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Cohomological dimension : first results

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EXPOSÉ XVIII$_{\mathrm{A}}$

COHOMOLOGICAL DIMENSION: FIRST RESULTS

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In this exposé we establish Gabber's bound on cohomological dimension stated in the introduction (in the comments on the proof of the finiteness theorem).

1. Bound in the strictly local case and applications

1.1. Theorem. — *Let X be a strictly local, noetherian scheme of dimension $d > 0$, and let ℓ be a prime number invertible on X. Then, for any open subset U of X, we have*

(1.1.1) $$\mathrm{cd}_\ell(U) \leq 2d - 1.$$

Recall that, for a scheme S, $\mathrm{cd}_\ell(S)$ (ℓ-**cohomological dimension** of S) denotes the infimum of the integers n such that for all ℓ-torsion abelian sheaves F on S, and all $i > n$, $\mathrm{H}^i(S, F) = 0$.

1.2. Corollary. — *Let $X = \operatorname{Spec} A$ be as in 1.1, and assume A is a domain, with fraction field K. Then*

(1.2.1) $$\mathrm{cd}_\ell(K) \leq 2d - 1.$$

Indeed, it suffices to show that if F is a finitely generated $\mathbf{F}_\ell$-module over $\eta = \operatorname{Spec} K$, then $\mathrm{H}^i(\eta, F) = 0$ for $i > 2d - 1$. But η is a filtering projective limit of affine open subsets U_α of X, F is induced from a locally constant constructible $\mathbf{F}_\ell$-sheaf F_{α_0} on U_{α_0}, and $\mathrm{H}^i(\eta, F) = \varinjlim \mathrm{H}^i(U_\alpha, F_\alpha)$, where $F_\alpha = F_{\alpha_0}|U_\alpha$ for $\alpha \geq \alpha_0$ ([SGA 4 VII 5.7]).

1.3. Remark. — (a) The proof shows that, given X as in 1.1, with X integral, then, if (1.1.1) holds for any affine open subset U, (1.2.1) holds, too.

(b) If X is an integral noetherian scheme of dimension d, with generic point $\operatorname{Spec} K$, and ℓ is a prime number invertible on X, then $\mathrm{cd}_\ell(K) \geq d$ ([SGA 4 x 2.5]). Gabber can prove that under the assumptions of 1.2 one has $\mathrm{cd}_\ell(K) = d$ (see XVIII$_{\mathrm{B}}$).

1.4. Corollary. — *Let Y be a noetherian scheme of finite dimension, $f : X \to Y$ a morphism of finite type, and ℓ a prime number invertible on Y. Then*

$$\mathrm{cd}_\ell(\mathrm{R}f_\star) < \infty,$$

i.e. there exists an integer N such that for any ℓ-torsion abelian sheaf F on X, $\mathrm{R}^q f_\star F = 0$ for $q > N$.

Proof of 1.4. We may assume Y affine. Covering X by finitely many open affine subsets U_i $(0 \leq i \leq n)$, and using the alternating Čech spectral sequence

$$E_1^{pq} = \bigoplus \mathrm{R}^q(f|U_{i_0\cdots i_p})_\star(F|U_{i_0\cdots i_p}) \Rightarrow \mathrm{R}^{p+q} f_\star F,$$

where $U_{i_0\cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$, we may assume f separated. Repeating the procedure, we may assume X affine. Choose an immersion $X \to \mathbf{P}^n_Y$, and replace $\mathbf{P}^n_Y$ by the scheme-theoretic closure of X. We get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & \overline{X} \\ {\scriptstyle f}\big\downarrow & \swarrow{\scriptstyle g} & \\ Y & & \end{array}$$

with j open and g projective of relative dimension $\leq n$. By the proper base change theorem we have $\mathrm{cd}_\ell(Rg_\star) \leq 2n$. By the Leray spectral sequence $\mathrm{R}^p g_\star \mathrm{R}^q j_\star F \Rightarrow \mathrm{R}^{p+q} f_\star F$ it thus suffices to prove 1.4 for j, in other words, we may assume that f is an open immersion. Let d be the dimension of Y. Let y be a geometric point of Y, and let $U = Y_{(y)} \times_Y X$ be the corresponding open subset of the strictly local scheme $Y_{(y)}$ (of dimension $\leq d$, that we may assume to be > 0). Then

$$\mathrm{R}^i f_\star(F)_y = \mathrm{H}^i(U, F)$$

(where we still denote by F its inverse image on U). The conclusion follows from 1.1.

