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EXPOSÉ $\mathrm{XII_B}$

ON HYPER BASE CHANGE

Weizhe Zheng [i]

In this note, we sketch a short proof of Gabber's hyper base change theorem $\mathrm{XII_A}$-2.2.5 without using oriented products. We will first treat the Abelian case and then briefly comment on the non-Abelian case. We also include an analogue, due to Gabber, for topological spaces.

1. A descent formalism

We let $\mathscr{S}$ denote the category of schemes. We fix a diagram $X \to S \leftarrow Y$ in $\mathscr{S}$ and a complex $K \in \mathsf{D}^+(Y_{\text{ét}}, \mathbf{Z})$ whose cohomology sheaves are torsion (see [SGA 4 IX 1.1] for the definition of torsion sheaves).

We regard $X \to S \leftarrow Y$ as an object of the category of functors $\mathrm{Fun}(V, \mathscr{S})$, where V denotes the diagram category $\bullet \to \bullet \leftarrow \bullet$. We let $\mathscr{B}$ denote the full subcategory of $\mathrm{Fun}(V, \mathscr{S})$ spanned by diagrams $X' \to S' \leftarrow Y'$ such that $Y' \to S'$ is coherent (namely, quasi-compact quasi-separated). We let $\mathscr{C}$ denote the overcategory $\mathscr{B}_{/(X\to S\leftarrow Y)}$. We will drop the letter R in derived direct image functors.

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1.1. Definition. — (i) We say that an augmented simplicial object $f_\bullet$ in $\mathscr{C}$ corresponding to the diagram

(1.1.1)
$$\begin{array}{ccccc} X_\bullet & \xrightarrow{a_\bullet} & S_\bullet & \xleftarrow{b_\bullet} & Y_\bullet \\ {\scriptstyle f_{X\bullet}}\downarrow && \downarrow{\scriptstyle f_{S\bullet}} && \downarrow{\scriptstyle f_{Y\bullet}} \\ X_{-1} & \xrightarrow{a_{-1}} & S_{-1} & \xleftarrow{b_{-1}} & Y_{-1} \\ {\scriptstyle x}\downarrow && \downarrow && \downarrow{\scriptstyle y} \\ X & \longrightarrow & S & \longleftarrow & Y \end{array}$$

is **of descent** (relative to K) if the composite map

(1.1.2)
$$x_\star a^\star_{-1}b_{-1\star}y^\star K \xrightarrow{A_Y} x_\star a^\star_{-1}b_{-1\star}f_{Y\bullet\star}f^\star_{Y\bullet}y^\star K \simeq x_\star a^\star_{-1}f_{S\bullet\star}b_{\bullet\star}f^\star_{Y\bullet}y^\star K \xrightarrow{C_X} x_\star f_{X\bullet\star}a^\star_\bullet b_{\bullet\star}f^\star_{Y\bullet}y^\star K$$

is an isomorphism in $\mathsf{D}^+(X_{\text{ét}},\mathbf{Z})$, where A_Y is the adjunction map induced by $f_{Y\bullet}$ and C_X is the base change map induced by the top left square of (1.1.1).

(ii) We say that a morphism $f\colon T_0 \to T_{-1}$ in $\mathscr{C}$ is **of descent** if its Čech nerve $(\operatorname{cosk}_0^{T_{-1}}T_0)_\bullet \to T_{-1}$ is of descent. We say that f is **of universal descent** if it is of descent after arbitrary base change in $\mathscr{C}$.

(iii) We say that an augmented simplicial object $T_\bullet \to T_{-1}$ in $\mathscr{C}$ is a **hypercovering for universal descent** if for every $n \geq 0$, the morphism $T_n \to (\operatorname{cosk}_{n-1}^{T_{-1}}T_\bullet)_n$ is of universal descent.

