primitive Selmer group of A_f is defined by

$$S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; A_f, F^+A_f).$$

We assume that $2 \le k \le p-1$ and

(Assumption) ψ is unramified at p and odd, and
φ is ramified at p and even.

Hence $\psi(\mathrm{Frob}_p) \equiv a(p,f) (\mathrm{mod}\ \varpi)$.

LEMMA 3.11. — *We assume that φ and ψ are as above and p is odd. Then we have*

$$H^0(\mathbb{Q}, A_f[\varpi]) = 0.$$

Proof. — Since φ is ramified at p, $H^0(\mathbb{Q},\Phi) \subset H^0(G_{\mathbb{Q}_p},\Phi) = 0$. Since ψ is odd and p is odd, $H^0(\mathbb{Q},\Psi) \subset H^0(\langle c\rangle,\Psi) = 0$, where $c \in G_{\mathbb{Q}}$ is the complex conjugation. □

LEMMA 3.12. — *Suppose that ψ is odd and p is odd. Then,*

$$H^0(\mathbb{Q}_\infty,\Psi) = 0.$$

Proof. — Since ψ is odd and p is odd, $H^0(\mathbb{Q}_\infty,\Psi) \subset H^0(\langle c\rangle,\Psi) = 0$, where $c \in G_{\mathbb{Q}_\infty}$ is the complex conjugation. □

LEMMA 3.13. — *Assume that p is odd. Then,*

$$H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,\Phi) = 0.$$

Proof. — For a Galois module $A \cong F_p/\mathcal{O}$ via the character $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty) \xrightarrow{\varphi} \kappa^\times \hookrightarrow \mathcal{O}^\times$, we have

$$0 \to \Phi \to A \xrightarrow{\varpi} A \to 0.$$

Therefore, in order to prove the lemma, it is enough to show that

$$\text{(i) } H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,A) \text{ is divisible, and}$$
$$\text{(ii) } H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,A) = 0.$$

Indeed, we have an exact sequence

$$H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,A) \xrightarrow{\varpi} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,A) \to H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,\Phi) \to H^2(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,A)$$

as a part of the cohomology long exact sequence. The proof of (i) and (ii) can be found in [16, p.46] just after the equation (16) under the assumptions that φ is even and non-trivial. □

Therefore, using Lemma 3.12 and Lemma 3.13, we have an exact sequence

$$0 \to H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,\Phi) \xrightarrow{\alpha} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,A_f[\varpi]) \xrightarrow{\beta} H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,\Psi) \to 0.$$

By this exact sequence and the definition of $S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)$, $S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty)$, and $S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty)$, we have

$$S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)/S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty) = S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty).$$

Here, by the definition, $S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty) = \ker(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,\Psi) \to H^1(I_{\eta_p},\Psi))$ and $S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty) = H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty,\Phi)$. Hence we have

$$\dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)) = \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty)) + \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty)).$$

We compute the Selmer groups for one-dimensional representations V_p with some assumptions. The Galois group $\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$ acts on V_p via a continuous homomorphism $\theta\colon \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}) \to \mathcal{O}^\times$. Then, θ factors through $G = \mathrm{Gal}(K_\infty/\mathbb{Q})$, where K_∞ is a certain finite extension of $\mathbb{Q}_\infty$ such that K_∞ is an abelian extension over $\mathbb{Q}$. We put $\Delta = \mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)$ and assume that

$(p, \sharp\Delta) = 1$. We can identify Γ with a subgroup of G such that $G = \Delta \times \Gamma$. This decomposition is unique for our case $(p, \sharp\Delta) = 1$. We have $\mathbb{Z}_p[[G]] = \mathbb{Z}_p[\Delta][[\Gamma]]$.

Let $X_\infty = \mathrm{Gal}(M_\infty/K_\infty)$ and $Y_\infty = \mathrm{Gal}(L_\infty/K_\infty)$. Here M_∞ denotes the maximal abelian pro-p extension of K_∞ which is unramified outside $\{p, \infty\}$, and L_∞ denotes the maximal abelian pro-p extension of K_∞ which is unramified everywhere. Let $\xi = \theta|_\Delta$ be the restriction of θ to Δ and $\Sigma_0 = \Sigma - \{p, \infty\}$. If θ is even (resp. odd), then $d^+ = 1$ (resp. $d^+ = 0$) and we have $F^+V_p = V_p$ (resp. $F^+V_p = 0$).

PROPOSITION 3.14 ([16], p.45, 46). — *The Selmer groups for one-dimensional representations have the following properties.*

(1)

$$S_A(\mathbb{Q}_\infty) \simeq \begin{cases} \mathrm{Hom}_{\mathcal{O}}((X_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A) \text{ if } \theta \text{ is even,} \\ \mathrm{Hom}_{\mathcal{O}}((Y_\infty \otimes_{\mathbb{Z}_p} \mathcal{O})^\xi, A) \text{ if } \theta \text{ is odd.} \end{cases}$$

(2) *The Λ-modules $S_A(\mathbb{Q}_\infty)$ and $S_A^{\Sigma_0}(\mathbb{Q}_\infty)$ are cotorsion, and we have*

$$\mu(S_A(\mathbb{Q}_\infty)^\vee) = \mu(S_A^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = 0.$$

.

(3) *Assume that ξ is non-trivial if θ is even, and $\xi \neq \omega$ if θ is odd. Then we have*

$$\begin{aligned} \dim_{\mathcal{O}/\varpi\mathcal{O}}(S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)) &= \mathrm{corank}_{\mathcal{O}}(S_A^{\Sigma_0}(\mathbb{Q}_\infty)) \\ &= \mathrm{corank}_{\mathcal{O}}(S_A(\mathbb{Q}_\infty)) + \sum_{l \in \Sigma_0} \mathrm{corank}_{\mathcal{O}}(\mathcal{H}_l(\mathbb{Q}_\infty, A)). \end{aligned}$$

In particular, $S_{A[\varpi]}^{\Sigma_0}(\mathbb{Q}_\infty)$ is finite.

We can apply these results to $(A, \theta) = (A_\varphi, \widetilde{\varphi})$ (resp. $(A_\psi, \widetilde{\psi})$) for a Galois module $A_\varphi \cong F_p/\mathcal{O}$ (resp. $A_\psi \cong F_p/\mathcal{O}$) via the character $\widetilde{\varphi} = \chi_p^{k-1}\varepsilon\widetilde{\psi}^{-1} : G_{\mathbb{Q}} \to \kappa^\times \hookrightarrow \mathcal{O}^\times$ (resp. $\widetilde{\psi} : G_{\mathbb{Q}} \to \kappa^\times \hookrightarrow \mathcal{O}^\times$). We remark that $A_\varphi[\varpi] = \Phi$ and $A_\psi[\varpi] = \Psi$.

(i) We consider $S_\Phi^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; \Phi, \Phi)$.
Since φ is even and ramified at p by our assumption, it is non-trivial. Therefore we have $\mu(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = 0$ and

$$\dim_{\mathcal{O}/\varpi\mathcal{O}}(S_\Phi^{\Sigma_0}(\mathbb{Q}_\infty)) = \mathrm{corank}_{\mathcal{O}}(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)).$$

(ii) We consider $S_\Psi^{\Sigma_0}(\mathbb{Q}_\infty) = S^{\Sigma_0}(\mathbb{Q}_\infty; \Psi, 0)$.
Since ψ is odd and unramified at p, we have $\psi \neq \omega$. Therefore, we have $\mu(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = 0$ and

$$\dim_{\mathcal{O}/\varpi\mathcal{O}}(S_\Psi^{\Sigma_0}(\mathbb{Q}_\infty)) = \mathrm{corank}_{\mathcal{O}}(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)).$$

We define the Iwasawa λ-invariants by

$$\lambda_{\varphi,\Sigma_0} = \mathrm{corank}_{\mathcal{O}}(S^{\Sigma_0}_{A_\varphi}(\mathbb{Q}_\infty)),\ \lambda_{\psi,\Sigma_0} = \mathrm{corank}_{\mathcal{O}}(S^{\Sigma_0}_{A_\psi}(\mathbb{Q}_\infty)).$$

By Proposition 3.14 (3), using the exact sequence

$$0 \to S^{\Sigma_0}_{A_\varphi}(\mathbb{Q}_\infty) \to S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty) \to S^{\Sigma_0}_{A_\psi}(\mathbb{Q}_\infty) \to 0,$$

$S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty)$ is finite. Therefore, by combining these results, Proposition 3.10 and Lemma 3.11, we see that $S_{A_f}(\mathbb{Q}_\infty)$ and $S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)$ are Λ-cotorsion. Thus we can define the algebraic Iwasawa invariants by

$$\begin{aligned}
\lambda_f^{\mathrm{alg}} &= \lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee) = \lambda(S(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee) = \deg(f^{\mathrm{alg}}(T)),\\
\mu_f^{\mathrm{alg}} &= \mu(S_{A_f}(\mathbb{Q}_\infty)^\vee) = \mu(S(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee),\\
\lambda_{f,\Sigma_0}^{\mathrm{alg}} &= \lambda(S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)^\vee) = \lambda(S^{\Sigma_0}(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee) = \deg(f^{\mathrm{alg}}_{\Sigma_0}(T)),\\
\mu_{f,\Sigma_0}^{\mathrm{alg}} &= \mu(S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)^\vee) = \mu(S^{\Sigma_0}(\mathbb{Q}_\infty; A_f, F^+A_f)^\vee),
\end{aligned}$$

where $f^{\mathrm{alg}}(T)$ (resp. $f^{\mathrm{alg}}_{\Sigma_0}(T)$) is the distinguished polynomial corresponding to $S_{A_f}(\mathbb{Q}_\infty)^\vee$ (resp. $S^{\Sigma_0}_{A_f}(\mathbb{Q}_\infty)^\vee$) via the Weierstrass preparation theorem. Again by using Proposition 3.10 and Lemma 3.11, we obtain

$$\mu_f^{\mathrm{alg}} = \mu_{f,\Sigma_0}^{\mathrm{alg}} = 0 \tag{3.3}$$

and

$$\begin{aligned}
\lambda_{f,\Sigma_0}^{\mathrm{alg}} &= \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{A_f[\varpi]}(\mathbb{Q}_\infty))\\
&= \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Phi}(\mathbb{Q}_\infty)) + \dim_{\mathcal{O}/\varpi\mathcal{O}}(S^{\Sigma_0}_{\Psi}(\mathbb{Q}_\infty))\\
&= \lambda_{\varphi,\Sigma_0} + \lambda_{\psi,\Sigma_0}.
\end{aligned} \tag{3.4}$$

3.3. *p*-adic *L*-functions. — We recall p-adic L-functions of modular forms. These functions have been constructed by Amice-Vélu [1], Vishik [40], Mazur, Tate, and Teitelbaum [26]. Also, we recall non-primitive p-adic L-functions of modular forms in the sense of Greenberg. Let K be an abelian number field, $2 \le k \le p-1$, and $f(z) = \sum_{n=1}^\infty a(n,f)e(nz) \in S_k(\Gamma_0(N), \varepsilon, \mathcal{O})$ a p-ordinary normalized Hecke eigenform which satisfies (RR), (3.1), and (Assumption). We assume that K is unramified at all primes dividing the level N, and tamely ramified at p. Put $G = \mathrm{Gal}(K/\mathbb{Q})$, and fix a character χ of G. We write $\Gamma = \mathrm{Gal}(K_\infty/K)$, where K_∞ denotes the cyclotomic $\mathbb{Z}_p$-extension of K. We can identify Γ with the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Let γ denote a fixed topological generator of Γ. Put $\Lambda = \mathcal{O}[\chi][[\Gamma]] \cong \mathcal{O}[\chi][[T]]\,;\, \gamma \mapsto 1+T$. For a finite order character $\rho\colon \Gamma \to \mathbb{C}^\times$, we define $\zeta \in \mu_{p^\infty}$ by $\zeta = \rho(\gamma)$. The p-adic L-function $\mathscr{L}_p(f,\chi,T) \in \Lambda$ is the power series characterized by the following interpolation property: for every non-trivial p-adic character $\rho\colon \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order with conductor p^{ν_ρ},

$$\mathscr{L}_p(f,\chi,\zeta-1) = \tau(\chi^{-1}\rho^{-1})\alpha(p,f)^{-\nu_\rho}\frac{L(f,\chi\rho,1)}{(-2\pi\sqrt{-1})\Omega_f^\alpha} \in \mathcal{O}[\chi,\rho],$$

where $\tau(\chi^{-1}\rho^{-1})$ is the Gauss sum of $\chi^{-1}\rho^{-1}$, $\alpha(p,f)$ is a unit root of $X^2 - a(p,f)X + \varepsilon(p)p^{k-1} = 0$, Ω_f^α is the canonical period defined by (2.5), and $\alpha = \chi(-1)$. By the Weierstrass preparation theorem, this interpolation property characterizes $\mathscr{L}_p(f,\chi,T)$. Also, for any finite set of primes Σ_0 with $p \notin \Sigma_0$, the non-primitive p-adic L-function $\mathscr{L}_p^{\Sigma_0}(f,\chi,T) \in \Lambda$ is characterized by the interpolation property

$$\mathscr{L}_p^{\Sigma_0}(f,\chi,\zeta-1) = \tau(\chi^{-1}\rho^{-1})\alpha(p,f)^{-\nu_\rho}\frac{L_{\Sigma_0}(f,\chi\rho,1)}{(-2\pi\sqrt{-1})\Omega_f^\alpha} \in \mathcal{O}[\chi,\rho],$$

where $L(f,\chi,s)\prod_{l\in\Sigma_0} E_l(f,\chi,s) = L_{\Sigma_0}(f,\chi,s)$. Here, $E_l(f,\chi,s)$ is the Euler factor of $L(f,\chi,s)$ at l. Then, putting $\chi =$ trivial character, we have

$$\mathscr{L}_p^{\Sigma_0}(f,T) = \mathscr{L}_p(f,T)\prod_{l\in\Sigma_0}\mathcal{P}_l(T),$$

where $\mathcal{P}_l(T)$ is defined by Proposition 3.9. We define the analytic Iwasawa invariants by

$$\begin{aligned}\lambda_f^{\mathrm{anal}} &= \lambda(\mathscr{L}_p(f,T)) = \deg(f^{\mathrm{anal}}(T)),\\ \mu_f^{\mathrm{anal}} &= \mu(\mathscr{L}_p(f,T)),\\ \lambda_{f,\Sigma_0}^{\mathrm{anal}} &= \lambda(\mathscr{L}_p^{\Sigma_0}(f,T)) = \deg(f_{\Sigma_0}^{\mathrm{anal}}(T)),\\ \mu_{f,\Sigma_0}^{\mathrm{anal}} &= \mu(\mathscr{L}_p^{\Sigma_0}(f,T)),\end{aligned}$$

where $f^{\mathrm{anal}}(T)$ (resp. $f_{\Sigma_0}^{\mathrm{anal}}(T)$) is the distinguished polynomial corresponding to $\mathscr{L}_p$ (resp. $\mathscr{L}_p^{\Sigma_0}$) via the Weierstrass preparation theorem.

3.4. The Iwasawa main conjecture. — In this subsection, we assume that $2 \le k \le p-1$ and a normalized Hecke eigenform $f \in S_k(\Gamma_0(N),\varepsilon,\mathcal{O})$ is p-ordinary and satisfies (RR), (3.1), and (Assumption). Let $\mathrm{Char}_\Lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee)$ be the characteristic ideal of the Pontryagin dual of the Selmer group $S_{A_f}(\mathbb{Q}_\infty)$. The following main conjecture for ρ_f is formulated by Greenberg.

CONJECTURE 3.15. — *We have*

$$\mathrm{Char}_\Lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee) = (\mathscr{L}_p(f,T)) \text{ in } \Lambda.$$

Kato has proven the following deep theorem in [21].

