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OFER GABBER

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APPENDICE A

FACSIMILÉ : ORSAY

Transparents de l'exposé d'Ofer Gabber, fait le 27 juin 2005 à la conférence en l'honneur de Luc Illusie ([Gabber, 2005a]).

A finiteness theorem for
non abelian H^1 of excellent
schemes

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1. Statements

Theorem 1.0 Let $f: X \rightarrow Y$ be a morphism of finite type of noetherian schemes and F a constructible sheaf of sets on X_{et} .

Then $f_* F$ is constructible.

The proof is by reducing to the case f proper (SGA 4) and f an open immersion (easier when normalizations are finite)

Theorem 1.1 Let $\mathbb{L}$ be a finite set of primes, $f: X \rightarrow Y$ a morphism of finite type between quasi-excellent schemes on which $\bigwedge_{p \in \mathbb{L}} \mathbb{L}$ is invertible. Let F be a constructible $\mathbb{L}$ -torsion sheaf of groups on X .

Then $R^1 f_* F$ is constructible.

Also stack version.

Proved here using ultraproducts.

Theorem 1.2. $X, Y, \mathbb{L}, f$ as in 1.1.

Y qc. If F is a constructible

$\mathbb{L}$ -torsion abelian sheaf on X , then

the sheaves $R^i f_* F$ are constructible

and 0 for $i \gg 0$.

(Planned for Deligne's conference,
Using alterations)

2. Reductions.

Th. 1.1 known for proper f , so enough to consider open immersions. Reduce to Y normal affine, $X = Y - Z$ (Z reduced) and F constant.

Enough to find $Y' \rightarrow Y$, normalization of Y in a finite extension of $R(Y)$ which kills $R^1 f_* F$.

This exists outside $\text{Sing}(Y) \cup \text{Sing}(Z)$ (Abh. Lemma). Reduce to

Th. 2.1. Let Y be a normal excellent scheme, Z a closed subscheme of $\text{cod} \geq 2$, $j: Y - Z \rightarrow Y$, G a finite group. Then $R^1 j_* G$ is constructible.

(The order of G not necessarily inv. on Y)

This will be reduced to the following case:

Th. 2.2. Let A be a strictly henselian excellent normal local ring of dimension 2.

For every finite group G

$H^1(\text{Spec}(A) - \{m\}, G)$ is finite.

3. Lefschetz

Recall (SGA 2 XIII §2)

Th. 3.0. A c.n.l.r. $f \in m$, f nonzerodivisor,

$X = \text{Spec}(A)$ $X' = X - \{m\}$, $\text{depth } \mathcal{O}_{X',x} \geq 2$

for closed points of X' . Then

$\Gamma(X', \mathcal{O}) \cong \Gamma(\hat{X}', \mathcal{O})$ so

$\pi_0(X' \cap V(f)) \cong \pi_0(X')$.

Cor. 1. A excellent normal n.l.r. of $\dim \geq 3$

$f \in A$ nonzero nonunit, then the punctured

spectrum of $\underset{Y}{\text{Spec}}(A/fA)$ is connected.

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Y is connected in cod. 1, i.e. if $Z \subset Y$ is of cod ≥ 2 then $Y - Z$ is connected.

Cor. 2. Let X be normal excellent scheme
 $D \subset X$ effective Cartier divisor
 $Z \subset D$ closed, $\text{cod}_X Z \geq 3$

$$\begin{array}{ccc} D - Z & \xrightarrow{j'} & D \\ \downarrow & & \downarrow \\ X - Z & \xrightarrow{j} & X \end{array}$$

If L is a locally ~~isotrivial~~ constant constr sheaf on $X - Z$

$$(j_* L)|_D \xrightarrow{\sim} j'_*(L|_{D-Z}).$$

$(R^1 j_* G)|_D$ injects into $R^1 j'_* G$
 for every finite group G .