1.2. Remark. — The map (1.1.2) is equal to the composite map

$$x_\star a^\star_{-1}b_{-1\star}y^\star K \xrightarrow{A_X} x_\star f_{X\bullet\star}f^\star_{X\bullet}a^\star_{-1}b_{-1\star}y^\star K \simeq x_\star f_{X\bullet\star}a^\star_\bullet f^\star_{S\bullet}b_{-1\star}y^\star K \xrightarrow{C_Y} x_\star f_{X\bullet\star}a^\star_\bullet b_{\bullet\star}f^\star_{Y\bullet}y^\star K,$$

where A_X is the adjunction map induced by $f_{X\bullet}$ and C_Y is the base change map induced by the transpose of the top right square of (1.1.1). In fact, the diagram

$$\begin{array}{ccccccc}
 & & x_\star a^\star_{-1}b_{-1\star}f_{Y\bullet\star}f^\star_{Y\bullet}y^\star K & \xrightarrow{\sim} & x_\star a^\star_{-1}f_{S\bullet\star}b_{\bullet\star}f^\star_{Y\bullet}y^\star K & & \\
 & \nearrow^{A_Y} & & \nearrow^{C_Y} & & \searrow^{C_X} & \\
x_\star a^\star_{-1}b_{-1\star}y^\star K & \xrightarrow{A_S} & x_\star a^\star_{-1}f_{S\bullet\star}f^\star_{S\bullet}b_{-1\star}y^\star K & & & & x_\star f_{X\bullet\star}a^\star_\bullet b_{\bullet\star}f^\star_{Y\bullet}y^\star K \\
 & \searrow_{A_X} & & \searrow_{C_X} & & \nearrow_{C_Y} & \\
 & & x_\star f_{X\bullet\star}f^\star_{X\bullet}a^\star_{-1}b_{-1\star}y^\star K & \xrightarrow{\sim} & x_\star f_{X\bullet\star}a^\star_\bullet f^\star_{S\bullet}b_{-1\star}y^\star K & &
\end{array}$$

is commutative, where A_S is the adjunction map induced by $f_{S\bullet}$.

We say that a morphism $g\colon Z_0 \to Z$ in $\mathscr{S}$ is **of cohomological descent relative to** $L \in \mathsf{D}^+(Z_{\text{ét}}, \mathbf{Z})$ if the adjunction map $L \to g_{\bullet\star}g_\bullet^\star L$ is an isomorphism, where $g_\bullet$ is the Čech nerve of g. By classical cohomological descent [SGA 4 V$^{\text{bis}}$ 4.3.2, 4.3.3], g is of cohomological descent relative to L if either g is proper surjective and the cohomology sheaves of L are torsion or g is flat surjective locally of finite presentation.

1.3. Proposition. — *Let f be a morphism in $\mathscr{C}$ corresponding to the diagram*

$$\begin{array}{ccccc} X_0 & \xrightarrow{a_0} & S_0 & \xleftarrow{b_0} & Y_0 \\ \downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_S} & & \downarrow{\scriptstyle f_Y} \\ X_{-1} & \xrightarrow{a_{-1}} & S_{-1} & \xleftarrow{b_{-1}} & Y_{-1} \\ \downarrow{\scriptstyle x} & & \downarrow & & \downarrow{\scriptstyle y} \\ X & \longrightarrow & S & \longleftarrow & Y. \end{array} \tag{1.3.1}$$

Then f is of descent if it satisfies either of the following conditions:

(i) *f_Y is of cohomological descent relative to $y^\star K$, f_S is proper, and the top left square of* (1.3.1) *is Cartesian;*

(ii) *f_X is of cohomological descent relative to $a_{-1}^\star b_{-1\star}y^\star K$, f_S is étale, and the top right square of* (1.3.1) *is Cartesian.*

Note that the assumption that b_n is coherent implies that $b_{n\star}f_{Y,n}^\star y^\star K$ has torsion cohomology sheaves.

Proof. — In case (i), A_Y is an isomorphism by assumption and C_X is an isomorphism by the proper base change theorem [SGA 4 XII 5.1]. In case (ii), A_X is an isomorphism by assumption and C_Y is an isomorphism by étale base change, and we conclude by Remark 1.2. □

The following lemma on comparison of augmentations is similar to [SGA 4 V$^{\text{bis}}$ 3.1.1].