1.5. Remarks. — (a) Under the assumptions of 1.1, if X is quasi-excellent and U is affine, then by Gabber's affine Lefschetz theorem (XV-1.2.4) we have $\mathrm{cd}_\ell(U) \leq d$. More generally, see XVII-3.2.1 for a proof of 1.1 for X quasi-excellent.

(b) Gabber can show that, under the assumptions of 1.1, one has $\mathrm{cd}_\ell(U) \geq d$ if U is not empty and does not contain the closed point and that for each n such that $d \leq n \leq 2d-1$, there exists a pair (X, U) as in 1.1, with U affine, such that $\mathrm{cd}_\ell(U) = n$ (by (a), for $n > d$, X is not quasi-excellent). These results are proved in XVIII$_{\mathrm{B}}$.

2. Proof of the main result

2.1. Lemma. — *Let X be as in 1.1, and let $\{x\}$ be the closed point of X. Then (1.1.1) holds for $U = X - \{x\}$.*

Proof. — It suffices to show that for any constructible $\mathbf{F}_\ell$-sheaf F on U, $\mathrm{H}^i(U, F) = 0$ for $i \geq 2d$. Let $\widehat{X}$ be the completion of X at $\{x\}$ and set $\widehat{U} := \widehat{X} \times_X U = \widehat{X} - \{x\}$. Let $\widehat{F}$ be the inverse image of F on $\widehat{U}$. By Gabber's formal base change theorem ([Fujiwara, 1995, 6.6.4]), the natural map

$$\mathrm{H}^i(U, F) \to \mathrm{H}^i(\widehat{U}, \widehat{F})$$

is an isomorphism for all i. Therefore we may assume X complete, and in particular, excellent. Let $(f_1, \ldots, f_d)$ be a system of parameters of X, and let $U_i = X_{f_i}$, so that $U = \bigcup_{1 \leq i \leq d} U_i$. Consider the (alternate) Čech spectral sequence

$$E_1^{pq} = \bigoplus \mathrm{H}^q(U_{i_0 \cdots i_p}, F) \Rightarrow \mathrm{H}^{p+q}(U, F),$$

with $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$ as above. By definition, $E_1^{pq} = 0$ for $p \geq d$. On the other hand, as X is excellent, by Gabber's affine Lefschetz theorem (XV-1.2.4), we have $E_1^{pq} = 0$ for $q \geq d+1$. Therefore $E_1^{pq} = 0$ for $p + q \geq 2d$, hence $\mathrm{H}^i(U, F) = 0$ for $i \geq 2d$. □

2.2. Lemma. — *Let X be a noetherian scheme of finite dimension, Y a closed subset, ℓ a prime number invertible on X. Then, for any ℓ-torsion sheaf F on X,*

$$\mathrm{H}^i_Y(X, F) = 0$$

for

$$i > \sup_{x \in Y}(\mathrm{cd}_\ell(k(x)) + 2 \dim \mathscr{O}_{X,x}).$$

In particular,

$$\mathrm{cd}_\ell(X) \leq \sup_{x \in X}(\mathrm{cd}_\ell(k(x)) + 2 \dim \mathscr{O}_{X,x}).$$

Proof. — For $p \geq 0$, let Φ^p be the set of closed subsets of Y of codimension $\geq p$ in X. We have $\Phi^p = \varnothing$ for $p > \dim(X)$. Consider the (biregular) coniveau spectral sequence of the filtration (Φ^p) (cf. [Grothendieck, 1968, 10.1]),

(2.2.1) $$E_1^{pq} = \mathrm{H}^{p+q}_{\Phi^p/\Phi^{p+1}}(X, F) \Rightarrow \mathrm{H}^{p+q}_Y(X, F).$$