THEOREM 3.16. — *We have*

$$\mathrm{Char}_\Lambda(S_{A_f}(\mathbb{Q}_\infty)^\vee) \supset (\mathscr{L}_p(f,T)) \text{ in } \Lambda\otimes_{\mathbb{Z}_p}\mathbb{Q}_p.$$

Therefore, in order to confirm the Iwasawa main conjecture, we will show that

$$\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}.$$

Non-primitive objects $S_{A_f}^{\Sigma_0}(\mathbb{Q}_\infty)$ and $\mathscr{L}_p^{\Sigma_0}(f,T)$ will behave well under congruences, where $\Sigma_0 = \{l \text{ is a prime number} | l | N\}$. Then the following theorem obtained by [16, Theorem 1.5] is crucial for our proof.

THEOREM 3.17. — *The following statements hold:*
(1) $\mu_f^{\mathrm{alg}} = \mu_f^{\mathrm{anal}}$ *if and only if* $\mu_{f,\Sigma_0}^{\mathrm{alg}} = \mu_{f,\Sigma_0}^{\mathrm{anal}}$.
(2) $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$ *if and only if* $\lambda_{f,\Sigma_0}^{\mathrm{alg}} = \lambda_{f,\Sigma_0}^{\mathrm{anal}}$.
(3) $f^{\mathrm{alg}}(T) = f^{\mathrm{anal}}(T)$ *if and only if* $f_{\Sigma_0}^{\mathrm{alg}}(T) = f_{\Sigma_0}^{\mathrm{anal}}(T)$.

Now, we analogously define the p-adic L-functions for the Galois representations A_φ and A_ψ appearing in the previous subsection.

(i) The p-adic L-function $\mathscr{L}_p(A_\varphi, T) \in \Lambda$ is defined by the interpolation property

$$\mathscr{L}_p(A_\varphi, \zeta - 1) = L(\varepsilon\psi^{-1}\rho, 2-k) = L(\chi_p^{k-1}\varepsilon\psi^{-1}\rho, 1)$$

for every non-trivial p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order and $\zeta = \rho(\gamma)$. Here we remark that, by (Assumption), $\omega^{k-1}\varepsilon\psi^{-1}$ is non-trivial character and hence $L(\chi_p^{k-1}\varepsilon\psi^{-1}\rho, s)$ is holomorphic for $s \in \mathbb{C}$. Then, $\mathscr{L}_p(A_\varphi, T)$ is related to the Kubota-Leopoldt p-adic L-function by

$$L_p(\varepsilon\chi_p^{k-2}\omega\psi^{-1}, s) = \mathscr{L}_p(A_\varphi, \kappa(\gamma)^{-s} - 1)$$

for any $s \in \mathbb{Z}_p$. Here, $\kappa(\gamma)$ is the element of $1+p\mathbb{Z}_p$ which induces the action of γ on μ_{p^∞} when we identify Γ with $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$. The Ferrero-Washington theorem (Theorem 3.5) and the Mazur-Wiles theorem assert that $\mathscr{L}_p(A_\varphi, T) \notin \varpi\Lambda$ and the λ-invariant of $\mathscr{L}_p(A_\varphi, T)$ is equal to $\mathrm{corank}_{\mathcal{O}}(S_{A_\varphi}(\mathbb{Q}_\infty))$, which is denoted by $\lambda_{\varepsilon\omega^{k-1}\psi^{-1}} = \lambda_\varphi$. In addition, the non-primitive p-adic L-function $\mathscr{L}_p^{\Sigma_0}(A_\varphi, T)$ is defined by

$$\mathscr{L}_p^{\Sigma_0}(A_\varphi, T) = \mathscr{L}_p(A_\varphi, T) \prod_{l \in \Sigma_0} (1 - \varepsilon\psi^{-1}(l) l^{k-2}(1+T)^{f_l}).$$

Here $f_l \in \mathbb{Z}_p$ is determined by $\gamma_l = \gamma^{f_l}$, where γ_l is the Frobenius element corresponding to the prime l in Γ.

(ii) The p-adic L-function $\mathscr{L}_p(A_\psi, T) \in \Lambda$ is defined by the interpolation property

$$\mathscr{L}_p(A_\psi, \zeta - 1) = \tau(\psi^{-1}\rho^{-1}) \frac{L(\psi\rho, 1)}{2\pi\sqrt{-1}} = \frac{1}{2} L(\psi^{-1}\rho^{-1}, 0)$$

for every non-trivial p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order and $\zeta = \rho(\gamma)$. Here we remark that, by (Assumption), $\omega\psi^{-1}$ is non-trivial character. Then $\mathscr{L}_p(A_\psi, T)$ is related to the Kubota-Leopoldt p-adic L-function by

$$L_p(\omega\psi^{-1}, s) = \frac{1}{2}\mathscr{L}_p(A_\psi, \kappa(\gamma)^s - 1)$$

for any $s \in \mathbb{Z}_p$. The μ-invariant of $\mathscr{L}_p(A_\psi, T)$ is again zero and its λ-invariant is $\lambda_{\omega\psi^{-1}} = \lambda_\psi$, which is equal to $\mathrm{corank}_{\mathcal{O}}(S_{A_\psi}(\mathbb{Q}_\infty))$ by the Mazur-Wiles theorem. In addition, the non-primitive p-adic L-function $\mathscr{L}_p^{\Sigma_0}(A_\psi, T)$ is defined by

$$\mathscr{L}_p^{\Sigma_0}(A_\psi, T) = \mathscr{L}_p(A_\psi, T) \prod_{l \in \Sigma_0} (1 - \psi(l)l^{-1}(1+T)^{f_l}).$$

To state our theorem, we need to recall some facts about Eisenstein series. The following theorem is obtained from the results in [28, Theorem 4.7.1].

THEOREM 3.18 (Lifting). — *Let ε_i be a primitive Dirichlet character modulo M_i for $i = 1, 2$. We put $M = M_1M_2$ and $\varepsilon = \varepsilon_1\varepsilon_2$. If $\varepsilon(-1) = (-1)^k$, there exists an Eisenstein series $G = E_k(\varepsilon_1, \varepsilon_2) \in M_k(\Gamma_0(M), \varepsilon, \mathbb{C})$ such that*

$$L(G, s) = L(\varepsilon_1, s)L(\varepsilon_2, s - k + 1).$$

Moreover, $a(0, G) = 0$ if $k \neq 1$ and ε_1 is non-trivial.

Let $G = \sum_{n=0}^\infty a(n, G)e(nz) \in M_k(\Gamma_0(M), \varepsilon, \mathcal{O})$ be the Eisenstein series of weight k determined by

$$L(G, s) = L_{\Sigma_0}(\psi, s)L_{\Sigma_0}(\varepsilon\psi^{-1}, s - k + 1).$$

Note that Theorem 3.18 assures the existence of such G.

We define the p-adic L-function $\mathscr{L}_p(G, T)$ by the interpolation property

$$\begin{aligned}\mathscr{L}_p(G, \zeta - 1) &= \tau(\psi^{-1}\rho^{-1})\frac{L(G, \rho, 1)}{2\pi\sqrt{-1}} \\ &= L_{\Sigma_0}(\varepsilon\psi^{-1}\rho, 2 - k)\tau(\psi^{-1}\rho^{-1})\frac{L_{\Sigma_0}(\psi\rho, 1)}{2\pi\sqrt{-1}}\end{aligned}$$

for every non-trivial p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order and $\zeta = \rho(\gamma)$. Then clearly we have

$$\mathscr{L}_p(G, T) = \mathscr{L}_p^{\Sigma_0}(A_\varphi, T)\mathscr{L}_p^{\Sigma_0}(A_\psi, T).$$

Therefore, the μ-invariant of $\mathscr{L}_p(G, T)$ is zero and the λ-invariant of $\mathscr{L}_p(G, T)$ is equal to $\lambda_{\varphi,\Sigma_0} + \lambda_{\psi,\Sigma_0}$.

We define an eigenform $g(z) \in S_k(\Gamma_0(M), \varepsilon, \mathcal{O})$ by

$$(f \otimes \mathbf{1}_N)(z) = \sum_{(n,N)=1} a(n, f)e(nz),$$

where $\mathbf{1}_N$ denotes the trivial character on $(\mathbb{Z}/N\mathbb{Z})^\times$.

THEOREM 3.19. — *With the notation and the assumptions above, we have the congruence*

$$\frac{\Omega_f^+}{\Omega_g^+}\mathscr{L}_p^{\Sigma_0}(f,T) \equiv u(1+T)^{-n_\psi}\mathscr{L}_p(G,T) \bmod \varpi\Lambda,$$

where u is a unit in $\mathcal{O}$ and $(1+T)^{n_\psi} \in \Lambda^\times$ is the image of the conductor m_ψ under $\mathbb{Z}_p \twoheadrightarrow 1+p\mathbb{Z}_p \simeq \Gamma \hookrightarrow \Lambda$. Here α (explicitly given by (A.27)*) is equal to $+1$ by our assumption of φ and ψ.*

Proof. — We remark that

$$L(f,s) = \prod_{l\nmid N}(1-a(l,f)l^{-s}+\varepsilon(l)l^{k-1-2s})^{-1} \times \sum_{(n,N)\neq 1} a(n,f)n^{-s},$$
$$L(g,s) = \prod_{l\nmid N}(1-a(l,f)l^{-s}+\varepsilon(l)l^{k-1-2s})^{-1}.$$

Thus, we have $\mathscr{L}_p(g,\zeta-1) = \frac{\Omega_f^+}{\Omega_g^+}\mathscr{L}_p^{\Sigma_0}(f,\zeta-1)$ for every $\zeta \neq 1$ and hence

$$\mathscr{L}_p(g,T) = \frac{\Omega_f^+}{\Omega_g^+}\mathscr{L}_p^{\Sigma_0}(f,T).$$

For any l with $l \nmid Np$,

$$\begin{aligned} a(l,g) = a(l,f) &= \mathrm{Tr}(\rho_f(\mathrm{Frob}_l)) \\ &\equiv \psi(\mathrm{Frob}_l) + \varphi(\mathrm{Frob}_l) \\ &\equiv \psi(\mathrm{Frob}_l) + \det(\rho_f)\psi^{-1}(\mathrm{Frob}_l) \\ &= \psi(\mathrm{Frob}_l) + \varepsilon\chi_p^{k-1}\psi^{-1}(\mathrm{Frob}_l) \\ &= \psi(l) + \varepsilon(l)l^{k-1}\psi^{-1}(l) \\ &= a(l,G) \pmod{\varpi}. \end{aligned}$$

Also, by (3.1), (3.2), and (Assumption), we obtain

$$a(p,g) = a(p,f) \equiv \psi(\mathrm{Frob}_p) \equiv a(p,G) \pmod{\varpi}.$$

Therefore we have

$$g \equiv G \pmod{\varpi}.$$

We have $a(0,G) = 0$ since ψ is non-trivial. Since $(p,M) = 1$, the assumption (2) of Theorem 1.9 follows immediately from the q-expansion principle. Next we will check the assumptions (3) and (4) of Theorem 1.9. We claim that, under the assumption $k < p$,

$$\tau(\bar{\chi})D(G,\chi,1) = \sum_{i=1}^{\varphi(m)} \bar{\chi}(b_i)P_i \tag{3.5}$$

for some $P_i \in L_{k-2}(\mathcal{O})$ depending only on the parity $\chi(-1)$.

For the moment, we admit the claim (3.5). For $\delta \in \{0,1\}$, note that

$$\sum_{\substack{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \\ \chi(-1)=(-1)^\delta}} \chi(c) = \begin{cases} \varphi(m)/2 & \text{if } c \equiv 1 \pmod m, \\ (-1)^\delta \varphi(m)/2 & \text{if } c \equiv -1 \pmod m, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.5), we have

$$\begin{aligned} \sum_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) &= \sum_{\substack{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \\ \chi(-1)=1}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) \\ &\quad + \sum_{\substack{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times} \\ \chi(-1)=-1}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) \\ &= \varphi(m)P_i \in L_{k-2}(\mathcal{O}). \end{aligned}$$

On the other hand, by Lemma 1.8, we have

$$\sum_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}} \chi(b_i)\tau(\bar{\chi})D(G,\chi,1) = \varphi(m) \begin{pmatrix} 1 & -\frac{b_i}{m} \\ 0 & 1 \end{pmatrix} \star D_{\begin{pmatrix} 1 & \frac{b_i}{m} \\ 0 & 1 \end{pmatrix}}(G,1).$$

Therefore, the assumptions (3) and (4) of Theorem 1.9 are satisfied.

Thus it remains to prove the claim (3.5). The coefficient of $X^{k-2-j}Y^j$ in $\tau(\bar{\chi})D(G,\chi,1)$ is equal to

$$\begin{aligned} &-\tau(\bar{\chi})\binom{k-2}{j} j! \left(\frac{1}{2\pi\sqrt{-1}}\right)^{j+1} L(G,\chi,j+1) \\ &\qquad = -\binom{k-2}{j}\tau(\bar{\chi})\frac{j! L_{\Sigma_0}(\psi\chi,j+1)}{(2\pi\sqrt{-1})^{j+1}} \cdot L_{\Sigma_0}(\varepsilon\chi\psi^{-1}, 1-(k-j-1)). \end{aligned}$$

Then the existence of such polynomials P_i in $L_{k-2}(F_p)$ follows from the functional equation for $L_{\Sigma_0}(\psi\chi,s)$ (see, for example, [28, Theorem 3.3.1, page 93] or [18, Theorem 2, page 47]) and the Siegel-Klingen theorem. We prove that P_i belongs to $L_{k-2}(\mathcal{O})$. In order to do it, we show that $\tau(\overline{\chi})D(G,\chi,1) \in L_{k-2}(\mathcal{O}[\mathfrak{X}_S])$. For any Dirichlet character χ and any positive integer n, we have

$$L(\chi,1-n) = -\frac{1}{n}\sum_{a=1}^{m_\chi} \chi(a) m_\chi^{n-1} B_n\left(\frac{a}{m_\chi}\right).$$

Here recall that the n-th Bernoulli polynomial $B_n(X)$ is characterized by

$$B_n(X) = \sum_{j=0}^{n} \binom{n}{j} B_j X^{n-j},$$

where B_j is the j-th Bernoulli number. The von Staudt-Clausen theorem implies that, for a positive integer j,

$$B_j + \sum_{(l-1)|j} \frac{1}{l} \in \mathbb{Z}.$$

Here the sum runs over prime numbers l such that $l-1$ divides j. Hence, for each $1 \leq n < p-1$ and Dirichlet character χ with $(p, m_\chi) = 1$,

$$L(\chi, 1-n) \in \mathbb{Z}_p[\chi]. \tag{3.6}$$

Hence the integrality $\tau(\overline{\chi})D(G, \chi, 1) \in L_{k-2}(\mathcal{O}[\mathfrak{X}_S])$ follows from (3.6) and the functional equation for $L_{\Sigma_0}(\psi\chi, s)$, where we use the assumption that the conductor of $\psi\chi$ is prime to p. Therefore, the integrality $P_i \in L_{k-2}(\mathcal{O})$ follows from $\varphi(m)P_i \in L_{k-2}(\mathcal{O}[\mathfrak{X}_S])$, which is obtained by the same argument mentioned after (3.5), and $(\varphi(m), p) = 1$.