1.4. Lemma. — *Let $f_\bullet\colon T_\bullet \to T_{-1}$, $f'_\bullet\colon T'_\bullet \to T_{-1}$ be augmented simplicial objects of $\mathscr{C}$ and let $\alpha_\bullet\colon T_\bullet \to T'_\bullet$ be a morphism over T_{-1}. Then the diagram*

$$\begin{array}{ccccccc} x_\star a_{-1}^\star b_{-1\star}y^\star K & & & & & \searrow & \\ \downarrow & & & & & & \\ x_\star f'_{X\bullet\star}a'^\star_\bullet b'_{\bullet\star}f'^\star_{Y\bullet}y^\star K & \xrightarrow[A_Y]{} & x_\star f'_{X\bullet\star}a'^\star_\bullet b'_{\bullet\star}\alpha_{Y\bullet\star}f^\star_{Y\bullet}y^\star K & \xrightarrow[\sim]{} & x_\star f'_{X\bullet\star}a'^\star_\bullet \alpha_{S\bullet\star}b_{\bullet\star}f^\star_{Y\bullet}y^\star K & \xrightarrow[C_X]{} & x_\star f_{X\bullet\star}a^\star_\bullet b_{\bullet\star}f^\star_{Y\bullet}y^\star K \end{array}$$

where the vertical and oblique arrows are given by (1.1.2), *is commutative. Moreover, the composite $\eta_{\alpha_\bullet}$ of the horizontal arrows depends only on the simplicial homotopy class of $\alpha_\bullet$.*

Proof. — The first assertion is trivial: the diagram

$$\begin{array}{ccccc}
x_\star a_{-1}^\star b_{-1\star} y^\star K & & & & \\
\downarrow{\scriptstyle A_{f'_{Y\bullet}}} & \searrow^{A_{f_{Y\bullet}}} & & & \\
x_\star a_{-1}^\star b_{-1\star} f'_{Y\bullet\star} f'^\star_{Y\bullet} y^\star K & \xrightarrow{A_{\alpha_{Y\bullet}}} & x_\star a_{-1}^\star b_{-1\star} f_{Y\bullet\star} f^\star_{Y\bullet} y^\star K & & \\
\downarrow{\scriptstyle C_{f'_{X\bullet}}} & & \downarrow{\scriptstyle C_{f'_{X\bullet}}} & \searrow^{C_{f_{X\bullet}}} & \\
x_\star f'_{X\bullet\star} a'^\star_\bullet b'_{\bullet\star} f'^\star_{Y\bullet} y^\star K & \xrightarrow{A_{\alpha_{Y\bullet}}} & x_\star f'_{X\bullet\star} a'^\star_\bullet b'_{\bullet\star} \alpha_{Y\bullet\star} f^\star_{Y\bullet} y^\star K & \xrightarrow{C_{\alpha_{X\bullet}}} & x_\star f_{X\bullet\star} a^\star_\bullet b_{\bullet\star} f^\star_{Y\bullet} y^\star K
\end{array}$$

is commutative. To show the second assertion, let $\beta_\bullet\colon T_\bullet \to T'_\bullet$ be a morphism over T_{-1} and let $h\colon \alpha_\bullet \to \beta_\bullet$ be a simplicial homotopy over T_{-1}. As in the proof of [SGA 4 v$^{\mathrm{bis}}$ 3.1.1.9], consider the category D whose objects are objects of $[1] \times \Delta$ and whose morphisms are pairs $(\xi, \delta)\colon (m, [n]) \to (m', [n'])$, where $\xi\colon [n] \to [m, m']$ and $\delta\colon [n] \to [n']$. Here $[m, m']$ denotes the subset of $[1]$ spanned by i with $m \leq i \leq m'$. The homotopy h defines a functor $ThT'\colon D^{\mathrm{opp}} \to \mathscr{C}$, augmented over T_{-1}. Let J be a resolution of $(aha')^\star(bhb')_\star(fhf')^\star_Y y^\star K$ such that the sheaves $J^k_{m,[n]}$ are flabby. We let $r_m\colon \Delta \to D$ denote the functor carrying $[n]$ to $(m, [n])$. Then J induces a cosimplicial homotopy between the maps $f'_{X\bullet\star} r_0^\star J \rightrightarrows f_{X\bullet\star} r_1^\star J$ of cosimplicial complexes on X_{-1}, which give rise to $\eta_{\alpha_\bullet}$ and $\eta_{\beta_\bullet}$. Therefore, $\eta_{\alpha_\bullet} = \eta_{\beta_\bullet}$. □

The proof of the following proposition is similar to [SGA 4 v$^{\mathrm{bis}}$ 3.3.1 a), b)] (see also Remark 2.5 below).