4. Use of desingularization of 2 dimensional schemes.

Lemma 4.1. In the situation of Th. 2.1

if Z_i is an irreducible component of Z of codim 2 in Y then there is an open dense $U_i \subset Z_i$ s.t. the sheaf of pointed sets $F = R^1 j_* G$ on U_i has specialization maps $F_s \rightarrow F_t$ (for $t \rightarrow s$ map of geom. pts of U_i
 "topos pts") with trivial kernel.

Proof. w.m.a. $Z = Z_i$ and that there is $p: Y' \rightarrow Y$ proper, birational, isomorphism outside Z , Y' regular, $p^{-1}(Z)_{\text{red}}$ is a normal crossings divisor with $\checkmark$ components dominating Z and $p^{-1}(Z) \rightarrow Z$ is (univ) locally 0-acyclic. (connected Milnor fibres)

5. Proof of Th. 2.1 assuming Th. 2.2.

WMA $Y = \text{Spec}(A)$, A excellent normal domain and that the result is known

for ~~(proper subschemes of Y and for)~~

normalizations of $Y' \subsetneq Y$, Y' irr, in

finite extensions of $R(Y')$.

Using section 4 there is

$\text{Spec}(A') \xrightarrow[\text{finite}]{\quad} \text{Spec}(A)$ which kills

$R^1 j_* G$ outside a cod ≥ 3 locus,

so reduce to $\mathbb{Z}$ of cod ≥ 3 and

use section 3.

6. Discriminants.

Lemma 6.1. Let $A = (A_{ij})$ be an $n \times n$ matrix of $m \times m$ matrices over a commutative ring k , with commuting A_{ij} . Let A be the resulting block $nm \times nm$ matrix.

Then $\det A = \det \det A$.

[Hint: replace A_{11} by $A_{11} + t$, invertible over $k' = k[t]$.]

Lemma 6.2. Let $A \subset B \subset C$ be comm. rings with C a free B -module with basis f_j ($1 \leq j \leq n$) and B a free A -module with basis e_i ($1 \leq i \leq m$).

(so $(e_i f_j)$ is an A basis of C). We use these bases to define discriminants, e.g.

$$\text{disc}_{B/A} = \det \text{Tr}_{B/A}(e_i e_j).$$

Then $\text{disc}_{C/A} = (\text{disc}_{B/A})^n \text{Norm}_{B/A}(\text{disc}_{C/B})$.

7. Rigidity.

Th. 7.1. Let (A, I) be a henselian pair with I finitely generated, $U \subset \operatorname{Spec}(A)$ a quasi-compact open $U \supset \operatorname{Spec}(A) - V(I)$, $\hat{A}$ the I -adic completion of A , $\hat{U} \subset \operatorname{Spec}(\hat{A})$ the inverse image of U . Let F be a sheaf of sets (resp. an ind-finite sheaf of groups) on U . Then $H^0(U, F) \xrightarrow{\sim} H^0(\hat{U}, F)$ (resp. $H^1(U, F) \xrightarrow{\sim} H^1(\hat{U}, F)$).

version for stacks.

Can reduce to case F constant, which holds by Elkies's approximation when A is noetherian and also when I is principal and $U = \operatorname{Spec}(A) - V(I)$. In general blow up a f.g. ideal defining the complement of U and use proper base change for stacks (Giraud + extension to non noeth. case) and affine base change to reduce to the principal ideal case.

Th. 7.2. (close to SGA4 XV)

$$\begin{array}{ccc} p: & X' & \longrightarrow X \\ & \downarrow j' & \downarrow j \\ & U' & \downarrow j \\ & & U \end{array}$$

p smooth, X normal excellent, $U' = p^{-1}(U)$,
 U ^{open} containing all points of cod ≤ 1 .

Then $p^* R^1 j_* G \xrightarrow{\sim} R^1 j'_* G$ (G finite).

Lemma 7.3. Let Z be a nowhere dense closed subscheme of a noetherian scheme X .