Therefore, by applying the proof of Theorem 2.10 to the triple (g, G, ρ) instead of (f, G, χ), there exists a p-adic unit u' in $\mathcal{O}^\times$ such that $[\delta_g]^+ = u'[\pi_G]^+$ in $H^1_{\mathrm{par}}(\Gamma_1(M), L_{k-2}(\mathcal{O}/\varpi))$ (by the same argument mentioned at the beginning of the proof), and it gives the congruence for L-functions.

$$\tau(\bar{\rho})\frac{L(g, \rho, 1)}{(2\pi\sqrt{-1})\Omega_g^+} \equiv u'\tau(\bar{\rho})\frac{L(G, \rho, 1)}{2\pi\sqrt{-1}} \pmod{\varpi}$$

for every non-trivial, and non-exceptional p-adic character $\rho : \Gamma \to \overline{\mathbb{Q}}_p^\times$ of finite order whose conductor $m_\rho = p^{\nu_\rho}$. An explicit calculation with $(m_\rho, m_\psi) = 1$ shows that

$$\tau(\bar{\psi})\tau(\bar{\rho}) = \bar{\psi}(m_\rho)^{-1}\bar{\rho}(m_\psi)^{-1}\tau(\bar{\psi}\bar{\rho}).$$

We remark that $\bar{\psi}(m_\rho) \equiv \alpha(p, g)^{-\nu_\rho} \pmod{\varpi}$. Therefore we obtain

$$\begin{aligned} \mathscr{L}_p(g, \zeta - 1) &= \tau(\bar{\rho})\alpha(p, g)^{-\nu_\rho}\frac{L(g, \rho, 1)}{(2\pi\sqrt{-1})\Omega_g^+} \\ &\equiv u'\tau(\bar{\rho})\alpha(p, g)^{-\nu_\rho}\frac{L(G, \rho, 1)}{2\pi\sqrt{-1}} \\ &\equiv u'\tau(\bar{\psi})^{-1}\bar{\rho}(m_\psi)^{-1}\tau(\bar{\psi}\bar{\rho})\frac{L(G, \rho, 1)}{2\pi\sqrt{-1}} \\ &\equiv u'\tau(\bar{\psi})^{-1}(1+T)^{-n_\psi}\mathscr{L}_p(G, \zeta - 1) \pmod{\varpi}, \end{aligned}$$

for every $\zeta = \rho(\gamma) \neq 1$. This proves the theorem. □

Finally, we prove Theorem 0.2. By Theorem 3.19 and $\mu(\mathscr{L}_p(G, T)) = 0$, we obtain

$$\lambda^{\mathrm{anal}}_{f,\Sigma_0} = \lambda(\mathscr{L}_p(G, T)) = \lambda(\mathscr{L}_p^{\Sigma_0}(A_\varphi, T)) + \lambda(\mathscr{L}_p^{\Sigma_0}(A_\psi, T)).$$

By the definition,

$$\lambda(\mathscr{L}_p^{\Sigma_0}(\tilde{\varphi},T)) = \lambda(\mathscr{L}_p(A_\varphi,T)) + \sum_{l\in\Sigma_0} \lambda(1-\tilde{\varphi}(l)l^{-1}(1+T)^{f_l}),$$
$$\lambda(\mathscr{L}_p^{\Sigma_0}(\tilde{\psi},T)) = \lambda(\mathscr{L}_p(A_\psi,T)) + \sum_{l\in\Sigma_0} \lambda(1-\tilde{\psi}(l)l^{-1}(1+T)^{f_l}).$$

On the other hand, by Proposition 3.14 (3), we have

$$\lambda(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \lambda(S_{A_\varphi}(\mathbb{Q}_\infty)^\vee) + \sum_{l\in\Sigma_0} \lambda(\mathcal{H}_l(\mathbb{Q}_\infty,A_\varphi)),$$
$$\lambda(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee) = \lambda(S_{A_\psi}(\mathbb{Q}_\infty)^\vee) + \sum_{l\in\Sigma_0} \lambda(\mathcal{H}_l(\mathbb{Q}_\infty,A_\psi)).$$

Moreover, by Proposition 3.9, for $l\in\Sigma_0$,

$$\lambda(\mathcal{H}_l(\mathbb{Q}_\infty,A_\varphi)) = \lambda(1-\tilde{\varphi}(l)l^{-1}(1+T)^{f_l}),$$
$$\lambda(\mathcal{H}_l(\mathbb{Q}_\infty,A_\psi)) = \lambda(1-\tilde{\psi}(l)l^{-1}(1+T)^{f_l}).$$

Thus, by the Mazur-Wiles theorem, we get

$$\lambda(\mathscr{L}_p^{\Sigma_0}(A_\varphi,T)) = \lambda(S_{A_\varphi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee), \qquad \lambda(\mathscr{L}_p^{\Sigma_0}(A_\psi,T)) = \lambda(S_{A_\psi}^{\Sigma_0}(\mathbb{Q}_\infty)^\vee).$$

Combining these results with Theorem 3.19, we obtain

$$\lambda_{f,\Sigma_0}^{\mathrm{anal}} = \lambda(\mathscr{L}_p(G,T)) = \lambda(\mathscr{L}_p^{\Sigma_0}(A_\varphi,T)) + \lambda(\mathscr{L}_p^{\Sigma_0}(A_\psi,T)) = \lambda_{\varphi,\Sigma_0} + \lambda_{\psi,\Sigma_0}.$$

Thus, by (3.4), $\lambda_{f,\Sigma_0}^{\mathrm{alg}} = \lambda_{f,\Sigma_0}^{\mathrm{anal}}$, which by Theorem 3.17 implies that $\lambda_f^{\mathrm{alg}} = \lambda_f^{\mathrm{anal}}$. We have completed the proof of Theorem 0.2.

Appendix A. Comparison theorem for torsion cohomology in the $\mathbf{GL_2}(\mathbb{Q})$ case

In this section, we retain the notation as before. Let p be an odd prime number and $N\geq 4$ a positive integer with $(p,N)=1$. Let $C=X_1(N)$ be the modular curve over $\mathbb{Z}[1/N]$ parametrizing generarized elliptic curves with $\Gamma_1(N)$-structure. The cuspidal subscheme $Z=Z_N$ is étale over $\mathbb{Z}[1/N]$ and set $C^\circ = C-Z$. We write $\pi\colon \mathcal{E}\to C$ for the universal generalized elliptic curve with $\Gamma_1(N)$-structure. The map π is smooth away from Z and the fibers of π over Z are the standard Neron N'-gon, where N' divides N. Let $f\colon X\to C$ be the k-fold fiber product of $\mathcal{E}$ over C. If $k\geq 2$, X is singular and proper. Let $\tilde{f}\colon \tilde{X}\to C$ denote the desingularization of X constructed by Deligne [7], and explained by Scholl [31], [29], Ulmer [37], and Subsection A.2 in this paper. Put $X^\circ = \tilde{f}^*(C^\circ)$. Then X° is smooth and not proper. Note that $f^\circ\colon X^\circ\to C^\circ$ is smooth.

Let $G_k = (\mathbb{Z}/N\mathbb{Z}\rtimes\mu_2)^k\rtimes\mathfrak{S}_k$ and $H_k=\mu_2^k\rtimes\mathfrak{S}_k$, where $\mathfrak{S}_k$ is the symmetric group and the action of $\mu_2=\{\pm1\}$ on $\mathbb{Z}/N\mathbb{Z}$ is by the multiplication and the action of $\mathfrak{S}_k$ is by permutation. Moreover, $\mathbb{Z}/N\mathbb{Z}$ acts on X° by translation by points of order N, μ_2 acts on X° by inversion in the fibers and $\mathfrak{S}_k$ acts on X° by permuting the factors of the fiber product. Then both G_k and H_k act on X°. This action extends to X and $\tilde{X}$ by definition. Let $\varepsilon\colon G_k \to \{\pm 1\}$ be the homomorphism which is trivial on each factor $\mathbb{Z}/N\mathbb{Z}$, the identity on each factor μ_2 and the sign character sgn^k on $\mathfrak{S}_k$. Let $\Pi := \frac{1}{|G_k|}\sum_{g\in G_k}\varepsilon(g)g^{-1} \in \mathbb{Z}[\frac{1}{2N\cdot k!}][G_k]$ be the projector attached to ε. Also we denote by $\varepsilon_k = \varepsilon|_{H_k}$ the restriction of the character ε to the subgroup H_k, and $\Pi_k := \frac{1}{|H_k|}\sum_{g\in H_k}\varepsilon_k(g)g^{-1} \in \mathbb{Z}[\frac{1}{2N\cdot k!}][H_k]$ the projector associated to ε_k. If $p \geq 3$, $k < p$, and p is prime to N, then $\Pi \in \mathcal{O}[G_k]$ and $\Pi_k \in \mathcal{O}[H_k]$. We denote by $V(\varepsilon)$ the ε-eigenspace for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][G_k]$-module V, and $W(\varepsilon_k)$ the ε_k-eigenspace for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][H_k]$-module W. Note that $V(\varepsilon) = \mathrm{im}[\Pi\colon V \to V]$ for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][G_k]$-module V and $W(\varepsilon_k) = \mathrm{im}[\Pi_k\colon W \to W]$ for any $\mathbb{Z}[\frac{1}{2N\cdot k!}][H_k]$-module W.

A.1. The Hecke correspondence and the Atkin correspondence. — We define the Hecke correspondence T_l and the Atkin correspondence U_l on the curves $X_1(N)$ and $Y_1(N)$ over $\mathbb{Z}[1/N]$.

First, we assume that l is prime to N. Let $Y_1(N,l)$ be the fine moduli scheme over $\mathbb{Z}[1/N]$ which represents the functor of triples (E,P,C), where $E \to S$ is an elliptic curves over a $\mathbb{Z}[1/N]$-scheme S, P a point of exact order N on E, and C a finite locally free subgroup scheme of order l in $E[l]$. The morphism $p_1\colon Y_1(N,l) \to Y_1(N)$ defined by

$$p_1\colon (E,P,C) \longmapsto (E,P)$$

is finite flat. Since $(l,N)=1$, we can define a morphism $p_2\colon Y_1(N,l) \to Y_1(N)$ of schemes over $\mathbb{Z}[1/N]$ by

$$p_2\colon (E,P,C) \longmapsto (E/C, P \,(\mathrm{mod}\, C)).$$

We define a morphism $\psi\colon Y_1(N,l) \to Y_1(N,l)$ of schemes over $\mathbb{Z}[1/N]$ by

$$\psi\colon (E,P,C) \longmapsto (E/C, P\,(\mathrm{mod}\,C), E[l]/C).$$

Since $\psi^2(E,P,C) = (E, lP, C)$, ψ is an automorphism of $Y_1(N,l)$. Hence $p_2 = p_1 \circ \psi$ implies that p_2 is also finite flat.

Then we have a commutative diagram

$$\begin{tikzcd} \mathcal{E}^k \arrow[d] \arrow[dr, phantom, "\square"] & p_1^*\mathcal{E}^k \arrow[l, "\phi_1^k"'] \arrow[r, "\psi^k"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & p_2^*\mathcal{E}^k \arrow[r, "\phi_2^k"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E}^k \arrow[d] \\ Y_1(N) & Y_1(N,l) \arrow[l, "p_1"] \arrow[r, equal] & Y_1(N,l) \arrow[r, "p_2"'] & Y_1(N), \end{tikzcd}$$

where the first and third squares are cartesian. Thus we define the Hecke correspondence T_l on X° by scheme-theoretic image of the morphism

$$(\phi_1^k, \phi_2^k \circ \psi^k)\colon p_1^*\mathcal{E}^k \to \mathcal{E}^k \times \mathcal{E}^k,$$

which induces an endomorphism of $H^*_?(X^\circ)$ for $? =$ ét or dR (see §A.2 and §A.3). We also define the Hecke correspondence T'_l on $Y_1(N)$ by

$$(p_1, p_2)\colon Y_1(N, l) \to Y_1(N) \times Y_1(N).$$

Then T'_l and ψ^k induce an endomorphism of $H^*_?(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3). If (E, P) is a $\overline{\mathbb{Q}}$-valued point of $Y_1(N)$, then

$$T'_l(E, P) = \sum_{\varphi} (\varphi E, \varphi P),$$

where the sum runs over the l-isogenies φ with source E.

Similarly, we define the Hecke correspondence T'_l on $X_1(N)$ and it induces an endomorphism of compact support cohomologies $H^*_{?,\mathrm{c}}(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3).

Next we assume that l divides N. Let $X_1(N, l)$ be the fine moduli scheme over $\mathbb{Z}[1/N]$ which represents the functor of triples (E, P, C), where $E \to S$ is a generalized elliptic curves over a $\mathbb{Z}[1/N]$-scheme S, P a point of exact order N on E, and C a finite locally free subgroup scheme of order l in $E[l]$ which is not contained in the subgroup generated by P. The morphism $p_1\colon X_1(N, l) \to X_1(N)$ defined by

$$p_1\colon (E, P, C) \longmapsto (E, P)$$

is finite. Since C is not contained in the subgroup generated by P, we can define a finite morphism $p_2\colon X_1(N, l) \to X_1(N)$ of schemes over $\mathbb{Z}[1/N]$ by

$$p_2\colon (E, P, C) \longmapsto (E/C, P \,(\mathrm{mod}\, C)).$$

Then, we define the Atkin correspondence U_l on X by scheme-theoretic image of the map

$$(\phi_1^k, \phi_2^k \circ \psi^k)\colon p_1^* \mathcal{E}^k \to \mathcal{E}^k \times \mathcal{E}^k,$$

which induces an endomorphism of $H^{k+1}_?(X^\circ)$, and U'_l on $X_1(N)$ by

$$(p_1, p_2)\colon Y_1(N, l) \to Y_1(N) \times Y_1(N),$$

which induces an endomorphism of $H^*_?(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3). If (E, P) is a $\overline{\mathbb{Q}}$-valued point of $Y_1(N)$, then

$$U'_l(E, P) = \sum_{\varphi} (\varphi E, \varphi P),$$

where the sum runs over the l-isogenies φ with source E such that $\ker(\varphi)$ is not contained in the subgroup generated by P.

Similarly, we define the Hecke correspondence U'_l on $X_1(N)$ and it induces an endomorphism of compact support cohomologies $H^*_{?,\mathrm{c}}(Y_1(N))$ for $? =$ ét or dR (see §A.2 and §A.3).

We now define the Hecke correspondence $\tilde{T}_l$ on $\tilde{X}$ as the closure of T_l in $\tilde{X} \times \tilde{X}$ and the Atkin correspondence $\tilde{U}_l$ on $\tilde{X}$ as the closure of U_l in $\tilde{X} \times \tilde{X}$. These induce an endomorphism of $H^*_?(\tilde{X})$ for $? =$ ét or dR (see §A.2 and §A.3).

A.2. Comparison with p-adic étale cohomology. — In this subsection, we assume that $2 \leq k < p$. Let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and $\varpi \in \mathcal{O}$ a uniformizer. Let $A = \mathcal{O}$ or $\mathcal{O}/\varpi^n$. For any scheme T over $\mathcal{O}$, we denote by $T_{\overline{\mathbb{Q}}_p} = T \times_{\mathcal{O}} \overline{\mathbb{Q}}_p$ its base change to $\mathrm{Spec}(\overline{\mathbb{Q}}_p)$. The aim of this subsection is to prove the following proposition which gives an isomorphism between p-adic torsion étale cohomology for the modular curve with non-constant coefficients and for the Kuga-Sato variety with constant coefficients.

Proposition A.1. — *Assume that $k < p$. Then there exists the canonical exact sequence*

$$\begin{aligned}(\mathrm{A.1}) \quad 0 &\to H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \to H^{k+1}_{\text{ét}}(X^{\circ}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to H^0_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, A)(-k-1) \\ &\to H^{k+2}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \to H^{k+2}_{\text{ét}}(X^{\circ}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to 0\end{aligned}$$

and canonical isomorphisms

$$(\mathrm{A.2}) \qquad H^1_{\text{ét}}(C^{\circ}_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \simeq H^{k+1}_{\text{ét}}(X^{\circ}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k),$$

$$(\mathrm{A.3}) \qquad H^1_{\text{ét},\mathrm{par}}(C^{\circ}_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \simeq H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon),$$

$$(\mathrm{A.4}) \qquad H^n_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0 \ \textit{if}\ n \neq k+1, k+2 \ \textit{and}\ k > 0,$$

as Hecke modules endowed with a continuous $\mathbb{Q}_p$-linear action of $G_{\mathbb{Q}_p}$. Here the parabolic cohomology group in p-adic theories was defined by Deligne as

$$\begin{aligned}&H^1_{\text{ét},\mathrm{par}}(C^{\circ}_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A) \\ &\qquad = \mathrm{im}\left(H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}_p}, j_! \mathrm{Sym}^k R^1\pi_* A) \to H^1_{\text{ét}}(C^{\circ}_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^k R^1\pi_* A)\right),\end{aligned}$$

where j denotes the open immersion $j : C^{\circ} \hookrightarrow C$.