1.5. Proposition. — (i) *A morphism f in $\mathscr{C}$ is of universal descent if it admits a section.*

(ii) *Let*

$$\begin{array}{ccc}
T' & \xrightarrow{f'} & U' \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
T & \xrightarrow{f} & U
\end{array}$$

be a Cartesian square in $\mathscr{C}$ such that the base change of f by $(U'/U)^n \to U$ is of descent for all $n \geq 1$ and the base change of g by $(T/U)^n \to U$ is of descent for all $n \geq 0$. Then f is of descent.

In (ii), the assumptions on f and g are satisfied if g is of descent and f', g' are of universal descent. In particular, if in (ii) f' and g are of universal descent, then f is of *universal* descent.

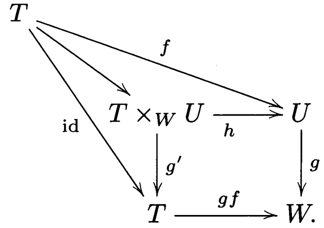
The following corollary is similar to [SGA 4 v$^{\mathrm{bis}}$ 3.3.1 c), d)].

1.6. Corollary. — *Let $T \xrightarrow{f} U \xrightarrow{g} W$ be a sequence of morphisms in $\mathscr{C}$.*

(i) *If gf is of universal descent, then g is of universal descent.*

(ii) *If f is of universal descent and g is of descent, then gf is of descent.*

Proof. — Consider the commutative diagram



Since g' admits a section, g' is of universal descent by Proposition 1.5 (i). Case (i) then follows from Proposition 1.5 (ii). In case (ii), h is of universal descent by (i), and the assertion follows again from Proposition 1.5 (ii). □

The following analogue of [SGA 4 $\mathrm{V}^{\mathrm{bis}}$ 3.3.1 f)] is obvious.

1.7. Proposition. — *Let $(f_\alpha\colon T_\alpha \to U_\alpha)_{\alpha\in A}$ be a family of morphisms of $\mathscr{C}$ of descent. Then $\coprod_{\alpha\in A} f_\alpha\colon \coprod_{\alpha\in A} T_\alpha \to \coprod_{\alpha\in A} U_\alpha$ is of descent.*

The following is a consequence of Corollary 1.6 and Proposition 1.7, applied to constant diagrams of the form $Z \to Z \leftarrow Z$.

1.8. Corollary. — *Let $(Z_\alpha \to Z)_{\alpha\in A}$ be a covering family for the h-topology ($\mathrm{XII_A}$-2.1.3) on $\mathscr{S}$. Then the morphism $\coprod_{\alpha\in A} Z_\alpha \to Z$ is of cohomological descent relative to every $L \in \mathsf{D}^+(Z_{\text{ét}}, \mathbf{Z})$ whose cohomology sheaves are torsion.*

We consider the oriented h-topology on $\mathscr{B}$ generated by families of types (i) through (v) ($\mathrm{XII_A}$-2.1.4).

1.9. Corollary. — *Let $(T_\alpha \to T)_{\alpha\in A}$ be a family of morphisms in $\mathscr{C}$ that is covering for the oriented h-topology on $\mathscr{B}$. Then the morphism $\coprod_{\alpha\in A} T_\alpha \to T$ is of universal descent. In particular, if f is a morphism in $\mathscr{C}$ corresponding to the diagram* (1.3.1) *such that both top squares are Cartesian and f_S is a covering morphism for the h-topology, then f is of universal descent.*

Proof. — The second assertion follows from the first one, as f is a covering family of type (iii) for the oriented h-topology. To show the first assertion, we may assume, by Corollary 1.6 and Proposition 1.7, that the family $(T_\alpha \to T)_{\alpha\in A}$ is of one of the five types in the definition of the oriented h-topology. Moreover, for type (iii), we may assume that the family is induced by either a proper surjective morphism or an étale covering. The case of a proper surjective morphism being a composition of types (ii) and (iv), we may further restrict to the case of an étale covering. We write $T_\alpha = (X_\alpha \to S_\alpha \leftarrow Y_\alpha)$, $T = (X' \to S' \leftarrow Y')$. For type (ii), localizing on S', we may assume that A is finite, because every h-covering of a quasi-compact scheme admits a finite subfamily that is an h-covering. The finiteness of A implies that $\coprod_{\alpha\in A} S_\alpha = \coprod_{\alpha\in A} S' \to S'$ is proper. We conclude by Proposition 1.3 (ii) for types (i), (iii), (v), and by Proposition 1.3 (i) for types (ii) and (iv). □

The proof of the following proposition is similar to [SGA 4 $\mathrm{v^{bis}}$ 3.3.3].

