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ON p-ADIC SPEH REPRESENTATIONS

BY ALEXANDRU IOAN BADULESCU

À la Virgen del Rocío

ABSTRACT. — This note contains simple proofs of some known results (unitarity, character formula) on Speh representations of a group $GL_n(D)$ where D is a local non Archimedean division algebra of any characteristic.

RÉSUMÉ (*Sur les représentations de Speh p-adiques*). — Cette note contient des preuves simples de certains faits connus (unitarisabilité, formule des caractères) concernant les représentations de Speh d'un groupe $GL_n(D)$, où D est une algèbre à division locale non-archimédienne de caractéristique quelconque.

Introduction

In this note I give simple proofs of some known results on Speh representations of groups $GL_n(D)$ where D is a central division algebra of finite dimension over a local non archimedean field F of any characteristic. The new idea is to use the Moeglin-Waldspurger algorithm (MWA) for computing the dual of an

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irreducible representation. For the unitarizability of Speh representations, previous proofs were based either on the trace formula and the close fields theory, or on deep results of type theory. The proof here is "combinatoric", independent of D, the characteristic, and Bernstein's (also called U0) theorem. Short general proofs using MWA are also given for other known facts. The proof is always the same: one wants to prove a relation (R) involving some representations (for example an induced representation π is irreducible). One starts by writing the "naive" relation (R$'$) between these representations, known from standard theory, but not as strong as (R) (for example the semi simplification of π is a sum with non negative coefficients of some irreducible representations). Usually (R$'$) has more terms that (R), because it is weaker, and one wants to prove that some terms which are supposed to appear in (R$'$) are actually not there (for example all the subquotients of π except of the expected one have coefficient zero). The method then is to consider also the dual relation (R$''$) to (R$'$), and to play with the MWA, in order to show that the mild constraints one has on (R$'$) and (R$''$) are enough to show that the extra terms are not there and (R$'$) reduces actually to (R).

All the results in this paper are already known, and here I only give new short proofs. So I kindly ask the reader, when using one of these facts, to quote, at least in first place, the original reference (see the historical notice at the end of the paper). Beside the Zelevinsky and Tadić classification of the admissible dual ([24], [18]), the proofs here rely on [1] (dual of an irreducible representation is irreducible), [12] and [7] (algorithm for computing the dual) and some easy tricks from [16], [4] and [10], and do not involve any complicated technique.

The idea of searching for "simple proof" for classification of unitary representations, together with a list of basic tricks to use, are due to Marko Tadić ([19] for example). He also was the first to formulate some properties of Speh representations (*formula for ends of complimentary series, character formula, Speh representations are prime elements in the ring of representations, dual of a Speh representation is Speh*) and to prove them when $D = F$. The starting point of my proof here of the assertion *Speh representations are unitary* is also due to Tadić who found the simple but brilliant trick reducing the problem of unitarity to a problem of irreducibility.

I would like to thank Guy Henniart who read the paper and made useful observations.

Notation. — Let $(F, |\ |_F)$ be a local non archimedean field of any characteristic $ch(F)$ and D a central division algebra of finite dimension d^2 over F. Let $\nu_n : GL_n(D) \to \mathbb{C}^\times$ be the character $g \mapsto |\det(g)|_F$, denoted simply ν when no confusion may occur (det is the reduced norm). All representations here will be admissible of finite length. If π, π' are representations of $GL_a(D)$ and $GL_b(D)$

such that $a+b=n$, let P be the parabolic subgroup of $GL_n(D)$ containing the upper triangular matrices and of Levi factor $GL_a(D)\times GL_b(D)$ and set $\pi\times\pi' := \mathrm{ind}_P^{GL_n(D)}\pi\otimes\pi'$.

The results in this section for which no other reference is given may be found in [24] (for $D=F$) and [18] (for general D). In [18] the characteristic zero is assumed, but this restriction may be removed ([5, 2.2]). Let Gr_n denote the Grothedieck group of admissible representations of finite length of $GL_n(D)$, and, by convention, $Gr_0=\mathbb{Z}$. Set $\mathscr{R} := \oplus_{n\in\mathbb{N}} Gr_n$. The composition law $\times$ induces a composition law $*$ in $\mathscr{R}$ which makes $\mathscr{R}(+,*)$ into a commutative $\mathbb{Z}$-algebra ($\times$ is not commutative, but $*$ is). We write *representation* for *isomorphy class of representations* where no confusion may occur. The set Irr of irreducible representations of $GL_n(D)$, $n\in\mathbb{N}^\times$, is a natural linear basis of $\mathscr{R}$.