In order to prove this proposition, we strictly follow the arguments in [29].

First we construct the isomorphism (A.2). There exists the Leray spectral sequence for $f^{\circ} : X^{\circ} \to C^{\circ}$:

$$E_2^{i,j} = H^i_{\text{ét}}(C^{\circ}_{\overline{\mathbb{Q}}_p}, R^j f^{\circ}_* A) \Rightarrow H^{i+j}_{\text{ét}}(X^{\circ}_{\overline{\mathbb{Q}}_p}, A).$$

Since C° is affine, we have $E_2^{i,j} = 0$ for $i \geq 2$. By using the Künneth formula, we have

$$R^j f^{\circ}_* A \simeq \bigoplus_{r_1 + \cdots + r_k = j} R^{r_1}\pi_* A \otimes \cdots \otimes R^{r_k}\pi_* A.$$

Note that $-1 \in \mu_2$ acts as $(-1)^r$ on $R^r\pi_* A$. Hence, if $k < p$, then we have

$$R^r f^{\circ}_* A(\varepsilon_k) = \begin{cases} \mathrm{Sym}^k R^1\pi_* A & \text{if } r = k, \\ 0 & \text{if } r \neq k. \end{cases}$$

Therefore we get

$$H^i_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_*A) \simeq H^{i+k}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \tag{A.5}$$

for $i = 0, 1$. This proves (A.2).

Secondly we prove (A.1). We begin by considering the cuspidal fibers of $\tilde{X}$.

We remark that, for any smooth scheme of finite type S over $\overline{\mathbb{Q}}_p$, the Künneth map induces an isomorphism

$$H^*_{\text{ét}}(\mathbb{G}^k_m \times_{\bar{\mathbb{Q}}_p} S, A)(k) \simeq H^*_{\text{ét}}(S, A) \otimes \bigwedge(At_1 + \cdots + At_k), \tag{A.6}$$

where $t_i = \operatorname{pr}^*_i(t) \in H^1(\mathbb{G}_m, A)(1)$ and $\bigwedge(At_1+\cdots+At_k)$ is the exterior algebra on $At_1 + \cdots + At_k$.

The symmetric group $\mathfrak{S}_k$ acts on $\mathbb{G}^k_m$ by permuting the coordinates and μ_2^k acts on $\mathbb{G}^k_m$ by $(x_i) \mapsto (x_i^{a_i})$ for any $(a_i)_i \in \mu_2^k$. Then the group H_k acts on $\mathbb{G}^k_m$.

Proposition A.2. — *Cup product with $t_1 \cup \cdots \cup t_k$ defines isomorphisms*

$$H^*_{\text{ét}}(S, A)(-k) \simeq H^{*+k}_{\text{ét}}(\mathbb{G}^k_m \times_{\bar{\mathbb{Q}}_p} S, A)(\varepsilon_k)$$

for any smooth scheme of finite type S over $\overline{\mathbb{Q}}_p$.

Proof. — We denote by Π_k the projector associated to ε_k. Then, by (A.6), it is enough to show that $\Pi_k(t_1 \cup \cdots \cup t_r) = 0$ for each $r < k$. Since μ_2 acts on k-th component of $\mathbb{G}^k_m$ by $x_k \mapsto x_k^{a_k}$, it acts trivially on $t_1 \cup \cdots \cup t_r$ and $\varepsilon_k|_{\mu_2}$ is non-trivial. Hence we obtain the assertion as required. □

Let $P_k = \operatorname{Proj} \mathcal{O}[x_1, y_1, \ldots, x_k, y_k]/(x_1y_1 = x_2y_2 = \cdots = x_ky_k)$ be the closed subscheme of the projective space $\mathbb{P}^{2k-1}_{\mathcal{O}}$ over $\mathcal{O}$ defined by the equations $x_1y_1 = x_2y_2 = \cdots = x_ky_k$. Note that H_k acts on P_k. We define a subscheme P_k^{reg} as

$$P_k^{\text{reg}} = \{(x_i, y_i) \in P_k | \text{there are no two pairs } (x_i, y_i) \text{ simultaneously vanish}\}.$$

As in the proof of [31, Proposition 2.4.1] or [29, Proposition 7.2.3.1], we obtain the following result.

Proposition A.3. — *Assume that $k < p$. Then $H^*_{\text{ét}}(P^{\text{reg}}_{k,\overline{\mathbb{Q}}_p} \times_{\bar{\mathbb{Q}}_p} S, A)(\varepsilon_k) = 0$ for any smooth scheme of finite type S over $\overline{\mathbb{Q}}_p$.*

Let X^{reg} be the regular locus of X and $X^* = X^\circ \cup (Z \times \mathbb{G}^k_m)$ the open variety whose fiber over $x \in C$ is the connected component of the Néron model of $X^\circ \to C^\circ$.

Proposition A.4. — (1) $H^j_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^j_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon)$.

(2) $H^j_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^j_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k)$.

Proof. — (1). We define $V = X \times_C Z = X - X^\circ$ and a filtration of V by closed subschemes

$$V = V_k \supset V_{k-1} \supset \cdots \supset V_0 \supset V_{-1} = \emptyset,$$

where V_i is the set of $(x_1, x_2, \ldots, x_k)$ such that at least $(k-i)$ of the components x_i are singular points of corresponding Néron polygon. We define a desingularization $\tilde{X} = X\langle k-1\rangle$ of X and a filtration on $\tilde{V} = \tilde{X} \times_C Z$. We put $X\langle 0\rangle = X$ and $P\langle 0\rangle = V_0$. We define inductively $X\langle j\rangle$ and $P\langle j\rangle$ as follows: Let $\phi_j \colon X\langle j\rangle \to X\langle j-1\rangle$ be the blowing-up with center $P\langle j-1\rangle$ and let $P\langle j\rangle \subset X\langle j\rangle$ be the strict transform of V_j. We write $\tilde{X} = X\langle k-1\rangle$ and

$$\psi_j \colon \tilde{X} = X\langle k-1\rangle \xrightarrow{\phi_{k-1}} X\langle k-2\rangle \xrightarrow{\phi_{k-2}} \cdots \xrightarrow{\phi_{j+1}} X\langle j\rangle$$

for the composition $\phi_{j+1} \circ \cdots \circ \phi_{k-2} \circ \phi_{k-1}$. We will show that

$$H^*(\tilde{V}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0.$$

We define a filtration on $\tilde{V}$ by

$$\tilde{V} \supset W_0 \supset W_1 \supset \cdots \supset W_{k-2} \supset W_{k-1} = \emptyset$$

given by

$$W_j = \psi_j^{-1}(X\langle j\rangle^{\mathrm{sing}}).$$

Here $X\langle j\rangle^{\mathrm{sing}}$ is the singular locus of $X\langle j\rangle$.

We claim that

$$H^*((W_j - W_{j+1})_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0$$

for all $0 \leq j \leq k-2$. The proof of this claim is same as [31, Theorem 3.1.0,(ii)] using Proposition A.3 instead of [31, Proposition 2.4.1]. Thus we have, for $j \geq 0$,

$$H^*_{\text{ét}}(W_{j,\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(W_{j+1,\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

Since $\tilde{V} - W_0 \simeq V^{\mathrm{reg}}$,

$$H^*_{\text{ét}}(\tilde{V}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(W_{0,\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

Therefore we get

$$H^*_{\text{ét}}(\tilde{V}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(W_{0,\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq \cdots \simeq H^*_{\text{ét}}(W_{k-1,\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0.$$

Since $\tilde{X} - \tilde{V} \simeq X^{\mathrm{reg}}$, we obtain

$$H^*_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(X^{\mathrm{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon),$$

as required.

(2). Fix a cusp $x \in Z$. Then we have, on $f^{-1}(x)$,

$$\begin{aligned} V_{k-1} - V_{k-2} &= \{(x_1, \ldots, x_k) \in V \mid \text{there exists one pair such that } x_i \text{ is singular}\} \\ &= \bigcup_{\sigma \in G_k} \sigma\{(x_1, \ldots, x_k) \in V \mid x_1 \text{ is singular, } x_i \text{ is non-singular for any } i \neq 1\} \\ &= \coprod_{\sigma \in G_k/\mu_2 \times (\mu_2^{k-1} \rtimes \mathfrak{S}_{k-1})} \sigma T. \end{aligned}$$

Here T is the component

$$T := \{(x_1, \ldots, x_k) \in V \mid x_1 \text{ is singular and } x_i \text{ is non-singular for any } i \neq 1\}$$

and $\mu_2 \times (\mu_2^{k-1} \rtimes \mathfrak{S}_{k-1})$ is the stabilizer of T under G_k. Note that the first factor μ_2 acts on T trivially. Therefore

$$H^*_{\text{ét}}((V_{k-1} - V_{k-2})_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = \operatorname{Ind}^{G_k}_{\mu_2 \times (\mu_2^{k-1} \rtimes \mathfrak{S}_{k-1})} H^*_{\text{ét}}(T_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) = 0$$

by Frobenius reciprocity. Then by using the Gysin sequence for $X - V_{k-1} \hookrightarrow X^{\text{reg}} = X - V_{k-2}$, we have

$$H^*_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}((X - V_{k-1})_{\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

Note that

$$(X - V_{k-1}) \times Z \simeq (\mathbb{G}_m \times \mathbb{Z}/N\mathbb{Z})^k \times Z \text{ and } X^* - (X - V) = X^* \times Z = \mathbb{G}_m^k \times Z.$$

Then by using the Gysin sequence for $X^\circ = X - V \hookrightarrow X^{\text{reg}}$ and $X^\circ = X - V \hookrightarrow X^*$, we see that

$$H^*_{\text{ét}}(X^{\text{reg}}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \simeq H^*_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k). \qquad \square$$

By the Gysin sequence for $X^\circ \hookrightarrow X^*$, we have the exact sequence

(A.7)
$$\begin{aligned} \cdots \to H^{j-2}_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p} \times \mathbb{G}_m^k, A)(-1)(\varepsilon_k) &\to H^j_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to H^j_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \\ \to H^{j-1}_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p} \times \mathbb{G}_m^k, A)(-1)(\varepsilon_k) &\to H^{j+1}_{\text{ét}}(X^*_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to \cdots. \end{aligned}$$

Therefore (A.1) follows from (A.7), Proposition A.2, and Proposition A.4. Also (A.4) follows from (A.7), Proposition A.2, (A.5), and $H^0_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_* A) = 0$ if $(p, N) = 1$ and $k > 0$ (by (2.4)).

Thirdly, we construct the isomorphism (A.3). Let j (resp. $\tilde{j}$) denote the open immersion $C^\circ \hookrightarrow C$ (resp. $X^\circ \hookrightarrow \tilde{X}$). There exists the Leray spectral sequence

$$E_2^{a,b} = H^a_{\text{ét}}(C_{\overline{\mathbb{Q}}_p}, R^b \tilde{f}_* \tilde{j}_! A) \Rightarrow H^{a+b}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \tilde{j}_! A).$$

As before in the proof of (A.5), by using the proper base change theorem and the Künneth formula, we see that

$$R^b\tilde{f}_*\tilde{j}_!A(\varepsilon) \simeq j_!R^bf^\circ_*A(\varepsilon) \simeq \begin{cases} j_! \operatorname{Sym}^k R^1\pi_*A & \text{if } \ b=k, \\ 0 & \text{if } \ b\neq k. \end{cases}$$

Therefore we have a commutative diagram

$$\begin{tikzcd} H^1_{\text{ét}}(C_{\overline{\mathbb{Q}}_p}, j_! \operatorname{Sym}^k R^1\pi_*A) \arrow[r, "\simeq"] \arrow[dd] & H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \tilde{j}_!A)(\varepsilon) \arrow[d] \\ \circlearrowleft & H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \arrow[d] \\ H^1_{\text{ét}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_*A) \arrow[r, "\simeq"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \end{tikzcd} \tag{A.8}$$

from the functoriality of the Leray spectral sequence and (A.2). By Poincaré duality, we see that

$$H^{k+1}_{\text{ét,c}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \times H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k^{-1}) \xrightarrow{\text{trace}} \mathbb{Q}/\mathbb{Z},$$
$$H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \times H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon^{-1}) \xrightarrow{\text{trace}} \mathbb{Q}/\mathbb{Z}$$

are perfect pairings. Note that $\varepsilon = \varepsilon^{-1}$ and

$$H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon) \to H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k)$$

is an injection by (A.1). Thus we see that

$$H^{k+1}_{\text{ét,c}}(X^\circ_{\overline{\mathbb{Q}}_p}, A)(\varepsilon_k) \to H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon)$$

is a surjection by duality. Therefore we have

$$H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^k R^1\pi_*A) \simeq H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A)(\varepsilon).$$

This proves (A.3).

We prove that the isomorphisms (A.2) and (A.3) are compatible with the Hecke operator and the Atkin operator. From the Leray spectral sequence and its functoriality for $A \to p_{1*}p_1{}^*A$, $A \to \psi_*\psi^*A$, and the trace map $\operatorname{tr}_{p_2} : p_{2*}p_2{}^*A \to A$, the diagram

$$\begin{tikzcd} \mathcal{E}^k \arrow[d] \arrow[r, phantom, "\square", shift right=6] & p_1^*\mathcal{E}^k \arrow[l, "\phi_1^k"'] \arrow[r, "\psi^k"] \arrow[d] \arrow[r, phantom, "\circlearrowleft", shift right=6] & p_2^*\mathcal{E}^k \arrow[r, "\phi_2^k"] \arrow[d] \arrow[r, phantom, "\square", shift right=6] & \mathcal{E}^k \arrow[d] \\ Y_1(N) & Y_1(N,l) \arrow[l, "p_1"] \arrow[r, equal] & Y_1(N,l) \arrow[r, "p_2"'] & Y_1(N) \end{tikzcd}$$

implies that the isomorphism $(A.2)$ is an isomorphism of Hecke modules as desired.

In order to prove that the isomorphism (A.3) is compatible with the Hecke operator and the Atkin operator, from the diagram (A.8) it suffices to show that the commutative diagram

$$\begin{tikzcd} \tilde{T}_l \arrow[r] & \tilde{X} \times \tilde{X} \arrow[r, "\mathrm{pr}_i"] & \tilde{X} \\ T_l \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & X^\circ \times X^\circ \arrow[u, "\tilde{j}\times\tilde{j}"'] \arrow[r, "\mathrm{pr}_i"'] \arrow[ur, phantom, "\square"] & X^\circ \arrow[u, "\tilde{j}"'] \end{tikzcd}$$

induces the following commutative diagram

$$\begin{tikzcd} H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A) \arrow[r] \arrow[d, "\mathrm{pr}_1^*"] \arrow[dr, phantom, "\circlearrowright"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A) \arrow[d, "\mathrm{pr}_1^*"] \\ H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p} \times \tilde{X}_{\overline{\mathbb{Q}}_p}, A) \arrow[r] \arrow[d, "\cup \mathrm{cl}(\tilde{T}_l)"] \arrow[dr, phantom, "\circlearrowright"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p} \times X^\circ_{\overline{\mathbb{Q}}_p}, A) \arrow[d, "\cup \mathrm{cl}(T_l)"] \\ H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p} \times \tilde{X}_{\overline{\mathbb{Q}}_p}, A)(k+1) \arrow[r] \arrow[d, "\mathrm{pr}_{2*}"] \arrow[dr, phantom, "\circlearrowright"] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p} \times X^\circ_{\overline{\mathbb{Q}}_p}, A)(k+1) \arrow[d, "\mathrm{pr}_{2*}"] \\ H^{k+1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, A) \arrow[r] & H^{k+1}_{\text{ét}}(X^\circ_{\overline{\mathbb{Q}}_p}, A). \end{tikzcd}$$

Here cl is the cycle map. The first square is compatible by the smooth base change theorem and the second square is compatible by the semi-purity theorem. The compatibility of third square follows from the fact that trace maps are compatible with base change. This completes the proof of Proposition A.1.