Then the following conditions are equivalent

(1) Let $p: X' \rightarrow X_{\text{red}}$ be the normalization.

Then $p^{-1}(Z)$ is of codim ≥ 2 in X' .

(2) for every $y \in Z$, the irreducible components of $\text{Spec}(\hat{\mathcal{O}}_{X,y}^h)$ are of dim ≥ 2

(3) same for $\hat{\mathcal{O}}_{X,y}$.

If this holds we say that Z is

$\subset 2$ in X .

Prop. 7.4. Let $A \longrightarrow B$ be a local homomorphism of nlr, formally smooth for m-adic topologies. Then there is a direct system of local essentially smooth A-algebras B_i with

$$\hat{B} \simeq (\varinjlim B_i)^\wedge$$

$$B_i \longrightarrow B_j \quad \text{for} \quad m_i B_j = m_j.$$

Th. 7.5. Let Z be c2 in X

$p: X' \longrightarrow X$ a flat morphism of noetherian schemes, the fibres of p above points of Z are geometrically regular.

(conclusion as in 7.2.)

8. Separable projection.

Lemma 8.1. Let R be a complete noetherian local ~~ring~~^{domain}. Then R has a regular subring R_0 , R finite over R_0 with $\text{Frac}(R)$ separable over $\text{Frac}(R_0)$.

Only problem in equal characteristic $p > 0$.
Let k be the residue field of R and $\{b_i\}_{i \in I}$ a p -basis of k . There is a bijection

$\{\text{coefficient fields of } R\} \longleftrightarrow \{\text{liftings of } b_i\}.$
Fix a coefficient field.

By the proof of Nagata's Jacobian criterion there is a finite subset $J \subset I$ s.t. if $k' = k^p(b_i, i \notin J)$ then

$\Omega = \Omega_R^1 / \overline{k' R^p}$ has generic rank

$\dim(R) + \text{card}(J).$

Then change the liftings of b_i $i \in J$ s.t. their differentials are linearly independent in Ω . With this coefficient field

one can take $k' = k$.

Let $t_1, \dots, t_d$ be a system of parameters for R and let $f_1, \dots, f_d \in R$ be s.t. df_i form a basis of $\Omega \otimes_R \text{frac}(R)$.

WMA $f_i \in \mathfrak{m}$.

Let $t'_i = t_i (1 + f_i)$.

Can take

$$R_0 = k[[t'_1, \dots, t'_d]].$$

9. Ultraproducts

Let I be a set. There is a bijection

$$\{\text{ultrafilters on } I\} \longleftrightarrow \text{Spec}((\mathbb{Z}/2\mathbb{Z})^I)$$

$$F \longmapsto \mathfrak{p} = \{ \chi_A \mid A \notin F \}.$$

If R_i ($i \in I$) are rings then the ultraproduct $\prod_{i \in I} R_i$ is

$\prod_{i \in I} R_i / \sim$ where

$$(r_i) \sim (s_i) \iff \{i \mid r_i = s_i\} \in F,$$

In the commutative case there is a map of topological spaces

$$f: \text{Spec}(\prod_i R_i) \longrightarrow \text{Spec}((\mathbb{Z}/2)^I)$$

defined by " $f^* \chi_A = \chi_A$ ".

The fibres of f (with the restriction of the structure sheaf) are Spec of the ultraproduct.

R_i : all fields, domains, local rings,
 $\Rightarrow$ same for $\prod_{i \in I} R_i = R_\infty$

(*) If R_i are local rings whose maximal ideals are generated by n elements then R_∞ has the same property, so its completion is a complete noetherian local ring.

• If furthermore F is not ω complete

R_∞ maps onto $\hat{R}_\infty$

• If F is ω complete and R_i are noetherian
 (10),

then R_∞ is noetherian.