Let δ be an essentially square integrable (i.e. twist of a square integrable by a character) representation of $GL_n(D)$. Let $s(\delta)$ be the smallest positive real number x such that $\delta\times\nu^x\delta$ is reducible. Then $s(\delta)\in\mathbb{N}^*$ and $s(\delta)|d$ (so $s(\delta)=1$ if $D=F$). Set $\nu_\delta := \nu^{s(\delta)}$. Then δ may be obtained as the unique irreducible subrepresentation of a representation of type $\nu_\delta^a\rho\times\nu_\delta^{a-1}\rho\times\nu_\delta^{a-2}\rho\times\cdots\times\nu_\delta^{a-m+1}\rho$, with $a\in\mathbb{R}$, m a divisor of n, and ρ a cuspidal unitary representation of $GL_{\frac{n}{m}}(D)$. Also, a,m and ρ are determined by δ. We then write $\delta=Z(\rho,a,m)$. We have $s(\rho)=s(\delta)$. We set $e(\delta) := a-\frac{m-1}{2}$. Then δ is unitary (i.e. is square integrable) if and only if $e(\delta)=0$. In this case we simply write $\delta=Z(\rho,m)$, understood as: δ is the unique subrepresentation of $\nu_\delta^{\frac{m-1}{2}}\rho\times\nu_\delta^{\frac{m-1}{2}-1}\rho\times\nu_\delta^{\frac{m-1}{2}-2}\rho\times\cdots\times\nu_\delta^{-\frac{m-1}{2}}\rho$. We make the convention that $Z(\rho,a,0)$ is the unit element of $\mathscr{R}$.

To any irreducible representation π of $GL_n(D)$ we may associate a unique multiset $[\delta_1,\delta_2,\dots,\delta_k]$ such that

- δ_i is an essentially square integrable representations of some $GL_{n_i}(D)$, $n_i\geq 1$,
- $\sum_i n_i = n$ and
- if we reorder the d_i such that $e(\delta_i)$ is decreasing with i, then π is the unique irreducible quotient of $\delta_1\times\delta_2\times\cdots\times\delta_k$.

We write $\pi=L([\delta_1,\delta_2,\dots,\delta_k])$. We call *standard representation* an element of $\mathscr{R}$ which is a product $\delta_1*\delta_2*\cdots*\delta_k$ with d_i essentially square integrable representations. It is known that $\delta_1*\delta_2*\cdots*\delta_k$ determines the multiset $[\delta_1,\delta_2,\dots,\delta_k]$, so we may write $\pi=L(\delta_1*\delta_2*\cdots*\delta_k)$ instead of $\pi=L([\delta_1,\delta_2,\dots,\delta_k])$. Let $\mathscr{D}$ be the set of essentially square integrable representations in $\mathscr{R}$ and Std be the set of standard representations. Then L realizes a bijection from Std to Irr (and its restriction to $\mathscr{D}$ is the identity). If Θ is a standard representation and $\pi=L(\Theta)$, we then say that π is *the Langlands quotient* of Θ.

If δ is square integrable and $k \in \mathbb{N}^*$, then $u(\delta, k)$ (resp. $\bar{u}(\delta, k)$) will denote the Langlands quotient of $\nu_\delta^{\frac{k-1}{2}}\delta \times \nu_\delta^{\frac{k-3}{2}}\delta \times \nu_\delta^{\frac{k-5}{2}}\delta \times \cdots \times \nu_\delta^{-\frac{k-1}{2}}\delta$ (resp. of $\nu^{\frac{k-1}{2}}\delta \times \nu^{\frac{k-3}{2}}\delta \times \nu^{\frac{k-5}{2}}\delta \times \cdots \times \nu^{-\frac{k-1}{2}}\delta$). The representations $u(\delta, k)$ are called *Speh representations.*

If $\delta = Z(\rho, l)$, then δ_+ (and δ_-) will stand for $Z(\rho, l+1)$ (and $Z(\rho, l-1)$).