1.10. Proposition. — *A hypercovering for universal descent in $\mathscr{C}$ is an augmentation of descent.*

2. Variants and counterexamples

2.1. Remark. — Let Λ be a ring annihilated by a positive integer. For $K \in \mathsf{D}^+(Y_{\text{ét}}, \Lambda)$, Section 1 holds with $\mathscr{B}$ replaced by the larger category $\mathrm{Fun}(V, \mathscr{S})$, and the oriented h-topology in the first assertion of Corollary 1.9 replaced by the **restricted oriented h-topology** on $\mathrm{Fun}(V, \mathscr{S})$, generated by families of types (i) through (v), where for type (ii) we restrict to families of finite index sets. In view of the second assertion of Corollary 1.9, Proposition 1.10 in this setting implies the hyper base change theorem $\mathrm{XII_A}$-2.2.5 (without the assumption that g is coherent).

2.2. Remark. — In Section 1, we may replace K by an ind-finite stack on $Y_{\text{ét}}$. For proper base change and proper cohomological descent in this setting, we refer to XX-7.1 and [Orgogozo, 2003, Proposition 2.5] (the latter extends easily to stacks not necessarily in groupoids). Thus we obtain another proof of the non-Abelian hyper base change theorem (see $\mathrm{XII_A}$-2.2.6.2).

***2.3. Remark* (Gabber).** — Section 1 also holds with $\mathscr{S}$ replaced by the category of topological spaces, $\mathscr{B}$ replaced by the category $\mathrm{Fun}(V, \mathscr{S})$, and K replaced by either a complex in $\mathsf{D}^+(Y, \mathbf{Z})$ (not necessarily of torsion cohomology sheaves), or a stack on Y (not necessarily ind-finite), and the oriented h-topology replaced by the *restricted oriented h-topology.* Proper morphisms are defined to be separated and universally closed, and étale morphisms are defined to be local homeomorphisms. The analogue of the proper base change theorem for topological spaces in the Abelian case is shown in [SGA 4 $\mathrm{v^{bis}}$ 4.1.1]. See Section 3 for the case of stacks.

Note that the first assertion of Corollary 1.9 in this setting does not hold for the oriented h-topology. Indeed, the family $(T_i \to T)_{i>0}$, where $T_i = (\{0\} \to [0,1] \leftarrow \xi_i)$, $T = (\{0\} \to [0,1] \leftarrow \coprod_{i>0} \xi_i)$, $\xi_i = \{1/i\}$, is covering for the oriented h-topology, but the morphism $\coprod_{i>0} T_i \to T$ is not of descent for general K.

2.4. Remark. — Classical cohomological descent and the descent formalism in this note can be dealt with uniformly using the language of ∞-categories. In fact, Proposition 1.5, Corollary 1.6, and Proposition 1.10 (resp. their non-Abelian analogues in Remark 2.2) incarnate abstract descent properties in ∞-categories ([Liu & Zheng, 2012, Lemma 3.1.2], [Liu & Zheng, 2014, Section 4.1]) applied to the functor from the nerve of $\mathscr{C}$ to the derived ∞-category $\mathscr{D}(X_{\text{ét}}, \mathbf{Z})$ (resp. to the 2-category of stacks on $X_{\text{ét}}$) sending an object of $\mathscr{C}$ corresponding to the commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{a} & S' & \xleftarrow{b} & Y' \\ {\scriptstyle x}\downarrow & & \downarrow & & \downarrow{\scriptstyle y} \\ X & \longrightarrow & S & \longleftarrow & Y \end{array}$$

to $x_\star a^\star b_\star y^\star K$.