We have

$$E_1^{pq} = \bigoplus_{x \in Y^{(p)}} \mathrm{H}^{p+q}_{\{x\}}(X_x, F|X_x),$$

where $Y^{(p)}$ denotes the set of points of Y of codimension p in X, and $X_x = \operatorname{Spec} \mathscr{O}_{X,x}$. For $x \in Y^{(p)}$ (i.e. $\dim \mathscr{O}_{X,x} = p$), let $\overline{x}$ be a geometric point above x. Consider the diagram

$$\begin{array}{ccccc} \{\overline{x}\} & \xrightarrow{i_{\overline{x}}} & X_{(\overline{x})} & \xleftarrow{\overline{j}} & \overline{U}\ , \\ \downarrow & & \downarrow & & \downarrow \\ \{x\} & \xrightarrow{i_x} & X_x & \xleftarrow{j} & U \end{array}$$

where $U = X_x - \{x\}$, $\overline{U} = X_{(\overline{x})} - \{\overline{x}\}$. We have

$$R\Gamma_{\{x\}}(X_x, F|X_x) = R\Gamma(\{x\}, \mathrm{R}i_x^!(F|X_x)).$$

The stalk of $\mathrm{R}i_x^!(F|X_x)$ at $\overline{x}$ is

$$\mathrm{R}i_x^!(F|X_x)_{\overline{x}} = \mathrm{R}i_{\overline{x}}^!(F|X_{(\overline{x})}),$$

as $(\mathrm{R}j_\star(F|U))_{\overline{x}} = R\overline{j}_\star(F|\overline{U})_{\overline{x}}$. We thus have a spectral sequence

(2.2.2) $$E_2^{rs} = \mathrm{H}^r(k(x), \mathrm{R}^s i_{\overline{x}}^!(F|X_{(\overline{x})})) \Rightarrow \mathrm{H}^{r+s}_{\{x\}}(X_x, F|X_x),$$

It suffices to show that, in the initial term of (2.2.1),

(2.2.3) $$\mathrm{H}^{p+q}_{\{x\}}(X_x, F|X_x) = 0$$

for $p + q > \mathrm{cd}_\ell(k(x)) + 2p$. If $p = 0$, then $\mathrm{R}i_{\overline{x}}^!(F|X_{(\overline{x})}) = F_{\overline{x}}$, and, in (2.2.2), $E_2^{rs} = 0$ for $s > 0$, $E_2^{r0} = 0$ for $r > \mathrm{cd}_\ell(k(x))$, so (2.2.3) is true in this case. Assume $p > 0$. We have

(2.2.4) $$\mathrm{R}^s i_{\overline{x}}^! F = \mathrm{H}^{s-1}(\overline{U}, F|\overline{U})$$

for $s \geq 2$, where, as above, $\overline{U} = X_{(\overline{x})} - \{\overline{x}\}$. By 2.1, $\mathrm{H}^{s-1}(\overline{U}, F|\overline{U}) = 0$ for $s - 1 \geq 2p$, hence, by (2.2.4), $\mathrm{R}^s i_{\overline{x}}^!(F|X_{(\overline{x})}) = 0$ and $E_2^{rs} = 0$ for $s \geq 2p+1$. If $r+s \geq \mathrm{cd}_\ell(k(x)) + 2p + 1$ and $s \leq 2p$, then $r > \mathrm{cd}_\ell(k(x))$, hence $E_2^{rs} = 0$ as well. Therefore, by (2.2.2), (2.2.3) holds, which finishes the proof. □

Proof of 1.1. We prove 1.1 by induction on d. For $n \geq 0$ consider the assertion

G_n : *For every strictly local, noetherian scheme X of dimension n, all open subsets U of X and any prime number ℓ invertible on X, we have* $\mathrm{cd}_\ell(U) \leq \sup(0, 2n-1)$.