A.3. Comparison with algebraic de Rham cohomology. — The aim of this subsection is to prove Proposition A.8, which gives an isomorphism between mod p de Rham cohomology for the modular curve with non-constant coefficients and for the Kuga-Sato variety with constant coefficients. In order to do it, we use the terminology of logarithmic structures in Kato [19].

Let $\mathcal{Y}$ be a regular scheme and $\mathcal{D}$ a reduced divisor with normal crossings on $\mathcal{Y}$. Then the subsheaf L of monoids on $\mathcal{Y}_{\text{ét}}$ defind by

$$L(\mathcal{U})=\{g \in \mathcal{O}_{\mathcal{Y}}(\mathcal{U}) | g \text{ is invertible outside } \mathcal{D} \times_{\mathcal{Y}} \mathcal{U}\} \tag{A.9}$$

for each étale $\mathcal{Y}$-scheme $\mathcal{U}$ is a fine log structure ([19, (2.5)]).

We fix an algebra $A_0 = \mathbb{Z}[1/N]$. We define a log scheme $C^\times$ over A_0 to be the scheme C over A_0 endowed with the log structure $L{=}\{g \in \mathcal{O}_C | g$ is invertible outside $Z\}$, and $\mathcal{E}^\times$ the scheme $\mathcal{E}$ over A_0 endowed with the log structure $M{=}\{g \in \mathcal{O}_{\mathcal{E}} | g$ is invertible outside $\pi^{-1}(Z)\}$. Then the morphism of log schemes $\mathcal{E}^\times \to C^\times$ over A_0 is log smooth ([19, Theorem 3.5]) and hence the $\mathcal{O}_C$-module $\Omega^i_{\mathcal{E}^\times/C^\times} = \Omega^i_{\mathcal{E}/C}(\log(M/L))$ is locally free of finite type ([19, Theorem 3.10]).

For any A_0-algebra A and A_0-scheme $\mathcal{Y}$, we denote by $\mathcal{Y}_A$ its base change to $\mathrm{Spec}(A)$. Moreover, for any A_0-algebra A and A_0-log scheme $\mathcal{Y}^\times = (\mathcal{Y}, L)$, we denote by $\mathcal{Y}^\times_A = (\mathcal{Y}, L)_A$ its base change to $(\mathrm{Spec}(A), \mathrm{triv})$ with the trivial log structure.

In this subsection, let $\mathcal{O}$ be the ring of integers of a finite extension over $\mathbb{Q}_p$ and κ the residue field of $\mathcal{O}$.

We define the de Rham cohomology sheaf on C_κ by

$$\mathcal{L}_\kappa = R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa}.$$

We have the invertible sheaf

$$\boldsymbol{\omega}_\kappa = \pi_*\Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa}$$

([8, II.1.6], [24, §10.13]). The exact sequence

$$0 \to \Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa}[-1] \to \Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa} \to \mathcal{O}_{\mathcal{E}^\times_\kappa} \to 0$$

induces an exact sequence

$$0 \to \boldsymbol{\omega}_\kappa \to \mathcal{L}_\kappa \to \boldsymbol{\omega}_\kappa^{-1} \to 0 \tag{A.10}$$

(cf. [23, A1.2.1, page 163]). This sequence (A.10) defines the Hodge filtration

$$\mathcal{L}_\kappa = F^0(\mathcal{L}_\kappa) \supset F^1(\mathcal{L}_\kappa) = \boldsymbol{\omega}_\kappa \supset F^2(\mathcal{L}_\kappa) = 0.$$

We have the canonical integrable Gauss-Manin connection

$$\nabla_\kappa \colon \mathcal{L}_\kappa \to \mathcal{L}_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}.$$

For a non-negative integer k, we denote by $\mathcal{L}_{\kappa,k}$ the k-th symmetric tensor $\mathrm{Sym}^k\,\mathcal{L}_\kappa$ of $\mathcal{L}_\kappa$ and by $\nabla_{\kappa,k} \colon \mathcal{L}_{\kappa,k} \to \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$ the k-th symmetric power of ∇_κ. Explicitly, it is given by

$$\nabla_{\kappa,k}(x_1, \ldots, x_k) = \sum_{r=1}^{k} x_1 \cdots x_{r-1}x_{r+1} \cdots x_k \nabla_\kappa(x_r). \tag{A.11}$$

We define a complex of sheaves $\Omega^\bullet(\mathcal{L}_{\kappa,k})$ by

$$\Omega^0(\mathcal{L}_{\kappa,k}) = \mathcal{L}_{\kappa,k},\ \Omega^1(\mathcal{L}_{\kappa,k}) = \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$$

and the cohomology $H^m(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$ by

$$H^m(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) = H^m(C_\kappa, \Omega^\bullet(\mathcal{L}_{\kappa,k})).$$

Let R_κ denote the canonical residue map in the sense of Deligne, which gives an exact sequence

$$0 \to \Omega^1_{C_\kappa/\kappa} \to \Omega^1_{C_\kappa^\times/\kappa} \xrightarrow{\mathrm{R}_\kappa} \mathcal{O}_{Z_\kappa} \to 0.$$

Since $\mathrm{R}_\kappa(\nabla_{\kappa,k}(ax)) = \mathrm{R}_\kappa(a\nabla_{\kappa,k}(x)) + \mathrm{R}_\kappa(x \otimes \mathrm{d}a) = \mathrm{R}_\kappa(a\nabla_{\kappa,k}(x))$ on $\mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}$ for any $a \in \mathcal{O}_{C_\kappa}$ and $x \in \mathcal{L}_{\kappa,k}$, the morphism R_κ induces an $\mathcal{O}_{C_\kappa}$-linear morphism

$$\mathcal{L}_{\kappa,k} \xrightarrow{\nabla_{\kappa,k}} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \xrightarrow{\mathrm{R}_\kappa} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}.$$

We define a complex of sheaves $\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})$ by

$$\Omega^0_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) = \mathcal{L}_{\kappa,k},\ \Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) = \nabla_{\kappa,k}(\mathcal{L}_{\kappa,k}) + \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}$$

and the parabolic cohomology $H^m_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$ in the sense of Scholl [30] by

$$H^m_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) = H^m(C_\kappa, \Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})).$$

Proposition A.5. — *Assume that $k < p$. Then, the morphism R_κ induces an exact sequence*

$$0 \to H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \to H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \xrightarrow{\mathrm{R}_\kappa} H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes \mathcal{O}_{Z_\kappa}).$$

Proof. — Fix a cusp s. The level structure on $\mathrm{Tate}_{N'}(q) = \mathbb{G}_m/(q^{1/N'})^{\mathbb{Z}}$ defines a morphism

$$\psi\colon \mathrm{Spec} A_0[[q^{1/N'}]] \to C$$

identifying $A_{N'} = A_0[[q^{1/N'}]]$ with the formal completion of C along the cusp s, where $N'|N$. Then $\psi^*(\boldsymbol{\omega}_{A_0})$ has the nowhere vanishing section dt/t on the formal completion of C along the cusp s, where t is the parameter on $\mathbb{G}_m$ (cf. [8, VII,1.16.2], [23, A1.3.18]). Let ω be the canonical generator. Since $(p, N) = 1$, ∇_{A_0} induces

$$\nabla_{A_0} : \psi^*\mathcal{L}_{A_0} \to \psi^*\mathcal{L}_{A_0} \cdot d\log(q^{1/N'}) = \psi^*\mathcal{L}_{A_0} \cdot \frac{dq}{q},$$

and we have

$$\psi^*\mathcal{L}_{A_0} = A_{N'} \cdot \omega \oplus A_{N'} \cdot \xi,$$

where $\nabla_{A_0}(\omega) = \xi \cdot \frac{dq}{q}$ and $\nabla_{A_0}(\xi) = 0$ (cf. [23, A.1.3]). Then we get

$$\psi^*\mathcal{L}_{A_0,k} = \bigoplus_{r=0}^{k} A_{N'} \cdot \omega^{k-r}\xi^r \tag{A.12}$$

and, by (A.11),

$$\nabla_{A_0,k}(\omega^{k-r}\xi^r) = \sum_{i=1}^{k-r} \omega^{k-r-1}\xi^r \nabla_{A_0}(\omega) + \sum_{j=k-r+1}^{k} \omega^{k-r}\xi^{r-1}\nabla_{A_0}(\xi)$$
$$= \begin{cases} (k-r)\omega^{k-r-1}\xi^{r+1}\frac{dq}{q} & \text{if } r \neq k, \\ 0 & \text{if } r = k. \end{cases}$$

Since $k < p$, we obtain the exact sequence

$$\text{(A.13)} \qquad 0 \to \Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) \to \Omega^\bullet(\mathcal{L}_{\kappa,k}) \xrightarrow{\mathrm{R}_\kappa} \boldsymbol{\omega}^k_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}[-1] \to 0.$$

This proves the theorem. □

We denote by $X^\times$ the k-fold fiber product of $\mathcal{E}^\times$ over $C^\times$.

Proposition A.6. — *Assume that $k < p$. Then there exists a canonical isomorphism*

$$H^m(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \simeq H^{m+k}(X_\kappa, \Omega^\bullet_{X^\times_\kappa/\kappa})(\varepsilon) \textit{ for all } m.$$

Proof. — Similarly as in the proof of (A.2), by using the Künneth formula, we see that

$$R^j f_* \Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}(\varepsilon) \simeq \begin{cases} \mathcal{L}_{\kappa,k} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Thus, the Leray spectral sequence ([22, Remark 3.3]) implies the assertion as required. □

We define $\tilde{X}^\times$ to be the scheme $\tilde{X}$ endowed with the log structure defined by the subsheaf of functions invertible on the cuspidal fibers as (A.9).

Proposition A.7. — *The morphism $g\colon \tilde{X}^\times \to X^\times$ induces isomorphisms*

$$Rg_*\Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa} \simeq \Omega^\bullet_{X^\times_\kappa/\kappa} \quad \textit{and} \quad Rg_*\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa} \simeq \Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}.$$

Proof. — In virtue of [20, Theorem 11.3], $Rg_*\mathcal{O}_{\tilde{X}_\kappa} \simeq \mathcal{O}_{X_\kappa}$. Since g is log étale, applying [19, (3.12)] to $\tilde{X}^\times_\kappa \xrightarrow{g} X^\times_\kappa \to (\kappa, \mathrm{triv})$ (resp. $\tilde{X}^\times_\kappa \xrightarrow{g} X^\times_\kappa \to C^\times_\kappa$), we obtain

$$g^*\Omega^i_{X^\times_\kappa/\kappa} = \Omega^i_{\tilde{X}^\times_\kappa/\kappa} \text{ (resp. } g^*\Omega^i_{X^\times_\kappa/C^\times_\kappa} = \Omega^i_{\tilde{X}^\times_\kappa/C^\times_\kappa})$$

for all i. Hence the projection formula implies that

$$Rg_*\Omega^j_{\tilde{X}^\times_\kappa/\kappa} \simeq \Omega^j_{X^\times_\kappa/\kappa} \text{ and } Rg_*\Omega^j_{\tilde{X}^\times_\kappa/C^\times_\kappa} \simeq \Omega^j_{X^\times_\kappa/C^\times_\kappa}.$$ □

Proposition A.8. — *Assume that $k < p$. Then there exist canonical isomorphisms*

$$H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \simeq H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon), \tag{A.14}$$

$$H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \simeq H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon), \tag{A.15}$$

as filtered Hecke modules. Here the filtration on $H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$ (resp. $H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$) is induced by the filtration (A.21) *(resp.* (A.23)*) and the filtration on $H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon)$ and $H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)$ are defined by the Hodge filtration.*

Proof. — First, the isomorphism (A.14) is obtained by Proposition A.6 and Proposition A.7.

Secondly, we construct the isomorphism (A.15) by using the Leray spectral sequence [22]. In order to do it, we make a general observation on logarithmic differential. Let $\mathcal{Y}$ be a regular scheme and suppose that $\mathcal{D}$, $\mathcal{D}'$, and $\mathcal{D}+\mathcal{D}'$ are reduced divisors with normal crossings on $\mathcal{Y}$. Let M be the log structure associated to $\mathcal{D}$ as (A.9). Étale locally on $\mathcal{Y}$, we can write $\mathcal{D} = \sum_{i=1}^r \mathcal{C}_i$, where $\mathcal{C}_i$ is a regular closed subsheme of $\mathcal{Y}$ defined by $\pi_i = 0$ for a non-zero divisor $\pi_i \in \Gamma(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ and M is isomorphic to the log structure associated to $(\mathbb{N}^r)_\mathcal{Y} \to \mathcal{O}_\mathcal{Y} : (n_i) \mapsto \Pi\pi_i^{n_i}$. The residue map Res from $\Omega^\bullet_\mathcal{Y}(\log(\mathcal{D}+\mathcal{D}'))$ to $\Omega^{\bullet-1}_{\mathcal{C}_i}(\log(\mathcal{C}_i \cap \mathcal{D}'))$ is defined by the formula

$$\mathrm{Res}(d\log(\pi_i) \wedge \omega) = \omega|_{\mathcal{C}_i}.$$

Summing over all components, we get the morphism

$$\mathrm{Res}\colon \Omega^\bullet_\mathcal{Y}(\log(\mathcal{D}+\mathcal{D}')) \to \alpha_*(\Omega^{\bullet-1}_{\tilde{\mathcal{D}}}(\log(\alpha^*\mathcal{D} \cap \mathcal{D}')))$$

for the normalization $\alpha : \tilde{\mathcal{D}} \to \mathcal{D}$ of $\mathcal{D}$.

We define D_j as the strict transform of the exceptional divisor $\phi_{j+1}^{-1}(P\langle j\rangle)$ in $\tilde{X}_\kappa$ for $j = 0, 1, \ldots, k-2$ and D_k as the strict transform in $\tilde{X}_\kappa$ of the cuspidal fibers of $f\colon X_\kappa \to C_\kappa$ over the cusps. We write $\tilde{D}_j$ for the normalization of D_j for all j. Put $D = D_0 + \cdots + D_{k-2} + D_k$ and $E_j = D_0 + \cdots + D_j$ for $0 \le j \le k-2$. Let $\tilde{D}_k^\times$ be the scheme $\tilde{D}_k$ endowed with the log structure associated to the normal crossing divisor $\sum_{j=0}^{k-2}(\tilde{D}_k \cap D_j)$.