Let ι be the Aubert involution on $\mathscr{R}$ from [1] (see also [13]) which we will call here *duality.* It takes an irreducible representation of $GL_n(D)$ to plus or minus an irreducible representation of $GL_n(D)$. We will denote i the involution of the set of irreducible representations induced by ι (i.e. forgetting the sign; for $D = F$ it is the Zelevinsky involution from [24]). We recall that $\iota(\pi) * \iota(\pi') = \iota(\pi * \pi')$.

We call *segment* a subset of $\mathbb{R}$ of the form $S = \{a, a+1, a+2, \ldots, a+m-1\}$, $m \in \mathbb{N}$, with the convention $S = \emptyset$ if $m = 0$. Set $l(S) = m$ and call it the *length of* S. We say a is the *beginning* of S and $a + m - 1$ is the *ending* of S.

We call *multisegment* a non empty multiset $M = [S_1, S_2, \ldots, S_k]$ of non empty segments. Set $m(M) := \max_{S \in M} l(S)$ (the *maxlength* of M) and $t(M) := k$ (the *thickness* of M). Let $B(M)$ (resp. $E(M)$) be the multiset of the beginnings (resp. endings) of segments in M. We call *support* of M the multiset reunion of all the segments in M. Two segments S_1 and S_2 are said to be *linked* if $S_1 \not\subseteq S_2$, $S_2 \not\subseteq S_1$ and $S_1 \cup S_2$ is a segment. If M and M' are multisegments we say M' is obtained from M by an *elementary operation* if there exists a pair of linked segments S_1, S_2 in M such that M' is obtained from M by replacing $[S_1, S_2]$ with $[S_1 \cup S_2, S_1 \cap S_2]$ where it is understood we forget $S_1 \cap S_2$ in case it is empty. One introduces then a partial order on multisegments by putting $M \leq M'$ if M is obtained from M' through a finite number of elementary operations. It is easy to see that on the partially ordered set of multisegments m is decreasing, while t, B and E are increasing (for the relation on sets $X \leq Y$ if $X \subset Y$).

Let R be the set of unitary cuspidal representations in $\mathscr{R}$. If $\rho \in R$, let K_ρ be the set $\{\nu_\rho^\alpha \rho, \alpha \in \mathbb{R}\}$. Let Irr_ρ (Std_ρ, $\mathscr{D}_\rho$) be the subset of Irr (Std, $\mathscr{D}$) made of representations with cuspidal support included in K_ρ. If $\rho, \rho' \in R$ and $\rho \neq \rho'$, if $\pi \in \mathrm{Irr}_\rho$ and $\pi' \in \mathrm{Irr}_{\rho'}$, then $\pi * \pi' \in$ Irr ([24], [18]). Let $\mathscr{R}_\rho$ be the linear span of Irr_ρ in $\mathscr{R}$. Then $\mathscr{R}_\rho$ is a subalgebra of $\mathscr{R}$ and one has $\mathscr{R} = \times_{\rho \in R} \mathscr{R}_\rho$. Each $\mathscr{R}_\rho$ is stable by the involution ι.

Let us fix $\rho \in R$ and for the rest of this paragraph consider only representations in $\mathscr{R}_\rho$. Then essentially square integrable representations are parametrized by segments and so standard representations are parametrized by multisegments. The (partial) order relation defined on the set of multisegments induces an order relation on Std_ρ. The bijection $L : \mathrm{Std} \to \mathrm{Irr}$ induces a bijection from

Std_ρ to Irr_ρ and we get by transfer a partial order on Irr_ρ. We will use this bijection L to compare for example an irreducible representation to a standard representation. Also we will use $E(\pi)$, $t(\pi)$ etc. for an irreducible representation π, being understood that this is $E(M)$, $t(M)$ etc., where M is the multisegment associated to the standard representation Θ such that $\pi = L(\Theta)$. For any standard representation Θ, one has in $\mathscr{R}$ (actually in $\mathscr{R}_\rho$)

$$\Theta = L(\Theta) + \sum_{\pi_j < L(\Theta)} a_i \pi_j,$$

$a_i \in \mathbb{N}^*$, $\pi_j \in \mathrm{Irr}$ (this is an avatar of the Langlands quotient theory explicitly stated in this stronger form in [24] and [18, Theorem 5.3]).

