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# ON CERTAIN SUBSOCLES OF A PRIMARY ABELIAN GROUP

BY

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Let $G$ be a $p$-primary abelian group. Then $G[p] = \{x \in G : px = 0\}$ is called the socle of $G$ and any subgroup of $G[p]$ will be referred to as a subsocle. In [3], the notion of a quasi-essential subsocle is introduced : A subsocle $S$ is said to be quasi-essential if $G = H + K$, whenever $H$ is a pure subgroup of $G$ containing $S$, and $K$ is maximal disjoint from $S$. Recall that $K$ will be maximal disjoint from $S$ if and only if $G[p] = K[p] \oplus S$ and $K$ is a neat subgroup of $G$ (that is, $pG \cap K = pK$). The purpose of this note is to prove the following proposition.

PROPOSITION. — *A subsocle $S$ of $G$ is quasi-essential if and only if either*

$$(1) \qquad S \subseteq G^1 = \bigcap_{n=1}^{\infty} p^n G$$

*or*

$$(2) \qquad (p^n G)[p] \supseteq S \supseteq (p^{n+1} G)[p]$$

*for some nonnegative integer $n$.*

That conditions (1) or (2) are sufficient is established in [3], but the converse is obtained there only when further conditions are placed either on $S$ or $G$. Our basic tool will be the following lemma :

LEMMA. — *If $G = Zb \oplus Za \oplus H$, where $o(b) = p^i$ and $o(a) \geq p^{i+2}$ and if $S$ is a subsocle of $G$ such that $S \subseteq Zb \oplus H$ and $S \cap Zb \neq 0$, then $S$ is not quasi-essential.*

*Proof.* — Write $S = (Zb)[p] \oplus S_1$ with $S_1 \subseteq H$, and choose $M$ maximal disjoint from $S_1$ with $a, b \in M$. Then $M$ is a neat subgroup of $G$, and

$M = Za \oplus Zb \oplus M_1$, where $M_1 = M \cap H$. Let $M' = Z(b + pa) \oplus M_1$, and note that $G[p] = M'[p] \oplus S$. $M'$ will therefore be maximal disjoint from $S$ provided it is neat in $G$. But the neatness of $M'$ is an easy consequence of the neatness of $Zb \oplus M_1$. To prove that $S$ is not quasi-essential it suffices to show that $G \neq M' + (Zb \oplus H)$. But, in fact, $a \notin M' + (Zb \oplus H)$. For suppose $a = t(b + pa) + m_1 + sb + h$, where $m_1 \in M_1$, $h \in H$ and $t, s \in Z$. Then $m_1 + h \in H \cap (Za \oplus Zb) = 0$, and we have the absurd equation $(1 - pt)a = (t + s)b \in Za \cap Zb = 0$.

We shall require the notion of a *center of purity*: A subgroup $H$ of $G$ is said to be a center of purity if every subgroup maximal disjoint from $H$ is pure in $G$. In [4], it is shown that a subsocle $S$ of a $p$-group $G$ is a center of purity if and only if either

(i) $$S \subseteq G^1$$

or

(ii) $$(p^n G)[p] \supseteq S \supseteq (p^{n+2} G)[p]$$

for some nonnegative integer $n$. Note the slight difference between (ii) and (2). In [3], it is actually proved that if a subsocle is both a center of purity and quasi-essential, then it satisfies (1) or (2). Consequently, we need only prove that every quasi-essential subsocle is a center of purity in order to establish our proposition.

Now if $S$ supports a pure subgroup $H$ (that is, $H[p] = S$) and if $S$ is quasi-essential, then clearly $G = M \oplus H$ whenever $M$ is maximal disjoint from $S$ and, since direct summands are pure, $S$ is a center of purity. The proof of our proposition thus reduces to showing that a quasi-essential subsocle that fails to support a pure subgroup is also a center of purity, or equivalently, that a subsocle $S$ which neither supports a pure subgroup nor is a center of purity cannot be quasi-essential.

By a standard technique, we can construct a basic subgroup $B = A \oplus C$ of $G$ where $C[p]$ is dense in $S$ (relative to the subspace topology induced on $G[p]$ by the $p$-adic topology of $G$). Since $S$ does not support a pure subgroup, $S \cap p^n G$ cannot be dense in $(p^n G)[p]$ for any $n$ (*see* [2]). This fact forces $A$ to be unbounded. But $S$ is not a center of purity and therefore $S \nsubseteq G^1$. Hence there is a minimal nonnegative $n$ such that $S \nsubseteq p^{n+1} G$. Then $S$ has an element of height exactly $n$ and, since $C[p]$ is dense in $S$, this element may be taken to be in $C$. Thus $C$ has a cyclic direct summand $Zb$ with $o(b) = p^{n+1}$. Recall that A is unbounded, and consequently has a cyclic summand $Za$ with $o(a) = p^k \geq p^{n+3}$. Exploiting the purity of $B$ and the fact that $C[p]$ is dense in $S$, one easily shows that $Za \cap (S + C + p^k G) = 0$. By Theorem 24.1 of [1], we then have a direct decomposition $G = Za \oplus M$, where $M \supseteq S + C$. But $Zb$ is a pure subgroup of $G$ and therefore $G = Za \oplus Zb \oplus H$, where $Zb \oplus H \supseteq S$ and $S \cap Zb \neq 0$. The conditions of our lemma are now satisfied, and we conclude that $S$ is not quasi-essential.

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