Filtrations on $\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}$ and $\Omega^\bullet_{\tilde{D}_k^\times/\kappa}$ are defined by

$$\begin{aligned} \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa} &= F^0(\Omega^\bullet_{X_\kappa^\times/\kappa}) \supset F^1(\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}) \supset F^2(\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}) = 0 \text{ and} \\ \Omega^\bullet_{\tilde{D}_k^\times/\kappa} &= F^0(\Omega^\bullet_{\tilde{D}_k^\times/\kappa}) = F^1(\Omega^\bullet_{\tilde{D}_k^\times/\kappa}) \supset F^2(\Omega^\bullet_{\tilde{D}_k^\times/\kappa}) = 0, \end{aligned} \tag{A.16}$$

respectively, where

$$F^1(\Omega^q_{\tilde{X}_\kappa^\times/\kappa}) = \mathrm{im}\left[\Omega^{q-1}_{\tilde{X}_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \to \Omega^q_{\tilde{X}_\kappa^\times/\kappa}\right]$$

(see, for example, [22, (3.2)]). The residue map Res: $\Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa} \to \alpha_*\Omega^{\bullet-1}_{\tilde{D}^\times_k/\kappa}$ for the canonical morphism $\alpha : \tilde{D}^\times_k \to \tilde{X}^\times_\kappa$ and the exact sequence $0 \to \mathrm{Gr}^1_F \to F^0 \to \mathrm{Gr}^0_F \to 0$ induce a commutative diagram

$$\begin{tikzcd}
\mathcal{L}_{\kappa,k} \arrow[r, "\nabla_{\kappa,k}"] \arrow[d, "\mathrm{Res}"'] & \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa} \arrow[d, "\mathrm{Res}"] \\
0 \arrow[r, "d"] & R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}^\times_k/\kappa}(\varepsilon)
\end{tikzcd} \tag{A.17}$$

obtained by the projection formula and cutting out by ε. Here we note that $R^k\tilde{f}_*\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa} \simeq R^kf_*\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}$ by Proposition A.7 and hence $R^k\tilde{f}_*\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa}(\varepsilon) \simeq R^kf_*\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}(\varepsilon) \simeq \mathcal{L}_{\kappa,k}$ by the proof of Proposition A.6. Thus the Leray spectral sequence [22, Remark 3.3] and its functoriality induce a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\ker(\mathrm{Res}) \arrow[r, "\simeq"] \arrow[d] & H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon) \arrow[d] \\
H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Res}"'] & H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa})(\varepsilon) \arrow[d, "\mathrm{Res}"] \\
H^1(C_\kappa, 0 \xrightarrow{d} R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}^\times_k/\kappa}(\varepsilon)) \arrow[r, "\simeq"] & H^k(\tilde{D}_k, \Omega^\bullet_{\tilde{D}^\times_k/\kappa})(\varepsilon).
\end{tikzcd}$$

Here, using

$$\begin{aligned} H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa}(\log(E_{k-2})))(\varepsilon) &\simeq H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa}(\log(E_{k-3})))(\varepsilon) \\ &\simeq \cdots \simeq H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon), \end{aligned}$$

obtained by an inductive argument with the help of the vanishing results [37, p.146] and $R^i\tilde{f}_*\Omega^\bullet_{\tilde{D}^\times_k/\kappa}(\varepsilon) = 0$ if $i \neq k$ by [37, p.145], we see that the second arrow in the right vertical sequence is an injection and the bottom horizontal morphism is an isomorphism. Since the image of $\Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa/\kappa}[-1]$ under $\mathrm{Gr}^1_F \mathrm{Res} : \mathrm{Gr}^1_F \Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa} = \Omega^\bullet_{\tilde{X}^\times_\kappa/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa}[-1] \to \mathrm{Gr}^1_F \Omega^\bullet_{\tilde{D}^\times_k/\kappa}[-1]$ is equal to 0, we have $\mathrm{Res}(\mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}) = 0$. Combining with (A.17), we get $\mathrm{Res}(\Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) = 0$. Thus, by the exact sequence (A.13), the map Res factors through $\Omega^1(\mathcal{L}_{\kappa,k})/\Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) \simeq \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}$. Then Res induces $\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa} \to R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon)$ and we have a commutative diagram

$$\begin{tikzcd} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[r, equal] \arrow[d, "\mathrm{R}_\kappa"] & \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[d, "\mathrm{Res}"] \\ \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa} \arrow[r] & R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon). \end{tikzcd} \tag{A.18}$$

Therefore, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r] \arrow[d] & \ker(\mathrm{Res}) \arrow[d] \\ H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, equal] \arrow[d, "\mathrm{R}_\kappa"'] & H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[d, "\mathrm{Res}"] \\ H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}) \arrow[r] & H^1(C_\kappa, 0 \xrightarrow{d} R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon)). \end{tikzcd}$$

Here the left vertical sequence is exact by Proposition A.5.

We claim that $\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa} \to R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon)$ is an injective morphism. Recall that from (A.12),

$$\psi^*\mathcal{L}_{\kappa,k} = \bigoplus_{r=0}^{k} \kappa_{_{N'}}\omega^{k-r}\xi^r,$$

where $\kappa_{_{N'}} = A_{N'} \otimes_{A_0} \kappa$. Since the claim is local on Z_κ, it is enough to show that $\psi^*(\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{O}_{Z_\kappa}) \to \psi^*(R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}(\varepsilon))$ is injective. Hereafter we drop the notation ψ^* and write $\mathcal{L}_{\kappa,k}$ for $\psi^*\mathcal{L}_{\kappa,k}$ and so on. For any $a = \sum_r b_r\omega^{k-r}\xi^r \in \mathcal{L}_{\kappa,k}$, $\mathrm{R}_\kappa(a d\log(q)) = \overline{b_0}\omega^k \otimes 1$, where $b_r \in \kappa_{_{N'}}$ and $\overline{b_0} \in \kappa_{_{N'}}/(q^{1/N'})$. Therefore, by (A.18), it suffices to show that $\mathrm{Res}(\omega^k \otimes d\log(q))$ is non-zero. Recall that Res is the composition of the following morphisms (A.19) and (A.20):

$$\begin{aligned} \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa} &\xrightarrow{\text{Künneth}} R^k\tilde{f}_*\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \\ &\simeq R^k\tilde{f}_*\left(\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \otimes_{\tilde{f}^*\mathcal{O}_{C_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa}\right); \end{aligned} \tag{A.19}$$

$$R^k\tilde{f}_*\left(\Omega^\bullet_{\tilde{X}_\kappa^\times/C_\kappa^\times} \otimes_{\tilde{f}^*\mathcal{O}_{C_\kappa}} \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa}\right) \xrightarrow{\mathrm{Res}} R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}_k^\times/\kappa}. \tag{A.20}$$

Here the morphism (A.20) is induced by Res : $\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa} \otimes_{\tilde{f}^*\mathcal{O}_{C_\kappa}} \tilde{f}^*\Omega^1_{C^\times_\kappa/\kappa} \to \Omega^\bullet_{\tilde{D}^\times_k/\kappa}$.

We shall compute the image of $\omega^k \otimes d\log(q)$ under (A.19). We denote p_i by the i-th projection $X^\times \to \mathcal{E}^\times$ and $\tilde{p}_i : \tilde{X}^\times \to \mathcal{E}^\times$ by $p_i \circ g$ for any i. Note that the image of $\omega^k \otimes d\log(q)$ under the map $\mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa} \to (R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa})^{\otimes k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$ is $\omega^{\otimes k} \otimes d\log(q)$ and the image of this element under the map $(R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa})^{\otimes k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa} \xrightarrow{\text{Künneth}} R^k f_*\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C^\times_\kappa/\kappa}$ is $p_1^*(\omega) \cup \cdots \cup p_k^*(\omega) \otimes d\log(q)$. The commutative diagram

$$\begin{tikzcd} p_i^{-1}\Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa}[-1] \arrow[r] \arrow[d] & \sigma_{\geq 1}\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa} \arrow[d] \\ p_i^{-1}\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa} \arrow[r] & \Omega^\bullet_{X^\times_\kappa/C^\times_\kappa} \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} \pi_*\Omega^1_{\mathcal{E}^\times_\kappa/C^\times_\kappa} \arrow[r, "h_i"] \arrow[d] & R^1 f_*(\sigma_{\geq 1}\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}) \arrow[d] \\ R^1\pi_*\Omega^\bullet_{\mathcal{E}^\times_\kappa/C^\times_\kappa} \arrow[r, "p_i^*"] & R^1 f_*\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}. \end{tikzcd}$$

Here we denote by h_i the upper horizontal morphism. We have $h_i(\omega) \in R^1 f_*(\sigma_{\geq 1}\Omega^\bullet_{X^\times_\kappa/C^\times_\kappa}) = \ker[f_*\Omega^1_{X^\times_\kappa/C^\times_\kappa} \to f_*\Omega^2_{X^\times_\kappa/C^\times_\kappa}]$. Similarly, we have $g^*h_i(\omega) \in \ker[\tilde{f}_*\Omega^1_{\tilde{X}^\times_\kappa/C^\times_\kappa} \to \tilde{f}_*\Omega^2_{\tilde{X}^\times_\kappa/C^\times_\kappa}]$. By putting $\tilde{h}_i(\omega) = g^*h_i(\omega)$, we have $\tilde{h}_1(\omega) \wedge \cdots \wedge \tilde{h}_k(\omega) \in \tilde{f}_*\Omega^k_{\tilde{X}^\times_\kappa/C^\times_\kappa} = R^k\tilde{f}_*(\sigma_{\geq k}\Omega^\bullet_{\tilde{X}^\times_\kappa/C^\times_\kappa})$. The image of $\omega^k \otimes d\log(q)$ under (A.19) is $\tilde{p}_1^*(\omega) \wedge \cdots \wedge \tilde{p}_k^*(\omega) \otimes d\log(q)$. Thus our claim follows from $\tilde{f}_*\Omega^k_{\tilde{D}^\times_k/\kappa}(\varepsilon) \simeq R^k\tilde{f}_*\Omega^\bullet_{\tilde{D}^\times_k/\kappa}(\varepsilon)$ obtained by [37, p.145] and $\tilde{h}_1(\omega) \wedge \cdots \wedge \tilde{h}_k(\omega) = \mathrm{d}t_1/t_1 \wedge \cdots \wedge \mathrm{d}t_k/t_k \neq 0$ on the smooth locus for the parameter t_i on $\mathbb{G}_m$.

Next, we prove that the isomorphisms (A.14) and (A.15) are filtered isomorphisms.

Case (A.14). — We construct the filtration of $H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})$. The Hodge filtration (A.10) on $\mathcal{L}_\kappa$ defines a decreasing filtration

$$F^r(\mathcal{L}_\kappa^{\otimes k}) = \sum_{\sigma \in \mathfrak{S}_k} \sigma \cdot [F^1(\mathcal{L}_\kappa)^{\otimes r} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-r)}]$$

on $\mathcal{L}_\kappa^{\otimes k}$ and

$$F^r(\mathcal{L}_{\kappa,k}) = \mathrm{im}(F^r(\mathcal{L}_\kappa^{\otimes k}) \xrightarrow{\mathrm{pr}} \mathcal{L}_{\kappa,k})$$

on $\mathcal{L}_{\kappa,k}$, where $\mathrm{pr}\colon \mathcal{L}_\kappa^{\otimes k} \to \mathcal{L}_{\kappa,k}$ is the canonical projection map. We can define a filtration on the complex $\Omega^\bullet(\mathcal{L}_{\kappa,k})$ by

$$\begin{aligned} F^r(\Omega^0(\mathcal{L}_{\kappa,k})) &= F^r(\mathcal{L}_{\kappa,k}), \\ F^r(\Omega^1(\mathcal{L}_{\kappa,k})) &= \Omega^1(\mathcal{L}_{\kappa,k}) \cap \left(F^{r-1}(\mathcal{L}_{\kappa,k}) \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right). \end{aligned} \tag{A.21}$$

In order to use the Hodge to de Rham spectral sequence

$$E_1^{i,j} = H^{i+j}(C_\kappa, \mathrm{Gr}^i(\Omega^\bullet(\mathcal{L}_{\kappa,k}))) \Rightarrow H^{i+j}(C_\kappa, \Omega^\bullet(\mathcal{L}_{\kappa,k})),$$

we compute the E_1-terms.

Proposition A.9. — *There is the canonical isomorphism*

$$\mathrm{Gr}^i(\mathcal{L}_{\kappa,k}) \simeq \boldsymbol{\omega}_\kappa^{2i-k} = \boldsymbol{\omega}_\kappa^i(\boldsymbol{\omega}_\kappa^{-1})^{k-i}.$$

Proof. — The canonical morphism

$$h\colon \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)} \to \mathrm{Gr}^i(\mathcal{L}_{\kappa,k}) = F^i(\mathcal{L}_{\kappa,k})/F^{i+1}(\mathcal{L}_{\kappa,k})$$

is surjective, since any element $\sum_\sigma \sigma \cdot m_\sigma \in F^i(\mathcal{L}_{\kappa,k})$ with $\sigma \in \mathfrak{S}_k$ and $m_\sigma \in \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)}$ is equal to $\sum_\sigma m_\sigma$ in $F^i(\mathcal{L}_{\kappa,k})/F^{i+1}(\mathcal{L}_{\kappa,k})$. The kernel of $\mathrm{pr}\colon \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)} \to \boldsymbol{\omega}_\kappa^{2i-k}$ is equal to

$$K := \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \boldsymbol{\omega}_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i-1)} + \cdots + \boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i-1)} \otimes_{\mathcal{O}_{C_\kappa}} \boldsymbol{\omega}_\kappa.$$

Indeed, over some open subset, if we fix a splitting of the exact sequence (A.10) and write e_1 and e_2 for a basis of $\boldsymbol{\omega}_\kappa$ and $\boldsymbol{\omega}_\kappa^{-1}$ respectively, then we can identify $\{e_1, e_2\}$ with a basis of $\mathcal{L}_\kappa$ and we see that ${e_1}^{\otimes i} \otimes {e_2}^{\otimes(k-i)}$ is a basis of $\boldsymbol{\omega}_\kappa^{\otimes i} \otimes_{\mathcal{O}_{C_\kappa}} \mathcal{L}_\kappa^{\otimes(k-i)}/K$. Since $K \subset \ker(h)$, we obtain a surjective morphism

$$\boldsymbol{\omega}_\kappa^{2i-k} \to \mathrm{Gr}^i(\mathcal{L}_{\kappa,k}),$$

and hence it is an isomorphism since $\mathrm{Gr}^i(\mathcal{L}_{\kappa,k})$ is free of rank 1 by the definition of $F^r(\mathcal{L}_{\kappa,k})$. $\square$

The Kodaira-Spencer map

$$\theta\colon \boldsymbol{\omega}_\kappa \hookrightarrow \mathcal{L}_\kappa \xrightarrow{\nabla_\kappa} \mathcal{L}_\kappa \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \to \boldsymbol{\omega}_\kappa^{-1} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa},$$

which is $\mathcal{O}_{C_\kappa}$-linear, induces an isomorphism

$$\boldsymbol{\omega}_\kappa^2 \simeq \Omega^1_{C_\kappa^\times/\kappa}$$

([23, A1.3.17], [8, VI §4.5], [24, Theorem 10.13.11]). Then θ induces

$$\begin{aligned} \mathrm{Gr}^0(\Omega^\bullet(\mathcal{L}_{\kappa,k})) &= [\boldsymbol{\omega}_\kappa^{-k} \to 0], \\ \mathrm{Gr}^r(\Omega^\bullet(\mathcal{L}_{\kappa,k})) &= \left[\boldsymbol{\omega}_\kappa^{(2r-k)} \xrightarrow{\theta\otimes\mathrm{id}} \boldsymbol{\omega}_\kappa^{(2r-k-2)} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right] \text{ (for } 1 \le r \le k), \\ \mathrm{Gr}^{k+1}(\Omega^\bullet(\mathcal{L}_{\kappa,k})) &= \left[0 \to \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right]. \end{aligned}$$

We claim that $\mathrm{Gr}^r(\mathcal{L}_{\kappa,k}\otimes_{\mathcal{O}_{C_\kappa}}\Omega^\bullet_{C_\kappa^\times/\kappa})=0$ for $1\le r\le k$, that is, $\theta\otimes\mathrm{id}$ is an isomorphism. Over some open set, if we write ω and η for a basis of $\boldsymbol{\omega}$ and $\Omega^1_{C_\kappa^\times/\kappa}$, respectively, then there is a basis ξ of $\boldsymbol{\omega}^{-1}$ such that $\theta(\omega)=\xi\otimes\eta$. Let $\bar{\xi}\in\mathcal{L}_\kappa$ such that $\nabla_\kappa(\omega)=\bar{\xi}\otimes\eta$. Since $\nabla_{\kappa,k}(ax)=a\nabla_{\kappa,k}(x)$ in $\mathrm{Gr}^{r-1}(\mathcal{L}_{\kappa,k})\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa}$ for any $a\in\mathcal{O}_{C_\kappa}$ and $x\in\mathrm{Gr}^r(\mathcal{L}_{\kappa,k})$,

$$\nabla_{\kappa,k}(\omega^r\bar{\xi}^{k-r})=\sum_{j=1}^{r}\omega^{r-1}\bar{\xi}^{k-r+1}\otimes\eta+\sum_{j=r+1}^{k}\omega^r\bar{\xi}^{k-r-1}\nabla_\kappa(\bar{\xi}).$$