As a consequence, Std_ρ is a linear basis of the sub-space $\mathscr{R}_\rho$. So $(\mathscr{R}_\rho, +, *)$ is isomorphic with the ring of polynomials over $\mathbb{Z}$ with (commutative) variables in the set $\mathscr{D}_\rho$. A standard representation $\Theta \in \mathscr{R}_\rho$ is then a monomial. Notice that its degree equals $t(L(\Theta))$ and is also $\geq t(\pi_j)$ for all the $\pi_j < L(\Theta)$ (because t is decreasing). Decomposing an irreducible representation on the standard basis comes to write it as a polynomial.

Important fact. — If $\pi \in \mathscr{R}_\rho$ has a decomposition $\pi = \sum_i m_i \pi_i$ with $m_i \neq 0$ and π_i distinct irreducible representations, then, as polynomials, we have $\deg(\pi) = \max_i \deg(\pi_i)$. To see this one writes $\pi_i = L(\Theta_i)$ and notice that the degree of π_i is also the degree of the monomial Θ_i. If, among the π_i *of maximal degree as polynomials*, π_{i_0} is maximal for the order relation on representations, then the monomial Θ_{i_0} appears with non zero coefficient in π, which shows that $\deg(\pi) \geq \max_i \deg(\pi_i)$. The inequality $\leq$ is obvious.

Using the relation $\mathscr{R} = \times_{\rho \in R} \mathscr{R}_\rho$ one may see that Std is a linear basis of $\mathscr{R}$ and $\mathscr{R}$ is a polynomial algebra with variables in $\mathscr{D}$. Every element of Std may be written in a unique way as a product of elements of Std_ρ, $\rho \in R$, which are almost all trivial. So we get a product order relation on Std. This order relation passes then to Irr. If $\Theta \in \mathrm{Std}$, one has then in $\mathscr{R}$

$$\Theta = L(\Theta) + \sum_{\pi_j < L(\Theta)} a_i \pi_j$$

with $a_i \in \mathbb{N}^*$ and $\pi_j \in \mathrm{Irr}$. And if $\Theta = \prod_{\rho \in A} \Theta_\rho$ where A is a finite subset of R and $\Theta_\rho \in \mathrm{Std}_\rho$,then $L(\Theta) = \prod_{\rho \in A} L(\Theta_\rho)$.

Let $\pi, \pi' \in \mathrm{Irr}$ and set $\pi = L(\Theta)$, $\pi' = L(\Theta')$ with $\Theta, \Theta' \in \mathrm{Std}$. Then

$$\pi * \pi' = x + \sum_{\pi_j < x} a_i \pi_j,$$

where $x = L(\Theta * \Theta')$, $a_i \in \mathbb{N}$, $\pi_j \in \mathrm{Irr}$. We will call x *the Langlands quotient* of $\pi * \pi'$.

The algorithm. — For this section we fix $\rho \in R$ and we parametrize Irr_ρ by multisegments. Let $\pi \in \mathrm{Irr}_\rho$. If M and $M^\#$ are the multisegments of π and $i(\pi)$, Moeglin and Waldspurger gave an algorithm for computing $M^\#$ from M ([12] for $D = F$, [7] for $D \neq F$). I will call here MWA this algorithm. Here is a description:

If $\delta = \{b, b+1, \dots, e\}$ is a non empty segment, we set $\delta^- = \{b, b+1, \dots, e-1\}$ (which may be empty).

Let M be a multisegment. We associate to M the multisegment $M^\#$ in the following way : let e be the greatest number in $E(M)$. Chose:

- a shortest segment $\delta_0 \in M$ with ending e.
- a shortest segment $\delta_1 \in M$ with ending $e-1$ and not included in δ_0
- a shortest segment $\delta_2 \in M$ with ending $e-2$ and not included in δ_1 etc.

Stop when it is not possible to find the next segment. Say δ_r is the last one. Then $S_0 = \{e-r, e-r+1, \dots, e\}$ is the first segment of $M^\#$. Let now M^- be the multisegment obtained from M after replacing δ_i with δ_i^- for all $i \in \{0, 1, \dots, r\}$ (if one of these is empty, just erase it). Starting with M^- what we have done with M, we find the second segment of $M^\#$, and so on (so in the end we have that $M^\#$ is the multiset union of $[S_0]$ and $(M^-)^\#$). As observed in [12], if the greatest number in $E(M)$ is e, then *all* the segments of $M^\#$ ending with e will be constructed exclusively with points from $E(M)$. As an easy consequence, we have the following property:

(P) *If $E(M)$ is included in some segment S of length k (i.e. $\forall x \in E(M), x \in S$), then $l(M^\#) \leq k$.* (Indeed, all the segments of $M^\#$ ending with the greatest point e of S will be included in S, so will have length $\leq k$. And once we are done with e, the set of endings will be (as a consequence) included in the shift of S with -1.)