2.5. Remark. — We conclude this section by making corrections to [SGA 4 v^{bis} 3.2.1, 3.3.1 b)]. As the referee points out, [SGA 4 v^{bis} 3.3.1 b)] would imply, by taking $g = f$ and applying [SGA 4 v^{bis} 3.3.1 a)] to f', that every morphism of cohomological G-descent is of universal cohomological G-descent. We have the following counterexample. We define a category B by taking for objects the pairs (M, w), where M is a set and $w\colon M \to \{0, 1, 2\}$ is a function, and for morphisms $(M, w) \to (M', w')$ the maps $\mu\colon M \to M'$ such that $w(m) \leq w'(\mu(m))$ for all $m \in M$. The category B admits small limits and small colimits. Small coproducts in B are disjoint and universal. Consider the functor $F\colon B \to \mathsf{Ens}$ to the category of sets carrying (M, w) to $M \amalg w^{-1}(1)$ and carrying $\mu\colon (M, w) \to (M', w')$ to the map $M \amalg w^{-1}(1) \to M' \amalg w'^{-1}(1)$ induced by the maps $M \xrightarrow{\mu} M' \to M' \amalg w'^{-1}(1)$ and $w^{-1}(1) \xrightarrow{\mu} w'^{-1}(2) \amalg w'^{-1}(1) \xrightarrow{\iota \amalg \mathrm{Id}} M' \amalg w'^{-1}(1)$, where $\iota\colon w'^{-1}(2) \to M'$ is the inclusion. For $i \in \{0, 1, 2\}$, we let $\bar{i}$ denote the object $(\{\star\}, i)$ of B, where $\{\star\}$ denotes a singleton. By construction, $F(\bar{0}) = F(\bar{2}) = \{\star\}$ and $F(\bar{1}) = \{\star\} \amalg \{\star\}$. We define a category $\mathscr{E}$ bifibered in duals of topoi over Ens by taking for objects the pairs $(N, \mathscr{F})$, where N is a set and $\mathscr{F}$ is a presheaf on the discrete category N, and for morphisms the pairs $(\nu, \phi)\colon (N, \mathscr{F}) \to (N', \mathscr{F}')$, where $\nu\colon N \to N'$ is a map and $\phi\colon \mathscr{F}' \to \nu_\star \mathscr{F}$ is a morphism of presheaves. Let Λ be a nonzero ring. We consider the category $\mathscr{E} \times_{\mathsf{Ens}} B$ bifibered in duals of topoi over B and the corresponding category G bifibered in the categories of Λ-modules over the opposite of B. Then, in the Cartesian square

$$\begin{array}{ccc} \bar{0} & \xrightarrow{f'} & \bar{0} \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ \bar{1} & \xrightarrow{f} & \bar{2}, \end{array}$$

f' is of universal effective cohomological G-descent, g is of effective cohomological G-descent, but neither f nor g' is of cohomological G-descent. The constant simplicial objects associated to $\bar{0}$ and $\bar{1}$ augmented over $\bar{2}$ also provide a counterexample to [SGA 4 $\mathrm{v}^{\mathrm{bis}}$ 3.2.1].

In [SGA 4 $\mathrm{v}^{\mathrm{bis}}$ 3.2.1], one should make the additional assumption that v remains an augmentation of cohomological G-descent after every base change $X_n \to S$, $n \geq 0$. In [SGA 4 $\mathrm{v}^{\mathrm{bis}}$ 3.3.1 b)], g should be assumed to be of *universal* cohomological G-i-descent.

3. Appendix: Proper base change for stacks on topological spaces

The results and proofs in this section are completely due to Gabber. Recall that a continuous map $f\colon X \to Y$ between topological spaces is said to be **separated** if the diagonal embedding $X \to X \times_Y X$ is closed. We say that a continuous map between topological spaces is **proper** if it is separated and universally closed, or, equivalently, separated and closed with compact fibers. We do *not* assume stacks to be in groupoids.

3.1. Theorem. — *Consider a Cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

in the category of topological spaces with f proper. Then, for every stack $\mathscr{C}$ on X, the base change morphism

$$g^\star f_\star \mathscr{C} \to f'_\star g'^\star \mathscr{C}$$

is an equivalence.

For a generalization to higher stacks in the case of locally compact Hausdorff spaces, we refer to [Lurie, 2009, Corollary 7.3.1.18].