Since the second term of the right hand side belongs to $F^r(\mathcal{L}_{\kappa,k})$, we have $(\theta\otimes\mathrm{id})(\omega^r\bar{\xi}^{k-r})=r\omega^r\xi^{k-r+1}\otimes\eta$. Thus we have $E_\infty^{i,j}=0$, $E_\infty^{k+2,j}=0$ if $1\le i\le k$ and any $j\in\mathbb{Z}$. Therefore we obtain

$$\begin{aligned}H^1(C_\kappa,\mathcal{L}_{\kappa,k},\nabla_{\kappa,k})&=F^0_\kappa\supset F^1_\kappa=\cdots=F^{k+1}_\kappa\\&=H^0(C_\kappa,\boldsymbol{\omega}^k_\kappa\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa})\supset F^{k+2}_\kappa=0.\end{aligned}$$

Here $F^{k+1}_\kappa=H^0(C_\kappa,\boldsymbol{\omega}^k_\kappa\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa})$ follows from the following long exact sequence induced by the exact sequence $0\to F^1\to F^0\to\mathrm{Gr}^0_F\to 0$, the quasi-isomorphism $F^k\to\cdots\to F^1$, and $\mathrm{Gr}^{k+1}_F=F^k$:

$$\begin{aligned}&0\to H^0(C_\kappa,\mathcal{L}_{\kappa,k},\nabla_{\kappa,k})\to H^0(C_\kappa,\boldsymbol{\omega}_\kappa^{-k})\\&\to H^0(C_\kappa,\boldsymbol{\omega}^k_\kappa\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa})\to H^1(C_\kappa,\mathcal{L}_{\kappa,k},\nabla_{\kappa,k})\to\cdots.\end{aligned}$$

Hence it suffices to show that the isomorphism (A.14) induces an isomorphism

$$F^{k+1}_\kappa\simeq F^{k+1}_{\kappa,\mathrm{Hdg}}(\varepsilon),$$

where $F^{k+1}_{\kappa,\mathrm{Hdg}}=H^0(\tilde{X}_\kappa,\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})$. Recall that the filtration $F^\bullet(\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})$ on $\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}$ is defined by (A.16).

We define the filtration $F^\bullet(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])$ on $\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)]$ by

$$\begin{aligned}\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)]&=F^0(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])\\&=F^1(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])\supset F^2(\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])=0.\end{aligned}$$

Similarly as the construction of (A.17), the canonical map $\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)]\to\Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}$ and the exact sequence $0\to\mathrm{Gr}^1_F\to F^0\to\mathrm{Gr}^0_F\to 0$ induce a commutative diagram

$$\begin{array}{ccc}0&\xrightarrow{d}&R^{k+1}\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)](\varepsilon)\\ \downarrow&&\downarrow\\ \mathcal{L}_{\kappa,k}&\xrightarrow{\nabla_{\kappa,k}}&\mathcal{L}_{\kappa,k}\otimes_{\mathcal{O}_{C_\kappa}}\Omega^1_{C_\kappa^\times/\kappa}.\end{array}$$

The Leray spectral sequence [22, Remark 3.3] and its functoriality induce a commutative diagram

$$\begin{tikzcd}
H^1(C_\kappa, 0 \xrightarrow{d} R^{k+1}\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)](\varepsilon)) \arrow[r, "\simeq"] \arrow[d] & H^{k+1}(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)])(\varepsilon) \arrow[d] \\
H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, "\simeq"] & H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon).
\end{tikzcd}$$

Hence, in order to prove that (A.14) is a filtered isomorphism, it suffices to show that

$$\begin{tikzcd}
H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}) \arrow[r, "\text{Künneth}"] \arrow[dr] & H^0(C_\kappa, R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon) \arrow[d] \\
& H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k})
\end{tikzcd}$$

is commutative and the horizontal morphism is an isomorphism. Since $\tilde{f} : \tilde{X} \to C$ is a log-smooth morphism, we have the exact sequence

$$0 \to \tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times} \to \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa} \to \Omega^{k+1}_{\tilde{X}_\kappa^\times/C_\kappa^\times} \to 0 \tag{A.22}$$

([19, (3.12)]).

The filtration $F^\bullet(\tilde{f}^*\Omega^1_{C_\kappa^\times/C_\kappa^\times} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)])$ on $\tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)]$ is defined by

$$\begin{aligned}
\tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)] &= F^0(\tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)]) \\
&= F^1(\tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)]) \\
&\supset F^2(\tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)]) = 0.
\end{aligned}$$

Then, similarly as in the proof of (A.17), the canonical diagram

$$\begin{tikzcd}
\tilde{f}^*\Omega^1_{C_\kappa^\times/\kappa} \otimes_{\mathcal{O}_{\tilde{X}_\kappa}} \Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}[-(k+1)] \arrow[r] \arrow[dr] & \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}[-(k+1)] \arrow[d] \\
& \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa}
\end{tikzcd}$$

and the exact sequence $0 \to \mathrm{Gr}_F^1 \to F^0 \to \mathrm{Gr}_F^0 \to 0$ induce a commutative diagram

$$\begin{tikzcd}
\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[r, "\text{Künneth}"] \arrow[drr] & R^0\tilde{f}_*\Omega^k_{\tilde{X}_\kappa^\times/C_\kappa^\times}(\varepsilon) \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \arrow[r] \arrow[dr] & R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}(\varepsilon) \arrow[d] \\
& & \mathcal{L}_{\kappa,k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}.
\end{tikzcd}$$

It remains to check that the composition of the horizontal arrows $\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \to R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}(\varepsilon)$ is an isomorphism.

Lemma A.10. — *There are canonical isomorphisms*

$$R^j f_*\Omega^i_{X_\kappa^\times/C_\kappa^\times}(\varepsilon) \simeq \begin{cases} \boldsymbol{\omega}_\kappa^{i-j} & \text{if } \ i+j=k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — We define complexes on $\mathcal{E}_\kappa^\times$ and $X_\kappa^\times$ as

$$\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} = \Omega^0_{\mathcal{E}_\kappa^\times/C_\kappa^\times} \oplus \Omega^1_{\mathcal{E}_\kappa^\times/C_\kappa^\times}[-1] \text{ and } \mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times} = \bigoplus_{i=0}^k \Omega^i_{X_\kappa^\times/C_\kappa^\times}[-i],$$

respectively. Then we have

$$\mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times} \simeq \oplus_{j=1}^k p_j^*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times}.$$

By using the Künneth formula, we obtain

$$\begin{aligned} R^n f_*\mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times} &\simeq \bigoplus_{n_1+\cdots+n_k=n} R^{n_1}\pi_*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} \otimes_{\mathcal{O}_{C_\kappa}} \cdots \otimes_{\mathcal{O}_{C_\kappa}} R^{n_k}\pi_*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} \\ &\simeq \bigoplus_{n_1+\cdots+n_k=n} (R^{n_1}\pi_*\mathcal{O}_{\mathcal{E}_\kappa} \oplus R^{n_1-1}\pi_*\Omega^1_{\mathcal{E}_\kappa^\times/C_\kappa^\times}) \\ &\qquad \otimes_{\mathcal{O}_{C_\kappa}} \cdots \otimes_{\mathcal{O}_{C_\kappa}} (R^{n_k}\pi_*\mathcal{O}_{\mathcal{E}_\kappa} \oplus R^{n_k-1}\pi_*\Omega^1_{\mathcal{E}_\kappa^\times/C_\kappa^\times}). \end{aligned}$$

As in the proof of (A.2) or Proposition A.6, we obtain

$$R^n f_*\mathcal{C}^\bullet_{X_\kappa^\times/C_\kappa^\times}(\varepsilon) \simeq \begin{cases} \mathrm{Sym}^k R^1\pi_*\mathcal{C}^\bullet_{\mathcal{E}_\kappa^\times/C_\kappa^\times} & \text{if } \ n=k, \\ 0 & \text{otherwise.} \end{cases}$$ □

By Lemma A.10, the long exact sequence of $R^\bullet\tilde{f}_*$ coming from (A.22), and $g_*\Omega^i_{\tilde{X}_\kappa^\times/C_\kappa^\times} = \Omega^i_{X_\kappa^\times/C_\kappa^\times}$ mentioned in the proof of Proposition A.7, we obtain an isomorphism

$$\boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa} \simeq R^0\tilde{f}_*\Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa}(\varepsilon).$$

Hence the isomorphism (A.14) is a filtered isomorphism.

Case (A.15). — We can define a filtration on the complex $\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})$ by

$$
\begin{aligned}
&F^r_{\mathrm{par}}(\Omega^0_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) = F^r(\mathcal{L}_{\kappa,k}),\\
&F^r_{\mathrm{par}}(\Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) = \Omega^1_{\mathrm{par}}(\mathcal{L}_{\kappa,k}) \cap \left(F^{r-1}(\mathcal{L}_{\kappa,k}) \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right).
\end{aligned}
\tag{A.23}
$$

Then we have

$$
\begin{aligned}
\mathrm{Gr}^0(\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= [\boldsymbol{\omega}_\kappa^{-k} \to 0],\\
\mathrm{Gr}^r(\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= \left[\boldsymbol{\omega}_\kappa^{(2r-k)} \xrightarrow{\theta\otimes\mathrm{id}} \boldsymbol{\omega}_\kappa^{(2r-k-2)} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}\right] \text{ (for } 1 \le r \le k),\\
\mathrm{Gr}^{k+1}(\Omega^\bullet_{\mathrm{par}}(\mathcal{L}_{\kappa,k})) &= \left[0 \to \boldsymbol{\omega}_\kappa^{k} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}\right].
\end{aligned}
$$

Thus we have $E_\infty^{i,j} = 0$, $E_\infty^{k+2,j} = 0$ if $1 \le i \le k$ and any $j \in \mathbb{Z}$. Therefore we obtain

$$
\begin{aligned}
H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) &= F^0_{\kappa,\mathrm{par}} \supset F^1_{\kappa,\mathrm{par}} = \cdots = F^{k+1}_{\kappa,\mathrm{par}}\\
&= H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}) \supset F^{k+2}_{\kappa,\mathrm{par}} = 0.
\end{aligned}
$$

Here $F^{k+1}_{\kappa,\mathrm{par}} = H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa})$ follows from the same argument as (A.14). Moreover we have the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \arrow[r] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon) \arrow[r, "\mathrm{Res}"] & H^0(\widetilde{D}_k, \Omega^k_{\tilde{D}_k^\times/\kappa})(\varepsilon) \\
& & H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}) \arrow[u, dash, "\simeq"'] \arrow[r, "\mathrm{R}_\kappa"] & H^0(Z_\kappa, \mathcal{O}_{Z_\kappa}) \arrow[u, dash, "\bigstar\ \simeq"']
\end{tikzcd}
$$

(cf. [37, p.150]). Here the isomorphism ★ is obtained by [37, p.145]. Then $\ker(\mathrm{Res}) = H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \simeq \ker(\mathrm{R}_\kappa) = H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa})$. Thus we obtain a commutative diagram

$$
\begin{tikzcd}
H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa}) \arrow[r, "\simeq"] \arrow[d] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa/\kappa})(\varepsilon) \arrow[d] \\
H^0(C_\kappa, \boldsymbol{\omega}_\kappa^k \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa^\times/\kappa}) \arrow[r, "\simeq"] \arrow[d] & H^0(\tilde{X}_\kappa, \Omega^{k+1}_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon) \arrow[d] \\
H^1(C_\kappa, \mathcal{L}_{\kappa,k}, \nabla_{\kappa,k}) \arrow[r, "\simeq"] & H^{k+1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa^\times/\kappa})(\varepsilon).
\end{tikzcd}
$$

In the same manner as in the proof of Proposition A.1, we see that the isomorphisms (A.14) and (A.15) are compatible with the Hecke operators and the Atkin operators. This completes the proof of Proposition A.8. □

In the next subsection, we will use the following lemma obtained by [37, Theorem 5.5] or [38, p.391].

LEMMA A.11. — *Assume that $k < p$. Then*

$$H^m(\tilde{X}_\kappa, \Omega^n_{\tilde{X}_\kappa/\kappa})(\varepsilon) \simeq 0 \text{ if } m+n \neq k+1.$$

A.4. Rank of parabolic cohomology. — We retain the notation as before. Let $\mathcal{O}$ be the ring of integers of a finite extension K over $\mathbb{Q}_p$, ϖ a uniformizer, and κ the residue field. We will use the results in §A.1, A.2, and A.3 by substituting $k-2$ for k.

Let $\Gamma = \Gamma_1(N)$, $k \geq 2$, $S_k = S_k(\Gamma, \mathcal{O})$, and $\bar{S_k} = H^0(C_\kappa, \boldsymbol{\omega}_\kappa^{k-2} \otimes_{\mathcal{O}_{C_\kappa}} \Omega^1_{C_\kappa/\kappa})$. We denote by $f \in S_k$ a normalized Hecke eigenform with character ε, and by $\mathfrak{M}_f$ a maximal ideal of the Hecke algebra generated by ϖ, $T_l - a(l,f)$ (for $(l,N)=1$), $U_l - a(l,f)$ (for $l|N$), and $\langle d\rangle - \varepsilon(d)$ over $\mathcal{O}$. The goal of this subsection is to understand the eigenspaces of the complex conjugation acting on the $\mathfrak{M}_f$-part $H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n))[\mathfrak{M}_f]$. We will prove the following theorem in this subsection. The author would like to express his deep gratitude to Professor Takeshi Tsuji whose guidance was crucial in proving the following theorem.

THEOREM A.12. — *Assume that $2 \leq k \leq p-1$ and the residual Galois representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\kappa)$ associated to f is reducible of the form*

$$\bar{\rho}_f \sim \begin{pmatrix} \xi_1 & * \\ 0 & \xi_2 \end{pmatrix}$$

satisfying that either ξ_1 or ξ_2 is unramified at p. Then, for any positive integer n and a parity $\alpha \in \{\pm 1\}$ as (A.27),

$$H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n))^\alpha[\mathfrak{M}_f] \simeq \mathcal{O}/\varpi^n,$$
$$H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})^\alpha[\mathfrak{M}_f] \simeq \mathcal{O}.$$

In order to prove this theorem, we need the following proposition. For each n, we write

$$\tilde{V}(n) = H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n))[\mathfrak{M}_f],$$
$$\tilde{V}(\infty) = H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})[\mathfrak{M}_f].$$

We put $\tilde{V} = \tilde{V}(1)$.