Also, every segment of $M^\#$ being constructed taking elements on distinct segments of M, we obviously have

(P′) : *$l(M^\#)) \leq t(M)$, i.e. the maxlength of $M^\#$ is less than or equal to the thickness of M.*

The results

In the sequel we will consider elements π of Irr which will actually be, most of the time, obviously in some Irr_ρ. For these, we will loosely say things like "let M be the multisegment of π", or apply MWA, being understood we parametrized by segments. The assertions **(b), (c), (d)** in the following theorem are the **(U1), (U2), (U3)** of [18].

THEOREM 1. — *Let δ be a square integrable representation of some $GL_n(D)$.*

(a) *If $\delta = Z(\rho, l)$ then $i(u(\delta, k)) = u(\tau, l)$ where $\tau := Z(\rho, k)$.*

(b) *The representation $u(\delta, k)$ is unitary.*

(c) *The representations $\pi(u(\delta, k), \alpha) := \nu_\delta^\alpha u(\delta, k) \times \nu_\delta^{-\alpha} u(\delta, k)$ are irreducible and unitary for $\alpha \in]-\frac{1}{2}, \frac{1}{2}[$.*

(d) *$u(\delta, k)$ is a prime element of the ring $\mathscr{R}$.*

Proof. — **(a)** is obvious by MWA.

(b): Tadić ([16]) reduced it by induction to the proof of :

(i) for all $k \geq 1$, $u(\delta, k) \times u(\delta, k)$ is irreducible and

(ii) for all $k \geq 2$, $u(\delta, k-1) \times u(\delta, k+1)$ is irreducible.

Here I prove **(i)** and **(ii)** (this trick of Tadić was also used in [5, 4.1]):

(i): let $\delta = Z(\rho, l)$ and set $\tau := Z(\rho, k)$. Write

$$u(\delta, k) * u(\delta, k) = \pi + \sum_{\pi_j < \pi} a_j \pi_j$$

where π is the Langlands quotient of the product and $a_j \in \mathbb{N}$. The same way:

$$u(\tau, l) * u(\tau, l) = \pi' + \sum_{\pi'_j < \pi'} b_j \pi'_j.$$

By **(a)**, $u(\tau, l)$ is the dual of $u(\delta, k)$. The second relation is then dual to the first. The MWA implies that the Langlands quotient π' is the dual of π. If some $a_j \neq 0$, then π_j appears effectively in the first sum and its dual, which we call π'_j, appears in the second sum by linear independence of the characters. As $l(\tau) = k$ and $\pi'_j < \pi'$, we have $m(\pi'_j) > k$ (at the first elementary operation the longest segment will already be longer than k; then m is decreasing). This is impossible by **(P)**, since $E(\pi_j) \subseteq E(u(\delta, k) * u(\delta, k))$ (E is increasing) which is included in a segment of length k. So $a_j = 0$ for all j.

(ii): write

$$u(\delta, k-1) * u(\delta, k+1) = \pi + \sum_{\pi_j < \pi} a_j \pi_j$$

where π is the Langlands quotient of the product and $a_j \in \mathbb{N}$. Then write the dual relation (using **(a)** for the left side) :

$$u(\tau_-, l) * u(\tau_+, l) = \pi' + \sum_{\pi'_j < \pi'} b_j \pi'_j$$

where MWA implies that the Langlands quotient π' is the dual of π. Let π_j, π'_j be two dual representations appearing effectively in the sums. As $l(\delta) = l$ we know $m(\pi_j) \geq l + 1$. Now

$$E(\pi'_j) \subset E(u(\tau_+, l) * u(\tau_-, l)) = x + [0, 1, 1, 2, 2, 3, 3, \ldots, l-1, l-1, l]$$

and

$$B(\pi'_j) \subset B(u(\tau_+, l) * u(\tau_-, l)) = y + [0, 1, 1, 2, 2, 3, 3, \dots, l-1, l-1, l]$$