The first reduction steps for Theorem 3.1 are similar to the Abelian case [SGA 4 $\mathrm{v}^{\mathrm{bis}}$ 4.1.1]. Theorem 3.1 is equivalent to Corollary 3.2 below. We obtain the corollary by taking Y' to be a point in the theorem. By [Giraud, 1971, III.2.1.5.8], to show the theorem, it suffices to show that the base change morphism is an equivalence after taking stalks, which follows from the corollary applied to f and f' by considering the sequence $(f_\star\mathscr{C})_y \to (f'_\star g'^\star\mathscr{C})_{y'} \to (j^\star\mathscr{C})(f^{-1}(y))$, where y' is a point of Y' and $y = g(y')$.

3.2. Corollary. — *Let $f\colon X \to Y$ be a proper map between topological spaces and let y be a point of Y. Then, for every stack $\mathscr{C}$ on X, the functor $(f_\star\mathscr{C})_y \to (j^\star\mathscr{C})(f^{-1}(y))$ is an equivalence. Here $j\colon f^{-1}(y) \to X$ denotes the inclusion.*

We have (cf. [Giraud, 1971, VII.2.1.5.2]) $(f_\star\mathscr{C})_y \simeq \mathrm{colim}_V\,\mathscr{C}(f^{-1}(V))$, where colim means 2-colimit, and V runs through open neighborhoods of y in Y. Since f is closed, the $f^{-1}(V)$ form a fundamental system of neighborhoods of $f^{-1}(y)$. Therefore, Corollary 3.2 follows from case (i) of the following theorem.

3.3. Theorem. — *Let X be a topological space and let K be a topological subspace satisfying any of the following conditions:*

(i) *K is compact and every pair of distinct points of K have disjoint neighborhoods in X;*

(ii) *X is paracompact Hausdorff and K is closed in X;*

(iii) *X is metrizable.*

We let $j\colon K \to X$ denote the inclusion. Then, for every stack $\mathscr{C}$ on X, the functor

$$\alpha\colon\ \mathrm{colim}_U\,\mathscr{C}(U) \to (j^\star\mathscr{C})(K),$$

induced by the restriction functors $\mathscr{C}(U) \to (j^\star\mathscr{C})(K)$, where U runs through open neighborhoods of K in X, is an equivalence.

The Abelian analogue of the theorem was proved in [SGA 4 $\mathrm{V}^{\mathrm{bis}}$ 4.1.3] for case (i) and in [Godement, 1973, Théorème II.4.11.1] for cases (ii) and (iii).

Proof. — The theorem holds for sheaves of sets: for cases (ii) and (iii), this is [Godement, 1973, Théorème II.3.3.1, Corollaire 1]; for case (i), the proof for H^0 of Abelian sheaves given in [SGA 4 $\mathrm{V}^{\mathrm{bis}}$ 4.1.3] works in general. Since taking sheaves of morphisms commutes with direct and inverse images [Giraud, 1971, II.3.1.5.3, II.3.2.8] and filtered colimits, it follows that α is fully faithful. Thus it suffices to show that α is essentially surjective.

Next we perform two reduction steps similar to XX-6.2. A given section s of $j^\star\mathscr{C}$ generates a maximal subgerbe, corresponding to a section of the sheaf of maximal subgerbes $\mathrm{Ger}(j^\star\mathscr{C})$. By [Giraud, 1971, III.2.1.5.5], $\mathrm{Ger}(j^\star\mathscr{C}) \simeq j^\star\mathrm{Ger}(\mathscr{C})$. By the known case of sheaves of sets of the theorem, the section of $j^\star\mathrm{Ger}(\mathscr{C})$ extends to a section of $\mathrm{Ger}(\mathscr{C})$ on an open neighborhood U, corresponding to a maximal subgerbe $\mathscr{G}$ of the restriction of $\mathscr{C}$ to U such that s is a section of the restriction of $\mathscr{G}$ to K. If the theorem holds for $\mathscr{G}$, then we obtain the desired extension of s. Therefore, we may assume that $\mathscr{C}$ is a gerbe.