PROPOSITION A.13. — *All of the constituents of $\tilde{V}$ are isomorphic to $\kappa(\xi_1)$ or $\kappa(\xi_2)$.*

Proof. — We denote by ρ the Galois representation $\tilde{V}$ of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Put $d = \dim_\kappa \tilde{V}$. Fix a rational prime number l with $(l, pN) = 1$. The Eichler-Shimura relations impose the relation $\rho(\mathrm{Frob}_l)^2 - a(l,f)\rho(\mathrm{Frob}_l) + \varepsilon(l)l^{k-1} = 0$. We denote by $\alpha(l)$ and $\beta(l)$ the solutions of $X^2 - a(l,f)X + \varepsilon(l)l^{k-1} = 0$. Let

$$\tilde{V}^* = \mathrm{Hom}(\tilde{V}, \kappa(\varepsilon\omega^{k-1}))$$

and

$$W = \tilde{V} \oplus \tilde{V}^*$$

the direct sum of $\tilde{V}$ and $\tilde{V}^*$. We consider the characteristic polynomial of Frob_l acting on W. Let G denote a finite quotient of $G_{\mathbb{Q}}$ through which the actions on W, $\kappa(\xi_1)$, and $\kappa(\xi_2)$ factor. We denote by $N_{\alpha(l)}$ and $N_{\beta(l)}$ the generalized eigenspaces of $\rho(\mathrm{Frob}_l)$ with respect to $\alpha(l)$ and $\beta(l)$ respectively. Then $\tilde{V} = N_{\alpha(l)} \oplus N_{\beta(l)}$. Since the operation $\mathrm{Hom}(*, \kappa(\varepsilon\omega^{k-1}))$ interchanges the eigenvalues of the action of Frob_l, the characteristic polynomial of Frob_l acting on W is $(T - \alpha(l))^d(T - \beta(l))^d$. On the other hand, the characteristic polynomial of Frob_l acting on $\kappa(\xi_1)^{\oplus d} \oplus \kappa(\xi_2)^{\oplus d}$, which is regarded as a G-module, is also $(T - \alpha(l))^d(T - \beta(l))^d$. By the Chebotarev density theorem, any element of G is the image of some Frob_l with $l \nmid pN$. Thus, by the Brauer-Nesbitt theorem,

$$W^{\mathrm{ss}} \simeq \kappa(\xi_1)^{\oplus d} \oplus \kappa(\xi_2)^{\oplus d},$$

where W^{ss} is the semi-simplification of W. Hence there exists a Jordan-Hölder filtration

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_d = \tilde{V} \tag{A.24}$$

of $\tilde{V}$ satisfying

$$V_i/V_{i-1} \simeq \kappa(\alpha_i),$$

where α_i is equal to ξ_1 or ξ_2 for each i. □

Using integral p-adic Hodge theory, we shall prove Theorem $A.12$ by determining a character such that the number of constituents of $\tilde{V}$ isomorphic to it is equal to one. The key ingredients in our proof are to restrict the action of $G_{\mathbb{Q}}$ on $\tilde{V}$ to $G_{\mathbb{Q}_p}$ and to use that the Hodge-Tate weights of ξ_1 and ξ_2 are distinct.

First we will briefly review the fully faithful functor from the category of finitely generated filtered φ-module to the category of $\mathcal{O}$-representations of $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of finite length, and state the comparison theorem between the parabolic étale cohomology and the parabolic log-crystalline cohomology, which we will use in this subsection.

For a non-negative integer r, let $\mathbf{MF}^r_{\mathcal{O}}$ denote the category whose objects are the following triples $(M, (\mathrm{Fil}^i M)_{i\in\mathbb{Z}}, (\varphi^i_M)_{i\in\mathbb{Z}})$:

(1) M is a finitely generated $\mathcal{O}$-module;

(2) $(\mathrm{Fil}^i M)_{i\in\mathbb{Z}}$ is a decreasing filtration on M by $\mathcal{O}$-submodules such that $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{r+1} M = 0$;

(3) $\varphi_M^i\colon \mathrm{Fil}^i M \to M$ is an $\mathcal{O}$-linear homomorphism such that $\varphi_M^i|_{\mathrm{Fil}^{i+1}M} = p\varphi_M^{i+1}$ and $\sum_{i=0}^r \varphi_M^i(\mathrm{Fil}^i M) = M$.

A morphism in $\mathbf{MF}^r_{\mathcal{O}}$ is a homomorphism of filtered $\mathcal{O}$-modules compatible with $\varphi^\bullet$. It is known that any morphism $\eta : M \to M'$ in $\mathbf{MF}^r_{\mathcal{O}}$ is strict with respect to the filtrations, that is, $\eta(\mathrm{Fil}^i M) = \mathrm{Fil}^i M' \cap \eta(M)$ for each $i \in \mathbb{Z}$ ([12, 1.10 (b)]). This implies that $\mathbf{MF}^r_{\mathcal{O}}$ is an abelian category as follows. Let $\eta : M \to M'$ be a morphism in $\mathbf{MF}^r_{\mathcal{O}}$, and let $\underline{\eta}$ denote η regarded as a homomorphism of underlying $\mathcal{O}$-modules. Then the $\mathcal{O}$-module $N := \ker(\underline{\eta})$ with $\mathrm{Fil}^i N$ and φ_N^i defined by $\mathrm{Fil}^i N = N \cap \mathrm{Fil}^i M$ and $\varphi_N^i = \varphi_M^i|_N$, respectively, belongs to $\mathbf{MF}^r_{\mathcal{O}}$ and gives the kernel of η in $\mathbf{MF}^r_{\mathcal{O}}$. Let N' denote $\mathrm{coker}(\underline{\eta})$. We define a filtration $\mathrm{Fil}^i N'$ and an $\mathcal{O}$-linear homomorphism $\varphi^i_{N'}$ by $\mathrm{Fil}^i N' = \mathrm{Fil}^i M'/\eta(\mathrm{Fil}^i M)$ and the homomorphism induced by φ_M^i and $\varphi^i_{M'}$, respectively. Note that $\mathrm{Fil}^i N' \to N'$ is injective because η is strict, and hence $\mathrm{Fil}^i N'$ may be regarded as an $\mathcal{O}$-submodule of N'. The triple $(N', (\mathrm{Fil}^i N')_{i\in\mathbb{Z}}, (\varphi^i_{N'})_{i\in\mathbb{Z}})$ belongs to $\mathbf{MF}^r_{\mathcal{O}}$ and gives the cokernel of η in $\mathbf{MF}^r_{\mathcal{O}}$. The strictness of η further shows that we have $\mathrm{Fil}^i(\mathrm{im}(\eta)) = \eta(M) \cap \mathrm{Fil}^i M' = \eta(\mathrm{Fil}^i M) \simeq \mathrm{Fil}^i(\mathrm{coim}(\eta))$ and hence $\mathrm{im}(\eta) = \mathrm{coim}(\eta)$ in $\mathbf{MF}^r_{\mathcal{O}}$.

Let $\mathbf{MF}^r_\kappa$ denote the full subcategory of $\mathbf{MF}^r_{\mathcal{O}}$ consisting of objects M satisfying $\varpi M = 0$. Let $\mathbf{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$ denote the category of representations of $G_{\mathbb{Q}_p}$ on $\mathcal{O}$-modules of finite length. For an integer r such that $0 \le r \le p-2$, there exists a fully faithful functor

$$T_{\mathrm{cris}}\colon \mathbf{MF}^r_{\mathcal{O}} \to \mathbf{Rep}_{\mathcal{O}}(G_{\mathbb{Q}_p})$$

given by J.-M. Fontaine and G. Laffaille ([12], [6], [41]). Let $\mathbf{Rep}^r_{\mathcal{O},\mathrm{cris}}(G_{\mathbb{Q}_p})$ denote the essential image of $\mathbf{MF}^r_{\mathcal{O}}$ by T_{cris}. For an object T of $\mathbf{Rep}^r_{\mathcal{O},\mathrm{cris}}(G_{\mathbb{Q}_p})$, the Hodge-Tate weights of T mean $s \in \mathbb{Z}$ for which $\mathrm{Gr}^s M \neq 0$, where M is an object of $\mathbf{MF}^r_{\mathcal{O}}$ such that $T_{\mathrm{cris}}(M) \simeq T$.

By (A.15), we have a filtered isomorphism

$$H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k-2}, \nabla_{\kappa,k-2}) \simeq H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon).$$

Here a filtration is given by

$$0 \subset \bar{S}_k = F^{k-1}_{\kappa,\mathrm{par}} = \cdots = F^1_{\kappa,\mathrm{par}} \subset F^0_{\kappa,\mathrm{par}} = H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k-2}, \nabla_{\kappa,k-2}).$$

Theorem A.14. — *Assume that $k-1 \le p-2$. Then there is an isomorphism of Hecke modules*

$$\begin{aligned} T_{\mathrm{cris}}(H^1_{\mathrm{par}}(C_\kappa, \mathcal{L}_{\kappa,k-2}, \nabla_{\kappa,k-2})) &\simeq T_{\mathrm{cris}}(H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)) \\ &\simeq H^{k-1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \kappa)(\varepsilon) \simeq H^1_{\text{ét},\mathrm{par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^{k-2} R^1\pi_*\kappa). \end{aligned}$$

Proof. — The first and last isomorphisms follow from Proposition A.8 (A.15) and Proposition A.1 (A.3) respectively. The second isomorphism is obtained by the comparison theorem for proper smooth varieties with constant coefficients (proved by Fontaine-Messing ([13, III 6.4]) and Faltings ([9, Theorem 5.3]) and improved by Breuil-Tsuji ([5, Theorem 3.2.4.6]=[35, Theorem 5.1] and [5, Theorem 3.2.4.7])). It remains to check that these morphisms are Hecke equivariant. By the Hodge to de Rham spectral sequence and Lemma A.11, we have

$$H^k(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon) = 0, \tag{A.25}$$

$$H^{k-2}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon) = 0. \tag{A.26}$$

By the long exact sequence of cohomology for an exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi} \mathcal{O} \to \kappa \to 0$, (A.25), and Nakayama's lemma, we obtain

$$H^k(\tilde{X}_\mathcal{O}, \Omega^\bullet_{\tilde{X}_\mathcal{O}/\mathcal{O}})(\varepsilon) = 0.$$

Therefore, for any integer $n \geq 1$, by the long exact sequence of cohomology for an exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi^n} \mathcal{O} \to \mathcal{O}/\varpi^n \to 0$, we obtain

$$\begin{aligned} H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})/\varpi^n H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O}) \\ \simeq H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*(\mathcal{O}/\varpi^n)). \end{aligned}$$

Moreover, (A.26) implies that $H^1_{\text{ét,par}}(C^\circ_{\overline{\mathbb{Q}}_p}, \operatorname{Sym}^{k-2} R^1\pi_*\mathcal{O})$ is torsion-free. Therefore the proof reduces to showing that the comparison isomorphism between $H^{k-1}_{\text{ét}}(\tilde{X}_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and $H^{k-1}(\tilde{X}_{\mathbb{Q}_p}, \Omega^\bullet_{\tilde{X}_{\mathbb{Q}_p}/\mathbb{Q}_p})$ is compatible with the Hecke correspondences and Atkin correspondences. This follows from the de Rham conjecture for proper smooth varieties with constant $\mathbb{Q}_p$-coefficients [36, Theorem A1]. □

Since $(p, N) = 1$, $\tilde{V}(n)$ is a crystalline representation of $G_{\mathbb{Q}_p}$.

Next we construct a filtration of $H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)[\mathfrak{M}_f]$ by using the filtration (A.24) of $\tilde{V}$. We put $\tilde{M} = H^{k-1}(\tilde{X}_\kappa, \Omega^\bullet_{\tilde{X}_\kappa/\kappa})(\varepsilon)[\mathfrak{M}_f]$.

Case 1. — ξ_1 is unramified at p.

Then there exists $M(\xi_1) \in \mathbf{MF}^{k-1}_\kappa$ satisfying $T_{\text{cris}}(M(\xi_1)) = \kappa(\xi_1)$ and $F^1 = 0 \subset F^0 = M(\xi_1)$. Similarly, there is $M(\xi_2) \in \mathbf{MF}^{k-1}_\kappa$ satisfying $T_{\text{cris}}(M(\xi_2)) = \kappa(\xi_2)$ and $F^k = 0 \subset F^{k-1} = M(\xi_2)$.

Thus we obtain $M(\alpha_1) \in \mathbf{MF}^{k-1}_\kappa$ satisfying $T_{\text{cris}}(M(\alpha_1)) = V_1$. Since the length of module is preserved under T_{cris}, we have $\dim_\kappa M(\alpha_1) = 1$. Since T_{cris} is fully faithful, the image of $M(\alpha_1)$ in $\tilde{M}$ is non-trivial. We write $M_1 = \operatorname{im}(M(\alpha_1) \to \tilde{M})$.

Similarly, by replacing V_1 by V_2/V_1, there exists $M(\alpha_2) \in \mathbf{MF}_\kappa^{k-1}$ satisfying $T_{\mathrm{cris}}(M(\alpha_2)) = V_2/V_1$. Let $M^1 = \tilde{M}/M_1$, $\overline{M_2} = \mathrm{im}(M(\alpha_2) \to M^1)$, and

$$M_2 = \ker(\tilde{M} \to M^1/\overline{M_2}).$$

Then we have

$$T_{\mathrm{cris}}(M_2) = V_2.$$

Repeating this arguments, we obtain a Jordan-Hölder filtration

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_d = \tilde{M}$$

of $\tilde{M}$ satisfying

$$T_{\mathrm{cris}}(M_i/M_{i-1}) = V_i/V_{i-1} = \kappa(\alpha_i)$$

where α_i is equal to ξ_1 or ξ_2. By noting that, for any integer r,

$$\dim_\kappa \mathrm{Gr}_F^r(\tilde{M}) = \sum_{j=1}^{d} \dim_\kappa \mathrm{Gr}_F^r(M_j/M_{j-1}),$$

we have the following proposition.

Proposition A.15. — *We have*

$$\dim_\kappa \mathrm{Gr}_F^r(\tilde{M}) = \begin{cases} 0 & \text{if } r \neq 0, k-1, \\ \sharp\{j|\alpha_j = \xi_1\} & \text{if } r = 0, \\ \sharp\{j|\alpha_j = \xi_2\} & \text{if } r = k-1. \end{cases}$$

Case 2. — ξ_2 is unramified at p.

Similarly as in Case 1, we have the following proposition.

Proposition A.16. — *We have*

$$\dim_\kappa \mathrm{Gr}_F^r(\tilde{M}) = \begin{cases} 0 & \text{if } r \neq 0, k-1, \\ \sharp\{j|\alpha_j = \xi_2\} & \text{if } r = 0, \\ \sharp\{j|\alpha_j = \xi_1\} & \text{if } r = k-1. \end{cases}$$

Now we can prove Theorem A.12. By the q-expansion principle [23, § 1.6],

$$\mathrm{Gr}_F^{k-1}(\tilde{M}) = \bar{S}_k[\mathfrak{M}_f] \simeq \kappa.$$

Then, by the above propositions, we have $\sharp\{j|\alpha_j = \xi_2\} = 1$ in Case 1, and $\sharp\{j|\alpha_j = \xi_1\} = 1$ in Case 2. This proves Theorem A.12 in the case $n = 1, \infty$. In particular,

$$\alpha = \xi_2(-1) \text{ in Case 1 and } \alpha = \xi_1(-1) \text{ in Case 2.} \tag{A.27}$$

Next, we prove Theorem A.12 for any $n > 1$. As noted in the proof of Theorem A.14, $H^1_{\text{ét},\mathrm{par}}(C^{\circ}_{\overline{\mathbb{Q}}_p}, \mathrm{Sym}^{k-2} R^1\pi_*\mathcal{O})$ is torsion free. Then the exact sequence $0 \to \mathcal{O} \xrightarrow{\times\varpi^n} \mathcal{O} \to \mathcal{O}/\varpi^n \to 0$ on $\tilde{X}_{\overline{\mathbb{Q}}_p,\text{ét}}$ with Proposition A.1 (A.3) and (A.4) induces an exact sequence

$$0 \to \tilde{V}(\infty) \xrightarrow{\times\varpi^n} \tilde{V}(\infty) \to \tilde{V}(n).$$

Similarly, using the exact sequence $0 \to \varpi^{n-1}/\varpi^n \to \mathcal{O}/\varpi^n \to \mathcal{O}/\varpi^{n-1} \to 0$ on $\tilde{X}_{\overline{\mathbb{Q}}_p,\text{ét}}$, we obtain an exact sequence

$$0 \to \tilde{V}(1) \to \tilde{V}(n) \to \tilde{V}(n-1).$$

Thus an inductive argument proves Theorem A.12.

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