Let $d > 0$. Assume G_n holds for $n < d$, and let us prove G_d. Let X be as in 1.1. If $(X_i)_{1 \leq i \leq r}$ are the reduced irreducible components of X and $U_i = U \times_X X_i$, we have $\mathrm{cd}_\ell(U) \leq \sup(\mathrm{cd}_\ell(U_i))$, hence we may assume X integral. Let x be the closed point of X, and $U = X - \{x\}$ the punctured spectrum. Let $j : V \to U$ be a nonempty open subset of U, and F be a constructible $\mathbf{F}_\ell$-sheaf on V. As $F = j^\star j_! F$, by 2.1 it suffices to show that, for any constructible $\mathbf{F}_\ell$-sheaf L on U, the restriction map

(∗) $$\mathrm{H}^i(U, L) \to \mathrm{H}^i(V, j^\star L)$$

is an isomorphism for $i \geq 2d$. Let $Y = U - V$. Consider the exact sequence

$$\mathrm{H}^i_Y(U, L) \to \mathrm{H}^i(U, L) \to \mathrm{H}^i(V, j^\star L) \to \mathrm{H}^{i+1}_Y(U, L).$$

By 2.2, we have $\mathrm{H}^i_Y(U, L) = 0$ for $i > \sup_{y \in Y}(\mathrm{cd}_\ell(k(y)) + 2\dim \mathscr{O}_{X,y})$. For $y \in Y$, denote by Z the closed, integral subscheme of X defined by the closure of $\{y\}$ *in* X. As X is integral and V nonempty, Z is a strictly local scheme of dimension $n < d$, with generic point y. By 1.3 (a) and G_n (inductive assumption), we have $\mathrm{cd}_\ell(k(y)) \leq 2n - 1$. We have $2n - 1 + 2\dim \mathscr{O}_{X,y} \leq 2d - 1$. Hence, for $i \geq 2d$, $\mathrm{H}^i_Y(U, L) = \mathrm{H}^{i+1}_Y(U, L) = 0$, and $(*)$ is an isomorphism, which finishes the proof.

2.3. Remark. — Gabber has an alternate proof of 1.1, based on the theory of Zariski-Riemann spaces. By 2.2, it suffices to show 1.2. Here is a sketch, pasted from an e-mail of Gabber to Illusie of 2007, Aug. 15 :

"For $Y \to X$ proper birational with special fiber Y_0, consider $i : Y_0 \to Y$ and $j : \eta \to Y$, η the generic point. We have by proper base change a spectral sequence

$$\mathrm{H}^p(Y_0, i^\star \mathrm{R}^q j_\star F) \to \mathrm{H}^{p+q}(\eta, F)$$

for F an ℓ-torsion Galois module. We take the direct limit and get a spectral sequence involving cohomologies on the étale topos of ZRS_0 defined as the limit of étale topoi of Y_0 or viewing ZRS_0 as a locally ringed topos and applying a universal construction in the book of M. Hakim. The limit of the $\mathrm{R}^q j_\star F$ is $\mathrm{R}^q(\eta \to ZRS)_\star F$. By a classical result of Abhyankar, also proved in Appendix 2 of the book of Zariski-Samuel Vol. II, if R is a noetherian local domain of dimension d and V a valuation ring of $\mathrm{Frac}(R)$ dominating R, the sum of the rational rank and the residue transcendence degree is at most d. For a strictly henselian valuation ring V with residue characteristic exponent p and value group Γ, the absolute Galois group of $\mathrm{Frac}(V)$ is an extension of the tame part (product for ℓ prime not equal to p of $\mathrm{Hom}(\Gamma, \mathbf{Z}_\ell(1))$) by a p-group, so the ℓ-cohomological dimension is the dimension of Γ tensored with the prime field $\mathbf{F}_\ell$, which is at most the dimension of Γ tensored with the rationals. If A is an ℓ-torsion sheaf on the étale topos of ZRS_0, let $\delta(A)$ be the sup of transcendence degrees of points where the stalk is non-zero. I claim that $\mathrm{H}^n(ZRS_0, A)$ vanishes for $n > 2\delta(A)$. One reduces it to the finite type case (passage to the limit [SGA 4 VI 8.7.4]) using that the δ of the direct image of A to Y_0 is at most $\delta(A)$. In Y_0 the transcendence degrees over the closed point of X are at most $d - 1$ by the dimension inequality. Summing up, for the limit spectral seequence the q-th direct image sheaf restricted to the special fiber has δ at most $\min(d-1, d-q)$, giving vanishing for certain $E_2^{p,q}$ and the result."