Lemma. If $R_i \rightarrow S_i^{(i \in I)}$ are finite maps of noetherian local rings, $\exists n$ s.t. V_i the maximal ideal of R_i and S_i have n generators as an R_i -module, F is an ultrafilter on I , $R_\infty \rightarrow S_\infty$ the corresponding map on ultraproducts then $R_\infty \rightarrow S_\infty$ ~~homomorphism~~ is finite and $\ker(R_\infty \rightarrow R_\infty^\wedge)$ generates $\ker(S_\infty \rightarrow S_\infty^\wedge)$.

The proof uses that in all cases

$R_{\infty, \text{sep}} = \text{Im}(R_\infty \rightarrow R_\infty^\wedge)$ is noetherian.

We will use this only for complete local rings, in which case it is easily seen that R_∞ maps onto $R_\infty^\wedge$,

10. Proof of Th. 2.2.

We may assume A is complete.

View A as a finite generically étale extension of a 2 dimensional regular complete local ring R . A/R finite free rk m .

If the assertion is false there is $n > 0$ and connected pairwise non isomorphic finite étale maps $E_i \rightarrow \text{Spec}(A) - \{m_A\}$ of degree n . Let B_i be the normalization of A in E_i . B_i is complete normal local ring and E_i is the punctured spectrum of B_i . B_i is finite free rk nm as an R -module.

$$\text{disc}_R B_i = (\text{disc}_R A)^n \text{ up to unit.}$$

Let F be an ultrafilter on $\mathbb{N}$. Consider the ultraproducts

$$R_\infty \longrightarrow A_\infty \longrightarrow B_\infty$$

$R_\infty \rightarrow A_\infty$ free rk m

$R_\infty \rightarrow B_\infty$ free rk nm

B_∞ a finitely presented A_∞ -module
information on $\text{disc}_R B$ retained

Same for $R_\infty^\wedge \rightarrow A_\infty^\wedge \rightarrow B_\infty^\wedge$.

Note $A \rightarrow A_\infty^\wedge$ is flat : mod m_A^ℓ this reduces to the fact that over a coherent ring infinite products of flat modules are flat.

The residue field of $A_\infty^\wedge$ is a regular (in particular separable) extension of the residue field of A . By "localization de la lissité formelle" the fibre rings of $A \rightarrow A_\infty^\wedge$ are geometrically regular. Hence $A_\infty^\wedge$ is normal.

$B_\infty^\wedge$ is torsion free over $R_\infty^\wedge$, hence over $A_\infty^\wedge$,

Hence $B_\infty^\wedge / A_\infty^\wedge$ is finite flat over the punctured spectrum, necessarily of rank n .

By the discriminant information we get that

$\hat{B}_\infty / \hat{A}_\infty$ is finite étale on the punctured spectrum. Hence $\forall \mathfrak{s} \in \text{Spec}(\hat{A}_\infty) - V(m_A)$, the criterion of flatness by fibres gives that $(B_\infty)_{\mathfrak{s}}$ is flat over $(A_\infty)_{\mathfrak{s}}$, hence finite étale. But every point of $\text{Spec}(A_\infty) - V(m_A)$ is a generalization of a point of $\text{Spec}(\hat{A}_\infty) - V(m_A)$.

[If $I \subset \text{Rad}(A)$ is a f.g. ideal in a ring A and $\hat{A}$ the I -adic completion of A then all closed points of $\text{Spec}(A) - V(I)$ are in the image of $\text{Spec}(\hat{A}) \rightarrow \text{Spec}(A)$.]

This proves

Theorem. $p: \text{Spec}(\prod B_i) \rightarrow \text{Spec}(\prod_i A_i)$ is finite étale of deg n on $V(m_A)^c$.

Let F be a non principal ultrafilter on $\mathbb{N}$. By the rigidity facts the restriction of p to $\text{Spec}(A_\infty) - V(m_A)$ comes from a finite étale cover of $\text{Spec}(A) - V(m_A)$. By passage to the limit $\exists T \in F$ s.t. same holds for $p|_{\text{Spec}(\prod_{i \in T} A_i) - V(m_A)}$.
Contradiction.