where for the sake of the reader I shifted with $x = \frac{k-l-1}{2}$ and $y = \frac{-k-l+1}{2}$. Notice that $x + l$ always belongs to $E(\pi'_j)$, so if M is the multisegment of π'_j, then the construction of (the first segment of) $M^{\#}$ starts with $x+l$. Now I will show that *the first segment S_0 of $M^{\#}$ (i.e. the one ending in $x + l$) does not contain x*. Indeed, the segment δ_0 of π'_j containing $x+l$ cannot be shorter than the one of $u(\tau_+, l) * u(\tau_-, l)$ containing $x + l$ (elementary operations cannot shorten *this one*). So the beginning of δ_0 is $\leq y + l - 1$. But the sequence of beginnings of segments $\delta_0, \delta_1, \delta_2, \dots$ used in the construction of S_0 has to be strictly decreasing, and there is no place in $B(\pi'_j) \backslash \{y + l\}$ for such a sequence with $l + 1$ terms. Hence the first segment of $M^{\#}$ is constructed only with points from $x + \{1, 2, \dots, l\}$ so its length is $\leq l$. But, as this one does not contain x, we have that $E(M^-)$ is included (in the sense of **(P)**) in the segment $x + \{0, 1, 2, \dots, l-1\}$, which implies by **(P)** that the other segments of $M^{\#}$ are also of length $\leq l$. This is in contradiction with $l(\pi_j) \geq l + 1$.

(c): Tadić showed in [18] irreducibility for $\alpha \in]-\frac{1}{2}, \frac{1}{2}[\backslash\{0\}$ and also that unitarity would follow from irreducibility in $\alpha = 0$, which has been proved in **(i)** (see the proof of **(b)**).

(d): Assume $u(\delta, k)$ is *not* a prime element of $\mathscr{R}$. Then it is not a prime element of $\mathscr{R}_\rho$ (easy) and we will work in this ring. As $\mathscr{R}_\rho$ is a polynomial ring over $\mathbb{Z}$, it is a UFD, hence $u(\delta, k)$ is reducible. Write

$$u(\delta, k) = S * T$$

where S and T are non scalar elements of $\mathscr{R}_\rho$. Then

$$u(\tau, l) = \iota(S) * \iota(T).$$

As a polynomial, $u(\delta, k)$ has degree k and so $deg(S), deg(T) < k$. Then, if we write S, T as linear combination of irreducible representations, every irreducible representation π in S or T has $t(\pi) < k$. So every irreducible representation $i(\pi)$ in $\iota(S)$ or $\iota(T)$ has $m(i(\pi)) < k$ (by **(P′)**). But, if α is an irreducible representation in $\iota(S)$, then $t(\alpha) \leq deg(\iota(S))$ and if β is an irreducible representation in $\iota(T)$ then $t(\beta) \leq deg(\iota(T))$ so finally $t(\alpha) + t(\beta) \leq l$ (because $deg(\iota(S)) + deg(\iota(T)) = deg(u(\tau, l)) = l$). But, as all the segments of α and β are strictly shorter than k, $\alpha * \beta$ lives on a group of smaller size than $u(\tau, l)$. It is impossible for $u(\tau, l)$ to be a sum of such representations. □

COROLLARY 1. — *Let $\mathscr{T}$ be the set of all finite products of (irreducible unitary) representations of type $u(\delta, k)$ and $\pi(u(\delta, k), \alpha)$ for $k \in \mathbb{N}^*$, δ square integrable and $\alpha \in]0, \frac{1}{2}[$. Let $\mathscr{U}$ be the set of all irreducible unitary representations of all $GL_m(D)$, $m \in \mathbb{N}^*$. Then $\mathscr{T} \subset \mathscr{U}$.*

Proof. — By prop. 2.13 in [4], a product $u_1 \times u_2 \times \cdots \times u_k$ of irreducible unitary representations is irreducible unitary if, for all $1 \leq i \leq k$, $u_i \times u_i$ is irreducible. By **(i)** (see the proof of **(b)**), $U = u(\delta,k) \times u(\delta,k)$ is indeed irreducible. To show that $\pi(u(\delta,k),\alpha) \times \pi(u(\delta,k),\alpha)$ is also irreducible we write it as $\nu_\delta^\alpha U \times \nu_\delta^{-\alpha} U$. Now $\nu_\delta^\alpha U$ and $\nu_\delta^{-\alpha} U$ are irreducible representations, and for $0 < \alpha < \frac{1}{2}$ no segment of the first one is linked to no segment of the second one. So $\nu_\delta^\alpha U \times \nu_\delta^{-\alpha} U$ is irreducible by Proposition 2.2 and Lemma 2.5 of [18]. □

If $\delta = Z(\rho,l)$ is a square integrable representation and $k \in \mathbb{N}^*$, set

$$F(\delta,k) := \nu^{-\frac{k+l}{2}} \Big(\sum_{w \in W_k^l} (-1)^{\mathrm{sgn}(w)} \prod_{i=1}^{k} Z(\rho, i, w(i)+l-i) \Big) \in \mathscr{R},$$

where W_k^l is the set of permutation w of $\{1,2,\ldots,k\}$ such that $w(i)+l \geq i$ for all $i \in \{1,2,\ldots,k\}$.