We claim that if $\beta\colon \mathscr{C} \to \mathscr{C}'$ is a faithful morphism of stacks on X and the theorem holds for $\mathscr{C}'$, then the theorem holds for $\mathscr{C}$. Let $s \in (j^\star\mathscr{C})(K)$ be a section. By the theorem for $\mathscr{C}'$, the image of s in $(j^\star\mathscr{C}')(K)$ extends to a section of $\mathscr{C}'$ on an open neighborhood U. The stack of liftings $U\times_{\mathscr{C}'}\mathscr{C}$ is a sheaf by the faithfulness of β. Thus, by the known case of sheaves of the theorem, the section of $j^\star(U\times_{\mathscr{C}'}\mathscr{C}) \simeq K\times_{j^\star\mathscr{C}'} j^\star\mathscr{C}$

defined by s extends to a section of $U \times_{\mathscr{C}'} \mathscr{C}$ on an open neighborhood of K in U, which gives an extension of s as claimed.

Let X^δ be X with the discrete topology and let $\varepsilon\colon X^\delta \to X$ be the identity map. For any sheaf of sets $\mathscr{F}$ on X, the adjunction map $\mathscr{F} \to \varepsilon_\star\varepsilon^\star\mathscr{F}$ is a monomorphism. Since taking sheaves of morphisms commutes with $\varepsilon_\star$ and $\varepsilon^\star$, the adjunction morphism $\mathscr{C} \to \varepsilon_\star\varepsilon^\star\mathscr{C}$ is faithful. By the above claim, it is enough to prove the theorem for $\varepsilon_\star\varepsilon^\star\mathscr{C}$. The gerbe $\varepsilon^\star\mathscr{C}$ on X^δ is necessarily equivalent to the gerbe of G-torsors for a sheaf of groups G on X^δ. Then $\varepsilon_\star\varepsilon^\star\mathscr{C}$ is equivalent to the gerbe of $\varepsilon_\star G$-torsors, and it suffices to show that every $j^\star\varepsilon_\star G$-torsor on K is trivial. Since $\varepsilon_\star G$ is flabby, $j^\star\varepsilon_\star G$ is soft in cases (i) and (ii) by [SGA 4 $\mathrm{V^{bis}}$ 4.1.5] and [Godement, 1973, Théorème II.3.4.2] and flabby in case (iii) by [Godement, 1973, Théorème II.3.3.1, Corollaire 2]. We conclude by the following lemma. □

3.4. Lemma. — *Let G be a flabby sheaf of groups on a topological space X, or a soft sheaf of groups on a paracompact Hausdorff space X. Then any G-torsor P on X has a section.*

Proof. — Consider an open covering $(U_i)_{i\in I}$ such that $P(U_i)$ is nonempty for all $i \in I$.

In the flabby case, consider the set of pairs (J,σ), where $J \subset I$ and $\sigma \in P(\bigcup_{i\in J} U_i)$, ordered by extension. By Zorn's lemma, there exists a maximal element (J_0,σ_0). Let $U_0 = \bigcup_{i\in J_0} U_i$. Let $i \in I$. Choose $\sigma_i \in P(U_i)$. The element of $G(U_0 \cap U_i)$ carrying the restriction of σ_i to the restriction of σ_0 extends to an element $g_i \in G(U_i)$. Patching σ_0 and $\sigma_i g_i$ produces an extension of σ_0 to $U_0 \cup U_i$. By the maximality of J_0, we get $i \in J_0$. Therefore, $J_0 = I$ and $\sigma_0 \in P(X)$.

The soft case is similar. Since X is paracompact, we may assume $(U_i)_{i\in I}$ locally finite. By a usual lemma on normal spaces, there exists an open covering $(V_i)_{i\in I}$ with $\overline{V_i} \subset U_i$ for all $i \in I$. Consider the set of pairs (J,σ), where $J \subset I$ and σ is a section of P restricted to $\bigcup_{i\in J}\overline{V_i}$, ordered by extension. By local finiteness, the hypothesis of Zorn's lemma is verified. Thus there exists a maximal element (J_0,σ_0). Let $V = \bigcup_{i\in J_0}\overline{V_i}$, which is closed in X. Let $i \in I$. Choose $\sigma_i \in P(\overline{V_i})$. The element of $G(V \cap \overline{V_i})$ carrying the restriction of σ_i to the restriction of σ_0 extends to an element $g_i \in G(\overline{V_i})$. Patching σ_0 and $\sigma_i g_i$ produces an extension of σ_0 to $V \cup \overline{V_i}$. By the maximality of J_0, we get $i \in J_0$. Therefore, $J_0 = I$ and $\sigma_0 \in P(X)$. □