THEOREM 2. — **(e)** *If $\delta = Z(\rho,l)$ then one has, in $\mathscr{R}$, $u(\delta,k) = F(\delta,k)$.*

(f) *One has* $\nu_\delta^{-\frac{1}{2}} u(\delta,k) * \nu_\delta^{\frac{1}{2}} u(\delta,k) = u(\delta,k-1) * u(\delta,k+1) + u(\delta_-,k) * u(\delta_+,k)$.

Proof of **(e)**. — The case $k = 1$ is trivial, $k = 2$ is known by [24] and [18]. Assume by induction that $u(\delta,k) = F(\delta,k)$ and $u(\delta,k-1) = F(\delta,k-1)$ for any square integrable representation δ.

Let us write

$$\begin{aligned} &\nu_\delta^{-\frac{1}{2}} u(\delta,k) * \nu_\delta^{\frac{1}{2}} u(\delta,k) \\ &\quad = u(\delta,k-1) * u(\delta,k+1) + u(\delta_-,k) * u(\delta_+,k) + \sum_{\pi_j < u(\delta,k-1) * u(\delta,k+1)} a_j \pi_j \end{aligned}$$

and taking the dual:

$$\begin{aligned} &\nu_\delta^{-\frac{1}{2}} u(\tau,l) * \nu_\delta^{\frac{1}{2}} u(\tau,l) \\ &\quad = u(\tau,l-1) * u(\tau,l+1) + u(\tau_-,l) * u(\tau_+,l) + \sum_{\pi'_j < u(\tau,l-1) * u(\tau,l+1)} b_j \pi'_j. \end{aligned}$$

These formulas need some explanation: as $u(\delta,k-1) * u(\delta,k+1)$ is irreducible by **(ii)** (see the proof of **(b)**), it is the Langlands quotient of $\nu_\delta^{-\frac{1}{2}} u(\delta,k) * \nu_\delta^{\frac{1}{2}} u(\delta,k)$. For the same reason, $u(\tau,l-1) * u(\tau,l+1)$ is the Langlands quotient of $\nu_\delta^{-\frac{1}{2}} u(\tau,l) * \nu_\delta^{\frac{1}{2}} u(\tau,l)$. Now, as $i(u(\tau,l-1) * u(\tau,l+1)) = u(\delta_-,k) * u(\delta_+,k)$, this representation appears with multiplicity one in $\nu_\delta^{-\frac{1}{2}} u(\delta,k) * \nu_\delta^{\frac{1}{2}} u(\delta,k)$ by duality. As $i(u(\delta,k-1) * u(\delta,k+1)) = u(\tau_-,l) * u(\tau_+,l)$, this representation appears with multiplicity one in $\nu_\delta^{-\frac{1}{2}} u(\tau,l) * \nu_\delta^{\frac{1}{2}} u(\tau,l)$ by duality. So the two Σ

are dual to one another and we also have the property **[A]**: *the coefficient* b_j *of* $\pi'_j = u(\tau_-, l) * u(\tau_+, l)$ *is zero.*

Chenevier and Renard ([10]) recognized in $F(\delta, k)$ a determinant with entries in $\mathscr{R}$ and proved using Lewis Carroll's identity that

$$[\mathbf{F}]\nu_\delta^{-\frac{1}{2}} F(\delta, k) * \nu_\delta^{\frac{1}{2}} F(\delta, k) = F(\delta, k-1) * F(\delta, k+1) + F(\delta_-, k) * F(\delta_+, k).$$

Using the induction assumption we replace to obtain:

$$\nu_\delta^{-\frac{1}{2}} u(\delta, k) * \nu_\delta^{\frac{1}{2}} u(\delta, k) = u(\delta, k-1) * F(\delta, k+1) + u(\delta_-, k) * u(\delta_+, k).$$

Subtracting from our first formula we get:

$$\sum_{\pi_j < u(\delta,k-1)*u(\delta,k+1)} a_j \pi_j = u(\delta, k-1)\big(F(\delta, k+1) - u(\delta, k+1)\big).$$

So taking the dual, $u(\tau_-, l)$ divides $\sum_{\pi'_j < u(\tau, l-1)*u(\tau,l+1)} b_j \pi'_j$. Write $\sum_{\pi'_j < u(\tau,l-1)*u(\tau,l+1)} b_j \pi'_j = u(\tau_-, l) * \sum c_p \theta_p$ with $c_p \in \mathbb{Z}^*$ and θ_p distinct irreducible representations. Chose a maximal θ_p. Then $L(u(\tau_-, l) * \theta_p)$ is one of the π'_j such that $b_j \neq 0$. But $t(\pi'_j) \leq 2l$ so $t(\theta_p) \leq l$. As $E(\pi_j)$ is included in a segment of length $k+1$, $m(\pi'_j) \leq k+1$ (by **(P)**) and so $m(\theta_p) \leq k+1$. As the support of θ_p has $l(k+1)$ elements and we proved it contains less then l segments, all of length less than $k+1$, the only possibility is $\theta_p = u(\tau_+, l)$. So $u(\tau_-, l) * \theta_p = u(\tau_-, l) * u(\tau_+, l)$. As this representation is irreducible, we have $L(u(\tau_-, l) * \theta_p) = u(\tau_-, l) * u(\tau_+, l)$.This is in contradiction with **[A]**. So what we called $\sum c_p \theta_p$ is empty, and this implies **(e)** at level $k+1$.

Proof of **(f)**. — The formula **[F]** implies that **(e)** for all k implies **(f)** ([10]). □

Corollary 2. — *Let* C *be the Jacquet-Langlands transfer for square integrable representations from* [11] *between* $GL_{nd}(F)$ *to* $GL_n(D)$ *and* LJ *the transfer of irreducible representations from* [3] *between* $GL_{ndk}(F)$ *to* $GL_{nk}(D)$. *If* δ *is a square integrable representation of* $GL_{nd}(F)$, *then* $\mathrm{LJ}(u(\delta, k)) = \bar{u}(\mathrm{C}(\delta), k)$.

Proof. — It is proved in [21] to be an easy consequence of the character formula **(e)**. Notice the [11] transfer is known also when $ch(F) \neq 0$ ([2]). □

Remark 0.1. — Bernstein's theorem asserts that *induced representations from unitary irreducible representations remain irreducible.* Bernstein proved this theorem for $D = F$ in [9]. Sécherre proved it for general D in [14]. Tadić showed ([16]) using Bernstein's theorem that actually $\mathscr{T} = \mathscr{U}$ (the Corollary 2 here is just one inclusion) and this is the only proof of the equality we have up to now. The proof of the Bernstein's theorem is difficult, especially for general D, and a new short one would be most welcome. This theorem has not been used in this paper.

Historical notice. — For $D = F$, Theorem 1 and Corollary 1 have been proved by Tadić in [16] (1986), **(f)** in [17] (1987), and **(e)** in [20] (1995). Tadić used Bernstein's theorem ([9]) and Zelevinsky derivatives. He conjectured this results should hold also for $D \neq F$ in [18] (1990), where he proved **(d)** in this case.

For $D \neq F$, $ch(F) = 0$:

- **(b)** has been proved in [6] (2004) for D of characteristic zero (as it requires a result from [3] based on the Arthur trace formula).
- Tadić gave then a proof of all the other results in this paper in [21] (2006) assuming only Bernstein's theorem were true for $D \neq F$ (but not involving anymore Zelevinsky's derivatives which are specific to $D = F$). I gave myself a proof of these facts in [4] (2008) without using the Bernstein's theorem which was not known for $D \neq F$ at that time. Eventually Sécherre proved Bernstein's theorem for $D \neq F$ in [14] (2009).

For proofs in the case $D \neq F$ and $ch(F) \neq 0$: [5] (2010).

A proof of similar assertions for D archimedean (i.e. $D = \mathbb{R}, \mathbb{C}, \mathbb{H}$) may be found in the literature. See [15], [23], [22], and [8]. It would be long to include the statement here because the analogue of the Zelevinsky classification is different, so notation and formulation change.

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